“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 5/6.***”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.***”,

which has been published under the **‘VIXRA’ Mathematics subheading:- ‘*Functions and Analysis*’**.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 5/6.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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are irrefutably continuous, being a characteristic property of all such real variable polynomial functions, and hence the quaternion hyper-complex function, $q'(t)$, is also continuous.

Evidently, the continuity of $q'(t)$ is dependent upon that of $q(t)$, in conjunction with the proviso that $q'(t) \neq 0$, $\forall t \in (a,b)$, so much so that it indeed epitomises the smoothness of such arcs and hence, for this reason, we shall explicitly refer to a smooth arc, C, whenever we talk about the derivatives of quaternion hypercomplex functions for the remainder of this section. Finally, we remark that the meaning of Definition DIV-6 should also be self-explanatory insofar as it purports to be nothing else other than a logical extension of its analogue from complex variable analysis.

3. Differentiation Formulae pertaining to Sums, Products and Quotients of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q-Space.

Various differentiation formulae pertaining to the sums, products and quotients of both real and complex variable functions have already been well established and this in turn motivates us to subsequently derive appropriate analogues thereof with respect to our ensuing analysis of the properties of quaternion hypercomplex functions. In so doing, we are naturally mindful of the provisos originating from the preceding Parts 1 and 2 of this section, as is evident from our next theorem which has been explicitly stated and likewise verified for that very purpose :-

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<u>Theorem TIV - 2</u>

Let there exist two quaternion hypercomplex functions, $\phi_1(q)$ and $\phi_2(q)$, which are restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \ \forall t \in [a, b] .$$

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Hence, it may be shown that, if $\phi_1(q)$ and $\phi_2(q)$ respectively are both differentiable in 't', $\forall t \in (a, b)$, then the following differentiation formulae shall apply, $\forall t \in (a, b)$;-

$$\text{(i)} \ \left[\frac{d}{dq}\right]_C (\phi_1(q) + \phi_2(q)) = \begin{cases} \dfrac{\left(\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}[\phi_2(q(t))]\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\left(\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}[\phi_2(q(t))]\right)}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} ,$$

$$\text{(ii)} \ \left[\frac{d}{dq}\right]_C (\phi_1(q)\phi_2(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} ,$$

where $\frac{d}{dt}[f(q(t))] = \phi_1(q(t))\frac{d}{dt}[\phi_2(q(t))] + \frac{d}{dt}[\phi_1(q(t))]\phi_2(q(t))$,

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(iii) $\left[\frac{d}{dq}\right]_C(\phi_2(q)\phi_1(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\overline{\frac{d}{dt}[q(t)]}}{|\frac{d}{dt}[q(t)]|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}[f(q(t))]}{|\frac{d}{dt}[q(t)]|^2} \end{cases},$

where $\frac{d}{dt}[f(q(t))] = \phi_2(q(t))\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}[\phi_2(q(t))]\phi_1(q(t))$,

(iv) $\left[\frac{d}{dq}\right]_C\left(\frac{\phi_1(q)}{\phi_2(q)}\right) = \begin{cases} \frac{d}{dt}[\psi_1(q(t))]\overline{\frac{d}{dt}[q(t)]}/|\frac{d}{dt}[q(t)]|^2 \\ \overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}[\psi_1(q(t))]/|\frac{d}{dt}[q(t)]|^2 \\ \frac{d}{dt}[\psi_2(q(t))]\overline{\frac{d}{dt}[q(t)]}/|\frac{d}{dt}[q(t)]|^2 \\ \overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}[\psi_2(q(t))]/|\frac{d}{dt}[q(t)]|^2 \end{cases},$

where the derivatives,

$$\frac{d}{dt}[\psi_1(q(t))] = \phi_1(q(t))\frac{d}{dt}\left[\overline{\phi_2(q(t))}/|\phi_2(q(t))|^2\right] + \frac{d}{dt}[\phi_1(q(t))]\left(\overline{\phi_2(q(t))}/|\phi_2(q(t))|^2\right) ,$$

$$\frac{d}{dt}[\psi_2(q(t))] = \left(\overline{\phi_2(q(t))}/|\phi_2(q(t))|^2\right)\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}\left[\overline{\phi_2(q(t))}/|\phi_2(q(t))|^2\right]\phi_1(q(t)) ,$$

$$\frac{d}{dt}\left[\overline{\phi_2(q(t))}/|\phi_2(q(t))|^2\right] = \frac{|\phi_2(q(t))|^2\frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] - \frac{d}{dt}\left[|\phi_2(q(t))|^2\right]\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^4}$$

and $\phi_2(q(t)) \neq 0$, $\forall t \in (a,b)$.

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PROOF :-

From Theorem T IV-1, we previously noted that the derivative of any function, 'f', thus constrained to a smooth arc, C, is generally given by

$$\left[\frac{d}{dq}\right]_C (f(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases}, \forall t \in (a,b).$$

Consequently, we are now in a position to directly apply this formula in those cases where 'f' is expressed consecutively as a sum, product and quotient of its component functions, 'ϕ_1' and 'ϕ_2'.

(i) Let us initially define a function,

$$f(q(t)) = \phi_1(q(t)) + \phi_2(q(t)) \Longleftrightarrow f(q) = \phi_1(q) + \phi_2(q).$$

We now construct a difference quotient for such a function which is

accordingly expressed as -

$$\frac{f(q(T)) - f(q(t))}{T-t} = \frac{\phi_1(q(T)) + \phi_2(q(T)) - (\phi_1(q(t)) + \phi_2(q(t)))}{T-t}$$

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$$= \frac{\phi_1(q(T)) - \phi_1(q(t)) + \phi_2(q(T)) - \phi_2(q(t))}{T-t}$$

$$= \frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} + \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}, \quad \forall t, T \in \mathbb{R}.$$

Furthermore, if $\phi_1(q)$ and $\phi_2(q)$ are differentiable functions, with respect to the real parameter, 't', and we also set

$T = t + h$,

then clearly

$$\lim_{T \to t}\left[\frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t}\right] = \frac{d}{dt}\left[\phi_1(q(t))\right],$$

$$\lim_{T \to t}\left[\frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}\right] = \frac{d}{dt}\left[\phi_2(q(t))\right],$$

and hence, in accordance with Theorem TIII − 6, it naturally follows that

$$\lim_{T \to t}\left[\frac{f(q(T)) - f(q(t))}{T-t}\right] = \lim_{T \to t}\left[\frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t} + \frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}\right]$$

$$= \frac{d}{dt}\left[\phi_1(q(t))\right] + \frac{d}{dt}\left[\phi_2(q(t))\right]$$

AND

$$\lim_{T \to t}\left[\frac{f(q(T)) - f(q(t))}{T-t}\right] = \frac{d}{dt}\left[f(q(t))\right]$$

$$\therefore \frac{d}{dt}\left[f(q(t))\right] = \frac{d}{dt}\left[\phi_1(q(t))\right] + \frac{d}{dt}\left[\phi_2(q(t))\right].$$

Finally, we deduce that

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$$\left[\frac{d}{dq}\right]_c(\phi_1(q)+\phi_2(q)) = \begin{cases} \dfrac{\left(\frac{d}{dt}[\phi_1(q(t))]+\frac{d}{dt}[\phi_2(q(t))]\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\left(\frac{d}{dt}[\phi_1(q(t))]+\frac{d}{dt}[\phi_2(q(t))]\right)}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} .$$

<u>Q.E.D.</u>

(ii) Let us initially define a function,

$$f(q(t)) = \phi_1(q(t))\,\phi_2(q(t)) \Longleftrightarrow f(q) = \phi_1(q)\,\phi_2(q).$$

We now construct a difference quotient for such a function, which is accordingly expressed as –

$$\frac{f(q(T))-f(q(t))}{T-t} = \frac{\phi_1(q(T))\,\phi_2(q(T)) - \phi_1(q(t))\,\phi_2(q(t))}{T-t}$$

$$= \frac{\begin{bmatrix}\phi_1(q(T))\,\phi_2(q(T)) - \phi_1(q(T))\,\phi_2(q(t)) + \\ \phi_1(q(T))\,\phi_2(q(t)) - \phi_1(q(t))\,\phi_2(q(t))\end{bmatrix}}{T-t}$$

$$= \frac{\phi_1(q(T))\left[\phi_2(q(T))-\phi_2(q(t))\right]+\left[\phi_1(q(T))-\phi_1(q(t))\right]\phi_2(q(t))}{T-t}$$

$$= \phi_1(q(T))\left[\frac{\phi_2(q(T))-\phi_2(q(t))}{T-t}\right]+\left[\frac{\phi_1(q(T))-\phi_1(q(t))}{T-t}\right]\phi_2(q(t)),$$

$\forall t, T \in \mathbb{R}$.

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Furthermore, if $\phi_1(q)$ and $\phi_2(q)$ are differentiable functions, with respect to the real parameter, 't', and we also set

$$T = t + h,$$

then clearly

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$$\lim_{T \to t}\left[\phi_1(q(T))\right] = \phi_1(q(t)),$$

$$\lim_{T \to t}\left[\frac{\phi_1(q(T)) - \phi_1(q(t))}{T-t}\right] = \frac{d}{dt}\left[\phi_1(q(t))\right],$$

$$\lim_{T \to t}\left[\phi_2(q(t))\right] = \phi_2(q(t)),$$

$$\lim_{T \to t}\left[\frac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}\right] = \frac{d}{dt}\left[\phi_2(q(t))\right],$$

and hence, in accordance with Theorem T III-6, it naturally follows that

$$\lim_{T \to t}\left[\frac{f(q(T)) - f(q(t))}{T-t}\right] = \lim_{T \to t}\left[\begin{array}{l} \phi_1(q(T))\left[\dfrac{\phi_2(q(T)) - \phi_2(q(t))}{T-t}\right] + \\ \left[\dfrac{\phi_1(q(T)) - \phi_1(q(t))}{T-t}\right]\phi_2(q(t)) \end{array}\right]$$

$$= \phi_1(q(t))\frac{d}{dt}\left[\phi_2(q(t))\right] + \frac{d}{dt}\left[\phi_1(q(t))\right]\phi_2(q(t))$$

AND

$$\lim_{T \to t}\left[\frac{f(q(T)) - f(q(t))}{T-t}\right] = \frac{d}{dt}\left[f(q(t))\right],$$

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as previously indicated in part (i) of this theorem.

Finally, we deduce that

$$\left[\frac{d}{dq}\right]_C (\phi_1(q)\,\phi_2(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases},$$

where

$$\frac{d}{dt}[f(q(t))] = \phi_1(q(t))\frac{d}{dt}[\phi_2(q(t))] + \frac{d}{dt}[\phi_1(q(t))]\,\phi_2(q(t)).$$ Q.E.D.

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(iii) The proof required for this part of the theorem is completely analogous with part (ii) thereof, insofar as the functions, $\phi_1(q)$ and $\phi_2(q)$, may be readily transposed with respect to the product functions, $\phi_1(q)\,\phi_2(q)$ and $\phi_2(q)\,\phi_1(q)$.

Q.E.D.

(iv) Let us initially define a function,

$$f(q(t)) = \phi_1(q(t))/\phi_2(q(t)) \iff f(q) = \phi_1(q)/\phi_2(q),$$

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$$\therefore f(q(t)) = \begin{cases} \phi_1(q(t))\left(\dfrac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right) \\ \\ \left(\dfrac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\phi_1(q(t)) \end{cases}$$

$$= \begin{cases} \psi_1(q(t)) \\ \\ \psi_2(q(t)) \end{cases} .$$

Hence, by virtue of the preceding parts (ii) and (iii) of this theorem, we perceive that

$$\left[\frac{d}{dq}\right]_C\left(\frac{\phi_1(q)}{\phi_2(q)}\right) = \begin{cases} \dfrac{\frac{d}{dt}[\psi_1(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[\psi_1(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\frac{d}{dt}[\psi_2(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[\psi_2(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} ,$$

where the derivatives,

$$\frac{d}{dt}[\psi_1(q(t))] = \frac{d}{dt}\left[\phi_1(q(t))\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\right]$$

$$= \phi_1(q(t))\frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] + \frac{d}{dt}[\phi_1(q(t))]\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right),$$

$$\frac{d}{dt}[\psi_2(q(t))] = \frac{d}{dt}\left[\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\phi_1(q(t))\right]$$

$$= \left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right]\phi_1(q(t)).$$

We now construct a difference quotient for the function, $\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}$, which is accordingly expressed as –

$$\left(\frac{\overline{\phi_2(q(T))}}{|\phi_2(q(T))|^2} - \frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\Big/(T-t) = \frac{|\phi_2(q(t))|^2\overline{\phi_2(q(T))} - |\phi_2(q(T))|^2\overline{\phi_2(q(t))}}{|\phi_2(q(T))\phi_2(q(t))|^2(T-t)}$$

$$= \frac{\left[\begin{array}{c}|\phi_2(q(t))|^2\overline{\phi_2(q(T))} - |\phi_2(q(t))|^2\overline{\phi_2(q(t))} + |\phi_2(q(t))|^2\overline{\phi_2(q(t))} \\ - |\phi_2(q(T))|^2\overline{\phi_2(q(t))}\end{array}\right]}{|\phi_2(q(T))\phi_2(q(t))|^2(T-t)}$$

$$= \frac{1}{|\phi_2(q(T))\phi_2(q(t))|^2}\left[\frac{|\phi_2(q(t))|^2(\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}) + (|\phi_2(q(t))|^2 - |\phi_2(q(T))|^2)\overline{\phi_2(q(t))}}{T-t}\right]$$

$$= \frac{1}{|\phi_2(q(T))\phi_2(q(t))|^2}\left[|\phi_2(q(t))|^2\left(\frac{\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}}{T-t}\right) + \left(\frac{|\phi_2(q(t))|^2 - |\phi_2(q(T))|^2}{T-t}\right)\overline{\phi_2(q(t))}\right]$$

$$= \frac{1}{|\phi_2(q(T))\phi_2(q(t))|^2}\left[|\phi_2(q(t))|^2\left(\frac{\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}}{T-t}\right) - \left(\frac{|\phi_2(q(T))|^2 - |\phi_2(q(t))|^2}{T-t}\right)\overline{\phi_2(q(t))}\right],$$

$\forall t, T \in \mathbb{R}$.

Furthermore, if $\phi_1(q)$ and $\phi_2(q)$ are differentiable functions, with respect to

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the real parameter, 't', and we also set

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$$T = t + h,$$

then clearly

$$\lim_{T\to t}\left[\phi_2(q(T))\right] = \phi_2(q(t)) \Longrightarrow \begin{cases} \lim_{T\to t}\left[|\phi_2(q(T))|\right] = |\phi_2(q(t))| \\ \lim_{T\to t}\left[\overline{\phi_2(q(T))}\right] = \overline{\phi_2(q(t))} \end{cases} \dagger ,$$

$$\lim_{T\to t}\left[\phi_2(q(t))\right] = \phi_2(q(t)) \Longrightarrow \begin{cases} \lim_{T\to t}\left[|\phi_2(q(t))|\right] = |\phi_2(q(t))| \\ \lim_{T\to t}\left[\overline{\phi_2(q(t))}\right] = \overline{\phi_2(q(t))} \end{cases} ,$$

$$\lim_{T\to t}\left[\frac{\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}}{T-t}\right] = \frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] ,$$

$$\lim_{T\to t}\left[\frac{|\phi_2(q(T))|^2 - |\phi_2(q(t))|^2}{T-t}\right] = \frac{d}{dt}\left[|\phi_2(q(t))|^2\right],$$

$$\lim_{T\to t}\left[\left(\frac{\overline{\phi_2(q(T))}}{|\phi_2(q(T))|^2} - \frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\Big/(T-t)\right] = \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] ,$$

and hence, in accordance with Theorem TIII-6, it naturally follows that

$$\lim_{T\to t}\left[\left(\frac{\overline{\phi_2(q(T))}}{|\phi_2(q(T))|^2} - \frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\Big/(T-t)\right]$$

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$$= \frac{|\phi_2(q(t))|^2 \frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] - \frac{d}{dt}\left[|\phi_2(q(t))|^2\right] \overline{\phi_2(q(t))}}{|\phi_2(q(t))|^4}$$

$$\therefore \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] = \frac{|\phi_2(q(t))|^2 \frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] - \frac{d}{dt}\left[|\phi_2(q(t))|^2\right] \overline{\phi_2(q(t))}}{|\phi_2(q(t))|^4}$$

† A formal justification for this statement is provided in Appendix A3.

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Finally, through utilising the appropriate substitutions, we deduce that

$$\left[\frac{d}{dq}\right]_C \left(\frac{\phi_1(q)}{\phi_2(q)}\right) = \begin{cases} \dfrac{\frac{d}{dt}\left[\psi_1(q(t))\right] \overline{\frac{d}{dt}\left[q(t)\right]}}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}\left[q(t)\right]} \frac{d}{dt}\left[\psi_1(q(t))\right]}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \\[2ex] \dfrac{\frac{d}{dt}\left[\psi_2(q(t))\right] \overline{\frac{d}{dt}\left[q(t)\right]}}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}\left[q(t)\right]} \frac{d}{dt}\left[\psi_2(q(t))\right]}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \end{cases} ,$$

where the derivatives,

$$\frac{d}{dt}\left[\psi_1(q(t))\right] = \phi_1(q(t)) \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] + \frac{d}{dt}\left[\phi_1(q(t))\right]\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right) ,$$

$$\frac{d}{dt}\left[\psi_2(q(t))\right] = \left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right) \frac{d}{dt}\left[\phi_1(q(t))\right] + \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] \phi_1(q(t)) ,$$

$$\frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] = \frac{|\phi_2(q(t))|^2 \frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] - \frac{d}{dt}\left[|\phi_2(q(t))|^2\right]\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^4},$$

and the function,

$\phi_2(q(t)) \neq 0, \ \forall t \in (a,b).$ <u>Q.E.D.</u>

From the remarks made in the preceding Parts 1 and 2 of this section, we surmise that, if a smooth arc, C, is alternatively expressed as —

$$q(t) = \begin{cases} x(t) + iy(t), \\ x(t) + j\hat{z}(t), \\ x(t) + k\hat{y}(t) \end{cases} \qquad (4-24),$$

then the differentiation formulae for the sums, products and quotients of the functions, $\phi_1(q)$ and $\phi_2(q)$, restricted to such arcs, are reduced to

$$\text{(a)}\ \left[\frac{d}{dq}\right]_C(\phi_1(q) + \phi_2(q)) = \frac{\frac{d}{dt}\left[\phi_1(q(t)) + \phi_2(q(t))\right]}{\frac{d}{dt}[q(t)]}$$

$$= \frac{\left(\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}[\phi_2(q(t))]\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$= \frac{\overline{\frac{d}{dt}[q(t)]}\left(\frac{d}{dt}[\phi_1(q(t))] + \frac{d}{dt}[\phi_2(q(t))]\right)}{\left|\frac{d}{dt}[q(t)]\right|^2},$$

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(b) $$\left[\frac{d}{dq}\right]_C(\phi_1(q)\phi_2(q)) = \left[\frac{d}{dq}\right]_C(\phi_2(q)\phi_1(q))$$

$$= \frac{\frac{d}{dt}[\phi_1(q(t))\phi_2(q(t))]}{\frac{d}{dt}[q(t)]}$$

$$= \frac{\frac{d}{dt}[\phi_1(q(t))\phi_2(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$= \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[\phi_1(q(t))\phi_2(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2},$$

where $$\frac{d}{dt}[\phi_1(q(t))\phi_2(q(t))] = \frac{d}{dt}[\phi_2(q(t))\phi_1(q(t))]$$

$$= \phi_1(q(t))\frac{d}{dt}[\phi_2(q(t))] + \frac{d}{dt}[\phi_1(q(t))]\phi_2(q(t)),$$

such that

$$\phi_1(q(t))\frac{d}{dt}[\phi_2(q(t))] = \frac{d}{dt}[\phi_2(q(t))]\phi_1(q(t)),$$

$$\frac{d}{dt}[\phi_1(q(t))]\phi_2(q(t)) = \phi_2(q(t))\frac{d}{dt}[\phi_1(q(t))],$$

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(c) $$\left[\frac{d}{dq}\right]_C\left(\frac{\phi_1(q)}{\phi_2(q)}\right) = \frac{\frac{d}{dt}\left[\frac{\phi_1(q(t))}{\phi_2(q(t))}\right]}{\frac{d}{dt}[q(t)]}$$

$$= \frac{\frac{d}{dt}\left[\frac{\phi_1(q(t))}{\phi_2(q(t))}\right]\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

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$$= \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left[\frac{\phi_1(q(t))}{\phi_2(q(t))}\right]}{\left|\frac{d}{dt}[q(t)]\right|^2} \quad ,$$

where the function,

$$\frac{\phi_1(q(t))}{\phi_2(q(t))} = \frac{\phi_1(q(t))\,\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2} = \frac{\overline{\phi_2(q(t))}\,\phi_1(q(t))}{|\phi_2(q(t))|^2} \quad ,$$

and the derivatives,

$$\frac{d}{dt}\left[\frac{\phi_1(q(t))}{\phi_2(q(t))}\right] = \phi_1(q(t))\frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] + \frac{d}{dt}[\phi_1(q(t))]\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)$$

$$\frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] = \frac{|\phi_2(q(t))|^2\frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] - \frac{d}{dt}\left[|\phi_2(q(t))|^2\right]\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^4}$$

such that

$$\phi_1(q(t))\frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right] = \frac{d}{dt}\left[\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right]\phi_1(q(t)) \quad ,$$

$$\frac{d}{dt}[\phi_1(q(t))]\left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right) = \left(\frac{\overline{\phi_2(q(t))}}{|\phi_2(q(t))|^2}\right)\frac{d}{dt}[\phi_1(q(t))] \quad ,$$

$$|\phi_2(q(t))|^2\frac{d}{dt}\left[\overline{\phi_2(q(t))}\right] = \frac{d}{dt}\left[\overline{\phi_2(q(t))}\right]|\phi_2(q(t))|^2 \quad ,$$

$$\frac{d}{dt}\left[|\phi_2(q(t))|^2\right]\overline{\phi_2(q(t))} = \overline{\phi_2(q(t))}\frac{d}{dt}\left[|\phi_2(q(t))|^2\right] \qquad (4-25) \quad ,$$

all of which are in accordance with Eq. (4-14).

Furthermore, if the quaternion hypercomplex functions, $f(x(t)+iy(t))$, $f(x+j\hat{x}(t))$, $f(x+k\hat{y}(t))$, are also <u>analytic</u> in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), we similarly deduce that the above mentioned formulae

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may now be rewritten respectively as -

$$\text{(a)}\ \left[\frac{d}{dq}\right]_C(\phi_1(q)+\phi_2(q)) = \frac{d}{dq}(\phi_1(q)+\phi_2(q))$$

$$= \frac{d}{dq}[\phi_1(q)] + \frac{d}{dq}[\phi_2(q)]\ ,$$

$$\text{(b)}\ \left[\frac{d}{dq}\right]_C(\phi_1(q)\phi_2(q)) = \frac{d}{dq}(\phi_1(q)\phi_2(q))$$

$$= \phi_1(q)\frac{d}{dq}[\phi_2(q)] + \frac{d}{dq}[\phi_1(q)]\,\phi_2(q)\ ,$$

$$\text{(c)}\ \left[\frac{d}{dq}\right]_C\left(\frac{\phi_1(q)}{\phi_2(q)}\right) = \frac{d}{dq}\left(\frac{\phi_1(q)}{\phi_2(q)}\right)$$

$$= \frac{\frac{d}{dq}[\phi_1(q)]\,\phi_2(q) - \phi_1(q)\frac{d}{dq}[\phi_2(q)]}{(\phi_2(q))^2} \qquad (4\text{-}26),$$

being completely analogous with their counterparts from both real and complex variable analysis.

4. Differentiation of the Composites of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q-Space.

From the calculus of real variable functions, we recall that any composite function, $(f\circ g)(x)$, which is differentiable in 'x', will invariably give rise to the derivative,

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$$\frac{d}{dx}[(f\circ g)(x)] = \frac{d}{dx}[f(g(x))]$$

$$= f'(g(x))\cdot g'(x) \qquad (4-27).$$

Similarly, we obtain the differential formula,

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$$\frac{d}{dz}[(f\circ g)(z)] = \frac{d}{dz}[f(g(z))]$$

$$= f'(g(z))\cdot g'(z) \qquad (4-28),$$

thus corresponding to a <u>composite</u> complex function, $(f\circ g)(z) \in \mathbb{C}$, which is likewise <u>differentiable over some arbitrary region of the 'z'-plane.</u>

However, bearing in mind the criteria previously enunciated in Parts 1 and 2 of this section, we immediately recognise that the formula,

$$\frac{d}{dq}[(f\circ g)(q)] = \frac{d}{dq}[f(g(q))]$$

$$= f'(g(q))\cdot g'(q) \qquad (4-29),$$

cannot, as a general rule, purport to be the <u>first derivative of every quaternion hypercomplex function, $(f\circ g)(q)$</u>. Consequently, we must once again invoke the notion of a smooth arc, C, embedded in q-space, before we proceed any further in deriving an appropiate differential formula which clearly pertains to this particular class of functions. Indeed, our first step towards achieving this objective is to formally define the first derivative of any quaternion hypercomplex <u>composite</u> function, $(\phi_1\circ\phi_2)(q)$, thus constrained to such arcs, as follows :-

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Definition DIV − 7

Let there exist a quaternion hypercomplex composite function, $(\phi_1 \circ \phi_2)(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \ \forall t \in [a, b].$$

Clearly, by virtue of Theorem TIV − 1, if $(\phi_1 \circ \phi_2)(q)$ is differentiable in 't', then the following differential formula pertaining to all such quaternion hypercomplex composite functions shall likewise apply, $\forall t \in (a, b)$:−

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$$\left[\frac{d}{dq}\right]_C ((\phi_1 \circ \phi_2)(q)) = \begin{cases} \dfrac{\frac{d}{dt}\left[(\phi_1 \circ \phi_2)(q(t))\right]\overline{\frac{d}{dt}\left[q(t)\right]}}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}\left[q(t)\right]}\frac{d}{dt}\left[(\phi_1 \circ \phi_2)(q(t))\right]}{\left|\frac{d}{dt}\left[q(t)\right]\right|^2} \end{cases}.$$

Whilst the existence of the above stated definition is certainly a necessary condition for the purposes of our analysis, it is by no means sufficient, since we have not explicitly stated a concise method for evaluating the parametric first derivative, $\frac{d}{dt}\left[(\phi_1 \circ \phi_2)(q(t))\right]$. Quite conceivably, one might even be tempted to regard Definition DIV − 7 as being little more than a trivial extension of Theorem TIV − 1, in which case we will need to examine the properties of $\frac{d}{dt}\left[(\phi_1 \circ \phi_2)(q(t))\right]$ more closely in order to overcome the apparent shortcomings of this definition.

Our chosen strategy to that effect thus consists of

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(i) resolving the function, $(\phi_1 \circ \phi_2)(q)$, into its constituent real and imaginary parts

AND

(ii) differentiating each of these components, in turn, with respect to the real parameter, 't',

as the reader will no doubt ascertain from our next theorem, which we shall henceforth enunciate and verify accordingly :-

Theorem TIV-3

Let there exist a quaternion hypercomplex composite function, $(\phi_1 \circ \phi_2)(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a, b] .$$

If $(\phi_1 \circ \phi_2)(q)$ is differentiable in 't', such that the functions;

$$\phi_2(q) = \sigma_1(x, y, \hat{x}, \hat{y}) + i\,\tau_1(x, y, \hat{x}, \hat{y}) + j\,\sigma_2(x, y, \hat{x}, \hat{y}) + k\,\tau_2(x, y, \hat{x}, \hat{y}),$$

$$(\phi_1 \circ \phi_2)(q) = \phi_1(\sigma_1(x, y, \hat{x}, \hat{y}) + i\,\tau_1(x, y, \hat{x}, \hat{y}) + j\,\sigma_2(x, y, \hat{x}, \hat{y}) + k\,\tau_2(x, y, \hat{x}, \hat{y}))$$

$$\begin{aligned}
= \ & \alpha_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) + \\
& i\beta_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) + \\
& j\alpha_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) + \\
& k\beta_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) \quad ,
\end{aligned}$$

are also differentiable in 't', then for the composite function, $(\phi_1 \circ \phi_2)(q)$, we obtain the corresponding differential formula which we accordingly express as –

$$\begin{aligned}
\frac{d}{dt}[(\phi_1 \circ \phi_2)(q(t))] = \ & \frac{d}{dt}[A_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + i\frac{d}{dt}[B_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] \\
& + j\frac{d}{dt}[A_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] + k\frac{d}{dt}[B_2(x(t), y(t), \hat{x}(t), \hat{y}(t))],
\end{aligned}$$

$\forall t \in (a, b)$,

where the functions,

$$\begin{aligned}
A_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) &= A_1(x,y,\hat{x},\hat{y}) \\
&= \alpha_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})),
\end{aligned}$$

$$\begin{aligned}
B_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) &= B_1(x,y,\hat{x},\hat{y}) \\
&= \beta_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})),
\end{aligned}$$

$$\begin{aligned}
A_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) &= A_2(x,y,\hat{x},\hat{y}) \\
&= \alpha_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})),
\end{aligned}$$

$$\begin{aligned}
B_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) &= B_2(x,y,\hat{x},\hat{y}) \\
&= \beta_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})),
\end{aligned}$$

and hence the derivatives,

$$\frac{d}{dt}[A_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] = \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[B_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] = \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[A_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] = \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[B_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] = \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{dX_1}{dt} = \frac{d}{dt}[\sigma_1(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_2}{dt} = \frac{d}{dt}[\tau_1(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\tau_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\tau_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\tau_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\tau_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_3}{dt} = \frac{d}{dt}[\sigma_2(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_4}{dt} = \frac{d}{dt}[\tau_2(x,y,\hat{x},\hat{y})]$$

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$$= \frac{\partial}{\partial x}(\tau_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\tau_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\tau_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\tau_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \;.$$

*   *   *

PROOF ;-

We commence our proof by initially defining a composite function,

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$$(\phi_1 \circ \phi_2)(q) = A_1(x,y,\hat{x},\hat{y}) + iB_1(x,y,\hat{x},\hat{y}) + jA_2(x,y,\hat{x},\hat{y}) + kB_2(x,y,\hat{x},\hat{y}) \;.$$

Now, by restricting this function to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t),\ \forall t \in [a,b] \;,$$

we further obtain

$$(\phi_1 \circ \phi_2)(q(t)) = A_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + iB_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + jA_2(x(t),y(t),\hat{x}(t),\hat{y}(t)) + kB_2(x(t),y(t),\hat{x}(t),\hat{y}(t)) \;,$$

whereupon, in accordance with Theorem TIV-1, it subsequently follows that, if $(\phi_1 \circ \phi_2)(q(t))$ is differentiable in 't', then

$$\frac{d}{dt}[(\phi_1 \circ \phi_2)(q(t))] = \frac{d}{dt}[A_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] + i\frac{d}{dt}[B_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] + j\frac{d}{dt}[A_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] + k\frac{d}{dt}[B_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] \;, \quad \forall t \in (a,b) \;.$$

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But since we had previously stated in the preamble to this proof that

$$A_1(x,y,\hat{x},\hat{y}) = \alpha_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) ,$$

$$B_1(x,y,\hat{x},\hat{y}) = \beta_1(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) ,$$

$$A_2(x,y,\hat{x},\hat{y}) = \alpha_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) ,$$

$$B_2(x,y,\hat{x},\hat{y}) = \beta_2(\sigma_1(x,y,\hat{x},\hat{y}), \tau_1(x,y,\hat{x},\hat{y}), \sigma_2(x,y,\hat{x},\hat{y}), \tau_2(x,y,\hat{x},\hat{y})) ,$$

we are now in a position to evaluate the derivatives, $\frac{d}{dt}A_1(x,y,\hat{x},\hat{y})$, $\frac{d}{dt}B_1(x,y,\hat{x},\hat{y})$, $\frac{d}{dt}A_2(x,y,\hat{x},\hat{y})$, $\frac{d}{dt}B_2(x,y,\hat{x},\hat{y})$, in terms of their hitherto mentioned component functions.

In order to do this, we shall henceforth utilise a particular differentiation

formula from the calculus of functions of several real variables (cf. Appendix A4) which thus pertains to the derivatives of composite functions of four such variables and is accordingly expressed as

$$\frac{dF}{dt} = \frac{\partial F}{\partial X_1}\cdot\frac{dX_1}{dt} + \frac{\partial F}{\partial X_2}\cdot\frac{dX_2}{dt} + \frac{\partial F}{\partial X_3}\cdot\frac{dX_3}{dt} + \frac{\partial F}{\partial X_4}\cdot\frac{dX_4}{dt} \qquad \underline{(i)} ,$$

where $F = F(X_1, X_2, X_3, X_4)$ and X_1, X_2, X_3, X_4 are all differentiable functions in 't'.

Finally, by setting

$$X_1 = \sigma_1(x,y,\hat{x},\hat{y}) \ , \quad x = x(t),$$
$$X_2 = \tau_1(x,y,\hat{x},\hat{y}) \ , \quad y = y(t),$$

$X_3 = \sigma_2(x, y, \hat{x}, \hat{y})$, $\hat{x} = \hat{x}(t)$,
$X_4 = \tau_2(x, y, \hat{x}, \hat{y})$, $\hat{y} = \hat{y}(t)$,

we furthermore deduce, in an analogous manner to Eq. (i) above, that

$$\frac{d}{dt}[A_1(x,y,\hat{x},\hat{y})] = \frac{d}{dt}[A_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]$$

$$= \frac{\partial}{\partial X_1}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_1}{dt} + \frac{\partial}{\partial X_2}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_2}{dt} + \frac{\partial}{\partial X_3}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_3}{dt} + \frac{\partial}{\partial X_4}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_4}{dt}$$

$$= \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\alpha_1(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[B_1(x,y,\hat{x},\hat{y})] = \frac{d}{dt}[B_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]$$

$$= \frac{\partial}{\partial X_1}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_1}{dt} + \frac{\partial}{\partial X_2}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_2}{dt} + \frac{\partial}{\partial X_3}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_3}{dt} + \frac{\partial}{\partial X_4}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_4}{dt}$$

$$= \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\beta_1(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[A_2(x,y,\hat{x},\hat{y})] = \frac{d}{dt}[A_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]$$

$$= \frac{\partial}{\partial X_1}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_1}{dt} + \frac{\partial}{\partial X_2}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_2}{dt} + \frac{\partial}{\partial X_3}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_3}{dt} + \frac{\partial}{\partial X_4}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_4}{dt}$$

$$= \sum_{k=1}^{4}\frac{\partial}{\partial X_k}(\alpha_2(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

$$\frac{d}{dt}[B_2(x,y,\hat{x},\hat{y})] = \frac{d}{dt}[B_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]$$

$$= \frac{\partial}{\partial X_1}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_1}{dt} + \frac{\partial}{\partial X_2}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_2}{dt} + \frac{\partial}{\partial X_3}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_3}{dt} + \frac{\partial}{\partial X_4}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_4}{dt}$$

$$= \sum_{k=1}^{4} \frac{\partial}{\partial X_k}(\beta_2(X_1,X_2,X_3,X_4))\frac{dX_k}{dt} ,$$

and similarly,

$$\frac{dX_1}{dt} = \frac{d}{dt}[\sigma_1(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\sigma_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_2}{dt} = \frac{d}{dt}[\tau_1(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\tau_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\tau_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\tau_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\tau_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_3}{dt} = \frac{d}{dt}[\sigma_2(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\sigma_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} ,$$

$$\frac{dX_4}{dt} = \frac{d}{dt}[\tau_2(x,y,\hat{x},\hat{y})]$$

$$= \frac{\partial}{\partial x}(\tau_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(\tau_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(\tau_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(\tau_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} .$$

Q.E.D.

The inclusion of this last theorem as an adjunct to Definition DIV-7 therefore not only serves to enhance the latter but also, more importantly, provides us with a rigorous method of evaluating any parametric derivative, $\frac{d}{dt}[(\phi_1 \circ \phi_2)(q(t))]$, in terms of its constituent real and imaginary parts.

In finalising our analysis of the properties of the first derivatives of quaternion hypercomplex composite functions, $(\phi_1 \circ \phi_2)(q)$, we perceive that, in the light of what was discussed in the preceding Parts 1 and 2 of this section, if a smooth arc, C, is alternatively expressed as –

$$q(t) = \begin{cases} x(t) + iy(t), \\ x(t) + j\hat{x}(t), \\ x(t) + k\hat{y}(t), \end{cases} \qquad (4-30),$$

then the differentiation formula, specified in Definition DIV-7, is subsequently reduced to

$$\left[\frac{d}{dq}\right]_C ((\phi_1 \circ \phi_2)(q)) = \frac{\frac{d}{dt}[(\phi_1 \circ \phi_2)(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$= \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[(\phi_1 \circ \phi_2)(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \qquad (4-31).$$

Furthermore, if the quaternion hypercomplex composite functions, $(\phi_1 \circ \phi_2)(x(t) + iy(t))$, $(\phi_1 \circ \phi_2)(x(t) + j\hat{x}(t))$, $(\phi_1 \circ \phi_2)(x(t) + k\hat{y}(t))$, are also <u>analytic</u> in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), we immediately recognise that Eq. (4-31) may now be rewritten as –

$$\left[\frac{d}{dq}\right]_C((\phi_1\circ\phi_2)(q)) = \frac{d}{dq}[(\phi_1\circ\phi_2)(q)]$$

$$= \frac{d}{dq}[\phi_1(\phi_2(q))]$$

$$= \phi_1'(\phi_2(q))\cdot\phi_2'(q) \qquad (4-32),$$

being completely analogous with its more familiar counterparts from both

real and complex variable analysis.

5. The Second Derivative of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q-Space.

The concept of a second derivative, needless to say, is well understood with respect to the analysis of both real and complex variable functions. We are naturally encouraged to extend this same notion into the realm of quaternion hypercomplex functions, notwithstanding any restrictions such as we previously encountered via the intermediary of a smooth arc, C, embedded in q-space.

Indeed, to facilitate our discussion along those lines, we shall firstly enunciate the following definition:-

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<u>*Definition DIV – 8*</u>

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t)\ ,\ \forall t \in [a,b]\,.$$

Now, if $f(q)$ is differentiable in 't', we may accordingly define the <u>second derivative</u> thereof by the formula,

$$\begin{aligned}\left[\tfrac{d^2}{dq^2}\right]_C (f(q)) &= \left[\tfrac{d}{dq}\circ\tfrac{d}{dq}\right]_C (f(q)) \\ &= \left(\left[\tfrac{d}{dq}\right]_C \circ \left[\tfrac{d}{dq}\right]_C\right)(f(q)) \\ &= \left[\tfrac{d}{dq}\right]_C \left(\left[\tfrac{d}{dq}\right]_C (f(q))\right),\quad \forall t \in (a,b)\,.\end{aligned}$$

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We now take the opportunity to briefly comment on our intended usage of the notation, $\left[\frac{d^2}{dq^2}\right]_C$. From the context of Definition DIV – 4, the reader may recall that we firstly introduced the notation, $\left[\frac{d}{dq}\right]_C$, which we then denoted as the <u>first order differential operator restricted to an arc, C, embedded in q–space</u>. It seems only logical that we should accordingly denote $\left[\frac{d^2}{dq^2}\right]_C$ as the <u>second order differential operator restricted to an arc, C, embedded in q–space</u>, thus having the property —

$$\left[\tfrac{d^2}{dq^2}\right]_C = \left[\tfrac{d}{dq}\circ\tfrac{d}{dq}\right]_C$$

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$$= \left[\frac{d}{dq}\right]_C \circ \left[\frac{d}{dq}\right]_C \qquad (4-33),$$

as prescribed above.

Apart from yielding a formula which incorporates this particular property, Definition DIV-8 does not, on the other hand, provide a concise method for evaluating any <u>second derivative</u>, $\left[\frac{d^2}{dq^2}\right]_C (f(q))$, by means of manipulating the differential operator, $\left[\frac{d}{dq}\right]_C$, and therefore requires further elucidation. Our next theorem, however, will assuredly fulfill this additional requirement :-

<u>Theorem TIV-4</u>

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a, b].$$

Here it may be shown that, if $f(q)$ is differentiable in 't', $\forall t \in (a, b)$, then the following differentiation formula for the <u>second derivative</u> thereof is likewise valid, $\forall t \in (a, b)$:-

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$$\left[\frac{d^2}{dq^2}\right]_C (f(q)) = \begin{cases} \dfrac{\frac{d}{dt}\left(\dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left(\dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\frac{d}{dt}\left(\dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left(\dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} ,$$

where the derivatives,

$$\frac{d}{dt}\left(\frac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \frac{d}{dt}[f(q(t))]\,\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] + \frac{d^2}{dt^2}[f(q(t))]\left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) ,$$

TO BE CONTINUED.

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$$\frac{d}{dt}\left(\frac{\overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\frac{d^2}{dt^2}[f(q(t))] \quad +$$

$$\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right]\frac{d}{dt}[f(q(t))] \quad ,$$

$$\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] = \frac{\left|\frac{d}{dt}[q(t)]\right|^2\frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\right] - \frac{d}{dt}\left[\left|\frac{d}{dt}[q(t)]\right|^2\right]\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^4} \quad .$$

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<u>PROOF</u>:-

In accordance with Definition DIV-8 and Theorem TIV-1, we initially perceive that, $\forall t \in (a,b)$,

$$\left[\frac{d^2}{dq^2}\right]_c(f(q)) = \left[\frac{d}{dq}\right]_c\left(\left[\frac{d}{dq}\right]_c(f(q))\right)$$

$$= \begin{cases} \dfrac{\frac{d}{dt}\left[\left[\frac{d}{dq}\right]_c(f(q))\right]\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\frac{d}{dt}\left[\left[\frac{d}{dq}\right]_c(f(q))\right]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases} , \forall t \in (a,b).$$

Furthermore, since we had previously deduced by virtue of the above stated theorem that

$$\left[\frac{d}{dq}\right]_C(f(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases}, \quad \forall t \in (a,b),$$

then, through making the appropriate substitutions, it naturally follows that

$$\left[\frac{d^2}{dq^2}\right]_C(f(q)) = \begin{cases} \frac{d}{dt}\left[\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}\Big/\left|\frac{d}{dt}[q(t)]\right|^2\right]\overline{\frac{d}{dt}[q(t)]}\Big/\left|\frac{d}{dt}[q(t)]\right|^2 \\ \overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left[\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}\Big/\left|\frac{d}{dt}[q(t)]\right|^2\right]\Big/\left|\frac{d}{dt}[q(t)]\right|^2 \\ \frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]\Big/\left|\frac{d}{dt}[q(t)]\right|^2\right]\overline{\frac{d}{dt}[q(t)]}\Big/\left|\frac{d}{dt}[q(t)]\right|^2 \\ \overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]\Big/\left|\frac{d}{dt}[q(t)]\right|^2\right]\Big/\left|\frac{d}{dt}[q(t)]\right|^2 \end{cases}{}^{\dagger},$$

† For *multi-valued functions*, it should be noted that the *order*, in which the *component functions* are listed, is not important, since all such permutations thereof are *equivalent* over the same domain.

whereupon, by virtue of the preceding parts (ii), (iii) and (iv) of Theorem T IV–2, we likewise deduce that

$$\frac{d}{dt}\left(\frac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \frac{d}{dt}\left[\frac{d}{dt}[f(q(t))]\left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\right]$$

$$= \frac{d}{dt}\left[f(q(t))\right]\frac{d}{dt}\left[\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] +$$

$$\frac{d^2}{dt^2}\left[f(q(t))\right]\left(\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) ,$$

$$\frac{d}{dt}\left(\frac{\overrightarrow{\frac{d}{dt}[q(t)]\frac{d}{dt}[f(q(t))]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \frac{d}{dt}\left[\left(\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\frac{d}{dt}[f(q(t))]\right]$$

$$= \left(\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\frac{d^2}{dt^2}[f(q(t))] +$$

$$\frac{d}{dt}\left[\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right]\frac{d}{dt}[f(q(t))]$$

and

$$\frac{d}{dt}\left[\frac{\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] = \frac{\left|\frac{d}{dt}[q(t)]\right|^2\frac{d}{dt}\left[\overrightarrow{\frac{d}{dt}[q(t)]}\right] - \frac{d}{dt}\left[\left|\frac{d}{dt}[q(t)]\right|^2\right]\overrightarrow{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^4} .$$

Q.E.D.

The above stated theorem in conjunction with Definition DIV-8 clearly demonstrates that any second derivative, $\left[\frac{d^2}{dq^2}\right]_c(f(q))$, may be obtained simply by applying the first order differential operator, $\left[\frac{d}{dq}\right]_c$, twice upon the function, $f(q)$, in accordance with Theorem TIV-1. As a matter of fact, we can generalise this whole procedure to include any 'nth-order' derivative, $\left[\frac{d^n}{dq^n}\right]_c(f(q))$, in which case we would utilise the operator, $\left[\frac{d}{dq}\right]_c$, 'n' consecutive times with respect to the same function.

Once again, we surmise that, on those occasions where we may choose to denote an arc, C, by the equations,

$$\left.\begin{aligned} q(t) \; = \; & x(t) + i\,y(t) \\ & x(t) + j\,\hat{z}(t) \\ & x(t) + k\,\hat{y}(t) \end{aligned}\right\} \qquad (4-34),$$

the formulae resulting from Theorem TIV-4 must therefore be rewritten as —

$$\left[\frac{d^2}{dq^2}\right]_C (f(q)) \;=\; \frac{\frac{d}{dt}\left(\dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$=\; \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left(\dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$=\; \frac{\frac{d}{dt}\left(\dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}$$

$$=\; \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left(\dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)}{\left|\frac{d}{dt}[q(t)]\right|^2} \quad ,$$

where the derivatives,

$$\frac{d}{dt}\left(\frac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \frac{d}{dt}[f(q(t))]\,\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] + \frac{d^2}{dt^2}[f(q(t))]\left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) ,$$

$$\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] = \frac{\left|\frac{d}{dt}[q(t)]\right|^2 \frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\right] - \frac{d}{dt}\left[\left|\frac{d}{dt}[q(t)]\right|^2\right]\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^4} ,$$

furthermore imply that

$$\frac{d}{dt}[f(q(t))]\,\frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right] = \frac{d}{dt}\left[\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right]\frac{d}{dt}[f(q(t))] ,$$

$$\frac{d^2}{dt^2}[f(q(t))]\left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right) = \left(\frac{\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}\right)\frac{d^2}{dt^2}[f(q(t))] ,$$

$$\left|\frac{d}{dt}[q(t)]\right|^2 \frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\right] = \frac{d}{dt}\left[\overline{\frac{d}{dt}[q(t)]}\right]\left|\frac{d}{dt}[q(t)]\right|^2 ,$$

$$\frac{d}{dt}\left[\left|\frac{d}{dt}[q(t)]\right|^2\right]\overline{\frac{d}{dt}[q(t)]} = \overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}\left[\left|\frac{d}{dt}[q(t)]\right|^2\right] \qquad (4\text{-}35).$$

Finally, if the quaternion hypercomplex functions, $f(x(t) + iy(t))$, $f(x(t) + j\hat{x}(t))$, $f(x(t) + k\hat{y}(t))$, are also analytic in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), it likewise follows that the second derivative,

$$\left[\frac{d^2}{dq^2}\right]_c (f(q)) = \frac{d^2}{dq^2}(f(q))$$

$$= f''(q) \qquad (4-36),$$

which is completely analogous with its more familiar counterparts from the calculus of both real and complex variable functions.

6. The Indefinite and Definite Integrals of Quaternion Hypercomplex Functions restricted to Smooth Arcs embedded in q–Space.

We conclude our discussion of the basic principles underlying the calculus of quaternion hypercomplex functions with some elementary definitions and

theorems pertaining to integration. Indeed, we had previously remarked in Part 5 of Section III that Churchill et al. (cf. Reference B2, Section VII) invoked the concept of an arc, C, defined on the complex z–plane, as a basis for analysing the behavioural properties of any complex function, f(z), under integration. Assuredly, this particular approach has enabled mathematicians to apply various techniques of integration upon complex functions which were originally formulated in relation to real variable analysis.

After taking all of these factors into consideration, we will proceed to derive similar techniques for quaternion hypercomplex functions, whereupon our first definition to that effect is accordingly stated as follows :–

Definition DIV–9

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) , \quad \forall t \in [a,b] .$$

Consequently, $\forall t \in [a,b]$, the <u>indefinite integral</u> or <u>anti-derivative</u> of 'f', with respect to the real parameter, 't', is defined by the formula,

$$F(q(t)) = F^*(t) = \int f(q(t))\, dt ,$$

such that the derivative,

$$\tfrac{d}{dt}[F(q(t))] = \tfrac{d}{dt}[F^*(t)] = \tfrac{d}{dt}\left[\int f(q(t))\, dt\right] = f(q(t)) .$$

Given the context of Definition DIV-9, we now wish to evaluate the <u>indefinite integral</u> mentioned therein in terms of its constituent real and imaginary parts. It is therefore not surprising that our next theorem should be concerned with just such an objective :-

<u>Theorem TIV-5</u>

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) , \quad \forall t \in [a,b] .$$

Furthermore, if we define the corresponding real and imaginary parts of 'f' such that

$$f(q(t)) = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t),$$

then it may be established that the <u>indefinite integral</u>,

$$\int f(q(t))\,dt = \int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt,$$

exists, $\forall t \in [a,b]$, provided that the indefinite integrals,

$$\int u_1^*(t)\,dt, \int v_1^*(t)\,dt, \int u_2^*(t)\,dt, \int v_2^*(t)\,dt,$$

are similarly defined with respect to the above mentioned interval.

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<u>PROOF</u>:-

We firstly construct an integral function,

$$F(q(t)) = U_1^*(t) + iV_1^*(t) + jU_2^*(t) + kV_2^*(t),$$

which is accordingly defined, $\forall t \in [a,b]$. Now, by virtue of Theorem TIV-1, we deduce that

$$\frac{d}{dt}[F(q(t))] = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t)$$

$$= f(q(t)), \ \forall t \in (a, b),$$

if and only if the derivatives,

$$\tfrac{d}{dt}[U_1^*(t)] = u_1^*(t), \quad \tfrac{d}{dt}[U_2^*(t)] = u_2^*(t),$$

$$\tfrac{d}{dt}[V_1^*(t)] = v_1^*(t), \quad \tfrac{d}{dt}[V_2^*(t)] = v_2^*(t),$$

also exist on the same interval.

Finally, it follows from Definition D IV-9 that

$$\tfrac{d}{dt}[F(q(t))] = f(q(t)) \Longleftrightarrow \int f(q(t))\,dt = F(q(t))$$

$$\therefore \int f(q(t))\,dt = U_1^*(t) + i V_1^*(t) + j U_2^*(t) + k V_2^*(t)$$

$$= \int u_1^*(t)\,dt + i \int v_1^*(t)\,dt + j \int u_2^*(t)\,dt + k \int v_2^*(t)\,dt,$$

$\forall t \in [a, b]$, since the Fundamental Theorem of Calculus, pertaining to functions of a single real variable, automatically gives rise to the equivalent statements —

To be continued via the author's next submission, namely -

<u>"An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 6/6."</u>

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