Intelligence of Crowd.

Michail Zak[a]

Jet Propulsion Laboratory California Institute of Technology, Pasadena, CA 91109, USA

A new class of dynamical systems with a preset type of interference of probabilities is introduced. It is obtained from the extension of the Madelung equation by replacing the quantum potential with a specially selected feedback from the Liouville equation. It has been proved that these systems are different from both Newtonian and quantum systems, but they can be useful for modeling spontaneous collective novelty phenomena when emerging outputs are qualitatively different from the weighted sum of individual inputs. Formation of language and fast decision-making process as potential applications of the probability interference is discussed.

1. Introduction

In Newtonian physics, the concept of probability ρ is introduced via the Liouville equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{F}) = 0 \qquad (1)$$

generated by the system of ODE

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\mathbf{v}_1(t), ... \mathbf{v}_n(t), t] \qquad (2)$$

where **v** is velocity vector.
It describes the continuity of the probability density flow originated by the error distribution

$$\rho_0 = \rho(t = 0) \qquad (3)$$

in the initial condition of ODE (2).
This equation is ***linear*** with respect to the probability density, and therefore, according to the superposition principle, the probabilities are combined by summation: when an event can occur in several alternative ways, the probability of the event is the sum of the probabilities for each way considered separately, i.e.

$$\rho = \rho_1 + \rho_2 \qquad (4)$$

In quantum physics, the probability is introduced via the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 \Psi - F\Psi = 0 \qquad (5)$$

[a] Senior research Scientist (Emeritus). Electronic mail:michail.zak@gmail.com

that is linear with respect to ***probability amplitude*** Ψ , i.e. with respect to the square root of the probability density. Therefore, when an event can occur in several alternative ways, the probability amplitude of the event is the sum of the probability amplitudes for each way considered separately

$$\psi = \psi_1 + \psi_2,\ \rho_i = |\psi_i|^2, \quad \rho = |\psi_1 + \psi_2|^2 \neq \rho_1 + \rho_2 \tag{6}$$

and this phenomenon is known as interference of probabilities: the probabilities are combined as the intensities of waves.
The objective of this paper is to introduce a new class of dynamical systems that has rules of summation of probabilities, which are different from those presented by Eqs. (4) and (6). Obviously such systems cannot belong to Newtonian or quantum physics.

2. The master equation

The starting point of our approach is the Madelung equation that is a hydrodynamical version of the Schrödinger equation (5)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\frac{\rho}{m} \nabla S) = 0 \tag{7}$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \tag{8}$$

Here ρ and S are the components of the wave function $\psi = \sqrt{\rho} e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π. The last term in Eq. (8) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (7) is the Liouville equation that expresses continuity of the flow of probability density, and Eq. (8) is the Hamilton-Jacobi equation for the action S of the particle. Actually the quantum potential in Eq. (8), as a feedback from Eq. (7) to Eq. (8), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

The Madelung equations (7), and (8) can be converted to the Schrödinger equation using the ansatz

$$\sqrt{\rho} = \Psi \exp(-iS / h) \tag{9}$$

where ρ and S being real function.

Let us rewrite Eq. (2) in the following form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}[\mathbf{v}_1(t), \ldots \mathbf{v}_n(t), t, \rho] \tag{10}$$

where **v** is a velocity of a hypothetical particle.

This is a fundamental step in our approach: in Newtonian dynamics, the probability never explicitly enters the equation of motion, [1,2,3]. In addition to that, the Liouville equation generated by Eq. (10) is nonlinear with respect to the probability density ρ

$$\frac{\partial\rho}{\partial t}+\nabla\bullet[\rho\mathbf{F}(\mathbf{v}_1,...\mathbf{v}_n,t,\rho)]=0 \qquad (11)$$

and therefore, the system (10),(11) departs from Newtonian dynamics. However although it has the same topology as quantum mechanics (since now the equation of motion is coupled with the Liouville equation), it does not belong to it either. Indeed Eq. (10) is more general than the Hamilton-Jacoby equation (8): it is not necessarily conservative, and **F** is not necessarily the quantum potential although further we will impose some restriction upon it that links **F** to the concept of information, [1]. The relation of the system (10), (11) to Newtonian and quantum physics is illustrated in Fig.1.

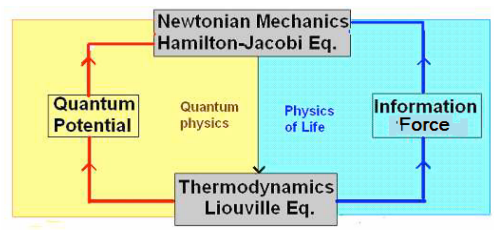

Figure 1. Classical physics, quantum physics, and physics of Life.

Prior to considering a specific form of the force **F,** we will make a comment concerning the normalization constrain satisfaction

$$\int_V \rho\, dV=1 \qquad (12)$$

in which V is the volume where Eqs. (10) and (11) are defined.
Turning to Eq. ((11) and integrating it over the volume V

$$\frac{\partial}{\partial t}\int_V \rho\, dV=-\int_V dV\nabla\bullet[\rho\mathbf{F}(\mathbf{v}_1,...\mathbf{v}_n,t,\rho)]=-\oint_\Phi d\Phi\nabla\bullet(\rho\mathbf{F})=0 \qquad (13)$$

if

$$\rho=0, \quad |\mathbf{F}|<\infty \quad at \quad \Phi \qquad (14)$$

where Φ is the surface bounding the volume V.
Therefore, if the normalization constraint (12) is satisfied at $t = 0$, it is satisfied for all the times.

3. Information force instead of quantum potential

In this section we propose the structure of the force **F** that plays the role of a feedback from the Liouville equation (11) to the equation of motion (10). For that purpose, we introduce an auxiliary variable φ as a preset function of the probability density ρ. We will call it the probability function, implying its similarity to probability amplitude known from quantum mechanics, and assume that

$\varphi(\rho)$ is differentiable, (15)

$$\varphi(0)=0, \quad \varphi(\infty)=\infty, \quad sign\,\varphi = sign\,\rho, \tag{16}$$

$$\frac{d\varphi}{d\rho} \neq 0 \tag{17}$$

The class of functions satisfying the conditions (15)-(17) includes wide range of elementary functions.

Starting with a one-dimensional case $\mathbf{F} \rightarrow f$, let us introduce the force f in the following form

$$f = \frac{\xi}{\rho(v,t)} \int_{-\infty}^{v} \frac{\varphi[\rho(\eta,t)] - \varphi[\rho^*(\eta)]}{d\varphi / d\rho} d\eta \tag{18}$$

Here $\rho^*(v)$ is a preset probability density satisfying the constraints (12), and ξ is a positive constant with dimensionality [1/sec]. As follows from Eq. (18), f has dimensionality of a force per unit mass that depends upon the probability density ρ, and therefore, it can be associated with the concept of information, so we will call it the *information force*. In this context, the coefficient ξ can be associated with the Planck constant that relates Newtonian and *information* forces, [2].

Now the system (10), (11) can be written as follows

$$\dot{v} = \frac{\xi}{\rho(v,t)} \int_{-\infty}^{v} \frac{\varphi[\rho(\eta,t)] - \varphi[\rho^*(\eta)]}{d\varphi / d\rho} d\eta \tag{19}$$

$$\frac{\partial \varphi}{\partial t} + \xi\{\varphi[\rho(\eta,t)] - \varphi[\rho^*(\eta)]\} = 0 \tag{20}$$

Thus due to the integral form of the information force (18), the Liouville equation is degenerated from PDE into ODE with respect to φ as a function of time, while the variable V plays the role of a parameter.

Remark. Here and below we make distinction between the random *variable v(t)* and its *values V* in probability space.

Eq. (20) has the analytical solution

$$\varphi = \{\varphi[\rho_0(V)] - \varphi[\rho^*(V)]\}e^{-\xi t} + \varphi[\rho^*(V)] \tag{21}$$

subject to the initial condition

$$\varphi[\rho(t=0)] = \varphi[\rho_0(V)] \tag{22}$$

while $\rho_0(V)$ satisfies the constraints (12).

This solution converges to a preset stationary distribution $\varphi[\rho^*(V)]$ representing a stochastic attractor. Obviously the normalization condition for ρ is satisfied if it is satisfied for ρ_0 and ρ^*, (see Eq. (13)).

Rewriting Eq. (21) in the form

$$\varphi(\rho)=\varphi(\rho_0)e^{-\xi t}+\varphi(\rho^*)(1-e^{-\xi t}) \tag{23}$$

one observes that $\varphi(\rho)\geq 0$ at all $t\geq 0$ and $-\infty > V > \infty$.

Hence according to the property Eq. (16), $\rho\geq 0$ at all $t\geq 0$ and $-\infty > V > \infty$ as well.

4. Emergence of randomness

In this section we will analyze the relationships between Eqs.(19) and (20) and illuminate the origin of randomness of solutions of Eq. (19). In order to deal with closed form analytical solutions, we have to specify the function $\varphi(\rho)$ assuming that

$$\varphi=\rho \tag{24}$$

Then following [1,2,3], and substituting the solution (21) with reference to Eq. (24) into Eq. (19), one arrives at the ODE that simulates the stochastic process with the probability distribution (21)

$$\dot{v}=\frac{\xi e^{-\xi t}}{[\rho_0(v)-\rho^*(v)]e^{-\xi t}+\rho^*(v)}\int_{-\infty}^{v}[\rho_0(\eta)-\rho^*(\eta)]\ d\eta \tag{25}$$

It is reasonable to assume that the solution (21) starts with a sharp initial condition

$$\rho_0(V)=\delta(V) \tag{26}$$

As a result of that assumption, all the randomness is supposed to be generated *only* by the controlled instability of Eq. (25). Substitution of Eq. (26) into Eq. (25) leads to two different domains of *v*: $v\neq 0$ and *v*=0 where the solution has two different forms, respectively

$$\int_{-\infty}^{v}\rho^*(\xi)d\xi=\left(\frac{C}{e^{-\xi t}-1}\right)^{1/\xi},\quad v\neq 0 \tag{27}$$

$$v\equiv 0 \tag{28}$$

Indeed, $\dot{v}=\dfrac{\xi e^{-\xi t}}{\rho^*(v)(e^{-\xi t}-1)}\displaystyle\int_{-\infty}^{v}\rho^*(\eta)]d\eta$

whence $\dfrac{\rho*(v)}{\int_{-\infty}^{v}\rho*(\eta)d\eta}dv=\dfrac{\xi e^{-\xi t}}{e^{-\xi t}-1}dt$. Therefore, $\ln\displaystyle\int_{-\infty}^{v}\rho*(\eta)d\eta=\ln\left(\frac{C}{e^{-\xi t}-1}\right)^{1/\xi}$

and that leads to Eq. (27) that presents an implicit expression for *v* as a function of time since ρ^* is the known function. Eq. (28) represents a singular solution, while Eq. (27) is a regular solution that includes arbitrary constant C . The regular solutions is discontinuous:

$$v\to\infty\quad at\quad t\to 0,\quad v=0\quad at\quad t=0 \tag{29}$$

the Lipschitz condition is violated

$$\left|\frac{\partial\dot{v}}{\partial v}\right|\to\infty\qquad at\qquad t\to 0\ ,\quad |v|\to 0 \tag{30}$$

and therefore, the uniqueness of the solution is lost thereby generating *randomness*.

As follows from Eq. (27), all the particular solutions for different values of *C* possess the same property (29), and that leads to non-uniqueness of the solution due to violation of the Lipchitz condition. Therefore, the same initial condition at $t \to 0$ yields infinite number of different solutions forming a family (27); each solution of this family appears with a certain probability guided by the corresponding Liouville equation (20). For instance, in cases plotted in Fig.2, a) and Fig.2, b), the “winner” solution is, respectively,

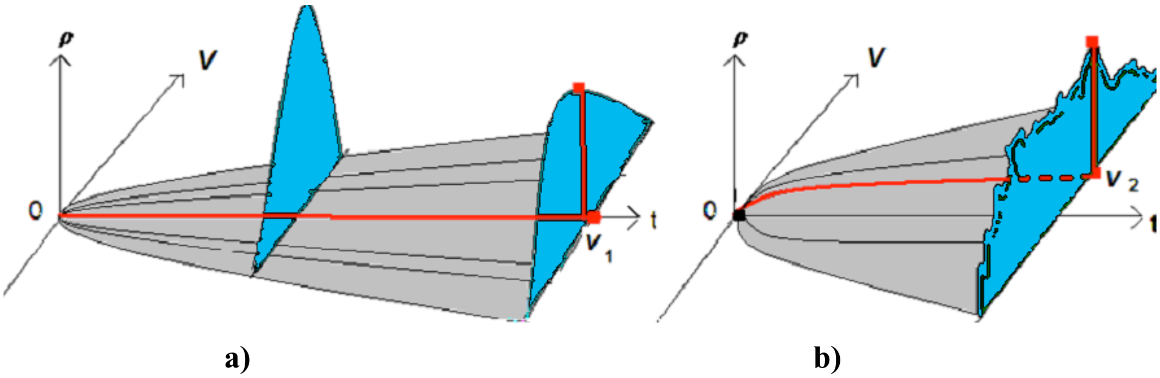

Figure 2. Stochastic processes and their attractors.

$$v_1 = \varepsilon \to 0, \quad \rho(v_1) = \rho_{\max}, \; and \quad v = v_2, \qquad \rho(v_2) = \sup\{\rho\}$$

since it passes through the maximum of the probability density . However, with lower probabilities, other solutions of the same family can appear as well. Obviously, this is a non-classical effect. Qualitatively, this property is similar to those of quantum mechanics: the system keeps all the solutions simultaneously and displays each of them “by a chance”, while that chance is controlled by the evolution of probability density (20).

Let us emphasize the connections between solutions of Eqs. (19) and (20): the solution of Eq. (19) is an one-parametrical family of trajectories (27), and each trajectory occurs with the probability described by the solution (21) of Eq. (20)). It should be recalled that the choice of displaying a certain solution is made by the particle only once, at *t=0*, i.e. when it departs from the deterministic to a random state; since than, it stays with this solution as long as the Liouville feedback is present.

Example 1. Let us start with the following normal distribution

$$\rho^*(V) = \frac{1}{\sqrt{2\pi}} e^{-\frac{V^2}{2}} \qquad (31)$$

Substituting the expression (31) and (26) into Eq. (27) at *V=v*, and $\xi = 1$ one obtains

$$v = erf^{-1}\left(\frac{C_1}{e^{-t} - 1}\right), \quad v \neq 0 \qquad (32)$$

Example 2. Let us choose the target density ρ^* as the Student’s distribution, or so-called power law distribution

$$\rho^*(V) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1 + \frac{V^2}{\nu}\right)^{-(\nu+1)/2} \qquad (33)$$

Substituting the expression (33) and (26) into Eq. (27) at *V=v*, ν=1, and $\xi = 1$one obtains

$$v = \cot(\frac{C}{e^{-t}-1})\ for \quad v \neq 0 \tag{34}$$

The 3D plot of the solutions of Eqs.(32) and (34), are presented in Figures 3a, and 3b, respectively.

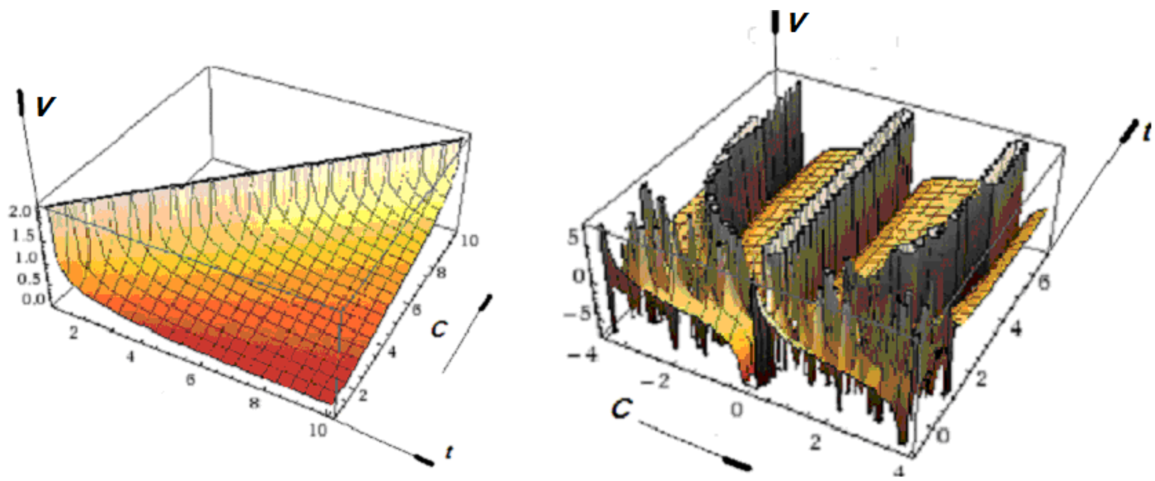

Figure 3. Dynamics driving random events to: (a) normal distribution; (b) power law.

5. Summation of probabilities: the case of linear superposition

In this section we will continue the case $\varphi = \rho$ and pose the following problem: what happens if the particle under consideration has a choice to approach two different stochastic attractors with the probability densities $\rho_1^*(V)$ and $\rho_2^*(V)$? Rewriting Eq. (23) with reference to Eq. (24) in the form expressed via ρ for two different targets one obtains, respectively

$$\rho_1 = \rho_0 e^{-\xi t} + \rho_1^*(1-e^{-\xi t}) \tag{35}$$

$$\rho_2 = \rho_0 e^{-\xi t} + \rho_2^*(1-e^{-\xi t}) \tag{36}$$

Therefore

$$\hat{\rho} = \frac{1}{2}(\rho_1+\rho_2) = \rho_0 e^{-\xi t} + \frac{1}{2}(\rho_1^*+\rho_2^*)(1-e^{-\xi t}) \tag{37}$$

Thus in the particular case

$$\varphi = \rho \tag{38}$$

when a particle has a choice to approach the different targets, the resulting probability density is the normalized sum of probability densities of each particle.

Further investigation of the case (24) was performed in [1,2]. However since there is no interference of probabilities in this case, we will not go in more details confining ourselves only by illumination of origin of randomness and summation of probabilities.

6. Generalization to n-dimensional case

The one-dimensional system (19), (20) is generalized to n-dimensional case simply by replacing v with a vector $v = v_1, v_2, \ldots v_n$, and the probability density $\rho(V,t)$ with the joint probability density $\rho(V_1, \ldots V_n, t)$ since Eq. (20) does not include space derivatives

$$\dot{v}_i = \frac{\xi}{\rho(v,t)} \int_{-\infty}^{v_i} \frac{\varphi[\rho(\eta,t)] - \varphi[\rho^*(\eta)]}{d\varphi / d\rho} d\eta_i \qquad (39)$$

$$\frac{\partial\varphi}{\partial t} + n\xi\{\varphi[\rho(\eta,t)] - \varphi[\rho^*(\eta)]\} = 0 \qquad (40)$$

The solution of Eq. (40) is similar to that of Eq. (21)

$$\varphi = \{\varphi[\rho_0(V_1, \ldots V_n)] - \varphi[\rho^*(V_1, \ldots V_n)]\} e^{-n\xi t} + \varphi[\rho^*(V_1, \ldots V_n)] \qquad (41)$$

However the solution of Eq. (39) can be obtained in a closed analytical form only for the case $\varphi = \rho$

$$\int_{-\infty}^{v_k} \rho^*(\eta, v_k) d\eta = \frac{C_k}{e^{-t} - 1}, \qquad v_k \neq 0\ , \qquad k=1,2,\ldots n \qquad (42)$$

7. Interference of probabilities

As follows from Eq. (41), the Liouville equation that governs the evolution of the probability density ρ, is linear with respect to the probability function $\varphi(\rho)$. Therefore it is nonlinear with respect to ρ, unless $\varphi = \rho$ as it was in the case Eq. (24).

Let us turn to the problem being posed in Section 5: what happens if a particle has choices to approach the same attractors with the probability densities $\rho_1^*(V)$ and $\rho_2^*(V)$? Now instead of Eq. (37) we have

$$\varphi = (\varphi_1 + \varphi_2) = 2\varphi_0^{(1)} e^{-\xi t} + (\varphi_1^* + \varphi_2^*)(1 - e^{-\xi t}) \qquad (43)$$

Let us introduce the inverse of the probability function $\varphi = \varphi(\rho)$

$$\rho = \theta(\varphi) \qquad (44)$$

Then the summation of probabilities is ruled by the following formula

$$\rho = \theta(\varphi_1 + \varphi_2) = \theta[2\varphi_0^{(1)} e^{-\xi t} + (\varphi_1^* + \varphi_2^*)(1 - e^{-\xi t})] \qquad (45)$$

or in normalized form

$$\hat{\rho} = \frac{1}{C}\theta(\varphi_1 + \varphi_2) = \frac{1}{C}\theta[2\varphi_0^{(1)}e^{-\xi t} + (\varphi_1^* + \varphi_2^*)(1 - e^{-\xi t})] \quad (46)$$

$$C = \int_{-\infty}^{\infty} \theta[2\varphi_0^{(1)}e^{-\xi t} + (\varphi_1^* + \varphi_2^*)(1 - e^{-\xi t})]dV \quad (47)$$

Let us select

$$\varphi = \ln(\rho + \sqrt{1+\rho^2}) \quad (48)$$

Then

$$\rho = \sinh\varphi \quad (49)$$

$\rho = \sinh\varphi$

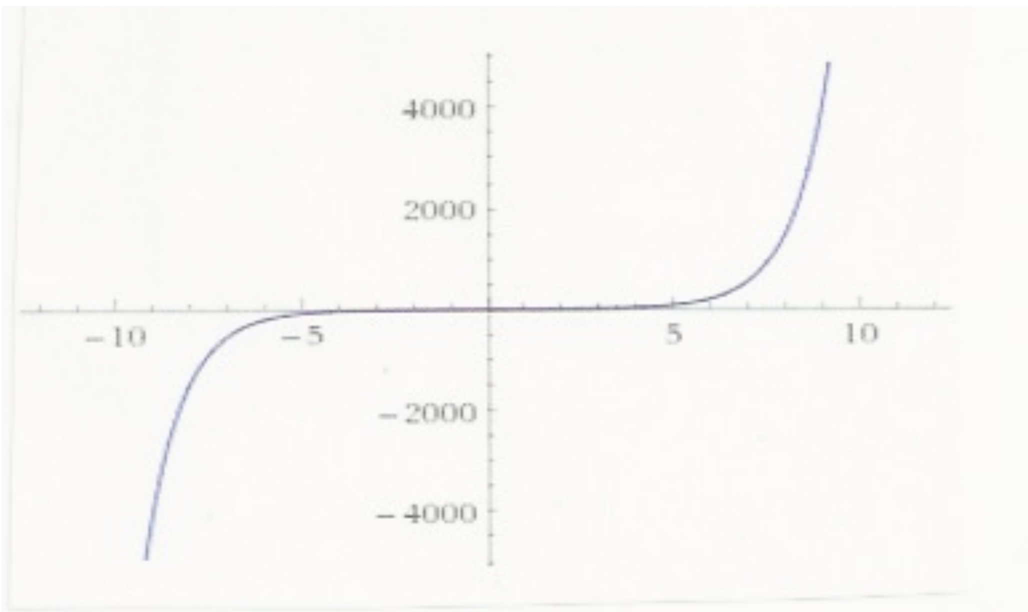

Figure 4. Selected hyperbolic sinus dependence ρ on φ.

As follows from Fig. 4, for small probability densities that are below the line $\rho = \varphi$, the interference is strongly destructive:

$$\rho_1 + \rho_2 << \varphi_1 + \varphi_2 \quad (50)$$

while for large probabilities that are above the line $\rho = \varphi$, the interference is strongly constructive:

$$\rho_1 + \rho_2 >> \varphi_1 + \varphi_2 \quad (51)$$

Therefore the sum of two interfering probabilities has a sharp peak that resembles "resonance", although its mechanism is totally different from the classical resonance. This phenomenon is qualitatively illustrated in Fig. (5). A less pronounced peak exists even in the double-slit experiment; however we have deliberately chosen the probability function $\varphi(\rho)$ in such a way that it has the largest deviation from the linear case Eq. (24), and that generates a strong constrictive interference effect for large densities that amplifies the maximum, and a strong destructive interference effect for small densities that suppress the “tails” of the distribution. From the information-processing viewpoint, the entropy of the resulting probability density sharply decreases and the dynamics becomes less random.

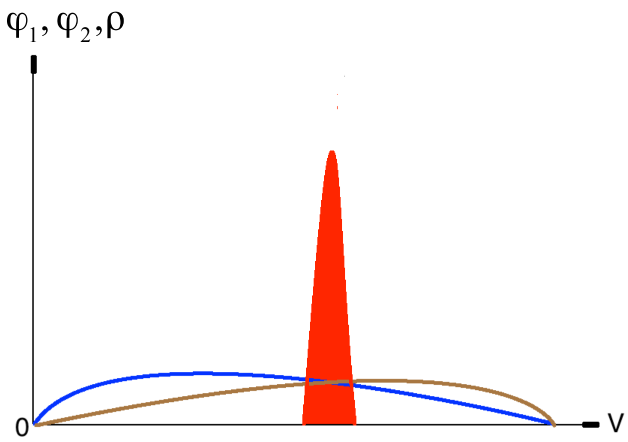

Figure 5. “Resonance” of the probabilities ρ_1 and ρ_2.

Obviously if we switch places φ and ρ in Eqs. (48) and (49), then for small probability densities that are above the line $\rho = \varphi$, the interference is strongly constructive while for large probabilities that are below the line $\rho = \varphi$, the interference is strongly destructive. Therefore the sum of two interfering probabilities will be flatter than the sum of the corresponding probability functions; that will lead to increase of entropy, and this case could be hardly exploited for applications.

Eqs. (43) and (46) can be generalized to interference of *n* target probabilities $\rho_i^*(V), \quad i = 1,2,...n$, respectively

$$\varphi = \sum_{i=1}^{n} a_i \varphi_i = n\varphi_0 e^{-\xi t} + \sum_{i=1}^{n} a_i \varphi_i^* (1 - e^{-\xi t}), \quad \sum_{i=1}^{m} a_i = 1 \tag{52}$$

$$\hat{\rho} = \frac{1}{C}\theta(\sum_{i=1}^{n} a_i \varphi_i) = \frac{1}{C}\theta[n\varphi_0 e^{-\xi t} + \sum_{i=1}^{n} a_i \varphi i (1 - e^{-\xi t})] \tag{53}$$

$$C = \int_{-\infty}^{\infty} \theta[n\varphi_0 e^{-\xi t} + \sum_{i=1}^{n} a_i \varphi_i (1 - e^{-\xi t})] dV \tag{54}$$

Here a_i constant weight coefficients.

8. From disorder to order

Prior to discussion of possible application of the dynamical systems with probability interference, we will derive a distinguish property of such systems that is associated with violation of the second law of thermodynamics. For that purpose, let us turn to Eq. (41) and notice that the probability densities ρ_0 and ρ^* are interchangeable

$$\rho_0 \Leftrightarrow \rho^* \tag{55}$$

i.e. the evolution of the probability density can start with ρ^* and approach ρ_0

$$\varphi_- = \{\varphi[\rho^*(V_1,...V_n)] - \varphi[\rho_0(V_1,...V_n)]\}e^{-n\xi t} + \varphi[\rho_0(V_1,...V_n)] \tag{56}$$

Such reversibility in ***probability space*** can happen neither in Newtonian, nor in quantum physics. Actually it violates the second law of thermodynamics. Indeed, if

$$H(\rho_0) = -\int_{-\infty}^{\infty} \rho_0 \ln \rho_0 dV > H(\rho^*) = -\int_{-\infty}^{\infty} \rho^* \ln \rho^* dV \tag{57}$$

then the reverse process Eq. (52) will have at least one interval at which the entropy decreases

$$\frac{dH(\rho)}{dt} < 0 \tag{58}$$

At the same time, the original system (10), (11) is isolated: it has no external interactions. Indeed the information force **F** is generated by the Liouville equation that, in turn, is generated by the equation of motion (10). Therefore the solution of Eqs. (10), and (11) can violate the second law of thermodynamics, and that confirms the conclusion that this class of dynamical systems does not belong to physics as we know it. This conclusion triggers the following question: are there any phenomena in Nature that can be linked to dynamical systems (10), (11)? In order to answer this question, let us turn to the Schrödinger paradox: in a world governed by the second law of thermodynamics, all isolated systems are expected to approach a state of maximum *disorder*; since life approaches and maintains a highly *ordered* state – one can argue that this violates the Second Law implicating a paradox, [4]. But livings are not isolated due to such processes as metabolism and reproduction: the increase of order inside an organism is compensated by an increase in disorder outside this organism, and that removes the paradox. Nevertheless it is still tempting to find a mechanism that drives livings from disorder to order in case when reproduction and metabolism are excluded from consideration: in this case the corresponding ***model*** becomes an isolated system, [2].

9. Emergence of novel patterns in information systems

Since the dynamical systems under consideration do not belong to the world of modern physics, we have to turn to phenomena that include a human factor: dynamics of social nets, dynamics of a crowd, dynamics of decision making process, dynamics of games in economics, etc. In such systems, the concept of energy that is fundamental in physics is replaced by the concept of information. This replacement started with the Madelung equation (8) in which the quantum potential is replaced by the information force (see Eq. (10)). Therefore application of the model introduced above targets ***information*** systems that support operations, management and decision making processes, and therefore, deals with human factor as a part of the dynamical system.

In this work, we will not go into specifics of possible applications, but rather generate a mathematical framework that would provide the rules of information processing. The phenomenon under consideration will be: emergence of novel patterns of behavior in information systems. For illustration, we will start with an information network described by the system of Eqs. (39) and (40). Recall that ODE (39) describe trajectories of n "agents" ; each "agent" can take randomly one trajectory out of the specific family of the trajectories. The joint probability density that controls randomness is described by the ODE (40).

The solution of Eqs. (39) eventually approaches the stochastic attractor

$$\rho^*(v_1), \rho^*(v_2)...\rho^*(v_n) \tag{59}$$

while the solution of Eq. (40), (see Eq. (41)) approaches the image of the stochastic attractor (59) in the probability space that is represented by the static attractor

$$\rho^*(V_1, V_2, ...V_n) \tag{60}$$

From the information-processing viewpoint, each kth component of the stochastic attractor (59) can be interpreted as an objective of the corresponding kth "agent", and the static attractor (60) - as an objective of the whole system.

Now let us assume that the system has a multiple objective, and importance of each of such an objective is measured by the weight coefficients a_i. This means that in physical space the original stochastic attractor (59) is replaced by a combination of the partial attractors

$$\rho^*(v_i) \to \tilde{a}_1\rho_1^*(v_l),...\tilde{a}_m\rho_m^*(v_i), \quad i = 1,2,...n \tag{61}$$

In probability space, the original static attractor (60) is replaced by the corresponding combination of static attractors

$$\rho^*(V_1, V_2, ...V_n) \to \tilde{a}_1\rho_1^*(V_1, ...V_n),...\tilde{a}_m\rho_m^*(V_1, ...V_n) \tag{62}$$

These attractors can be found from Eq. (39) and (41)

$$\dot{v}_i = \frac{\xi}{\rho(v,t)} \int_{-\infty}^{v_i} \frac{\varphi[\rho(\eta,t)] - \sum_{i=1}^{m} a_i \varphi[\rho_i^*(\eta)]}{d\varphi / d\rho} d\eta_i \tag{63}$$

$$\varphi = \sum_{i=1}^{n} a_i \varphi_i = n\varphi[\rho_0(V_1, ...V_n)]e^{-\xi t} + \sum_{i=1}^{m} a_i \varphi[\rho_i^*(V_1, ...V_n)(1 - e^{-\xi t}) \tag{64}$$

and therefore,

$$\hat{\rho} = \frac{1}{C}\theta(\sum_{i=1}^{m} a_i \varphi_i) =$$
$$\frac{1}{C}\theta\{n\varphi[\rho_0(V_1,...V_n)]e^{-\xi t} + \sum_{i=1}^{m} a_i \varphi[\rho_i^*(V_1,...V_n)(1-e^{-\xi t})\} \quad (65)$$

where C is expressed by Eq. (54).

Eqs. (63) describe random trajectories representing actions of the "agents". Their randomness is controlled by the probability density (65) found from the solution (64). Eq.(65) represents a static attractor (62) in the probability space that corresponds to the stochastic attractor (61) in physical space. But due to the nonlinearity of the probability function $\varphi(\rho)$, the partial stochastic attractors in Eq. (61) as well as the partial static attractors in Eq. (62) interfere, and that leads to creation of a fundamentally new attractor (65) that is different from the mean of the partial attractors.

In order to illustrate a potential application of the probability interference phenomenon, we will turn to the problem of language formation. Let us assume that we store ***letters*** of the alphabet in the form of the corresponding stochastic attractors ξ_η. Then if some of these letters, say $\xi_{\eta 1}..\xi_{\eta_l}$, are presented to the system Eq.(63) simultaneously, their processing will be accompanied by nonlinear interference in such a way that they will converge to a new attractor, say $\gamma_{1,2,...l}$ that can be associated with ***words***. This new attractor preserves the identities of the letters $\xi_{\eta 1}..\xi_{\eta_l}$, but at the same time, it is not a simple sum of these letters. Moreover, any additional letter $\xi_{\eta_{l+1}}$ may create a totally different new attractor $\delta_{1,2,...l,l+1}$.

Actually this phenomenon is similar to formation of words from letters, ***sentences*** from words, etc. In other words, the pattern interference creates a grammar by giving different meaning to different combinations of letters. This grammar is imposed by the form of the probability function $\varphi(\rho)$, and it can be varied if the probability function is presented as a polynomial

$$\varphi = b_1\rho + b_3\rho^3 + b_5\rho^5 + ...etc \quad (66)$$

Indeed, by changing the coefficients b_i in Eq. (66), one changes the way in which the patterns interfere and therefore, one language can be transformed into another one.

The procedure of creation of new patterns can be extended as following: collect the stochastic attractors γ_η obtained from the previous procedure, store the ***words*** $\gamma_{\eta_1}..\gamma_{\eta_l}$ and present them simultaneously to the system (63). Then as result of the next cycle of probability interference, the system converges to another attractor $\varepsilon_{1,2,...l}$ that can be associated with a new sentence, etc. The probability function $\varphi(\rho)$ can be different for each cycle.

Another potential application of interference of probabilities is modeling of a human crowd. It is a well-established fact that behavior of a crowd could be different from a weighted average of behavior of its members. In order to capture the phenomenon of emergence of new patterns, one should consider the crowd as a collection of members with different objectives. Then the spontaneous formation of an unexpected objective can be described by the system (63) after fitting the weight coefficients a_i and the coefficients b_i in Eq. (66).

10. Fast decision-making process

The decision-making process based upon the concept of a rational agent as well as upon psychological models has been discussed in [2]. In this work, we apply the interference of probability phenomena represented by the dynamical system introduced above to a ***fast*** decision-making process. Indeed in conflict situations, time is precious, and a late decision is as bad as a wrong decision; that leads to a possibility of a trade-off between the timing and the quality of decisions.

Any rational decision can be associated with maximization of a performance index expressed by the functional

$$J = \int_{t_0}^{t_f} \psi(\dot{v}_1, \ldots \dot{v}_n, v_1, \ldots v_n) dt \rightarrow Max \qquad (67)$$

Here $[t_0, t_f]$ is the time interval where the measure *J* is defined.

The problem is to find the optimal trajectory

$$v_i = v_i(t), \qquad i = 1, 2, \ldots n \qquad (68)$$

that delivers the global maximum to the performance measure (67) while satisfying the equality

$$\psi = \psi(v_1, \ldots v_n) \qquad at \qquad v_i \neq \alpha_{ij}, \qquad \beta_{ij} < \alpha_{ij} < \gamma_{ij}, \qquad (69)$$

and inequality

$$G_i(v_1, \ldots v_m) \leq 0, \qquad (70)$$

constraints.

This is a formidable problem that requires sophisticated methods such as dynamical programing or direct methods of calculus of variations, and even approximate solutions take time that is unacceptable in most of practical cases.

As shown in [5], the problem of maximization of the functional (67) can be reduced to finding the global maximum of a multi-dimensional function. But even that problem, in general, requires exponential computational resources.

The first step in a trade-off between timing and accuracy of the global maximum of a multi-dimensional function was proposed in [3]. The idea of the proposed algorithm is very simple: introduce a *positive* normalized function

$$\psi(v_1, v_2, \ldots v_n), \qquad |v_i| < \infty, \quad \int_{-\infty}^{\infty} \psi \, dv_1, \ldots dv_n = 1 \qquad (71)$$

to be maximized as the probability density

$$\rho^*(v_1, v_2, \ldots v_n), \quad \int_{-\infty}^{\infty} \rho^* dv_1, \ldots dv_n = 1, \qquad (72)$$

to which the solution of Eq. (40) is attracted. Then the larger value of this function will have the higher probability to appear. It was concluded that, after *polynomial* number of trials, one arrived at the solution to the problem (*unless the function* ψ *is flat*). The comment in the brackets is essential: usually the agent does not now the exact formulation of the performance measure (72); instead he has approximation that is relatively flat while the degree of flatness corresponds to the degree of incompleteness of his knowledge. As a result, his decision could take time even when he applies the Bernoulli trail described in [3].

Considering, for simplicity, one-dimensional version of the Eq. (72)

$$\rho^* = \rho^*(v), \quad \int_{-\infty}^{\infty} \rho^* dv = 1 \tag{73}$$

assume that the agent has several alternative versions to the curve (73) expressed in the form of the weighted sequence

$$\rho^*(v) \to \tilde{a}_1\rho_1^*(v),...\tilde{a}_m\rho_m^*(v), \quad i = 1,2,...n \tag{74}$$

while each of them is almost flat, i.e. taken alone it is not sufficient for making a fast decision.

Then the following question can be asked: what is the best strategy for making fast decision based upon incomplete knowledge presented by the sequence (74)?

It should be noticed that the sequence (9.139) does not suggest an exact formulation of the problem.

Let us assume that the agent's activity is described by the following equations

$$\dot{v} = \frac{\xi}{\rho(v,t)} \int_{-\infty}^{v_i} \frac{\varphi[\rho(\eta,t)] - \sum_{i=1}^{m} a_i \varphi[\rho_i^*(\eta)]}{d\varphi / d\rho} d\eta, \tag{75}$$

$$\varphi = \sum_{i=1}^{m} a_i \varphi_i = n\varphi[\rho_0(V_1,...V_n)]e^{-\xi t} + \sum_{i=1}^{m} a_i \varphi[\rho_i^*(V_1,...V_n)(1-e^{-\xi t}) \tag{76}$$

i.e. the agent has a capability to exploit the interference between the terms of the sequence (74) considered as alternative version of the stochastic attractor

$$\rho^* = \rho_i^*(v), \quad i = 1,2,...m \tag{77}$$

Then the resulting probability density as a result of interference of the densities (77) is

$$\hat{\rho} = \frac{1}{C}\theta(\sum_{i=1}^{m} a_i \varphi_i) = \\ \frac{1}{C}\theta\{\varphi[\rho_0(V)]e^{-\xi t} + \sum_{i=1}^{m} a_i \varphi[\rho_i^*(V)(1-e^{-\xi t})\} \tag{78}$$

The type of interference is imposed by the form of the probability function $\varphi(\rho)$, and, as in the previous Section, it can be varied if the probability function is presented as a polynomial

$$\varphi = b_1\rho + b_3\rho^3 + b_5\rho^5 + ...etc \qquad (79)$$

If the agent is capable to approach the probability function close to that in Fig. 4, then the resulting probability density will be closed to the delta-function (see Fig. 5). If the agent applies the Bernoulli trial described in [3], he is able to make the decision instantaneously. In addition to that, this decision is not only fast, but also reasonably good since the location of the maximum of the resulting probability density (78) is the weighted sum of locations of the maxima of the functions (77) that describe the whole available information.

The agent's capability of making fast and reasonably good decisions based upon interference of partial and incomplete information available to him can be associated with a "human factor", or intuition, and that distinguishes this agent from a rational agent that does not possesses such a capability.

11. Discussion and conclusion

Interference of probabilities as a physical phenomenon follows from quantum mechanics as a result of special property of the Schrödinger equation that is linear with respect to the probability amplitude rather than with respect to the probability itself. That leads to the rule (6) for summation of the probabilities, and this rule cannot be varied. In this paper, similar strategy is used for finding a new class of dynamical systems with interference of probabilities. Instead of the probability amplitude, a probability function $\varphi(\rho)$ has been introduced. Then starting with the Madelung version of the Schrödinger equation, the quantum potential has been replaced by the information force in such a way that the governing equations became linear with respect to the probability function $\varphi(\rho)$. Despite of a few restrictions imposed upon this function, (see Eqs. (15)-(17)), there is a lot of freedom in choosing it, and therefore, unlike the quantum rule, this function can be varied to extend of presetting the type of interference of probabilities.

Although this new class of dynamical systems does not violate ***mathematical*** rules, it does violate the ***physical*** rules, and in particular, it violates the second law of thermodynamics. That means that this class of systems does not belong to physics, as we know it: it rather belongs to extended physics that includes a human factor.

Thus a new class of dynamical systems with a preset type of interference of probabilities is introduced. It is obtained from the extension of the Madelung equation by replacing the quantum potential with a specially selected feedback from the Liouville equation. It has been proved that these systems are different from both Newtonian and quantum systems, but they can be useful for modeling spontaneous collective novelty phenomena when emerging outputs are qualitatively different from the weighted sum of individual inputs. Formation of language, and fast decision-making process as potential applications of the probability interference is discussed.

References

1. Zak, M., 2013, Information: Forgotten Variable in Physics Models, EJTP 10, No. 28, 199–220,

2. . Zak, M., 2013, Particle of life: mathematical abstraction or reality? , Nova publishers, NY.

3. Zak, M., 2008, Quantum-inspired maximizer, JOURNAL OF MATHEMATICAL PHYSICS **49,** 042702,
4. Schrödinger, E**.** What is life, Cambridge University Press, 1944.
5. Zak, M., 2009, Optimal Control Via Self-Generated Stochasticity, The Open Applied Mathematics Journal, Volume 3