“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 6/6.***”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.***”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 6/6.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$\left\{\begin{array}{l} \frac{d}{dt}[U_1^*(t)] = u_1^*(t) \\ \frac{d}{dt}[V_1^*(t)] = v_1^*(t) \\ \frac{d}{dt}[U_2^*(t)] = u_2^*(t) \\ \frac{d}{dt}[V_2^*(t)] = v_2^*(t) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} U_1^*(t) = \int u_1^*(t)\,dt \\ V_1^*(t) = \int v_1^*(t)\,dt \\ U_2^*(t) = \int u_2^*(t)\,dt \\ V_2^*(t) = \int v_2^*(t)\,dt \end{array}\right\}, \ \forall t \in [a, b]. \qquad \underline{\underline{Q.E.D.}}$$

Once again, we must emphasise that the properties of the <u>parametric indefinite integral</u> for any quaternion hypercomplex function, such as we have indicated via the preceding Definition DIV-9 and Theorem TIV-5, are completely analogous with the equations,

$$F(z(t)) = \int f(z(t))\,dt \Longleftrightarrow \frac{d}{dt}[F(z(t))] = f(z(t)) \qquad (4-37),$$

$$\int f(z(t))\,dt = \int (u^*(t) + i v^*(t))\,dt$$

$$= \int u^*(t)\,dt + i\int v^*(t)\,dt \qquad (4-38),$$

which predictably represent the <u>indefinite integral of a complex function, $f(z)$, restricted to an arc, C, with respect to the real parameter, 't'</u>. Thus, by expressing any smooth arc, C, embedded in q-space, as

$$q(t) = \left.\begin{array}{l} x(t) + iy(t) \\ x(t) + j\hat{z}(t) \\ x(t) + k\hat{y}(t) \end{array}\right\} \qquad (4\text{-}39),$$

the corresponding indefinite integral, $\int f(q(t))\,dt$, may be respectively expressed as –

$$\int f(x(t) + iy(t))\,dt = \int u_1^*(t)\,dt + i\int v^*(t)\,dt \quad (u_2^*(t) = v_2^*(t) = 0),$$

$$\int f(x(t) + j\hat{z}(t))\,dt = \int u_1^*(t)\,dt + j\int u_2^*(t)\,dt \quad (v_1^*(t) = v_2^*(t) = 0),$$

$$\int f(x(t) + k\hat{y}(t))\,dt = \int u_1^*(t)\,dt + k\int v_2^*(t)\,dt \quad (v_1^*(t) = u_2^*(t) = 0)$$

$$(4\text{-}40),$$

being completely analogous with Eq. (4-38) stated above.

The reader will also recall that various properties were established with respect to the <u>definite integral</u> from the calculus of real and complex variable functions and hence it seems only reasonable that we should wish to extend these same properties into the realm of quaternion hypercomplex functions. Bearing this in mind, we shall accordingly state the following definition for a <u>definite integral</u> in relation to this particular class of functions :–

<u>Definition DIV-10</u>

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

Henceforth, the <u>definite integral</u> of 'f', defined with respect to the real parameter, 't', over the interval, $[a,b]$, is accordingly given by the formula,

$$\int_a^b f(q(t))\,dt = F(q(b)) - F(q(a)) = F^*(b) - F^*(a).$$

The significance of the above stated formula will presumably become much clearer as a result of our next theorem, wherein we formally correlate such entities with their constituent real and imaginary parts.

<u>Theorem TIV-6</u>

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

Furthermore, if we define the corresponding real and imaginary parts of 'f' such that

$$f(q(t)) = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t),$$

then it may be established that the <u>definite integral</u>,

$$\int_a^b f(q(t))\,dt = \int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt,$$

exists, $\forall t \in [a, b]$, provided that the definite integrals,

$$\int_a^b u_1^*(t)\,dt, \int_a^b v_1^*(t)\,dt, \int_a^b u_2^*(t)\,dt, \int_a^b v_2^*(t)\,dt,$$

are similarly defined with respect to the above mentioned interval.

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PROOF :-

We firstly recall from Definition D IV - 10 the formula for a definite integral, namely -

$$\int_a^b f(q(t))\,dt = F(q(b)) - F(q(a)) .$$

Similarly, from Theorem T IV - 5, we likewise perceive that

$$\begin{aligned} \int f(q(t))\,dt &= F(q(t)) \\ &= U_1^*(t) + i V_1^*(t) + j U_2^*(t) + k V_2^*(t) \\ &= \int u_1^*(t)\,dt + i \int v_1^*(t)\,dt + j \int u_2^*(t)\,dt + k \int v_2^*(t)\,dt , \end{aligned}$$

$\forall t \in [a, b]$, and hence it naturally follows that

$$F(q(a)) = U_1^*(a) + i V_1^*(a) + j U_2^*(a) + k V_2^*(a) ,$$

$$F(q(b)) = U_1^*(b) + i V_1^*(b) + j U_2^*(b) + k V_2^*(b) .$$

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Finally, upon utilizing the appropriate substitutions, we subsequently obtain

$$\int_a^b f(q(t))\,dt = F(q(b)) - F(q(a))$$

$$= U_1^*(b) + iV_1^*(b) + jU_2^*(b) + kV_2^*(b) - (U_1^*(a) + iV_1^*(a) + jU_2^*(a) + kV_2^*(a))$$

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$$= U_1^*(b) - U_1^*(a) + i(V_1^*(b) - V_1^*(a)) + j(U_2^*(b) - U_2^*(a)) + k(V_2^*(b) - V_2^*(a))$$

$$= \int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt\,,$$

$\forall t \in [a, b]$,

since the Fundamental Theorem of Calculus, pertaining to functions of a single real variable, guarantees the validity of the equivalent statements –

$$\left\{\begin{aligned} U_1^*(t) &= \int u_1^*(t)\,dt \\ V_1^*(t) &= \int v_1^*(t)\,dt \\ U_2^*(t) &= \int u_2^*(t)\,dt \\ V_2^*(t) &= \int v_2^*(t)\,dt \end{aligned}\right\} \iff \left\{\begin{aligned} \int_a^b u_1^*(t)\,dt &= U_1^*(b) - U_1^*(a) \\ \int_a^b v_1^*(t)\,dt &= V_1^*(b) - V_1^*(a) \\ \int_a^b u_2^*(t)\,dt &= U_2^*(b) - U_2^*(a) \\ \int_a^b v_2^*(t)\,dt &= V_2^*(b) - V_2^*(a) \end{aligned}\right\},$$

$\forall t \in [a, b]$. Q.E.D.

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As a further consequence of both Definition D IV − 10 and Theorem T IV − 6, we will briefly examine the notion and concomitant properties of all indefinite and definite integrals of any function, $f(q)$, with respect to the quaternion hypercomplex variable 'q', for which we now present the last two definitions contained in this section of our analysis :-

Definition D IV − 11

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

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Consequently, $\forall t \in [a,b]$, the indefinite integral of 'f' with respect to the quaternion hypercomplex variable, 'q', is defined by the formula,

$$\int f(q)\,(dq)_C = \int f(q(t))\,\frac{d}{dt}[q(t)]\,dt.$$

Definition D IV − 12

Let there exist a quaternion hypercomplex function, $f(q)$, which is restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

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Henceforth, the definite integral of 'f', defined with respect to the quaternion hypercomplex variable 'q' over the real interval $[a,b]$, is accordingly given by the formula,

$$\int_C f(q)\,dq = \int_a^b f(q(t))\frac{d}{dt}[q(t)]\,dt .$$

Refering in particular to Definition DIV-11, it is instructive to note the usage of the differential element, $(dq)_C$. The author found it necessary to introduce this notation in preference to using 'dq', since we cannot, as a general rule, define an indefinite integral, ie. anti-derivative,

$$\Phi(q) = \int f(q)\,dq \iff \frac{d}{dq}[\Phi(q)] = \frac{d}{dq}\left[\int f(q)\,dq\right] = f(q) \quad (4-41).$$

The overriding difficulty here is that we have postulated the existence of the operator, $\frac{d}{dq}$, whose very deficiencies in that regard were previously outlined in Part 1 of this section. We must temporarily disregard Eq. (4-41) and instead revert to our original choice of notation. In so doing, we henceforth eliminate any ambiguities which might otherwise have arisen through failing to indicate that every integration, carried out with respect to 'q', shall be constrained to a smooth arc, C, embedded in q-space.

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After denoting any such arc by the equation,

$$\left.\begin{aligned} q(t) &= x(t) + iy(t) \\ & x(t) + j\hat{x}(t) \\ & x(t) + k\hat{y}(t) \end{aligned}\right\} \quad (4-42),$$

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we further surmise that the integral, $\int f(q)(dq)_c$, can now be expressed as —

$$\int f(q)(dq)_c = \int f(q(t))\frac{d}{dt}[q(t)]\,dt$$
$$= \int \frac{d}{dt}[q(t)]\,f(q(t))\,dt \qquad (4-43).$$

Allied to the foregoing, it also follows that if the above mentioned integrand, $f(q(t))\frac{d}{dt}[q(t)]$, is <u>analytic</u> in terms of Eqs. (4-17), (4-18), (4-19) and (4-20), then Eq. (4-43) may likewise be rewritten in the same form as Eq. (4-41), in other words —

$$\int f(q)(dq)_c = \int f(q)\,dq \Longleftrightarrow \frac{d}{dq}\left[\int f(q)\,dq\right] = f(q) \qquad (4-44),$$

upon recognising its algebraic structure as an analogue of the <u>analytic</u> complex integral,

$$\int f(z)\,dz = F(z) \Longleftrightarrow {}_K\!\int f(z)\,dz = 0 \qquad (4-45),$$

where K represents a <u>simple closed contour within a domain, D,</u> as specified by the <u>Cauchy — Goursat Theorem</u>.

Finally, we remark that the evaluation of any indefinite integral, $\int f(q)(dq)_c$, should be a reasonably straightforward procedure, insofar as the integrand,

$$f(q(t))\frac{d}{dt}[q(t)] = \left(u_1^*(t) + i v_1^*(t) + j u_2^*(t) + k v_2^*(t)\right) \times$$
$$\left(\frac{d}{dt}[x(t)] + i\frac{d}{dt}[y(t)] + j\frac{d}{dt}[\hat{x}(t)] + k\frac{d}{dt}[\hat{y}(t)]\right),$$

may be firstly expanded through utilising the established distributive laws

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for quatonian numbers and then each of its constituent real and imaginary parts integrated separately in accordance with the criteria laid down in Theorem TIV−5. Indeed, the same procedure may also be directly applied to the definite integral, $\int f(q)\,dq$, bearing in mind those provisions which were previously specified in Theorem TIV−6.

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V. Appraisal of Results with a View to Further Theoretical Development

In dealing with the subject matter presented throughout the preceding sections of this dissertation, the author has found it necessary to invoke a variety of concepts and techniques which, needless to say, have their origins in the analysis of real and complex variable functions. Granted the origin of these very same concepts and techniques, it is perhaps not surprising that one should accordingly venture to extend this methodology into the realm of quatenions and their corresponding functions — indeed, it could even be said that this dissertation has largely been an exercise with that very objective in mind.

Whilst not wishing to detract from the overall diversity ~~of~~ topics having been elucidated thus far, the author shall nevertheless draw the reader's attention to a number of unresolved problems which therefore warrant further investigation. We briefly outline these problems as follows :—

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(1) From Section II we learnt that the quotient of any two quaternions is not uniquely defined but instead takes on two distinct values. Indeed, the same argument also applies to quotients involving quaternion hypercomplex functions, as previously explained in Section III.

However, in the case of the first and second order derivatives with respect to 'q' of any quaternion hypercomplex function which is both restricted to a smooth arc, C, embedded in q-space, and is differentiable in 't', the situation becomes a little more complicated. We recall from Section IV that the first named entity predictably takes on two distinct values, whereas the latter takes on four separate values instead. A further complication arises with regard to the above mentioned first derivatives of quotient functions, since this particular class of derivatives initially assumes four different values. Evidently, any quaternion hypercomplex derivative of order, $n > 0$, becomes a multi-valued function whose total number of component functions will somehow be determined by its corresponding order. This being the case, we are immediately confronted with two separate problems, namely —

(a) Can we derive a general formula which specifies the total number of components contained in any 'nth' order quaternion hypercomplex derivative?

AND

(b) Can we likewise derive a formula or set of formulae which uniquely determines the algebraic structure of any such derivative?

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(2) The connection between *derivatives* and *indefinite integrals* is well understood by those already familiar with the basic principles underlying the analysis of real and complex functions. However, no such clear cut relationship has yet been established between the above mentioned 'nth' order derivatives of quaternion hypercomplex functions and the corresponding indefinite integrals thereof.

Bearing in mind that we are once again dealing with *multi-valued functions*, can we therefore derive a formula or set of formulae which rigorously defines this relationship, subject to the usual constraints imposed by the intermediary of a smooth arc, C, embedded in q-space?

(3) In addition to studying the general properties of real and complex functions as a whole, mathematicians have traditionally sub-divided the *universal set* of all such entities into a number of well defined sub-sets, that is to say –

(a) the set of *polynomial* functions (of finite degree),
(b) the set of *exponential* functions,
(c) the set of *logarithmic* functions,
(d) the set of *trigonometric* functions,
(e) the set of *hyperbolic* functions,
(f) the set of *inverse trigonometric* functions,
(g) the set of *inverse hyperbolic* functions,

and so on. Since all of the above stated sub-sets, with the exception of the set of polynomial functions (of finite degree), are often collectively referred

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to as the set of transcendental functions, can we readily formulate suitable quaternion-hypercomplex analogues thereof by utilizing algebraic techniques which were previously applied with respect to the analysis of real and complex variable functions?

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Assuredly, this list is by no means exhaustive — rather its intended purpose is to show that we have now established a logical precedent whereupon one might carry out further research with a view to extending our knowledge of the analytical properties of quaternion-hypercomplex functions. Even though the fulfillment of this task is beyond the immediate scope of our current discourse, one should not, on the other hand, be unduly deterred from undertaking a future study of any alternative mathematical methods which will effectively rationalize these idiosyncrasies such as we have already encountered either in relation to the general lack of commutativity of quaternion products or, for that matter, the generation of multi-valued quaternion-hypercomplex functions.

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VI. APPENDICES

A1. Some Definitions and Theorems pertaining to the Limits and Continuity of Real Variable Functions.

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Relevant commentary thus pertaining to —

(a) the definition for the limit of a single real variable function,
(b) a theorem on the uniqueness of such limits,
(c) theorems on the limits of the sums, products and quotients of a single real variable function,
(d) definitions for left-hand and right-hand limits thereof,
(e) a definition for the continuity of such functions,
(f) a theorem on the continuity of composite single real variable functions,
(g) a definition for one-sided continuity thereof,
(h) definitions for the neighbourhood of a point, the interior of a set, the boundary of a set, open sets, closed sets,
(i) a definition for the limit of a function of several real variables; the notion of continuity applied thereto,
(j) a theorem on the continuity of the composites of such functions,

may be found on pgs. 46, 56, 57–60, 69, 74, 76, 78, 682–687, respectively of Reference B3, Section VII.

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A2. Evaluation of the Limit, $\lim_{q \to q_0} \left[\frac{\overline{F(q)}}{|F(q)|^2}\right]$.

The evaluation of the limit, $\lim_{q \to q_0} \left[\frac{\overline{F(q)}}{|F(q)|^2}\right]$, is clearly dependant upon the value we initially assign to the limit, $\lim_{q \to q_0} [F(q)]$.
Hence, if we set

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$$\lim_{q \to q_0} [F(q)] = W_0 \qquad \text{(i)},$$

we accordingly surmise that

$$\lim_{q \to q_0} \left[\frac{\overline{F(q)}}{|F(q)|^2}\right] = \frac{\overline{W_0}}{|W_0|^2} \qquad \text{(ii)}.$$

In order to verify the existence of limit (ii), we recall from Theorem T III–2 both the function,

$$F(q) = U_1(x,y,\hat{x},\hat{y}) + i V_1(x,y,\hat{x},\hat{y}) + j U_2(x,y,\hat{x},\hat{y}) + k V_2(x,y,\hat{x},\hat{y}),$$

and the limiting value,

$$W_0 = U_{10} + i V_{10} + j U_{20} + k V_{20},$$

whereupon it likewise follows that

$$\frac{\overline{F(q)}}{|F(q)|^2} = \frac{U_1(x,y,\hat{x},\hat{y}) - i V_1(x,y,\hat{x},\hat{y}) - j U_2(x,y,\hat{x},\hat{y}) - k V_2(x,y,\hat{x},\hat{y})}{(U_1^2 + V_1^2 + U_2^2 + V_2^2)(x,y,\hat{x},\hat{y})},$$

$$\frac{\overline{W_0}}{|W_0|^2} = \frac{U_{10} - i V_{10} - j U_{20} - k V_{20}}{U_{10}^2 + V_{10}^2 + U_{20}^2 + V_{20}^2}.$$

Furthermore, in accordance with Theorem T III – 1, limit (ii) exists if and only if there exists the simultaneous pair of inequalities,

$$\left|(\overline{F(q)}/|F(q)|^2) - (\overline{W_0}/|W_0|)\right| < \epsilon \text{ and } 0 < |q - q_0| < \delta, \ \forall \delta, \epsilon > 0,$$

$$\therefore \left| \frac{U_1(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})} - \frac{U_{10}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} + i\left(\frac{V_{10}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} - \frac{V_1(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})}\right) + j\left(\frac{U_{20}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} - \frac{U_2(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})}\right) + k\left(\frac{V_{20}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} - \frac{V_2(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})}\right) \right| < \epsilon$$

$$\therefore \left. \begin{array}{l} \left| \dfrac{U_1(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})} - \dfrac{U_{10}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} \right| \\ \left| \dfrac{V_1(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})} - \dfrac{V_{10}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} \right| \\ \left| \dfrac{U_2(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})} - \dfrac{U_{20}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} \right| \\ \left| \dfrac{V_2(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})} - \dfrac{V_{20}}{U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2} \right| \end{array} \right\} < \epsilon,$$

whenever $0 < |q - q_0| < \delta$,

thus implying that

$$0 < |x - x_0| < \delta, \quad 0 < |\hat{x} - \hat{x}_0| < \delta,$$

$$0 < |y - y_0| < \delta, \quad 0 < |\hat{y} - \hat{y}_0| < \delta.$$

We instantly realise that these are the very conditions deemed necessary in the light of the relevant definitions and theorems listed in Appendix A1,

for the existence of the limits,

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[\frac{U_1(x,y,\hat{x},\hat{y})}{(U_1^2 + V_1^2 + U_2^2 + V_2^2)(x,y,\hat{x},\hat{y})} \right] = \frac{U_{10}}{U_{10}^2 + V_{10}^2 + U_{20}^2 + V_{20}^2}$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[\frac{V_1(x,y,\hat{x},\hat{y})}{(U_1^2 + V_1^2 + U_2^2 + V_2^2)(x,y,\hat{x},\hat{y})} \right] = \frac{V_{10}}{U_{10}^2 + V_{10}^2 + U_{20}^2 + V_{20}^2}$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[\frac{U_2(x,y,\hat{x},\hat{y})}{(U_1^2 + V_1^2 + U_2^2 + V_2^2)(x,y,\hat{x},\hat{y})} \right] = \frac{U_{20}}{U_{10}^2 + V_{10}^2 + U_{20}^2 + V_{20}^2}$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[\frac{V_2(x,y,\hat{x},\hat{y})}{(U_1^2 + V_1^2 + U_2^2 + V_2^2)(x,y,\hat{x},\hat{y})} \right] = \frac{V_{20}}{U_{10}^2 + V_{10}^2 + U_{20}^2 + V_{20}^2}$$

which further give rise to the limits,

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[U_1(x,y,\hat{x},\hat{y}) \right] = U_{10}, \quad \lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} \left[U_2(x,y,\hat{x},\hat{y}) \right] = U_{20},$$

$$\lim_{\substack{(x,y,\hat{x},\hat{y})\to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} \left[V_1(x,y,\hat{x},\hat{y}) \right] = V_{10} \,, \quad \lim_{\substack{(x,y,\hat{x},\hat{y})\to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} \left[V_2(x,y,\hat{x},\hat{y}) \right] = V_{20} \,,$$

being none other than the conditions required for the existence of limit (i), as originally anticipated.

A8. <u>Evaluation of the Limits, $\lim_{T\to t}\left[|\phi_2(q(T))|\right]$ and $\lim_{T\to t}\left[\overline{\phi_2(q(T))}\right]$</u>.

The evaluation of the limits, $\lim_{T\to t}\left[|\phi_2(q(T))|\right]$ and $\lim_{T\to t}\left[\overline{\phi_2(q(T))}\right]$, are clearly dependant upon the value we initially assign to the limit, $\lim_{T\to t}\left[\phi_2(q(T))\right]$. Hence, if we set

$$\lim_{T\to t}\left[\phi_2(q(T))\right] = \phi_2(q(t)) \qquad \underline{(i)} \,,$$

we accordingly surmise that

$$\lim_{T\to t}\left[|\phi_2(q(T))|\right] = |\phi_2(q(t))| \qquad \underline{(ii)} \,,$$

$$\lim_{T\to t}\left[\overline{\phi_2(q(T))}\right] = \overline{\phi_2(q(t))} \qquad \underline{(iii)} \,.$$

In order to verify the existence of limit (ii), we recall, in an analogous manner to Theorem TIII-5, that the existence of limit (i) requires the simultaneous existence of the inequalities,

$$|\phi_2(q(T)) - \phi_2(q(t))| < \epsilon \text{ and } 0 < |T-t| < \delta \,,\ \forall \delta, \epsilon > 0 \,.$$

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However, in an analogous manner to Theorem TII-13, we also perceive that the inequality,

$$||\phi_2(q(T))| - |\phi_2(q(t))|| \leqslant |\phi_2(q(T)) - \phi_2(q(t))| ,$$

and hence, after making the appropriate substitutions into the above stated pair of inequalities, it logically follows that

$$||\phi_2(q(T))| - |\phi_2(q(t))|| < \epsilon \text{ and } 0 < |T-t| < \delta , \ \forall \delta, \epsilon > 0,$$

which are precisely the conditions deemed necessary for the existence of limit (ii), as originally specified.

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Similarly, in order to verify the existence of limit (iii), we recall, in an analogous manner to Definition DII-7, that the modulus,

$$|\phi_2(q(T)) - \phi_2(q(t))| = |\overline{\phi_2(q(T)) - \phi_2(q(t))}|$$

$$= |\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}| \quad ,$$

and hence, after making the relevant substitutions, we further obtain the pair of inequalities,

$$|\overline{\phi_2(q(T))} - \overline{\phi_2(q(t))}| < \epsilon \text{ and } 0 < |T-t| < \delta , \ \forall \delta, \epsilon > 0,$$

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which are precisely the conditions deemed necessary for the existence of limit (iii), as required.

A4. The Derivative of a Composite Function in Four Real Variables.

From the calculus of multivariable functions, there exists a theorem pertaining to the differentiability of a composite function, $f \circ \underset{\sim}{r}$, which we accordingly state as follows :-

The Chain Rule for Differentiation

If a real function, $f = f(x, y, z)$, is continuously differentiable on an open set, U, and $\underset{\sim}{r}(t)$ is a differentiable curve contained in U, then the composite function, $f \circ \underset{\sim}{r}$, is also differentiable such that

$$\frac{d}{dt}\left[f(\underset{\sim}{r}(t))\right] = \nabla f(\underset{\sim}{r}(t)) \cdot \underset{\sim}{r}'(t) .$$

*   *   *

PROOF :-

A proof of this rule may be found on pgs. 710, 711 of Reference B3, Section VII.

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As an immediate consequence of the above mentioned 'chain rule', the derivative of any function, $f = f(x,y,z)$, with respect to 't', can also be rewritten as –

$$\frac{d}{dt}[f(x,y,z)] = \frac{d}{dt}[f(\underset{\sim}{r}(t))]$$

$$= \nabla f(\underset{\sim}{r}(t)) \cdot \underset{\sim}{r}'(t)$$

$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} \qquad \underline{(i)} \;.$$

In the two-variable case of a function, $f = f(x,y)$, the corresponding derivative, with respect to 't', is similarly written as –

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$$\frac{d}{dt}[f(x,y)] = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \qquad \underline{(ii)} \;.$$

Henceforth, in an analogous manner to Eqs. $\underline{(i)}$ and $\underline{(ii)}$ above, we perceive that the derivative of any four-variable function, $F = F(X_1, X_2, X_3, X_4)$, with respect to '$t$', is expressed as –

$$\frac{d}{dt}[F(X_1,X_2,X_3,X_4)] = \frac{\partial F}{\partial X_1}\cdot\frac{dX_1}{dt} + \frac{\partial F}{\partial X_2}\cdot\frac{dX_2}{dt} + \frac{\partial F}{\partial X_3}\cdot\frac{dX_3}{dt} + \frac{\partial F}{\partial X_4}\cdot\frac{dX_4}{dt} \qquad \underline{(iii)} \;,$$

after logically extending our original Chain Rule for Differentiation to include higher dimensional cases.

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VII. BIBLIOGRAPHY

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