Hyperspheres in Fermat's Last Theorem

Kelvin Kian Loong Wong[a,*]

[a]*School of Electrical and Electronic Engineering, University of Adelaide, Adelaide, SA 5005, Australia*

Abstract

This paper provides a potential pathway to a formal simple proof of Fermat's Last Theorem. The geometrical formulations of n-dimensional hypergeometrical models in relation to Fermat's Last Theorem are presented. By imposing geometrical constraints pertaining to the spatial allowance of these hypersphere configurations, it can be shown that a violation of the constraints confirms the theorem for n equal to infinity to be true.

Résumé

Cet article fournit une voie potentielle à une preuve formelle simple du dernier théorème de Fermat. Les formulations géométriques des modèles d'hypersphère dimensionnels en relation avec le dernier théorème de Fermat sont présentées. En imposant des contraintes géométriques relatives à l'allocation spatiale de ces configurations d'hypersphère, on peut montrer qu'une violation des contraintes confirme que le théorème de n égal à l'infini est vrai.

Keywords: Fermat's Last Theorem, n-dimensional hypersphere, Hypershell, Geometrical model

Mots-clé: le dernier théorème de Fermat, hypersphère n-dimensionnels, hypershell, Modèle géométrique

2010 MSC: 11D41, 11D25, 97G30

1. Introduction

For almost three hundred years, mathematicians have attempted to prove Fermat's Last Theorem. First proposed by Pierre de Fermat, the beauty of this theorem lies in its short expression of a seemingly simple equation. The statement of Fermat's Last Theorem is a simple mathematical proposition. It was first posed by the French mathematician Pierre de Fermat back in the seventeenth century as follows.

*Corresponding author
Email address: kwong@eleceng.adelaide.edu.au (Kelvin Kian Loong Wong)

Theorem 1.1. *Given that x, y, z and n represent positive integers $\mathbb{Z}^+$*

$$\nexists\, (x, y, z, n) \in \mathbb{Z}^+ : x^n + y^n = z^n \text{ for } n > 2. \tag{1.1}$$

Simply, the theorem states that when n is a whole number greater than 2, no number scaled to the power of n can be expressed as the sum of two smaller numbers scaled to the power of n.

Mathematicians began to prove specific exponents using Fermat's technique of infinite descent based on its original form [1, 2]. Leonard Euler was credited as the first to prove the case $n = 3$ using this method [3, 4]. The first proof for $n = 4$ is by Fermat using the infinite descent method. Subsequent proofs for this dimension have been developed by a number of mathematicians, such as de Bessy [5] and Euler [6]. Then, proof for $n = 6$ by Kausler [7], proof for $n = 7$ by Lebesgue [8], proof for $n = 10$ by Kapferer [7], and proof for $n = 14$ by Dirichlet [9] appeared over time. However, specific exponents became difficult to prove as n increased.

The main challenge was to develop a general theorem for all cases of n. Sophie Germain was credited as the first to contribute significant work on the general theorem [10]. Her approach required two cases of Fermat's Last Theorem to be proven; however, this was ultimately unsuccessful. For the first case, she was able to apply her approach to every odd prime exponent less than 100 and this became known as Sophie Germain's theorem [11]. In 1997, the first case was proved true for all even n by Guy Terjanian [12] and in 1985, the first case was proved true for infinitely many odd primes n by Leonard Adleman, Roger Heath-Brown and Etienne Fouvry [13].

In 1847, Gabriel Lame attempted to prove Fermat's last theorem using the cyclotomic field [14]. However, his proof was unsuccessful as it involved an incorrect assumption that complex numbers can be factorised uniquely into prime numbers [15]. Building upon Lame's approach, Ernst Kummer was successful in developing a proof of Fermat's Last theorem for all regular prime numbers [16, 17]. However, he was unable to prove the theorem for irregular primes.

Computational methods were developed by the 20th century to extend Kummer's method to irregular prime numbers. By 1993, the theorem for all prime numbers where $n < 4 \times 10^6$ was solved computationally. [18].

However, all of the previous works do not qualify for a general proof based on all possible n up to infinity. There was a need to start looking at solving the theorem based on the form of descent on elliptic curves. In 1995, Andrew Wiles became the first mathematician to develop a general proof of Fermat's Last Theorem using the modularity theorem, which states that elliptic curves over the field of rational numbers are related to modular forms [19]. Prior to this, the relationship between the Fermat's Last theorem and the modularity theorem (formerly known as the Taniyama-Shimura conjecture) had been proven by Ken Ribet [20].

This paper presents a pathway towards a greatly simplified proof of the theorem for n using a contradiction of geometry within the modelling of an n dimensional space, based on the concept of a hypersphere and its volume equivalent hyperspherical shell (hereby denoted as a hypershell). The $x^n + y^n = z^n$ equation can be constructed using hypersphere geometries with some predefined structural conditions in a multi-dimensional space.

2. Definition of Hypersphere and Hypershell

The general expression of a hypersphere can build a general model of Theorem 1.1 for any value of n. The formulation for the volume of an n-dimensional hypersphere can be generalized [21]. Here, let V_n denote the mass (i.e., n-dimensional volume) of an n-hypersphere of radius r_0 and S_n be the hypersurface area of an n-sphere of unit radius, and V_s denote the volume of a hypershell of radius r_s and shell width w. It is to be noted that the surface area of a sphere has infinitesimal thickness, whereas the volume of a hypershell has a finite width. Therefore, the equations for both geometrical entities are not the same. However, if $w \to 0$, then $V_s = S_n$. The model configuration is shown by Figure 1.

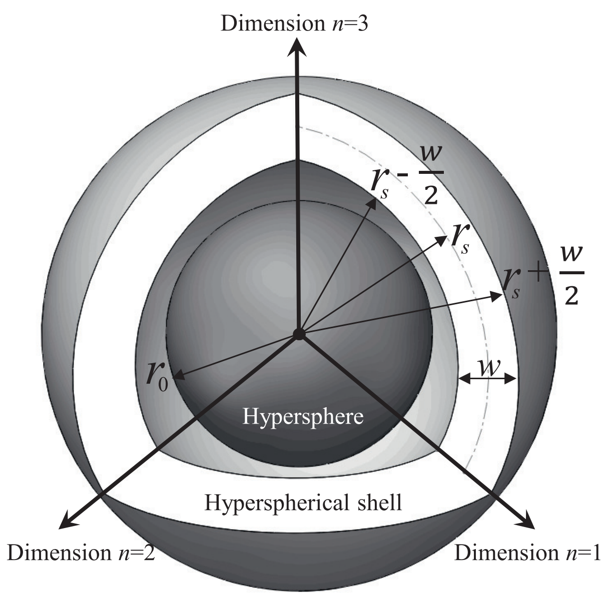

Figure 1: Model of hypersphere and hypershell in 3 dimensions based on geometrical parameters r_0, r_s, and w.

Definition 2.1. The surface area S_n and volume V_n of a hypersphere of general dimension n *cf.* [21, 22] are defined below as:

$$S_n = \begin{cases} \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{(n-2)!!} & \text{for } n \in \mathbb{O}^+ \\ \frac{2\pi^{\pi/2}}{\frac{1}{2}(n-1)!} & \text{for } n \in \mathbb{E}^+ \end{cases},$$

and

$$V_n = \frac{S_n r_h^n}{n}.$$

where $\mathbb{O}^+$ and $\mathbb{E}^+$ denote the set of positive odd and even integers respectively.

It should be noted that the double factorial !! involved in the equation for odd-dimensional volumes has been defined for odd integers $(2k+1)$ as $(2k+1)!! = 1 \times 3 \times 5 \times ... \times (2k-1) \times (2k+1)$. Based on Definition 2.1, the equations for a hypersphere and a hypershell will be able to be generated in Proposition 2.1 as follows. These equations are used in relation to Fermat's Last Theorem as stated in Theorem 1.1 to generate hypergeometrical models in relation to $x^n + y^n = z^n$.

Proposition 2.1. *Let V_s denote the volume of a hypershell with radius r_s and thickness w_s. Then,*

$$V_s = \frac{S_n}{n} \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(r_s^{n-(2k+1)}\right)\left(\frac{w}{2}\right)^{2k+1},$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases}.$$

Proof of Proposition 2.1. To compute the volume of a hypershell, formed between an external sphere V_e and inner sphere cavity V_i, subtract the volume of the inner cavity from that of its external structure, which is given as

$$V_s = V_e - V_i,$$

where

$$V_e = \frac{S_n}{n}\left(r_s + \frac{w}{2}\right)^n,$$
$$V_i = \frac{S_n}{n}\left(r_s - \frac{w}{2}\right)^n.$$

Case of odd n. The surface area and volume of a hypersphere is given by

$$S_n = \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{(n-2)!!}, V_n = \frac{2^{(n+1)/2}\pi^{(n-1)/2}r_0^n}{n(n-2)!!},$$

which results in the volume of the hypershell as

$$
\begin{aligned}
V_s &= \frac{S_n}{n}\left[\left(r_s+\frac{w}{2}\right)^n-\left(r_s-\frac{w}{2}\right)^n\right] \\
&= \frac{S_n}{n}\left[\begin{array}{l}\left(\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1+\binom{n}{2}\left(r_s^{n-2}\right)\left(\frac{w}{2}\right)^2+\ldots+\binom{n}{n-1}\left(r_s^{1}\right)\left(\frac{w}{2}\right)^{n-1}+\binom{n}{n}\left(r_s^{0}\right)\left(\frac{w}{2}\right)^n\right) \\ -\left(\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1-\binom{n}{2}\left(r_s^{n-2}\right)\left(\frac{w}{2}\right)^2+\ldots+\binom{n}{n-1}\left(r_s^{1}\right)\left(\frac{w}{2}\right)^{n-1}-\binom{n}{n}\left(r_s^{0}\right)\left(\frac{w}{2}\right)^n\right)\end{array}\right] \\
&= \frac{S_n}{n}\left(\begin{array}{c}2\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1+2\binom{n}{3}\left(r_s^{n-3}\right)\left(\frac{w}{2}\right)^3+ \\ \ldots+2\binom{n}{n-2}\left(r_s^{2}\right)\left(\frac{w}{2}\right)^{n-2}+2\binom{n}{n}\left(r_s^{0}\right)\left(\frac{w}{2}\right)^n\end{array}\right). \qquad (2.1)
\end{aligned}
$$

Case of even n. The surface area of volume of a hypersphere is given by

$$
S_n=\frac{2\pi^{n/2}}{\left(\frac{1}{2}n-1\right)!}, V_n=\frac{2\pi^{n/2}r_0^n}{n\left(\frac{1}{2}n-1\right)!},
$$

which results in the volume of the hypershell as

$$
\begin{aligned}
V_s &= \frac{S_n}{n}\left[\left(r_s+\frac{w}{2}\right)^n-\left(r_s-\frac{w}{2}\right)^n\right] \\
&= \frac{S_n}{n}\left[\begin{array}{l}\left(\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1+\binom{n}{2}\left(r_s^{n-2}\right)\left(\frac{w}{2}\right)^2+\ldots+\binom{n}{n-1}\left(r_s^{1}\right)\left(\frac{w}{2}\right)^{n-1}+\binom{n}{n}\left(r_s^{0}\right)\left(\frac{w}{2}\right)^n\right) \\ -\left(\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1-\binom{n}{2}\left(r_s^{n-2}\right)\left(\frac{w}{2}\right)^2+\ldots-\binom{n}{n-1}\left(r_s^{1}\right)\left(\frac{w}{2}\right)^{n-1}+\binom{n}{n}\left(r_s^{0}\right)\left(\frac{w}{2}\right)^n\right)\end{array}\right] \\
&= \frac{S_n}{n}\left(\begin{array}{c}2\binom{n}{1}\left(r_s^{n-1}\right)\left(\frac{w}{2}\right)^1+2\binom{n}{3}\left(r_s^{n-3}\right)\left(\frac{w}{2}\right)^3+2\binom{n}{5}\left(r_s^{n-5}\right)\left(\frac{w}{2}\right)^5+ \\ \ldots+2\binom{n}{n-3}\left(r_s^{3}\right)\left(\frac{w}{2}\right)^{n-3}+2\binom{n}{n-1}\left(r_s^{1}\right)\left(\frac{w}{2}\right)^{n-1}\end{array}\right). \qquad (2.2)
\end{aligned}
$$

Case of general n. Combining the solutions for odd n and even n,

$$
V_s=\frac{S_n}{n}\sum_{k=0}^{\frac{n-(\alpha+1)}{2}}2\binom{n}{2k+1}\left(r_s^{n-(2k+1)}\right)\left(\frac{w}{2}\right)^{2k+1},
$$

where

$$
\alpha=\begin{cases}0 & \text{for } n\in\mathbb{O}^+ \\ 1 & \text{for } n\in\mathbb{E}^+\end{cases}.
$$

This completes the proof of Proposition 2.1. □

3. Modelling Hypersphere in Relation to Theorem

In this section, a hyper-sphere model related to Theorem 1.1 is presented. This model relies solely on the re-expression of a hypersphere into a volume equivalent hypershell. Before creating the generalized multi-dimensional model, it can be demonstrated that the representation of Theorem 1.1 for n is based on n-dimensional hyperspheres.

The main result provided for is as follows.

Theorem 3.1. *There do not exist x, y and z in $x^n + y^n = z^n$ for $n \to \infty$, where $(x, y, z) \in \mathbb{Z}^+$.*

The expression of the governing equation for (x, y, z) in Theorem 3.1 into an equivalent equation for a single parameter λ is introduced. A new concept based on examining the range of λ for the different n values is presented. This may be achieved by volume equivalence of a hypersphere and a hypershell in an n-dimensional space described in Lemma 3.1.

Lemma 3.1. *Consider an n-dimensional hypersphere of radius r_0 and a hypershell of radius r_s and thickness w. Here, w is expressed as the product of λ and r_s. Then,*

$$\left(\frac{r_0}{r_s}\right)^n = \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1}, \tag{3.1}$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases},$$

and

$$w = \lambda r_s. \tag{3.2}$$

The constant λ is defined as the parameter for a feasible geometry. Taking this further, the representation of their geometrical volumes as x^n, y^n, and z^n appearing in Theorem 1.1 is based on

$$\begin{aligned} x &= r_0, \\ y &= \left(1 - \frac{\lambda}{2}\right) r_s, \\ z &= \left(1 + \frac{\lambda}{2}\right) r_s. \end{aligned} \tag{3.3}$$

Based on Eq. 3.2 and Eq. 3.3, $(\lambda, r_s) \in \mathbb{Q}$, whereby $\mathbb{Q}$ denotes rational numbers, such that the numerator of λ, and the denominator of r_s, each has a value of 2.

Note that the following geometrical constraints must be satisfied:

$$r_s - r_0 \geq 0, \tag{3.4}$$

where r_0, $w \in \mathbb{O}^+$.

Proof. To deduce this result, volume equivalence for a hypersphere and a hypershell of radii r_0 and r_s are applied respectively. Based on Proposition 2.1,

$$S_n = \begin{cases} \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{(n-2)!!} & \text{for } n \in \mathbb{O}^+ \\ \frac{2\pi^{\pi/2}}{\frac{1}{2}(n-1)!} & \text{for } n \in \mathbb{E}^+ \end{cases},$$

$$V_n = \frac{S_n r_0^n}{n}, \tag{3.5}$$

$$V_s = \frac{S_n}{n} \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(r_s^{n-(2k+1)}\right)\left(\frac{w}{2}\right)^{2k+1}. \tag{3.6}$$

The formulation of volume equivalence using Eq. 3.5 and Eq. 3.6 is performed as follows.

$$V_n = V_s,$$

$$\frac{S_n}{n} r_0^n = \frac{S_n}{n}\left(r_s + \frac{w}{2}\right)^n - \frac{S_n}{n}\left(r_s - \frac{w}{2}\right)^n, \tag{3.7}$$

$$r_o^n + \left(r_s - \frac{w}{2}\right)^n = \left(r_s + \frac{w}{2}\right)^n. \tag{3.8}$$

This essentially converts Eq. 3.7 and Eq. 3.8 into:

$$r_o^n = \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(r_s^{n-(2k+1)}\right)\left(\frac{w}{2}\right)^{2k+1}, \tag{3.9}$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases}.$$

Now, substituting Eq. 3.2 into Eq. 3.9,

$$r_o^n = \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1} r_s^n,$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases},$$

which then leads to Eq. 3.1.
This corresponds to Eq. 3.8 to give

$$\left(r_s + \frac{\lambda}{2} r_s\right)^n = \sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1} r_s^n + \left(r_s - \frac{\lambda}{2} r_s\right)^n. \tag{3.10}$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases} .$$

Note that Eq. 3.10 follows on from Eq. 3.8. Since each term in Eq. 3.8 and Eq. 3.10 gives an integer, $\left[r_0, \left(r_s - \frac{w}{2}\right), \left(r_s + \frac{w}{2}\right)\right]$ or $\left[r_0, \left(r_s - \frac{\lambda}{2}\right) r_s, \left(r_s + \frac{\lambda}{2}\right) r_s\right]$ can be represented as $[x, y, z]$ appearing in Theorem 1.1.

This completes the proof of Lemma 3.1. □

To complete the volume equivalence model, and to continue with determining the range of allowable λ values mentioned in Lemma 3.1, it is a necessity to establish the nature of numbers assigned to radius r_s and width w of a hypershell and to prove why it is so in Proposition 3.1.

Proposition 3.1. *Given that $w \in \mathbb{O}^+$, then for $\left(r_s - \frac{w}{2}\right)$ and $\left(r_s + \frac{w}{2}\right)$ in Eq. 3.7 to be integers, $r_s \in \mathbb{Q}$, where $\mathbb{Q}$ denotes a set of rational numbers that has a denominator of 2.*

Proof of Proposition 3.1. It follows that based on Eq. 3.2 where $w = \lambda r_s$, r_s being rational with a denominator of 2 requires that λ is rational with a numerator of 2. Define $r_s = \frac{\beta p}{2}$, $\lambda = \frac{2}{p}$, which implies that $p \in \mathbb{O}^+$, and $\beta \in \mathbb{N}$ and represents a constant (which is used as a factor to size the hypershell at discrete steps). Following Eq. 3.2, $w = \beta$. If $w \in \mathbb{O}^+$, then $\beta \in \mathbb{O}^+$, for which results r_s and λ to be rational. If $w \in \mathbb{E}^+$, then $\beta \in \mathbb{E}^+$, whereby $\mathbb{E}^+$ denotes even integers. This will result in $r_s \in \mathbb{N}$, which contradicts Proposition 3.1.

This completes the proof of Proposition 3.1. □

The last stage of our model construction is to establish that the determining constant for a feasible geometry becomes infinitesimal for an infinite number of dimensions.

Lemma 3.2. *Given that $(x, y, z, n) \in \mathbb{Z}^+$, it can be demonstrated that when $\lambda \to 0$ as $n \to \infty$, the hypergeometrical model results in diminishing values of the x, y, and z towards zero.*

Proof of Lemma 3.2. The range of λ is determined here, and then with the n exponent set to infinity.

Given our original constraint, $r_s - r_0 \geq 0$ and Lemma 3.1,

$$r_s - r_s \sqrt[n]{\sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1}} \geq 0,$$

$$r_s\left(1 - \sqrt[n]{\sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1}}\right) \geq 0,$$

$$\left(\sum_{k=0}^{\frac{n-(\alpha+1)}{2}} 2\binom{n}{2k+1}\left(\frac{\lambda}{2}\right)^{2k+1}\right) - 1 \leq 0, \tag{3.12}$$

where

$$\alpha = \begin{cases} 0 & \text{for } n \in \mathbb{O}^+ \\ 1 & \text{for } n \in \mathbb{E}^+ \end{cases}.$$

Here, as n increases, the value of λ decreases. As $n \to \infty$, $\lambda \to 0$. A solution fails to exist for Eq. 3.9 since substituting $\lambda = 0$ into $\left[r_0, \left(r_s - \frac{\lambda}{2}\right) r_s, \left(r_s + \frac{\lambda}{2}\right) r_s\right]$ generates $(x, y, z) = (0, 0, 0)$ in the expression

$$x^{\infty}_{\lambda \to 0} + y^{\infty}_{\lambda \to 0} = z^{\infty}_{\lambda \to 0}.$$

However, x, y, z are defined as positive integers, hence demonstrating that this is not a valid solution.

This completes the proof of Lemma 3.2. □

The validity of the hypergeometrical model can be tested for number of dimensions equal to two, as shown in Remark 3.1.

Remark 3.1. Given that $(x, y, z, n) \in \mathbb{Z}^+$, where the set (x, y, z) is piecewise coprime, $x^n + y^n = z^n$ is true for $n = 2$. Based on Lemma 3.1 for the case of $n = 2$,

$$r_0 = \sqrt{2 r_s w} = \sqrt{2\lambda} r_s,$$
$$\left(x^2, y^2, z^2\right) = \left[r_0^2, \left(r_s - \frac{w}{2}\right)^2, \left(r_s + \frac{w}{2}\right)^2\right].$$

Given our original constraint, $r_s - r_0 \geq 0$, the range of acceptable λ values can obtained based on Eq. 3.12:

$$0 < \lambda \leq \frac{1}{2}.$$

For the case of $n = 2$, assign $w = \lambda r_s = 1$ to give $\lambda = \frac{2}{9}$, and $r_s = \frac{9}{2}$. Then, since $r_0 = \frac{2}{3}\ r_s = 3$, $(x, y, z) = (3, 4, 5)$, which gives

$$3^2 + 4^2 = 5^2.$$

Another set of possible solution is $\lambda = \frac{2}{25}$, and $r_s = \frac{25}{2}$, $r_0 = \frac{2}{5}\, r_s = 5$, which leads to $(x, y, z) = (5, 12, 13)$ to give

$$5^2 + 12^2 = 13^2.$$

Assigning $w = 3$ leads to $\lambda = \frac{2}{9}$ and $r_s = \frac{27}{2}$, $r_0 = \frac{2}{3}\, r_s = 9$, and $(x, y, z) = (9, 12, 15)$ to give

$$9^2 + 12^2 = 15^2.$$

Another possible set based on $w = 9$ is $\lambda = \frac{2}{9}$ and $r_s = \frac{81}{2}$, $r_0 = \frac{2}{3}\, r_s = 27$, and $(x, y, z) = (27, 36, 45)$ to give

$$27^2 + 36^2 = 45^2.$$

Then, consider a series of shell thickness, such that $w \in \mathbb{O}^+$, leading to an infinite number of solutions for (x, y, z) and hence proving $x^n + y^n = z^n$ to be true for $n = 2$.

Next, the model is tested for other values of dimension numbers as in Remark 3.2.

Remark 3.2. For $n > 2$ dimensions, if and only if λ adopts an integer value of 2 can feasible solutions be obtained for Eq. 3.12, i.e. $\lambda = 2$ can give $r_0 = 2r_s$, and $w = 2r_s$, but this leads to an original hypersphere that is twice larger than its volume equivalent hypershell that has a thickness that is twice the shell radius. This presents a shell that is warped into the original hypersphere being smaller than itself, which violates our required geometrical constraints (i.e. $r_0 < r_s$, and $w < r_s$). Then, $(x, y, z) = \left[r_0, \left(1 - \frac{\lambda}{2}\right) r_s, \left(1 + \frac{\lambda}{2}\right) r_s\right] = (r_0, 0, 0)$. In this case, $(y, z) = 0$, and conflicts with the requirement that x, y and z are positive integers. To achieve a feasible geometry, the allowable range of λ in Figure 2 has to be adopted, and λ has to be a rational number. If $\frac{r_0}{r_s}$ in Eq. 3.1 is not a rational number, the failed geometry leads to w, r_0, and r_s being non-discrete, which implies that r_0, $\left(r_s - \frac{w}{2}\right)$, and $\left(r_s + \frac{w}{2}\right)$ from Eq. 3.8, which represents x, y, and z in (1.1), are irrational values.

Proof of Theorem 3.1. Based on Lemma 3.2, as number of dimensions, $n \to \infty$, then the governing parameter for the size of the hypersphere and hypershell, $\lambda \to 0$. This is verified by the graph of λ versus n (Figure 2, generated by program listed in Appendix A).

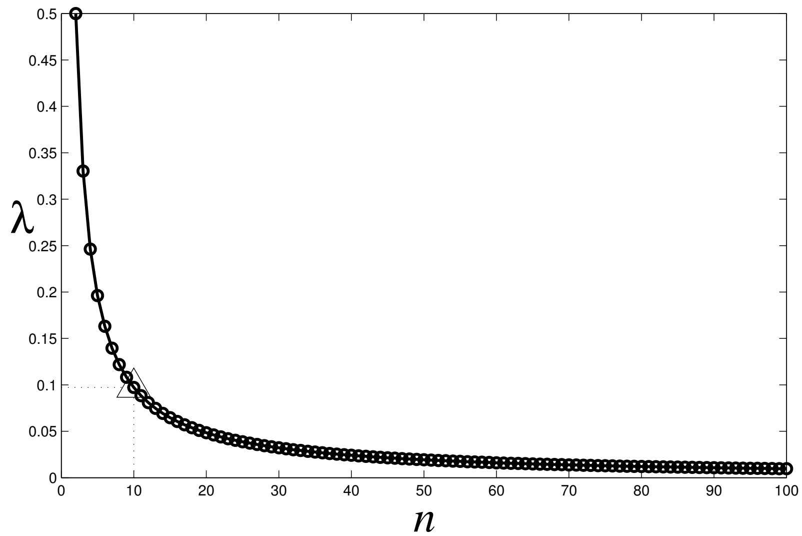

Figure 2: Graph of geometrical tolerance λ versus dimension n.

Then, w and r_0 assumes the value of 0 based on Eq. 3.1 and Eq. 3.2. Since for a feasible geometry to exist, $w > 0$ and $r_0 > 0$, therefore, no solution exists if $n \to \infty$. On the contrary, the range of λ is infinite for $n = 1$ when the number of dimension is reduced until the sphere is a dot in space.

This completes the proof of Theorem 3.1. □

Remark 3.3. The point of inflexion for λ versus n occurs at $n = 10$ as highlighted by the triangle in Figure 2. This asymptotic result is verified by plotting a graph of second derivative of λ with respect to n for n values up to 100. The curvature κ is computed as

$$\kappa = \frac{\left|\frac{d^2\lambda}{dn^2}\right|}{\left(1 + \left(\frac{d\lambda}{dn}\right)^2\right)^{3/2}}.$$

Then from Figure 3, it can be seen that the gradient becomes 0 in the graph of curvature versus geometrical tolerance at $n = 10$. This denotes the start of change in κ for the variation of λ in a space when the 10^{th} dimension is reached.

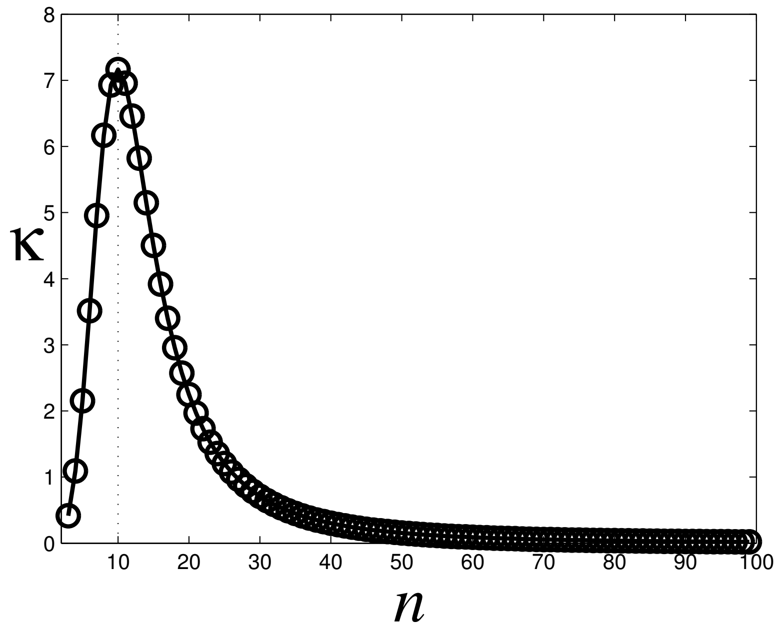

Figure 3: Graph of curvature κ versus dimension n.

The results may mean that as the number of dimensions of the hypersphere is increased, geometrical tolerance for construction of the hypergeometrical model begins to decrease rapidly until a point where it started to relax and give less accelerating tolerance to accommodate the hypergeometrical modelling.

It is well established that the mass of a hypersphere of fixed radius into the realm of high dimensional hypergeometry experience an inflexion point whereby its volume increment suddenly changed direction counterintuitively to become a decrement (*cf.* [22]). The unique behaviour of volume changing with space dimensions will affect the hypergeometrical modelling. The λ tolerance relaxation appears to be associated with this inflexion point but occurring at $n = 10$ instead. In this regard, the geometrical tolerance pertains to modelling of a hypersphere and its volume equivalent hypershell, and not a standard hypersphere of fixed radius. One may perceive a hypershell as a hypersphere that has a smaller hyperspherical core subtracted from within it.

Given that r is any positive integer, consider the case of hypersphere with an arbitrary external radius of r units, and for hypershell of shell radius $(r - 0.5)$ unit and width 1 unit. It is to be noted that that both structures have the same external radius r units, but the hypershell possesses a hollow spherical space having internal radius $(r - 1)$ units extended from its core; despite the structural difference, both their asymptotic points for volume variation coincide.

The asymptotes for volume V versus dimension n align at the same position for hypershell and non-hollow hypersphere such that their shapes of volume change are similar when both their external radii are the same. Figure 4 shows volumes of arbitrary hypersphere and hypershell versus dimension based on fixed radius $r = 2$ units, which illustrates that the different hypergeometries give rise to asymptotic points of volume variation at the same dimension point, i.e. $n = 24$. However, this does not imply that the volumes of these two different structures are equal.

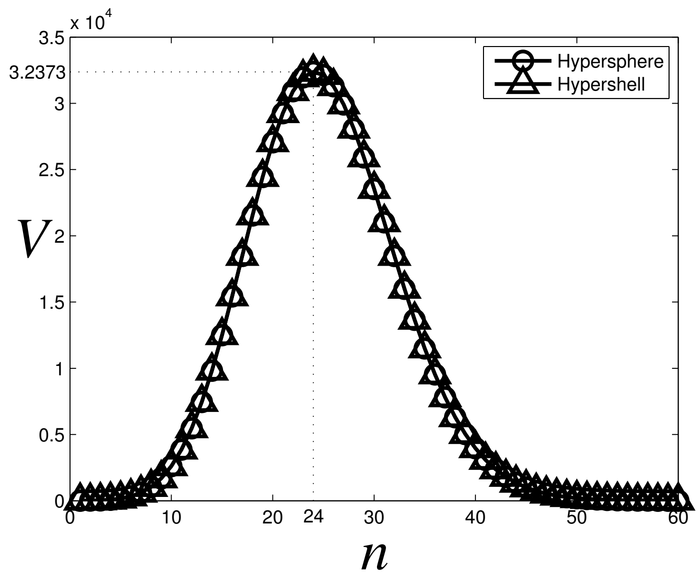

Figure 4: Graph of volume V versus dimension n for arbitrary hypersphere (with external radius 2 units) and hypershell (with external radius 2 units, and internal radius 1 unit).

4. Conclusion

Geometrical concepts are introduced into Fermat's Last Theorem by expressing volumetric equivalence of a hypersphere and a hypershell in n-dimensional space. Failing to define non-root values for the geometry of the hypersphere confirms that it is impossible to solve for infinite number of dimensions n in the theorem. Specifically, for $n \to \infty$, the margins for allowing the solution becomes infinitesimal, and a legitimate geometry fails to exist. There also exists a point of inflexion where the rate of decrease of this margin suddenly decelerates due to the counterintuitive decrease of volume for a hypersphere or hypershell entering into its realm of high dimensional hypergeometry.

References

[1] S. Dolan, Fermat's method of descente infinie, Mathematical Gazette 95 (2011) 269–271.

[2] M. Grant, Descending to the irrational, Mathematical Gazette 83 (1999) 263–267.

[3] L. Euler, Vollständige anleitung zur algebra, Royal Acaemy of Sciences, St Petersburg, 1770.

[4] J. J. Macys, On Euler's hypothetical proof, Mathematical Notes 82 (2007) 352–356.

[5] F. de Bessy, Traitè des Triangles Rectangles en Nombres, Reprinted in Mèm. Acad. Roy. Sci., 5, 1666-1699 (1729), 1676.

[6] L. Euler, Theorematum quorundam arithmeticorum demonstrationes, Comm. Acad. Sci. Petrop. 10 (1738) 125–146.

[7] H. Kapferer, Beweis des Fernatschen Satzes für die Exponenten 6 und 10, Arch. Math. Phys 21 (1913) 143–146.

[8] V. Lebesgue, Démonstration de l'impossibilité de résoudre l'équation $x^7 + y^7 + z^7 = 0$ en nombres entiers, J. Math. Pures Appl. 5 (1840) 348–349.

[9] P. Dirichlet, Démonstration du théorème de Fermat pour le cas des 14e puissances, J. Reine Angew. Math. 9 (1832) 390–393.

[10] A. D. Centina, Unpublished manuscripts of Sophie Germain and a revaluation of her work on Fermat's Last Theorem, Arch. Hist. Exact Sci. 62 (2008) 349–392.

[11] A. Legendre, Recherches sur quelques objets d'analyse indéterminée et particulièrement sur le théorème de Fermat, Mém. Acad. Roy. des Sciences de l'Institut de France 6 (1823) 1–60.

[12] G. Terjanian, Sur l'équation $x^2p + y^2p = z^2p$, Comptes rendus hebdomadaires des séances de l'Académie des sciences. Série a et B 285 (1977) 973–975.

[13] L. Adleman, D. Heath-Brown, The first case of Fermat's last theorem, Inventiones Mathematicae 79 (1985) 409–416.

[14] G. Lame, Demonstration generale du theoreme de Fermat sur l'impossibilite en nombres entiers de l'equation $2{+}yn = z$, C. R. Acad. Sci. Paris 24 (1847) 310–314.

[15] P. Ribenboim, 13 Lectures on Fermat's Last Theorem, Springer, 1979.

[16] H. M. Edwards, Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory, Springer, 1977.

[17] E. E. Kummer, Extrait d'une lettre de M. Kummer a M. Liouville, J. Math Pures et Appl. 12 (1847) 136.

[18] J. Buhler, R. Crandell, R. Ernvall, T. Metsänkylä, Irregular primes and cyclotomic invariants to four million, Math. Comp. America Mathematical Society. 61 (1993) 151–1552.

[19] A. Wiles, Modular elliptic curves and Fermat's Last Theorem, Annals of Mathematics 141 (1995) 443–551.

[20] K. Ribet, On modular representaitons of Gal(Q/Q) arising from modular forms, Inventiones Mathematicae 100 (1990) 432.

[21] R. P. Harold, The volumes of unit n-ball, Mathematics Magazine 86 (2013) 270–274.

[22] M. H. Peters, On the Shrinking Volume of the Hypersphere, The College Mathematics Journal 46 (2015) 178–180.

Appendix A- Program source code to plot λ versus n

Matlab program graphical plot demonstrating that as dimension n (set to be 1000 as a numerical approximation of an infinite n), then the parameter governing a feasible hypersphere and hypershell geometry $\lambda \to 0$ (note that λ is denoted as lambda in the source code). The original hypersphere shrinks to becoming a void, and the hyperspherical shell thickness approaches zero.

```
n_range=2:1000;       % Choose the range of n
for n=n_range
    alpha = ~rem(n,2);
    lower_limit = 0;
    upper_limit = 0.5*(n-(alpha+1));
    syms x                 % Create a symbolic variable x
    f=0;
    %%%%%%%%%%%%%%% Determine the function based on n %%%%%%%%%%%%%%%%%%
    for i=lower_limit+1:upper_limit+1
        k=i-1;
        A(i) = 2*nchoosek(n,2*k+1)*(0.5)^(2*k+1);
        B(i) = (2*k+1);
        f=eval(['f' '+' 'symfun(A(i)*x^B(i), [x])']);
    end
    f=eval(['f' '-' '1']);
    fprintf('\n\nThe function is:'); disp(f);
    %%%%%%%%%%%%%%% Calculate the root of the function %%%%%%%%%%%%%%%%%%
    xRoot = 0.5;                            % Initial guess for the root
    g = x-f/diff(f);                        % Create a Newton-Raphson approximation
    xNew = double(subs(g,'x',xRoot));       % Refined guess
    i=1;
    while abs(xNew-xRoot) > 1e-10           % Loop while they differ by more than 1e-10
        xRoot = xNew;
        xNew = double(subs(g,'x',xRoot));
        i=i+1;
    end
    lambda(n-1) = xNew;                      % Start from index=1 of lambda
    fprintf('Approximate Root is %.5f',lambda(n-1));
end
    %%%%%%%%%%%%%%% Plot of lambda versus n %%%%%%%%%%%%%%%%%%
figure,
plot(n_range, lambda,'o-','LineWidth',2,'MarkerSize',10,'color','k');
h1 =xlabel('n');
h2 =ylabel('\lambda');
set(h1, 'FontName', 'Times New Roman', 'FontAngle', 'italic', 'FontSize', 30);
set(h2, 'FontName', 'Times New Roman', 'FontAngle', 'italic', 'FontSize', 30);
set(h2, 'Rotation', 0);
```

Supplementary Material: hypergeometrical models for dimensions of $n = [2, 5]$

General steps	Procedural steps	*n=2*	*n=3*
Constraints	Define geometrical constraints.	$r_s - r_0 \geq 0$, where $(r_0, w) \in \mathbb{O}$, and $r_s \in \mathbb{Q}$.	
Volume of hypersphere and hypershell	Define volume equations (can be referenced in Definition 2.1).	$V_0 = \pi r_0^2$, $V_s = 2\pi r_s w$.	$V_0 = \frac{4}{3}\pi r_0^3$, $V_s = 4\pi r_s^2 w + \frac{\pi w^3}{3}$.
Conservation of volume	Perform volume equivalence, where $V_0 = V_s$.	$\pi r_0^2 = \pi\left(r_s + \frac{w}{2}\right)^2 - \pi\left(r_s - \frac{w}{2}\right)^2$, $\left(r_s + \frac{1}{2}w\right)^2 = r_0^2 + \left(r_s - \frac{1}{2}w\right)^2$.	$\frac{4}{3}\pi r_0^3 = \frac{4}{3}\pi\left(r_s + \frac{w}{2}\right)^3 - \frac{4}{3}\pi\left(r_s - \frac{w}{2}\right)^3$, $\left(r_s + \frac{1}{2}w\right)^3 = r_0^3 + \left(r_s - \frac{1}{2}w\right)^3$.
Expansion of equation and reformatting	Expand LHS and RHS of the power terms to summarize volume equivalence equation.	$r_0^2 = 2\, r_s w$.	$r_0^3 = 3r_s^2 w + \frac{w^3}{4}$.
Expressing thickness of shell as a function of its radius	Substitute $w = \lambda r_s$, where $(\lambda, r_s) \in \mathbb{Q}$.	$r_0 = \sqrt{2\lambda} r_s$.	$r_0 = \sqrt[3]{\left(3\lambda + \frac{\lambda^3}{4}\right)} r_s$.
	Express r_0 and w in terms of λ and r_s.	$\left(r_s + \frac{1}{2}\lambda r_s\right)^2 = 2\lambda r_s^2 + \left(r_s - \frac{1}{2}\lambda r_s\right)^2$.	$\left(r_s + \frac{1}{2}\lambda r_s\right)^3 = \left(3\lambda + \frac{\lambda^3}{4}\right) r_s^3 + \left(r_s - \frac{1}{2}\lambda r_s\right)^3$.
Determining range of λ	Given our original constraint: $r_s - r_0 \geq 0$, define the inequality equation governing λ.	$r_s - \sqrt{2\lambda} r_s \geq 0$, $1 - \sqrt{2\lambda} \geq 0$.	$r_s - \sqrt[3]{\left(3\lambda + \frac{\lambda^3}{4}\right)} r_s \geq 0$. $\lambda^3 + 12\lambda - 4 \leq 0$.
	For a viable geometry to exist, define the upper and lower bounds of λ.	$0 < \lambda \leq \frac{1}{2}$.	$0 < \lambda \leq \sqrt[3]{2 + 2\sqrt{17}} + \sqrt[3]{2 - 2\sqrt{17}}$.

General steps	Procedural steps	*n=4*	*n=5*
Constraints	Define geometrical constraints.	$r_s - r_0 \geq 0$, where $(r_0, w) \in \mathbb{O}$, and $r_s \in \mathbb{Q}$.	
Volume of hypersphere and hypershell	Define volume equations (can be referenced in Definition 2.1).	$V_0 = \frac{1}{2}\pi^2 r^4$, $V_s = \frac{1}{2}\pi^2\left(4wr_s^3 + w^3 r_s\right)$.	$V_0 = \frac{8}{15}\pi^2 r^5$, $V_s = \frac{8}{15}\pi^2\left(5r_s^4 w + \frac{5}{2}r_s^2 w^3 + \frac{1}{16}w^5\right)$.
Conservation of volume	Perform volume equivalence, where $V_0 = V_s$.	$\frac{1}{2}\pi^2 r_0^4 = \frac{1}{2}\pi^2\left(r_s + \frac{w}{2}\right)^4 - \frac{1}{2}\pi^2\left(r_s - \frac{w}{2}\right)^4$, $\left(r_s + \frac{1}{2}w\right)^4 = r_0^4 + \left(r_s - \frac{1}{2}w\right)^4$.	$\frac{8}{15}\pi^2 r_0^5 = \frac{8}{15}\pi^2\left(r_s + \frac{w}{2}\right)^5 - \frac{8}{15}\pi^2\left(r_s - \frac{w}{2}\right)^5$, $\left(r_s + \frac{1}{2}w\right)^5 = r_0^5 + \left(r_s - \frac{1}{2}w\right)^5$.
Expansion of equation and reformatting	Expand LHS and RHS of the power terms to summarize volume equivalence equation.	$r_0^4 = \left(4wr_s^3 + w^3 r_s\right)$.	$r_0^5 = 5r_s^4 w + \frac{5}{2}r_s^2 w^3 + \frac{1}{16}w^5$.
Expressing thickness of shell as a function of its radius	Substitute $w = \lambda r_s$, where $(\lambda, r_s) \in \mathbb{Q}$.	$r_0^4 = r_s^4(4\lambda + \lambda^3)$.	$r_0^5 = r_s^5(5\lambda + \frac{5}{2}\lambda^3 + \frac{1}{16}\lambda^5)$.
	Express r_0 and w in terms of λ and r_s.	$\left(r_s + \frac{1}{2}\lambda r_s\right)^4 = \left((4\lambda + \lambda^3) r_s^4\right) + \left(r_s - \frac{1}{2}\lambda r_s\right)^4$.	$\left(r_s + \frac{1}{2}\lambda r_s\right)^5 = \left(\left(5\lambda + \frac{5}{2}\lambda^3 + \frac{1}{16}\lambda^5\right) r_s^5\right) + \left(r_s - \frac{1}{2}\lambda r_s\right)^5$
Determining range of λ	Given our original constraint: $r_s - r_0 \geq 0$, define the inequality equation governing λ.	$r_s - r_s\sqrt[4]{(4\lambda + \lambda^3)} \geq 0$, $\lambda^3 + 4\lambda - 1 \leq 0$.	$r_0 - r_s\sqrt[5]{5\lambda + \frac{5}{2}\lambda^3 + \frac{1}{16}\lambda^5} \geq 0$, $\lambda^5 + 40\lambda^3 + 80\lambda - 16 \leq 0$.
	For a viable geometry to exist, define the upper and lower bounds of λ.	$0 < \lambda \leq \sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{283}{27}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{283}{27}}}$.	$0 < \lambda \leq 0.1962189$.