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ON INPOLARS

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Abstract: The aim of this article would be to show planar (whether polar, Cartesian or parametric) functions from a different, implicit viewpoint, hence the term *inpolars* (inpolar curves). The whole set of brand new planar curves can be seen from that perspective. Their generic mechanism is the so called *inpolar transformation* as well as its *inpolar inversion*. One entirely new geometric system is defined this way.

Introduction

The classical explicit form of the polar functions reads

$$r = f(\theta), \qquad \text{(I.1)}$$

where angle θ is an independent variable and radius r is dependent one. Thus, we often write $r(\theta)$. This, of course, always represents a planar curve. The other way around, let us assume a proposition like this one, i.e.

$$f(r) = \theta \quad . \qquad \text{(I.2)}$$

This form, however, makes a change and now we get a new dependency $\theta(r)$. It seems that relation (I.2) goes beyond our usual geometric experience. We may call it, e.g., *inpolar* form[1]. Sometimes it can be transformed into the explicit polar form $r(\theta)$. That is possible if and only if exists the inverse function (in explicit form)

$$r = f^{-1}(\theta) \quad . \qquad \text{(I.3)}$$

The proposed transformation will be discussed in more details (see the next chapter). Afterwards, in order to fully picture the basic idea, we will exploit the insight over different known as well as less known curves going through all the three kinds (whether polar, parametric or Cartesian) of planar functions.

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1 Likewise, throughout the article the prefix *in* will signify inpolar curves (e.g., *inspirals*, *inconics* etc.) or *inpolars.*

1. The inpolar transformation

Regarding the above, let us notice a symmetry (bijection) that each and every polar function possesses its *inpolar* picture as well. Moreover, we will see that the same is true for the entire set of planar functions, too. Next, observing a family of functions and related operation product of functions (∘), we may say that an inpolar function (*f'*) is inverse to a polar one, i.e.

$$f' \circ f(\theta)=\theta \quad . \qquad (1.1)$$

We may also conclude that the neutral element of the product is the special case of Archimedes' spiral *r=θ,* where *a*=1. Furthermore, both polar and inpolar functions are mutually identical when the condition

$$f(f(\theta))=\theta \qquad (1.2)$$

is fulfilled. Such is the case of hyperbolic spiral *r=a/θ.*

Polar functions can also be represented by the parametric form via well known relations

$$x=r*\cos(t) \qquad (1.3)$$
$$y=r*\sin(t) \quad .$$

In order to fulfill the basic idea (I.2) we now have to make the next inversion

$$x=t*\cos(r) \qquad (1.4)$$
$$y=t*\sin(r) \,,$$

which we rather call the *parametric inversion*[2]. In a way, (1.4) can be seen as the central point of this entire work because that in fact is the general inpolar mechanism whatsoever. Through the next few chapters we will present both the natural development and the geometrical application of the idea.

2. Inspirals

Spirals are the most natural curves to present in polar form. Partly because of that fact and partly from heuristic reasons of this work itself, we shell firstly explore few such curves, and all in the light of our basic premises.

Most well known spirals (polar form *r*(*θ*)) do possess an exact inversion which also would be their inpolar picture (*inspiral*). In the case of so called algebraic spirals that is pretty obvious by itself. Even for the logarithmic spiral $r=e^{\theta}$ that would be known $r=\ln(\theta)$, using *a,b*=1.

As for example, let us here present the *hyperspiral* $r=e^{(1/\theta)}$ [1], and its *inspiral* $e^{(1/r)}=\theta$, *a=1, b=1*. It is still an inverse form, i.e. $r=1/\ln(\theta)$ (see Fig. 1 below).

2 In GeoGebra solution reads Curve(t*cos(f(t)), t*sin(f(t)), t, t1, t2).

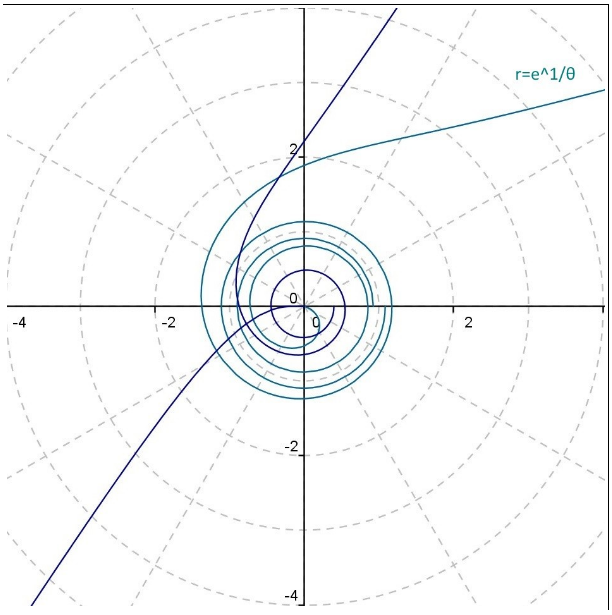

Fig. 1 *The inverse (inpolar) hyperspiral*

However, the most interesting case, in regard of the basic idea, would be such a spiraling curve which does not have its inverse function at all. We propose here a highly transcendental spiral, one of so called *thurals* [2], i.e. a super-spiraling curve $r=\theta^{\theta}$, where *a=1, b=1*. It was in fact the generic question of this very work: Does exist (and how looks like) the inspiral (*inthural*) of the form

$$r^{r}=\theta \quad ? \tag{2.1}$$

The positive answer, via the *parametric inversion* (1.4), leads towards the graph below (Fig. 2).

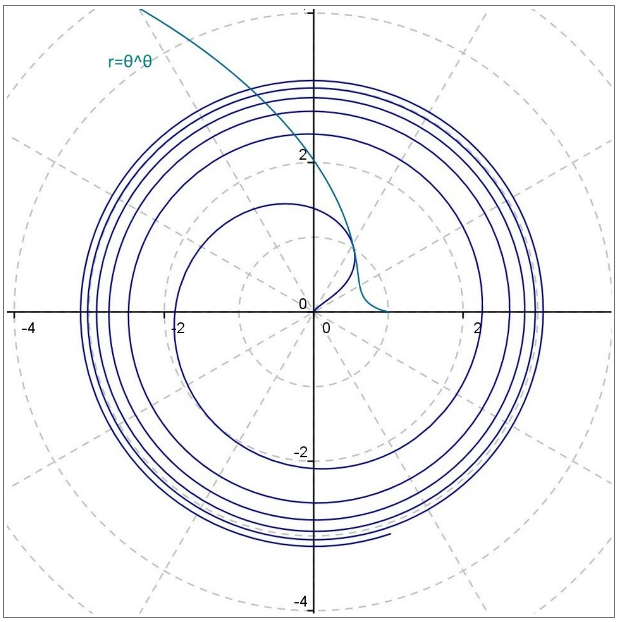

Fig. 2 *The inpolar s-spiral (inthural)*

Rest of *inthurals* (three of them) is shown as one common graph (Fig. 3).

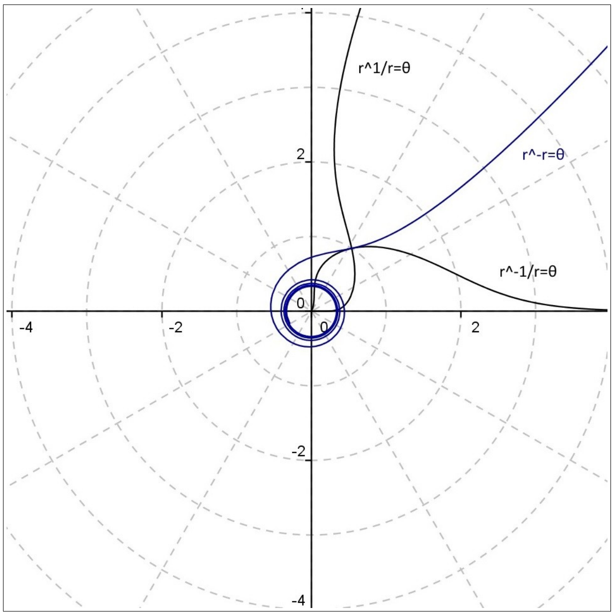

Fig. 3 *Three inthurals*

3. Three basic inpolars

Still staying in the frame of the polar system we would like to examine three basic shapes and their inpolar pictures, i.e. lines, conchoids as well as conics, respectively. The choice seems somewhat arbitrary but it is justified with at least two interesting common properties: 1. all the three do have related (and quite similar) inverse functions (always bearing in mind their intervals), and 2. for both the conchoids and conics there also exists a basic line (such as directrix). As we shall see, that nevertheless makes an interesting influence on their inpolar shapes.

3.1 Inlinears

The general polar equation which defines line in the plane is whether $r = a/\cos(\theta+\alpha)$ or $r = a/\sin(\theta+\alpha)$, equally. It is pretty obvious that, after application of the (I.2), we would obtain (e.g. for the former formula) the inverse explicit form

$$r = \arccos\frac{a}{\theta} - \alpha \quad . \qquad (3.1)$$

Assuming *a=1* and *α=0* follows *inlinear*[3] as in the first graph bellow (see Fig. 4). The next to it (Fig. 4a) represents the same inpolar line but now given directly via the inversion (1.4). Interestingly enough, the choice of limits in the parametric variable *t* does make quite a difference. In fact, we deal here with periodic, even asymptotic functions and should be aware of their transformed periods.

3 We may notice quite similarity with hyperspiral [1], so in a way this could be kind of a trigonometric hyperspiral.

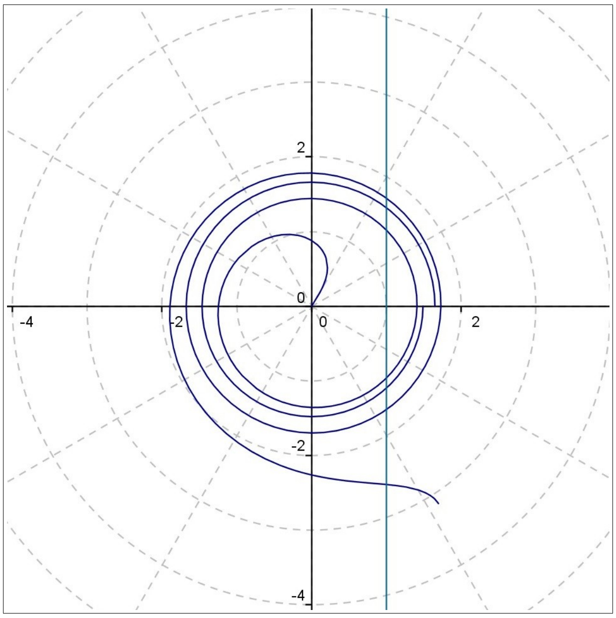

Fig. 4

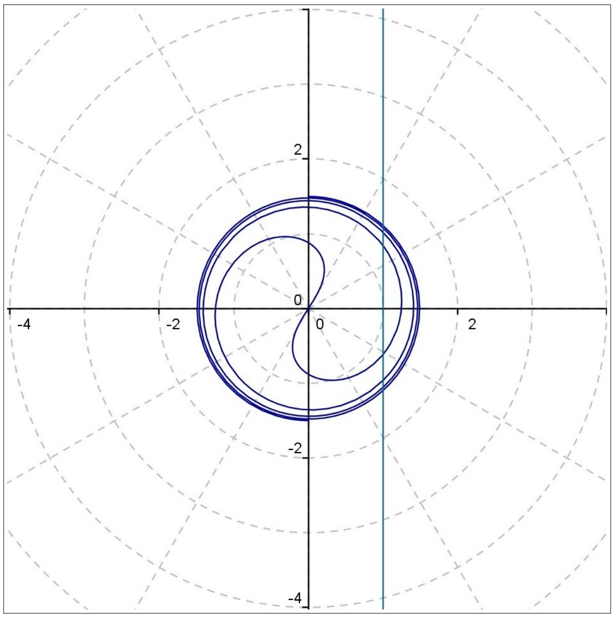

Fig. 4a

Next is the case (Fig. 5) where constant α is a non-zero angle, e.g. $\alpha=\pi/4$. Finally, there is the case of the horizontal line $r=1/\sin(\theta)$ and its inlinear $r=\arcsin\frac{1}{\theta}$ (see Fig. 6).

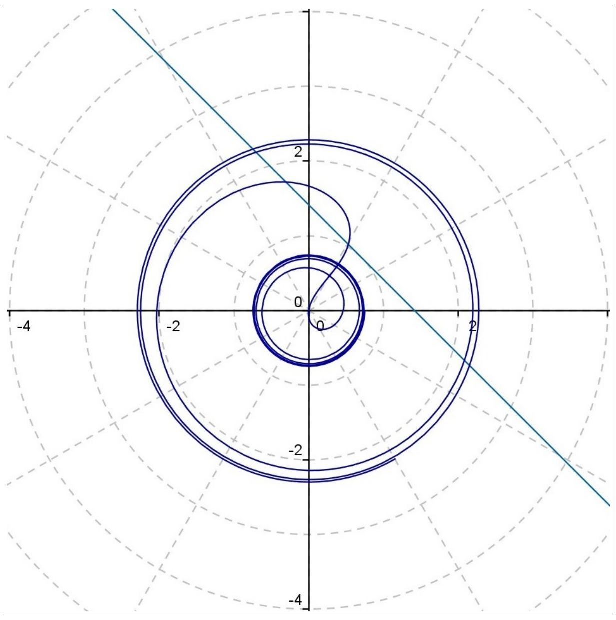

Fig. 5

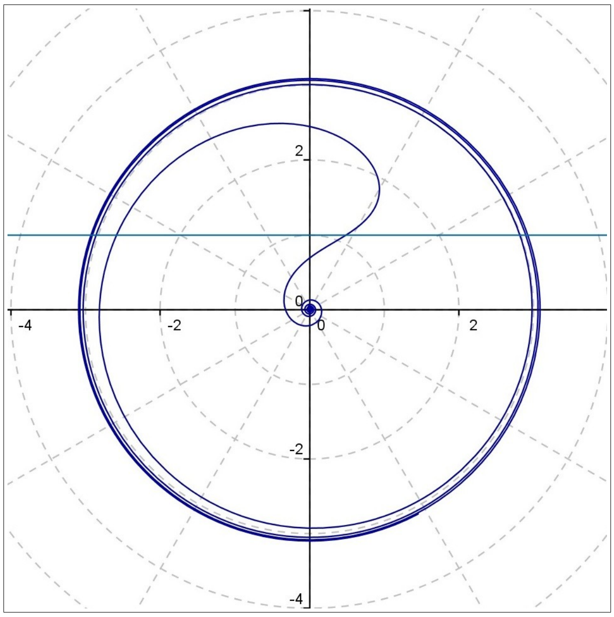

Fig. 6

Note, there is an interesting relationship between the slope of a line and its *inlinear*. We could say that *each and every polar angle is related to an exactly determined inpolar shape*.

3.2 Inconchoid

Nicomedes' conchoid itself is a remarkable curve given by the known polar equation $r = a / \sin(\theta + \alpha) + l$. Following application of the (I.2) we get its both the inverse and inpolar form as one

$$r = \arcsin \frac{1}{\theta - l} \quad . \qquad (3.2)$$

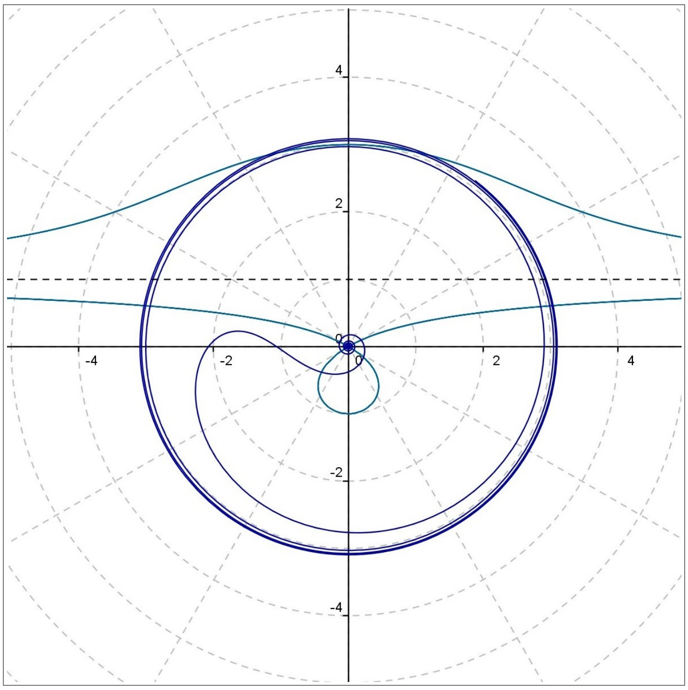

***Fig.** 7 Inconchoid as a rotated inlinear*

Within the frame of this transformation, and as a consequence of its basic line existence, the conchoid reveals a new inner remarkable property as well. Namely, noticing the equivalence with Fig. 6 we may say that *addition* (*for l*) *within polar frame is as same as rotation* (*for angle l*) *within inpolar one*.

3.3 Inconics

Conic sections are among the basic planar curves belonging to the very history of geometry, too. Following our line of reasoning we firstly present them in polar form[4] using known equation

$r = \dfrac{l}{1 - e * \cos \theta}$ and then, via (I.2), obtain the inverse, inpolar form

$$r = \arccos[\frac{1}{e}(1 - \frac{l}{\theta})], \qquad (3.3)$$

where *e* usually means conics' eccentricity and *l=p*e*. Depending on values for *e* and *l* follow next *inconics,* i.e. inpolars of ellipse, hyperbola and parabola, respectively (see Figs. 8, 9 and 10).

4 Inpolar circle will be specially analyzed later as a case of a parametric curve.

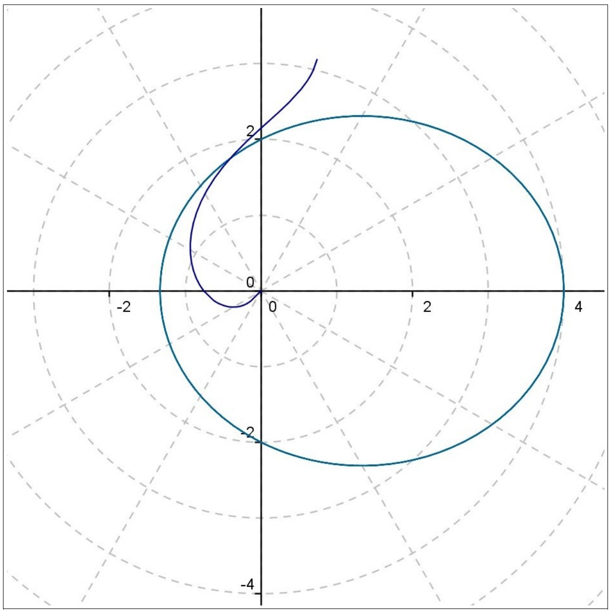

Fig. 8 *e=1/2, l=2*

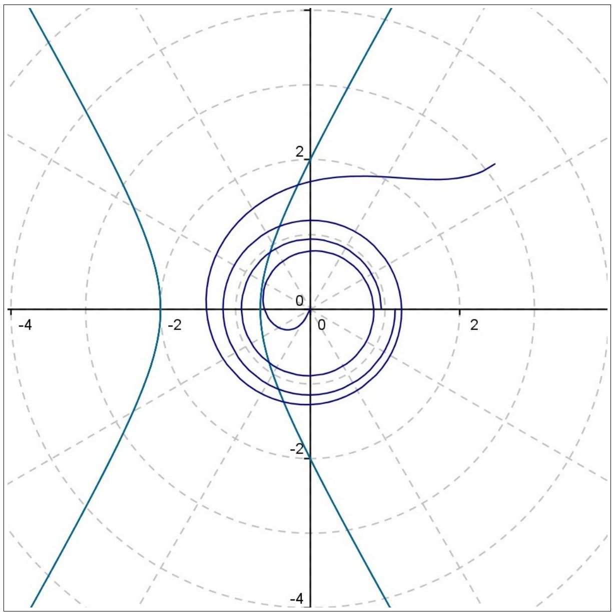

Fig. 9 *e=2, l=2*

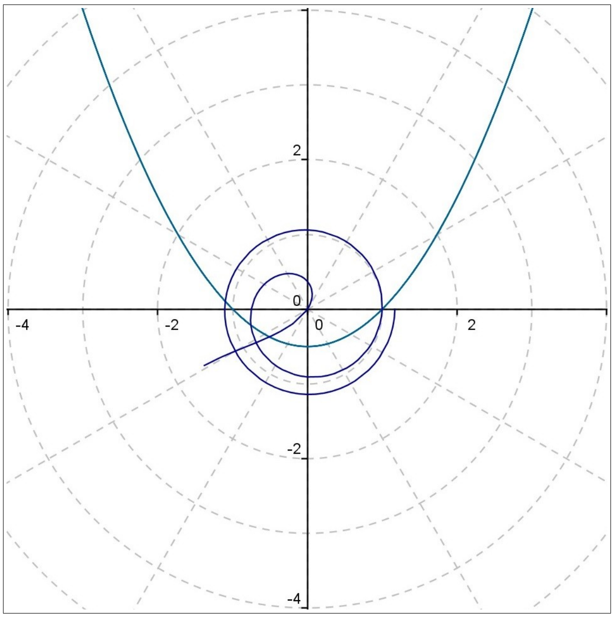

Fig. 10 *e=1, l=1*

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In conclusion of this chapter we also have to notice clear polar-inpolar relationship between circle and line (and vice versa). As we can see, even from the very definition (I. 2 and 3) of this idea follows that if *r=const* defines a circle then the basic inversion const=θ defines its inpolar half-line. The same follows from the above polar definition of a circle (where *e=0*) as well.

4. Parametric inpolars

The parametric form of planar functions, in a way, is very close to our basic idea of the inpolar transformation (1.4). Following the standard procedure we can always transform a point (*x, y*) of a given curve into the polar pair (*r, θ*). After that we should make the inpolar switch (*θ, r*) in order to reach a new inpolar pair (*x', y'*). Formally, it is finally as follows

$$x' = \arctan\left(\frac{y}{x}\right)\cos\left(\sqrt{x^2+y^2}\right)$$
$$y' = \arctan\left(\frac{y}{x}\right)\sin\left(\sqrt{x^2+y^2}\right) \quad . \tag{4.1}$$

The same procedure is applicable to the pure Cartesian functions as well. Accordingly, in order to properly read points of the primeval curve, in some cases we use pure *atan* function (or here *arctan*), in other cases the known *atan2* and sometimes our version of it, e.g. *atan2'*[5].

4.1 Inpolar circle

In order to explore the concept further we will start with the case of an eccentric circle but now given parametrically, i.e. $x = p + \rho * \cos(t)$ and $y = q + \rho * \sin(t)$, where (*p, q*) are the center's coordinates and *ρ* would be the circle's radius. Setting values *p*=16, *q*=16 and *ρ*=9, we get a case of a “far away” circle which inpolar coordinates, via (4.1), we read as

$$x' = \arctan\left(\frac{16+9\cos(t)}{16+9\sin(t)}\right)\cos\left(\sqrt{593+288(\sin(t)+\cos(t))}\right) \quad \text{and}$$

$$y' = \arctan\left(\frac{16+9\cos(t)}{16+9\sin(t)}\right)\sin\left(\sqrt{593+288(\sin(t)+\cos(t))}\right), \quad \text{(see Fig. 11).}$$

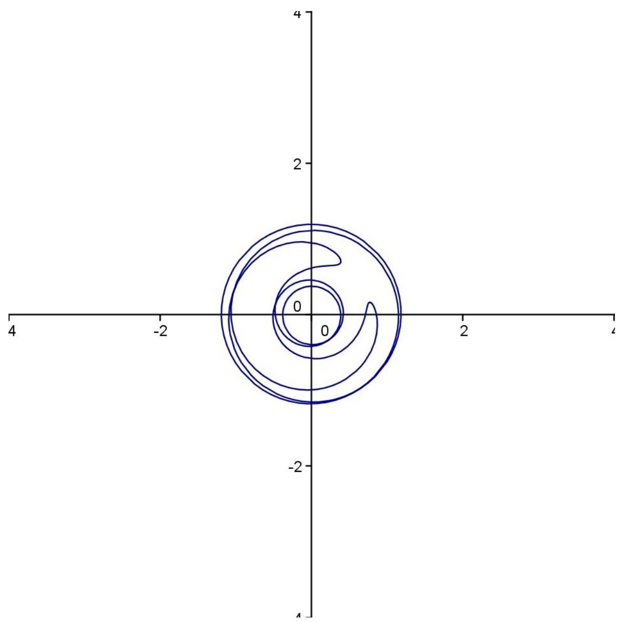

Fig. 11 *The eccentric in-circle*

5 atan2'(y,x) = atan2(y,x) – 2π*floor(sin(y)/2), $\theta \in (0, 2\pi)$

4.2 Incycloid

Cycloid, nevertheless the curve of a very interesting background, here is of importance as a clear parametric curve without any other definition. It has been given via parametric relations $x=\rho(t-\sin(t))$ and $y=\rho(1-\cos(t))$, where ρ would be radius of a rolling circle. Following again (4.1), but now with *atan2*, and choosing ρ=1, we reach the cycloid's inpolar picture (Fig. 12).

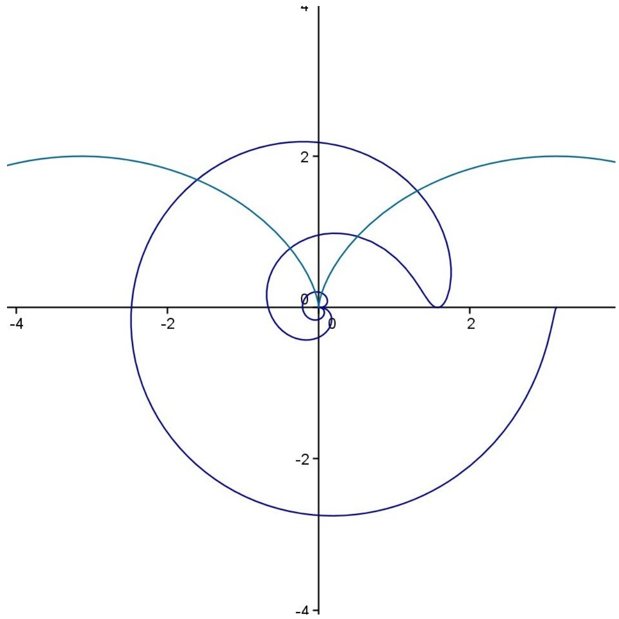

Fig. 12 *Incycloid, ρ=1*

4.3 Lissajous inpolars

Lissajous figures (or Bowdich curves) also are more than interesting purely parametric curves defined with related equations $x=A(a*\sin(t)+\alpha)$ and $y=B(b*\cos(t))$.

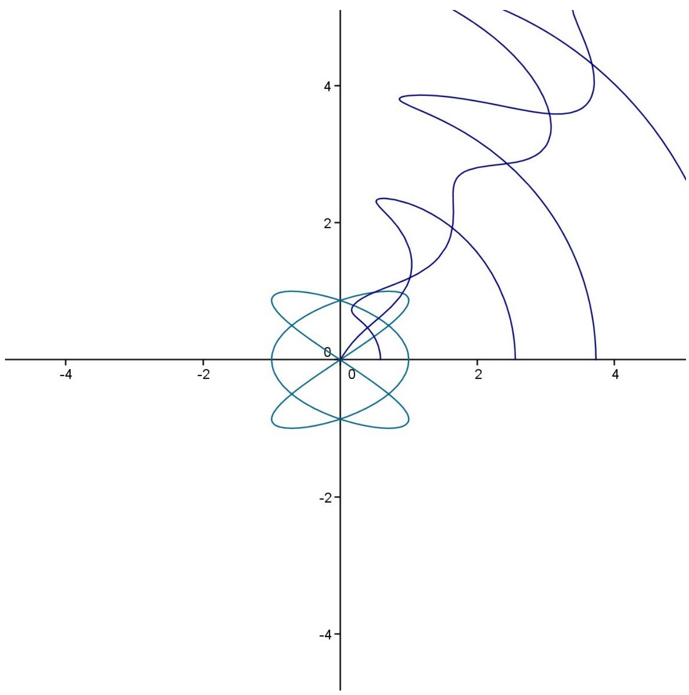

Fig. 13 *A=1, B=1, a=3, b=2, α=π/2*

Again, following the procedure (4.1) with different choices for *A, B, a, b* and *α* as well as using *atan2'* (footnote 5), we would picture different Lissajous inpolars (see above Fig. 13 and bellow Fig. 15). Interestingly enough, the change in *A* and *B* grossly reshapes the inpolar figure (Fig. 14).

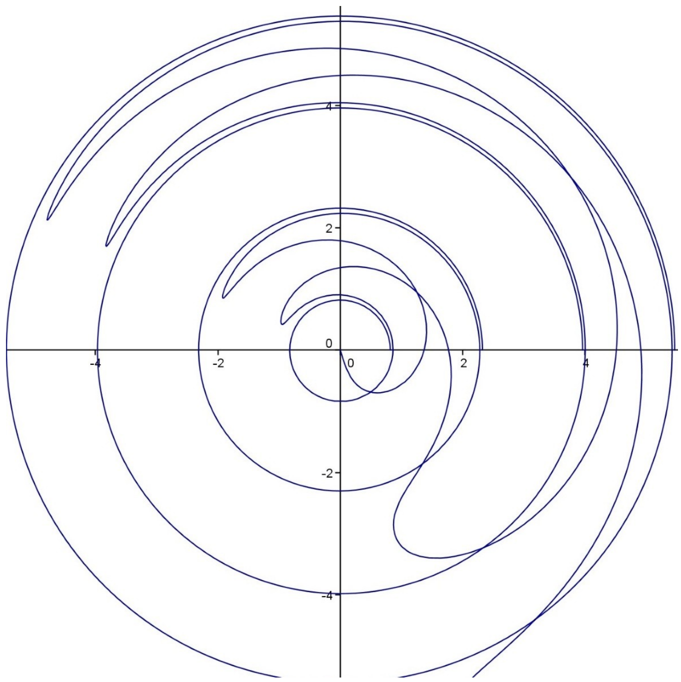

Fig. 14 *A=5, B=8, a=3, b=2, α=π/2*

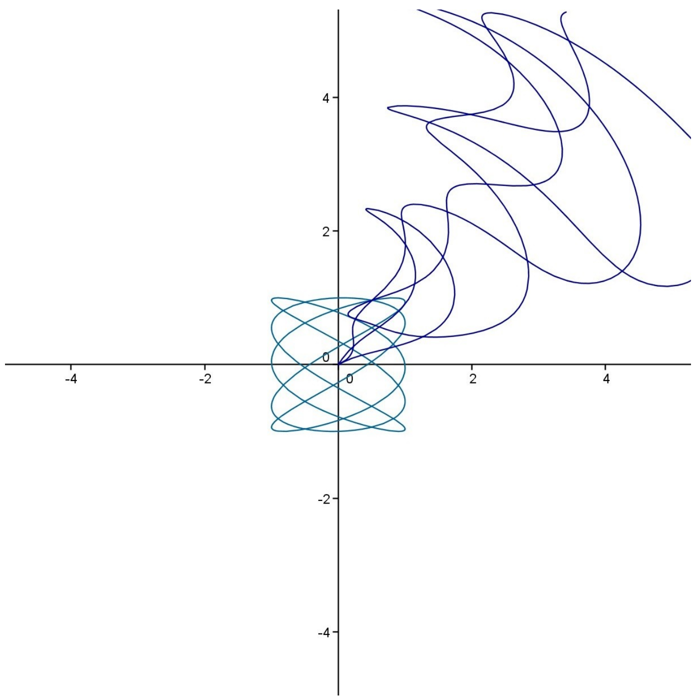

Fig. 15 *A=1, B=1, a=5, b=3, α=π/7*

5. Cartesian inpolars

Before we conclude with the idea of inpolar graphing we simply must examine at least some of the most important elementary planar functions. All of them are given in Cartesian system by the explicit expression *y=f(x)*. As we already mentioned, these functions can also be transformed into their inpolar forms following the same above procedure, i.e. the final relations (4.1); see next four graphs (Figs. 16, 17, 18 and 19).

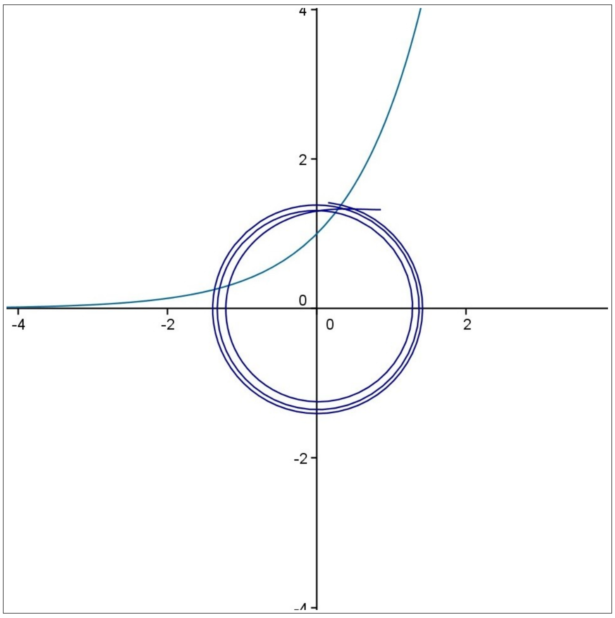

Fig. 16 *Exponential inpolar*

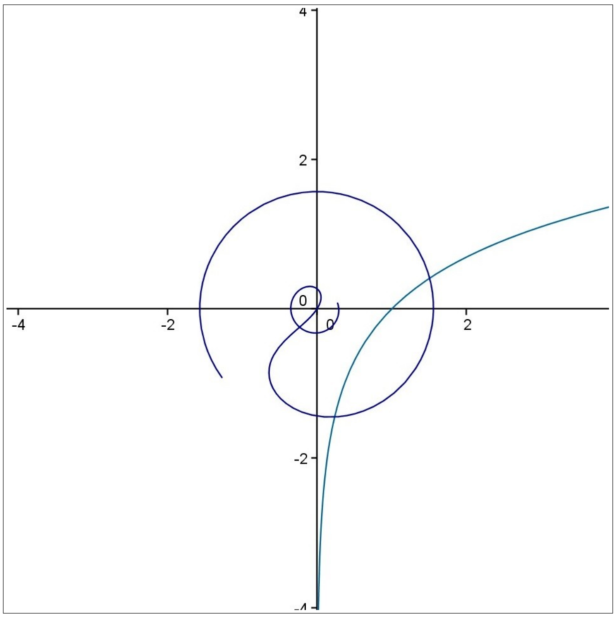

Fig. 17 *Logarithmic inpolar*

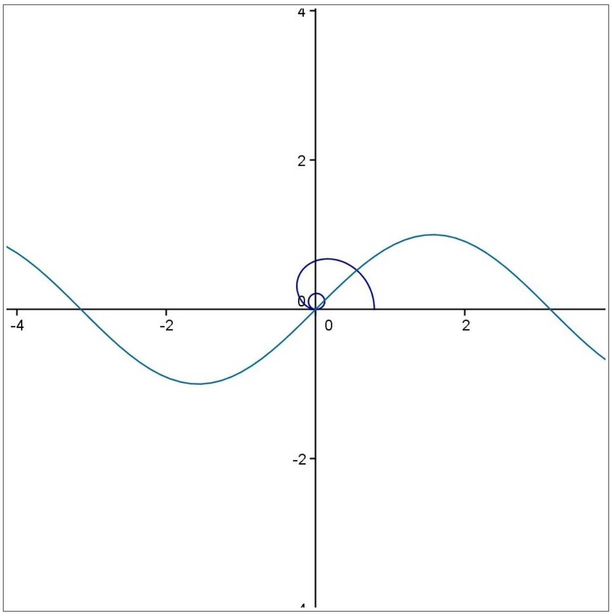

Fig. 18 *Trigonometric (sin) inpolar*

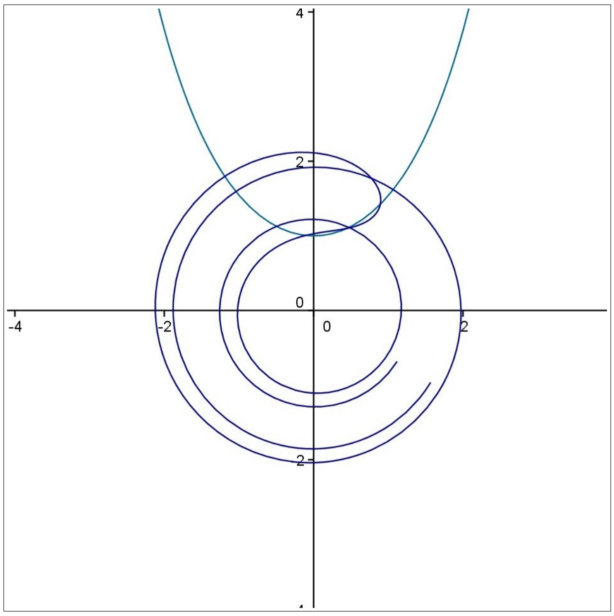

Fig. 19 *Incatenary (cosh inpolar)*

The final curve (Fig. 20) of which we here make an inpolar transformation is, in a way, interestengly connected with the initial one (Fig. 2).

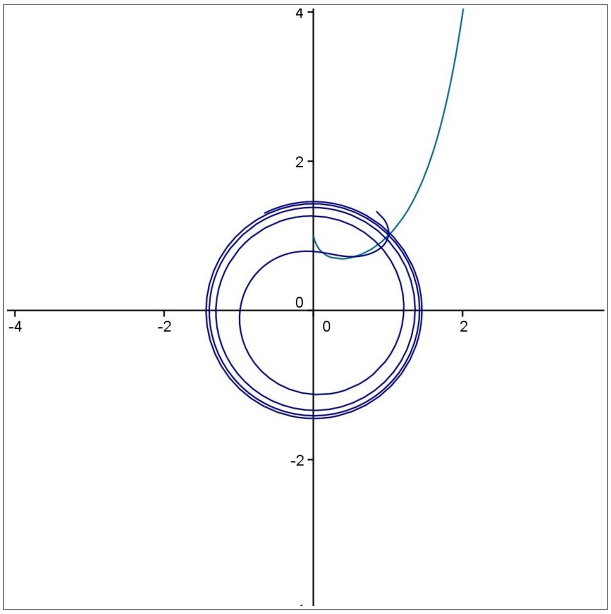

Fig. 20 *Inpolar of y=x^x*

6. The inpolar system

It seems that we have all reasons to believe that here exposed concept (of the inpolar transformation) leads towards an entirely new geometric system. Firstly, let us see that we could have started from a more general assumption than it was (I.1 and 2), i.e. it can be

$$g(r)=f(\theta) \quad . \qquad (6.1)$$

As next, we can easily generalize the so called parametric inversion (1.4) using the complex form $z=x+i*y,$ hence

$$z=\theta e^{ir} \quad . \qquad (6.2)$$

The assumption (6.1) is even more clear in complex plane because then we would have $z=g(\theta)e^{ir},$ thus follow the relations

$$\Re(z)=x=g(\theta)\cos(f(\theta)),$$
$$\Im(z)=y=g(\theta)\sin(f(\theta)) \quad . \qquad (6.3)$$

Also, we can recapitulate here, in a free manner, the basics of the mutual polar-inpolar transformation as such. Using our operators In(of polar), Rein[6](of inpolar) and $R^7_{C,\alpha}$ we get

$R_{C,\alpha}$(In(line)) = inconchoid (6.4)

6 "Rein" reads points of an in-curve and makes an exchange in places of θ and r (or ω and v like in Fig. 21).
7 $R_{C,\alpha}$ represents rotation around C for an angle α

or the other way around,

Rein(In($R_{C,\alpha}$(line))) = conchoid. (6.5)

Last but not least, let us emphasize here what we rather call the oscillatory (or wave) nature of the inpolar system (Fig. 21).

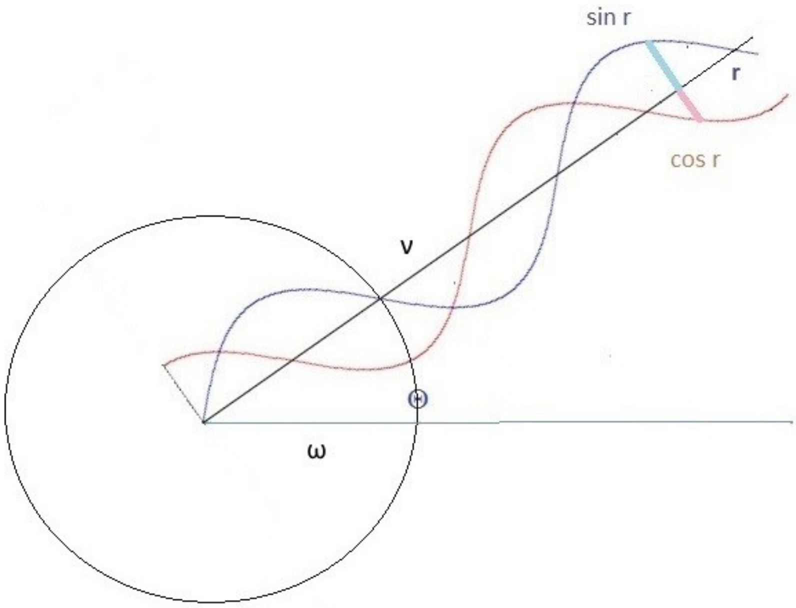

Fig. 21 *Inpolar coordinates ω and ν*

Even from the definition (1.4) itself, it is already clear that inpolar picture has something to do with "rotation" (for angle θ) and "oscillation" (along r). We can interpret that as two related "frequencies" ω and ν, which makes the new pair of coordinates (ω, ν). Assuming a "reading" operator over polar curves Pol(r, θ) as well as two new parameters T and τ, we can write down

Rein~(ν, ω) = Pol(ωT, ντ)[8]. (6.6)

The same is valid for the transformation of a curve from the inpolar system to the Cartesian one, i.e.

Rein~(ν, ω) = (ωT*cos(ντ), ωT*sin(ντ)). (6.7)

In a sense, T and τ might be considered as "periods" or even "time".

7. Conclusion

An initial, almost naive, question (2.1) about eventual existence of a new kind of "polar inverson" has surprisingly forced us to accept a change in our usual geometrical intuition of r and θ mutual dependency in general. Following that line of reasoning the *inpolar* transformation (1.4) has been a natural step, hence the entire set of *inpolars*. Moreover, the insight leads towards an entirely new "geometric" system as such. As if that system possesses some inherently wave (or oscillatory) nature. The authors do believe that many new roads of researching are opened this way.

8 T=τ=2π would be a possibility.

Acknowledgments

This work is dedicated to the memory of our dear friend Zoran Jovanović, a noble soul.

To Zoran and his ellipses.

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