Emergence of Space in Quantum Shape Kinematics

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A model universe is analyzed with N protons and electrons where there are electromagnetic and spin interactions in the Hamiltonian is investigated in the context of quantum shape kinematics. We have found that quantum shape space exists for $N \geq 4$ particles and has $2N-7$ functional degrees of freedom in the case of spin-1/2 particles. The emergence of space is associated with non-vanishing expectation value $\langle L^2 \rangle$. We have shown that for odd N space always emerges, and for large even N space almost always emerges because $\langle L^2 \rangle \neq 0$ for almost all states. In the limit $N \to \infty$ the density of states that yields $\langle L^2 \rangle = 0$ vanishes. Therefore we conclude that the space is almost always emergent in quantum shape kinematics.

INTRODUCTION

Shape dynamics [1, 2, 4, 5, 8, 10, 11] is a fully relational theory of gravitation. In the case of N-body problem, it states that only the relative distances and angles between them are dynamical [1]. In this scenario universe cannot have a non-vanishing angular momentum, otherwise it would define an absolute space in which the universe is rotating [2]. Similarly non-vanishing total energy implies an external absolute time according to which the universe evolves, therefore we require total energy to vanish [2]. Another constraint comes from working in the center of mass frame, the total momentum should be zero. Ref. [3] also puts another constraint on the system, *i.e.* the vanishing of the dilational momentum: $\sum_a \mathbf{r}^a \cdot \mathbf{p}^a = 0$. This is required for scale invariance [3], however we will not impose it. Overall, we have three constraints:

$$H = \sum_a E^a = 0 \tag{1}$$

$$\mathbf{P} = \sum_a \mathbf{p}^a = 0 \tag{2}$$

$$\mathbf{L} = \sum_a \mathbf{r}^a \times \mathbf{p}^a = 0 \tag{3}$$

There is a need to calculate Poisson brackets of constraints with each other in order to classify them. This distinction will be important when the theory is quantized. The list is as follows:

$$\{P_i, P_j\} = 0, \tag{4}$$

$$\{L_i, L_j\} = \varepsilon^{ijk} L_k, \tag{5}$$

$$\{L_i, P_j\} = \varepsilon^{ijk} P_k, \tag{6}$$

$$\{H, P_i\} = 0, \tag{7}$$

$$\{H, L_i\} = 0. \tag{8}$$

The results of these commutators are either zero or another constraint. Hence they vanish weakly. Therefore all the constraints are first class in the terminology of Dirac [7]. We did the Dirac analysis in order to show that our system is consistent and have seen that it is well defined.

QUANTIZATION OF THE MODEL

The model is quantized by promoting positions and momenta to quantum operators. The Poisson bracket $\{\cdot,\cdot\}$ is mapped to $i\hbar[\cdot,\cdot]$. Between position and momenta is the following expression:

$$[\hat{r}_i^a, \hat{p}_j^b] = i\hbar \delta_b^a \delta_j^i. \tag{9}$$

Momenta are represented by operators, $\hat{p}_i^a = -i\hbar\partial/\partial r_i^a$. It is time to consider what happens to constraints in this case. As the readers can verify easily, the constraint algebra survives the quantization. In particular there is no anomaly. In the presence of dilational momentum constraint ref. [3] reports the existence of scale anomaly. It is then argued in [3] that this anomaly may give rise to a gravitational arrow of time. However, the arrow of time is outside our scope in this study.

At the quantum level the constraints become operators acting on the quantum state of the system. For example the Hamiltonian constraint becomes:

$$H|\psi\rangle = 0, \tag{10}$$

where H includes kinetic terms, potential terms and angular momentum couplings. We see that we have obtained a time independent Schrödinger equation. Wavefunctions do not evolve in time and are static. This is similar to what happens with the Wheeler-DeWitt equation [6].

The momentum and angular momentum constraints, on the other hand, become:

$$\mathbf{P}|\psi\rangle = -i\hbar \sum_a \nabla_a|\psi\rangle = 0, \tag{11}$$

$$\mathbf{J}|\psi\rangle = -i\hbar \sum_a \mathbf{r}^a \times \nabla_a|\psi\rangle + \sum_a \mathbf{S}_a|\psi\rangle = 0, \tag{12}$$

where the spin operator for each particle $\mathbf{S}_a$ is added. We interpret equations (10) (11) and (12) as operator equations that determines the allowed kinematic states of the system.

EMERGENCE OF SPACE

The Hamiltonian constraint, momentum constraint and angular momentum constraint add up to total of 7 constraints in three dimensional space. For a single particle the momentum constraint implies that $|\psi\rangle$ is a constant spinor. Because it has no orbital angular momentum $\mathbf{L}|\psi\rangle = 0$, however $\mathbf{S}|\psi\rangle$ cannot vanish. Hence the theory disallows the existence of a single particle in the universe. In fact the theory disallows the existence of two and three particles as well, because there are at most six functional degrees of freedom. This is in contrast to classical shape dynamics, because there the shape space is defined for $N \geq 3$ [1]. The source of this mismatch is because in classical theory every particle is assigned a three dimensional vector where in the quantum theory every particle is assigned a *two* dimensional spinor.

For more than three particles quantum shape states are allowed since there are now at least eight functional degrees freedom. Quantum shape space is $2N-7$ dimensional, where as the classical shape space is $3N-7$ dimensional. Of course the dimension of the quantum shape space changes with species other than spin-1/2 particles, the result we give is for spin-1/2 particles.

Let $|\psi\rangle$ be an element of quantum shape space. Due to angular momentum constraint $\mathbf{L}|\psi\rangle = -\mathbf{S}|\psi\rangle$ holds, which implies $\langle L^2\rangle = \langle S^2\rangle$ where $\mathbf{L} = \sum_a \mathbf{L}_a, \mathbf{S} = \sum_a \mathbf{S}_a$. Hence for states $|\psi\rangle$ such that $\langle S^2\rangle \neq 0$, there is a non-vanishing classical+orbital angular momentum and absolute space emerges in which the universe has a non-vanishing classical angular momentum. We now focus our attention to existence of states that allows the emergence of space.

The spin Hilbert space is $\bigotimes_i^N 1/2$. For odd N there is no singlet state and the expectation value of $\langle S^2\rangle$ is always positive. For even N, there are there are singlet state(s) whose number is given by [9]:

$$f_0^N = \frac{N!}{(N/2+1)!(N/2)!}. \tag{13}$$

The density of singlet states which is the ratio of the number of singlet states to dimensionality of the spin Hilbert space (2^N) is:

$$f_0^N = \frac{N!}{2^N(N/2+1)!(N/2)!}, \tag{14}$$

whose limit is zero as $N \to \infty$. Therefore, for $N = 10^{80}$ as in our universe, space emerges for almost all the states.

CONCLUSION

In this study, quantum shape kinematics is studied for N electrons and protons in an otherwise empty universe for a general Hamiltonian that may include all types of conceivable interactions. For $N = 1, 2, 3$ no solution is found, hence there is no quantum shape space for these numbers of particles. Since the quantum shape space has $2N-7$ free functional degrees of freedom for spin-1/2 particles, solutions exist for $N \geq 4$.

The dimensionality of the quantum shape space crucially depends on the type of existing particle species. For scalars, quantum shape space has $N-7$ functional degrees of freedom. For a mixture of N spin-1/2 particles and M scalar particles, quantum shape space has $2N+M-7$ free functions.

The emergence of space is associated with non-vanishing value of classical angular momentum of the universe. Because if so, there exists a *container*, i.e. absolute space, in which universe has a classical angular momentum and is rotating. We have shown that for odd N space always emerges, and for large even N space almost always emerges because $\langle L^2\rangle \neq 0$ for almost all states. In the limit $N \to \infty$ the density of states that yields $\langle L^2\rangle = 0$ vanishes. Hence the space is almost always emergent.

Currently, the solutions are static and there is no time. The main result in this paper concerns the emergence of space, not time. Interesting phenomena can be expected to occur for solution of the to-be-developed theory of quantum shape *dynamics*. Because once the evolution of states becomes available, the expectation value $\langle L^2\rangle$ may vanish at certain times.

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