Modified standard Einstein field equations and cosmological constant

Faisal Amin Yassein Abdelmohssin[1]
Sudan Institute for Natural Sciences, P.O.BOX 3045, Khartoum, Sudan
January 22, 2017

Abstract: We have modified the standard Einstein field equations by introducing a general function that depends on Ricci's scalar without prior assumption of the mathematical form of the function. By demanding that the covariant derivative of the energy-momentum tensor should vanish and with application of Bianchi identity a first order ordinary differential equation in the Ricci's scalar has emerged. By integrating the resulting equation a constant of integration resulting from solving the equation is interpreted as the cosmological constant introduced by Einstein.

The form of function on Ricci's scalar and on the cosmological constant corresponds to the form of Einstein-Hilbert's Lagrangian appearing in the action integral.

PACS numbers: 04.20.-q, 04.20.Cv[2]
Keywords: general relativity, cosmological constant.

1. Introduction: Einstein presented his standard field equations (EFE) describing gravity in form of tensor equations, namely,

$$G_{ab} + kT_{ab} = 0 \tag{1.1}$$

where, k is the Einstein constant, T_{ab} is the energy-momentum, and G_{ab} is the Einstein tensor given by,

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{1.2}$$

where, R_{ab} is the Ricci curvature tensor, R is the Ricci's scalar curvature, and g_{ab} is the metric tensor.

In his search for a solution to his field equations he turned to cosmology and proposed a model of static universe filled with matter. Because he believed of the static model for the Universe, he introduced a constant term in his standard field equations to represent a kind of "anti gravity' to balance the effect of gravitational attractions of matter.

---

[1] Electronic address: f.a.y.abdelmohssin@gmail.com
[2] http://ufn.ru/en/pacs/all/

Einstein modified his standard equations by introducing a term to become

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 8\pi G c^{-4} T_{ab} \quad (1.3)$$

where, Λ is the cosmological constant (assumed to be very small), G is Newton's gravitational constant, c is the speed of light in vacuum, and T_{ab} is the stress–energy tensor.

Einstein rejected the cosmological constant for two reasons:
(1) The universe described by this theory was unstable.
(2) Observations by Edwin Hubble confirmed that the universe is expanding.

Recently, it has been believed that this cosmological constant might be one of the causes of the accelerated expansion of the Universe.

2. Modified standard Einstein field equations

We modify the EFE by introducing a general function $L(R)$ of Ricci's scalar into the standard EFE. We don't assume a concrete form of the function. The modified EFE then becomes,

$$R_{ab} + g_{ab} L(R) = 8\pi G c^{-4} T_{ab} \quad (2.1)$$

Taking covariant derivative (denoted by semicolon ;) of both sides, we get

$$R_{ab;b} + [g_{ab} L(R)]_{;b} = 8\pi G c^{-4} T_{ab;b} \quad (2.2)$$

Since covariant divergence of the metric vanishes, Equation (2.2) can be written

$$R_{ab;b} + g_{ab} \frac{\mathrm{d}L}{\mathrm{d}R} R_{;b} = 8\pi G c^{-4} T_{ab;b} \quad (2.3)$$

Substituting the Bianchi's identity

$$R_{;c} = 2 g^{ab} R_{ac;b} \quad (2.4)$$

Requiring the covariant divergence of the energy-momentum tensor to vanish (i.e. energy-momentum is conserved), namely,

$$T_{ab;b} = 0 \quad (2.5)$$

We arrive at

$$R_{ab;b} + g_{ab} \frac{\mathrm{d}L}{\mathrm{d}R} (2 g^{ac} R_{ab;c}) = 0 \quad (2.6)$$

This may be written as

$$R_{ab;b} + 2 \frac{\mathrm{d}L}{\mathrm{d}R} (g_{ab} g^{ac} R_{ab;c}) = 0 \quad (2.7)$$

Substituting

$$g_{ab} g^{ac} = \delta_b^c$$

In equation (2.7) we get

$$R_{ab;b} + 2\frac{\mathrm{d}L}{\mathrm{d}R}(\delta^{c}{}_{b} R_{ab;c}) = 0 \tag{2.8}$$

By changing the dummy indices, we arrive at

$$R_{ab;b}(1 + 2\frac{\mathrm{d}L}{\mathrm{d}R}) = 0 \tag{2.9}$$

We have either,

$$R_{ab;b} = 0 \tag{2.10}$$

Or,

$$1 + 2\frac{\mathrm{d}L}{\mathrm{d}R} = 0 \tag{2.11}$$

Equation (2.10) is doesn't always satisfied. While Equation (2.11) yields,

$$\frac{\mathrm{d}L}{\mathrm{d}R} = -\frac{1}{2} \tag{2.12}$$

which has a solution

$$L(R) = -\frac{1}{2}R + C \tag{2.13}$$

where C is a constant.

Interpreting the constant of integration, C as the cosmological constant Λ, the functional dependence on Ricci's scalar can be written as,

$$L(R) = -\frac{1}{2}(R - 2\Lambda) \tag{2.14}$$

Equation (2.14) is the Lagranian of the Einstein-Hilbert action with the cosmological constant.

3. Concluding remark

We arrived at Einstein field equations with the cosmological constant from a general function on Ricci scalar without a prior assumption of linear dependence on Ricci scalar.

References

[1] Misner, Charles W.*;* Thorne, Kip S.*;* Wheeler, John Archibald *(1973),* Gravitation*, San Francisco:* W. H. Freeman, ISBN 978-0-7167-0344-0.

[2] L.D. Landau, E.M. Lifshitz *(1975). The Classical Theory of Fields. Vol. 2 (4th ed.).* Butterworth-Heinemann. ISBN 978-0-7506-2768-9.

[3] J. B. Hartle (2003). Gravity: An Introduction to Einstein's General Relativity. Addison-Wesley. *ISBN 9780805386622*.

[4] S. Carroll (2003). Spacetime and Geometry: An Introduction to General Relativity. Addison-Wesley. p. 496. *ISBN 9780805387322*.

[5] P. Sharan (2009). *Spacetime, Geometry and Gravitation*. Springer. *ISBN 9780805387322*.
[6] R. D'Inverno (1992). *Introducing Einstein's Relativity*. Clarendon Press. *ISBN 9780198596868*.
[7] M. P. Hobson; G. P. Efstathiou; A. N. Lasenby (2006). *General Relativity: An Introduction for Physicists*. Cambridge University Press. *ISBN 9780521829519*.
[8] B. Schutz (1985). *A First Course in General Relativity*. Cambridge University Press. *ISBN 0521277035*.
[9] Foster, J; Nightingale, J.D. (1995). *A Short Course in General Relativity* (3nd ed.). Springer. *ISBN 0-03-063366-4*.
[10] R. M. Wald (1984). General Relativity. Chicago University Press. *ISBN 9780226870335*.
[11] S. W. Hawking; W. Israel (1987). *Three Hundred Years of Gravitation*. Cambridge University Press. *ISBN 9780521379762*.