Analytical prediction and representation of chaos.

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Abstract.

1. Introduction

The concept of randomness entered Newtonian dynamics almost a century ago: in 1926, Synge, J. introduced a new type of instability - orbital instability- in classical mechanics, [1], that can be considered as a precursor of chaos formulated a couple of decades later, [2]. The theory of chaos was inspired by the fact that in recent years, in many different domains of science (physics, chemistry, biology, engineering), systems with a similar strange behavior were frequently encountered displaying irregular and unpredictable behavior called chaotic. Currently the theory of chaos that describes such systems is well established. However there are still two unsolved problem remain: prediction of chaos (without numerical runs), and analytical description of chaos in term of the probability density that would formally follow from the original ODE. This paper proposes a contribution to the solution of these problems.

2 Randomness in chaotic systems

In this Section we present a sketch of general theory of chaos in context of existing analytical results starting with the flow generated by an autonomous ODE

$$\frac{dx_i}{dt} = V_i(\mathbf{x}), \quad i = 1,2,...m \tag{1}$$

and compare two neighboring trajectories in *m*-dimensional phase space with initial conditions x_0 and $x_0 + \Delta x_0$ denoting $\Delta x_0 = w$. These evolve with time yielding the tangent vector $\Delta x(x_0, t)$ with its Euclidian norm

$$d(x_0, t) = \left\| \Delta x(x_0, t) \right\| \tag{2}$$

Now the Liapunov exponent can be introduced as the mean exponential rate of divergence of two initially close trajectories

$$\tilde{\lambda}(x_0, w) = \lim_{\substack{t\to\infty \\ d(0)\to 0}} \left(\frac{1}{t}\right) \ln \frac{d(x_0, t)}{d(x_0, 0)} \tag{3}$$

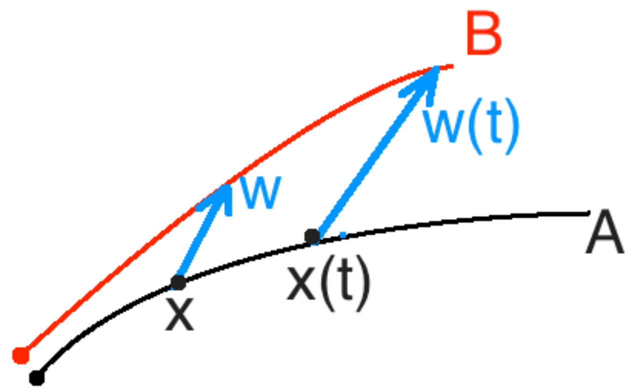
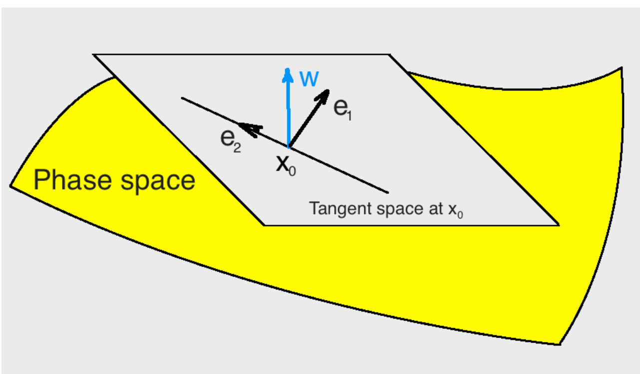


Figure 1. Two nearby trajectories that separate as time evolves.



Ficure 2. Tangent space for the Liapunov exponents.

Therefore in general the Lyapunov exponent cannot be analytically expressed via the parameters of the underlying dynamical system (as it was done in case of inertial motion on a pseudosphere), and that makes prediction of chaos a hard task. However some properties of the Liapunov exponents can be expressed in an analytical form. Firstly, it can be shown that in an *m*-dimensional space, there exist *m* Liapunov exponents

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \ldots \geq \tilde{\lambda}_m \tag{4}$$

while at least one of them must vanish. Indeed, as follows from Eqs. (1) and (2), **w** grows only linearly in the direction of the flow, and the corresponding Liapunov exponent is zero. Secondly it has been proven that the sum of the Liapunov exponents is equal to the average phase space volume contraction

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \Lambda_0 \tag{5}$$

where the instantaneous phase space volume contraction

$$\Lambda = \nabla \cdot \mathbf{V} \tag{6}$$

But

$$\Lambda_0 = \Lambda \quad (7)$$

when

$$\nabla \cdot \mathbf{V} = const \quad (8)$$

Therefore in case (20), the sum of the Liapunov exponents is expressed analytically

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \nabla \cdot \mathbf{V} \quad (9)$$

Thus the result we extracted from the theory of chaos, which can be used for comparison to quantum randomness is the following: the origin of randomness in Newtonian mechanics is instability of ignorable variables that leads to exponential divergence of initially adjacent trajectories; this divergence is measured by Liapunov exponents, which form a discrete spectrum of numbers that must include positive ones.

3. Orbital instability as a precursor of chaos

In this and the next Sections we take a non-traditional approach to chaos starting with orbital instability as its precursor.

Chaos is a special type of instability when the system does not have an alternative stable state and displays an irregular aperiodic motion. Obviously this kind of instability can be associated only with ignorable variables, i.e. with such variables that do not contribute into energy of the system. In order to demonstrate this kind of instability, consider an inertial motion of a particle M of unit mass on a smooth pseudosphere S having a constant negative curvature G_0, Fig. 3.

$$G_0 = const > 0 \quad (10)$$

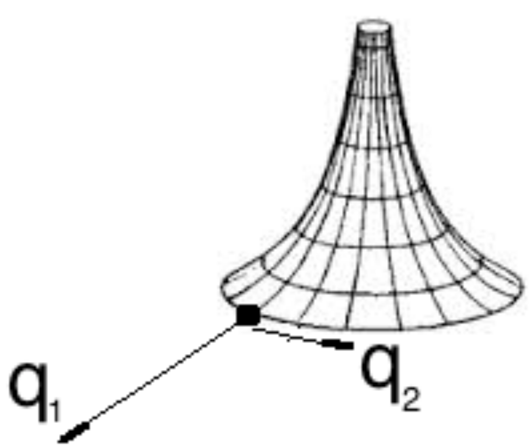


Figure 3. Inertial motion on a smooth pseudosphere.

Remembering that trajectories of inertial motions must be geodesics on S, compare two different trajectories assuming that initially they are parallel, and the distance ε_0 between them, are small (but not infinitesimal!),

$$0 < \varepsilon_0 << 1 \qquad (11)$$

As shown in differential geometry, the distance between these geodesics increases exponentially

$$\varepsilon = \varepsilon_0 e^{\sqrt{-G_0} t}, \quad G_0 < 0, \qquad (12)$$

Hence no matter how small the initial distance ε_0, the current distance ε tends to infinity.

Let us assume now that accuracy to which the initial conditions are known is characterized by the scale *L*. This means that any two trajectories cannot be distinguished if the distance between them is less than *L* i.e. if

$$\varepsilon < L \qquad (13)$$

The period during which the inequality (4) holds has the order

$$\Delta t \approx \frac{1}{\sqrt{|-G_0|}} \ln \frac{L}{\varepsilon_0} \qquad (14)$$

However for

$$t >> \Delta t \qquad (15)$$

these two trajectories diverge such that they can be easily distinguished and must be considered as two different trajectories. Moreover the distance between them tends to infinity no matter how small is ε_0. That is why the motion once recorded cannot be reproduced again (unless the initial condition are known exactly), and consequently it attains stochastic features. The Liapunov exponent for this motion is positive and constant

$$\sigma = \lim_{\substack{t \to \infty \\ \varepsilon_0 \to 0}} [\frac{1}{t} \ln \frac{\varepsilon_0 e^{\sqrt{-G_0} t}}{\varepsilon_0}] = \sqrt{-G_0} = const > 0 \qquad (16)$$

Remark. In theory of chaos, the Liapunov exponent measures divergence of initially close trajectories averaged over infinite period of time. But in this particular case, even “instantaneous” Liapunov exponent taken at a fixed time has the same value (16).

Let us introduce a system of coordinates on the surface S: the coordinate q_1 along the geodesic meridians and the coordinate q_2 along the parallels. In differential geometry such a system is called semigeodesic. The square distance between adjacent points on the pseudosphere is

$$ds = g_{11} dq_1^2 + 2q_{12} dq_1 dq_2 + g_{22} dq_2^2 \qquad (17)$$

where

$$g_{11} = 1, \quad q_{12} = 0, \quad g_{22} = -\frac{1}{G_0} e^{(-2\sqrt{-G_0}q_1)} \tag{18}$$

The Lagrangian for the inertial motion of the particle *M* on the pseudosphere is expressed via the coordinates and their temporal derivatives as

$$L = g_{ij}\dot{q}_i\dot{q}_j = \dot{q}_1^2 - \frac{1}{G_0} e^{(-2\sqrt{-G_0}q_1)} \dot{q}_2^2 \tag{19}$$

and consequently,

$$\frac{\partial L}{\partial q_2} = 0 \tag{20}$$

$$\frac{\partial L}{\partial q_1} \neq 0 \quad if \quad \dot{q}_2 \neq 0 \tag{21}$$

Hence q_1 and q_2 play the roles of position and ignorable coordinates, respectively, and therefore, the inertial motion of a particle on a smooth pseudosphere is unstable with respect to the ***ignorable*** coordinate. This instability known as orbital instability is not bounded by energy and it can persist indefinitely. As shown in [2], eventually orbital instability leads to stochasticity. Later on such motions were identified as chaotic.

4. Analytical criteria for prediction of chaos

a. Inertial motions. The results described above were related to inertial motions of a particle on a smooth pseudosphere. However they can be generalized to motions of any degree-of-freedom dynamical systems by using the concept of configuration space. Indeed if the dynamical system has *N* generalized coordinates q^i *(i= 1,2,…N)* and is characterized by the kinetic energy

$$W = a_{ij}\dot{q}^i\dot{q}^j \tag{22}$$

then the configuration space can be introduced as an *N*-dimensional space with the following metric tensor

$$g_{ij} = a_{ij} \tag{23}$$

while the motion of the system is represented by the motion of the unit- mass particle in this configuration space.

In order to continue the analogy to the motion of the particle on a surface in actual space we will consider only two-dimensional sub-spaces of the *N*-dimensional configuration space, without loss of generality. Indeed, a motion that is unstable in any such subspace has to be qualified as unstable in the entire configuration space.

Now the Gaussian curvature of a two-dimensional configuration space (q^1,q^2) follows from the Gauss formula

$$G = \frac{1}{a_{11}a_{22} - a_{12}^2}\left(\frac{\partial^2 a_{12}}{\partial q^1 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{11}}{\partial q^2 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{22}}{\partial q^1 \partial q^1}\right) - \Gamma_{12}^{\gamma}\Gamma_{12}^{\delta}a_{\gamma\delta} - \Gamma_{11}^{\alpha}\Gamma_{22}^{\beta}a_{\alpha\beta} \tag{24}$$

where the connection coefficients Γ_{sk}^{l} are expressed via the Christoffel symbols

$$\Gamma_{sk}^{l}=\frac{1}{2}a_{ip}(\frac{\partial a_{sp}}{\partial q^{k}}+\frac{\partial a_{kp}}{\partial q^{s}}-\frac{\partial a_{sk}}{\partial q^{p}}) \tag{25}$$

while

$$a^{\alpha\beta}a^{\beta\chi}=a_{\gamma}^{\alpha}=0 \quad if \quad \alpha\neq\gamma$$

(26)

$$a^{\alpha\beta}a^{\beta\chi}=a_{\gamma}^{\alpha}=1 \quad if \quad \alpha=\gamma$$

Thus the Gaussian curvature of these subspaces depends only on the coefficients a_{ij}, i.e ., it is fully determined by the kinematical structure of the system (see Eq. (22). In the case of inertial motions, the trajectories of the representative particle must be geodesics of the configuration space. If the Gaussian curvature (24) is negative

$$G<0 \tag{27}$$

then the trajectories of the inertial motions of the system that originated at close points of the configuration space diverge exponentially from each other, and the motion becomes chaotic, (see Fig. 3).

As proved in [1], orbital instability, and therefore, chaotic motion occurs if the Gaussian curvature is negative in each of two-dimensional subspace.

Example 1. Consider a double pendulum represented by a two bar linkage, i.e., a system of two rigid rods *AB* and *CD* connected by an ideal hinge B and rotating about a vertical axis *x* normal to the plane *ABC* (Fig.4a). Setting

$$q^{1}=\varphi_{1},\quad q^{2}=\varphi_{2} \tag{28}$$

one obtains their kinetic energy

$$2W=a_{11}\dot{\varphi}_{1}^{2}+a_{12}\dot{\varphi}_{1}\dot{\varphi}_{2}+a_{22}\dot{\varphi}_{2}^{2} \tag{29}$$

where

$$a_{11}=(I_{1}+mr^{2}),\quad a_{12}=mrl\cos(\varphi_{2}-\varphi_{1}),\quad a_{22}=I_{2} \tag{30}$$

While I_1 and I_2 are the moments of inertia of the rods *AB* and *BC* with respect to the vertical axes passing through the points *A* and *B*, respectively, m is the mass of the rod *BC*, *r* is the length of the rod *BC*, and l is the distance between point *B* and the center of inertia of the rod *BC*. Taking in the account that

$$\Gamma_{11}^{1}=\frac{-mrl\sin(\varphi_{2}-\varphi_{1})}{a^{2}}a_{12},\quad \Gamma_{22}^{1}=\frac{-mrl\sin(\varphi_{2}-\varphi_{1})}{a^{2}}a_{22}, \tag{31}$$

$$\Gamma_{11}^{2}=\frac{-mrl\sin(\varphi_{2}-\varphi_{1})}{a^{2}}a_{11},\quad \Gamma_{22}^{2}=\frac{-mrl\sin(\varphi_{2}-\varphi_{1})}{a^{2}}a_{12} \tag{32}$$

$$\Gamma_{12}^{1}=\Gamma_{12}^{2}=0,\quad a^{2}=a_{11}a_{22}-a_{12}^{2} \tag{33}$$

one arrives at the following expression for the Gaussian curvature of the configuration space (see Eq. (24))

$$G=\frac{mrl}{a^{2}}\{1+\frac{[mrl\sin(\varphi_{2}-\varphi_{1}]^{2}}{a^{2}}\}^{2}\cos(\varphi_{2}-\varphi_{1}) \tag{34}$$

where

$$G < 0 \quad if \quad \pi > |\varphi_2 - \varphi_1| > \frac{\pi}{2} \tag{35}$$

and

$$G > 0 \quad if\ |\varphi_2 - \varphi_1| < \frac{\pi}{2} \tag{36}$$

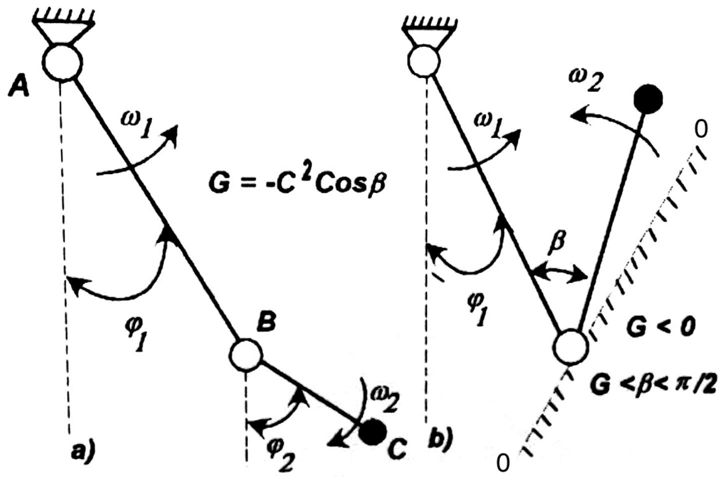


Figure 4. Chaotic oscillations of double pendulum.

The shaded line 0 – 0 separates the chaotic area (left) from the stable area (right) in Fig. 4b.

Example 2. Consider a symmetric rigid body rotating about its center of gravity, Fig.5.

Determining its position by Euler's angles

$$\theta = q^1, \psi = q^2, \quad \phi = q^3 \tag{37}$$

one obtains the following expression for the kinetic energy

$$W = \frac{1}{2}[A(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + C(\dot{\phi} + \dot{\psi}\cos\theta)^2] \tag{38}$$

in which *A* and *C* are the axial moments of inertia. Then the metric coefficients of the Christoffel symbols and the Gaussian curvature of two-dimensional subspaces are, respectively

$$a_{11} = A, a_{12} = 0, a_{22} = A\sin^2\theta + C\cos^2\theta, a_{13} = 0, a_{23} = C\cos\theta, a_{33} = 0, \tag{39}$$

$$\Gamma^1_{22} = \frac{C - A}{A}\sin\theta\cos\theta, \quad \Gamma^1_{23} = \frac{C}{2A}\sin\theta, \quad \Gamma^2_{21} = \frac{2A - C}{2A}\cos\theta, \tag{40}$$

$$\Gamma^2_{31} = \frac{C}{2A}\sin\theta, \quad \Gamma^2_{21} = -\frac{1}{2\sin\theta}(\frac{A - C}{A}\cos^2\theta + 1), \quad \Gamma^3_{31} = \frac{C}{2A}\cot\theta \quad (41),$$

$$G_{(12)} = \frac{1}{A(A\sin^2\theta + C\cos^2\theta)}[2(A-C)\cos 2\theta$$
$$+(\frac{2A-C}{2A}\cot\theta)^2(A\sin^2\theta + C\cos^2\theta)] \tag{42}$$

$$G_{(13)} = 0, \quad G_{(23)} = 0 \tag{43}$$

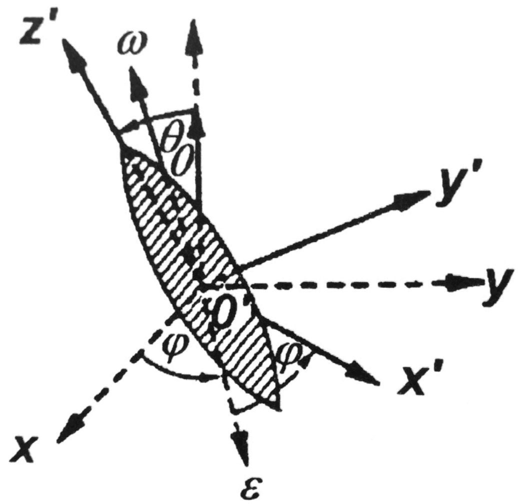


Figure 5. Chaotic rotating of a symmetric rigid body.

Now the condition for chaos can be presented as

$$G_{(12)} < 0 \tag{44}$$

where $G_{(12)}$ is the Gaussian curvature in the subspace θ, ψ ., (see eq. (42)).
Assuming, for simplicity, that *2A=C*, one reduces the condition (44) to the following

$$\cos 2\theta > 0 \tag{45}$$

Thus any motion in the subspace θ, ψ is chaotic if

$$0 < \theta < \frac{\pi}{4} \tag{46}$$

b. Potential motions. Turning back to the motion of the particle M on a smooth pseudosphere (Fig. 3), let us depart from inertia motions and introduce a force ***F*** acting on this particle. For noninertial motions the trajectories of the particle will not be geodesics, while the rate of their deviation from geodesics is characterized by the geodesic curvature χ. It is obvious that this curvature must depend of the forces *F*:

$$\chi = \chi(\mathbf{F}) \tag{47}$$

It has been shown in [1] that if the force ***F*** is potential

$$\mathbf{F} = -\nabla\Pi \tag{48}$$

where Π is the potential energy, then the condition (27) is replaced by

the following

$$G_0 + 3\chi^2 + \frac{1}{W}\left(\frac{\partial^2\Pi}{\partial q^i \partial q^j} - \Gamma^k_{ij}\frac{\partial\Pi}{\partial q^k}\right)n^i n^j < 0 \tag{49}$$

Here Γ^k_{ij} are defined by Eq. (25), and n^i are the contravariant components of the unit normal **n** to the trajectory**,** and G_0 is the Gaussian curvature for the case of inertial motion.

The geodesic curvature χ in Eq. (49) can be expressed via the potential force **F**

$$\chi = \frac{\mathbf{F}\cdot\mathbf{n}}{2W} = -\frac{\nabla\Pi}{2W} \tag{50}$$

As follows from Eqs. (49) and (50), the condition (49) is reduced toEq. (27) if $\boldsymbol{F} = 0$.

Example. Suppose that the elastic force

$$F = -\alpha^2\nu, \quad \alpha = const \tag{51}$$

proportional to the normal deviation ν from the geodesic trajectory is applied to the particle M moving on the smooth pseudosphere. If the initial velocity is directed along one of the meridians (which are all geodesics), the unperturbed motion will be inertial, and its trajectory will coincide with this meridian since there $\nu = 0$, and therefore, $F = 0$, In order to verify the orbital instability of this motion, let us turn to the criterion (49). Since

$$\chi = 0 \quad \text{and} \quad \frac{\partial\Pi}{\partial q^k} = F^k = 0 \tag{52}$$

for the unperturbed motion, one obtains the condition for chaos

$$G_0 + \frac{\alpha^2}{2W} < 0, \quad i.e. \quad \alpha^2 > -2WG_0, \quad G_0 < 0 \tag{53}$$

c. General case. So far we discussed the conservative chaos. But the main attention to chaotic motions was attracted by dissipative systems that possess so called "strange attractors". Following J.Synge, [1] the results for the orbital instability of inertial and potential motions can be generalized to arbitrary motions. For that purpose, instead of the

Gaussian curvature of two-dimensional subspaces (49) one has to introduce the Riemannian curvature that can be expressed by the following covariant curvature tensor

$$G_{msnl} = \frac{\partial \Gamma^m_{nl}}{\partial q^s} - \frac{\partial \Gamma^m_{ns}}{\partial q^l} + (\Gamma^u_{nl}\Gamma^v_{ms} - \Gamma^u_{ns}\Gamma^v_{ml}\}a^{uv}, \quad a^{uv}a_{vp} = \delta^p_u \qquad (53)$$

Here Γ^r_{mn} are the Christoffel symbols defined by Eq. (25) as well as the derivative of the generalized force Q^r

$$Q_{rs} = (\frac{\partial Q^r}{\partial q^l} + \Gamma^r_{\ln}Q^n)a_{ls} \qquad (54)$$

while the metric tensor of the configuration space is expressed by Eq. (22).

As shown in [1], the orbital instability, and therefore, chaotic motion occurs if the Riemannian curvature of the manifold of configurations corresponding to every two-space element containing the direction of the given trajectory is no positive, and if $Q_{mn}q^m q^n$ is no negative for arbitrary values of q^r at all points of the trajectory.

The significance of this result is in the fact that all the components of the criterion of chaos are uniquely defined by the coefficients of the governing ODE's. However this result gives sufficient, but not necessary condition for chaos. That is why we have concentrated on inertial motions for which the criterion of chaos is sufficient and necessary.

References

1. J. Synge, 1926, On the geometry of dynamics, Phil. Trans. R. Sos. Lond., Ser.A 226, 31-106.
2. V. Arnold, 1988, Mathematical methods of classical mechanics, Springer, New York