“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 3/6.”

by Stephen C. Pearson MRACI
[T/A S. C. Pearson Technical Services],
Dorset. ENGLAND.

Email Address:- *scpearson1952@talktalk.net* .

I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.”,

which has been published under the **‘VIXRA’ Mathematics subheading:- *‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 3/6.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

*   *   *   *   *   *   *   *

whence it immediately follows that

$$|q_2|-|q_1| \leqslant |q_1-q_2| .$$

Furthermore, it is evident that

$$|q_2|-|q_1| = ||q_2|-|q_1||, \text{ whenever } |q_2|>|q_1|,$$

and since

$$||q_2|-|q_1|| = |-(|q_1|-|q_2|)| = |-1|||q_1|-|q_2|| = ||q_1|-|q_2|| ,$$

we finally conclude that

$$||q_1|-|q_2|| = ||q_2|-|q_1|| = |q_2|-|q_1| \leqslant |q_1-q_2|$$

$$\Rightarrow ||q_1|-|q_2|| \leqslant |q_1-q_2|, \text{ whenever } |q_2|>|q_1|.$$

Hence, we have demonstrated that the inequality,

$$||q_1|-|q_2|| \leqslant |q_1-q_2| ,$$

is valid $\forall |q_1|,|q_2| \in \mathbb{R}$, as required. $\underline{\underline{\text{Q.E.D.}}}$

Finally, as previously agreed to, we shall now summarise, in tabular form, the relevant properties of moduli products and triangle inequalities

****

thus corresponding to complex and quaternion hypercomplex numbers by means of Table II(5) provided below :–

****

Description of Property	Concomitant Response of Variables
RESOLUTION OF MODULI	$\lvert z\rvert = \sqrt{x^2+y^2}$, $\forall x,y \in \mathbb{R}$. $\lvert q\rvert = \sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}$, $\forall x,y,\hat{x},\hat{y} \in \mathbb{R}$.
MODULI PRODUCTS	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$, $\forall z_1, z_2 \in \mathbb{C}$. $\lvert q_1 q_2\rvert = \lvert q_1\rvert\lvert q_2\rvert$, $\forall q_1, q_2 \in \mathbb{H}$.
TRIANGLE INEQUALITIES	$\left.\begin{array}{l} \lvert z_1+z_2\rvert \leqslant \lvert z_1\rvert+\lvert z_2\rvert \\ \lvert\lvert z_1\rvert-\lvert z_2\rvert\rvert \leqslant \lvert z_1+z_2\rvert \\ \lvert\lvert z_1\rvert-\lvert z_2\rvert\rvert \leqslant \lvert z_1-z_2\rvert \end{array}\right\}$, $\forall z_1, z_2 \in \mathbb{C}$. $\left.\begin{array}{l} \lvert q_1+q_2\rvert \leqslant \lvert q_1\rvert+\lvert q_2\rvert \\ \lvert\lvert q_1\rvert-\lvert q_2\rvert\rvert \leqslant \lvert q_1+q_2\rvert \\ \lvert\lvert q_1\rvert-\lvert q_2\rvert\rvert \leqslant \lvert q_1-q_2\rvert \end{array}\right\}$, $\forall q_1, q_2 \in \mathbb{H}$.

Table II(5)

****

****

III. Functions of a Single Quaternion Hypercomplex Variable; the Concepts of Limit and Continuity applied to Such Functions

In this section, we will both define and evaluate the notion and subsequent properties of a <u>function of a single quaternion hypercomplex variable</u> as well as the concepts of <u>limit</u> and <u>continuity</u> applied to such functions. To all intents and purposes, the approach we are about to undertake is very similar to that adopted by Churchill et al. (cf. Reference B2, Section VII) in the study of functions of a <u>single complex variable</u> and hence we may justifiably look upon the material covered in the preceding Sections I and II of this dissertation as a necessary prerequisite for the introduction of these hitherto described ideas.

However, prior to our commencing any formal analysis thereof, the author considers it only appropriate that we initially redefine any quaternion as an <u>ordered quadruplet</u> and thus correlate all relevant arguments concerning this alternative approach with the previously established Definition DI-1. Indeed, our reasons for doing so should become increasingly apparent to the reader as we progress through the latter stages of our ensuing analysis.

1. Representation of Quaternions as Ordered Quadruplets; Regions of Quaternion Hypercomplex Space.

From the second last paragraph of Section I, the reader will no doubt recall Eq. (I-4), namely

****

$$q = 1\cdot x + i\cdot y + j\cdot\hat{x} + k\cdot\hat{y},$$

as a defining equation for the quaternion, q, differing slightly in notation from that which was previously enunciated with respect to Definition DI-1, that is to say

$$q = x + iy + j\hat{x} + k\hat{y} \qquad (3-1).$$

We also made the point that some authors may indeed prefer to explicitly state the basis element '1' for the purposes of definition, whereas others, apart from the author of this dissertation, have instead chosen to suppress it. Quite obviously, any such variations in notation may cause some confusion to those who are otherwise unfamiliar with the subject matter and therefore require further clarification.

From Section I, we recall that the set, $\mathbb{H}$, of quaternions, was originally defined by Hamilton (1805 - 1865) as a two-dimensional vector space over $\mathbb{C}$ and hence a four-dimensional vector space over $\mathbb{R}$. This being the case, it naturally follows that the four basis vectors, $1, i, j, k$, are respectively defined as

$$\left.\begin{aligned} 1 &= (1,0,0,0) \\ i &= (0,1,0,0) \\ j &= (0,0,1,0) \\ k &= (0,0,0,1) \end{aligned}\right\} \qquad (3-2),$$

whereupon any given quaternion, q, may now be rewritten as

$$q = 1\cdot x + i\cdot y + j\cdot\hat{x} + k\cdot y$$

$$= x(1,0,0,0) + y(0,1,0,0) + \hat{x}(0,0,1,0) + \hat{y}(0,0,0,1)$$

$$= (x,0,0,0) + (0,y,0,0) + (0,0,\hat{x},0) + (0,0,0,\hat{y})$$

$$= (x,y,\hat{x},\hat{y}) = x + iy + j\hat{x} + k\hat{y} \qquad (3-3).$$

In a completely analogous manner to the above equation, Churchill et al. (cf. Reference B2, Section VII) have defined the complex number,

$$z = 1\cdot x + i\cdot y$$

$$= x(1,0) + y(0,1)$$

$$= (x,0) + (0,y)$$

$$= (x,y) = x + iy \qquad (3-4)$$

and so Eq. (3-3) is essentially a logical extension of Eq. (3-4) into $\mathbb{H}$. In describing the hitherto mentioned basis vectors, Heyting (cf. Reference B5, Section VII) has used the notation, e, j_1, j_2, j_3, instead of $1, i, j, k$, which was incidentally adopted by Griffiths and Hilton (cf. Reference B1, Section VII). On the other hand, Birkhoff and Mac Lane (cf. Reference B6, Section VII) have chosen to denote the quaternion, q, in the same way as was done in Definition DI−1. Granted that the algebra of quaternions is a relatively recent innovation in terms of the overall history of mathematics, it is therefore only reasonable to expect that these variations in notation will continue to persist for some time to come and hence one should not at this stage be overly dogmatic with regard to their respective usages.

Of more immediate interest to us is the fact that, with the exception of Churchill et al., the above mentioned authors have also introduced the concept of a complex and a quaternion hypercomplex number using set−theoretic terminology in addition to formally defining these same numbers by way of Eqs.(3−3) and (3−4). Consequently, by adopting a set−theoretic approach, we need to invoke the notion of an isomorphism − more specifically, we say that a complex number, $x + iy$, is isomorphic with the ordered pair, (x, y), via the mapping,

$$x + iy \longleftrightarrow (x, y) \qquad (3-5) \ ,$$

and similarly, with respect to the set of quaternions, $\mathbb{H}$, the real number, x, is isomorphic with the ordered quadruplet, $(x, 0, 0, 0)$, via the mapping,

$$x \longleftrightarrow (x, 0, 0, 0) \qquad (3-6) \ .$$

Moreover, there exists an isomorphism between the subset of quaternions, $\{q\} = \{x + iy\}$ with $\hat{x} = \hat{y} = 0$, and the set of all complex numbers, $\mathbb{C}$.

Fortunately, it turns out that both of these alternative approaches are equivalent, differing only in the notation employed, and indeed the reader may wish to verify this assertion by further consulting

(i) pgs. 399 – 405 incl. of Reference B1, Section VII;
(ii) pgs. 10 – 12 " " " B5, " ",
(iii) pgs. 222 – 224 " " " B6, " ".

As a means of summarising the content of our preceding commentary, we now state the following definition :–

Definition DIII – 1

A quaternion hypercomplex number,

$$q = x + iy + j\hat{x} + k\hat{y},$$

may also be represented as an <u>ordered quadruplet</u>, that is to say

$$q = (x, y, \hat{x}, \hat{y}) \in \mathbb{H},$$

the set of <u>all ordered quadruplets</u> comprising quaternion hypercomplex space, wherein

$$\begin{aligned} 1 &= (1,0,0,0), \\ i &= (0,1,0,0), \\ j &= (0,0,1,0), \\ k &= (0,0,0,1), \end{aligned}$$

are the constituent basis elements (vectors) thereof.

From Eq. (3-4), we perceive that the real advantage in formally defining complex numbers as ordered pairs is that they can be easily represented as points on a two-dimensional co-ordinate system often referred to as an Argand diagram (cf. Fig. (3-1) illustrated below), whereupon both individual points and whole regions (R) thereof may be depicted in this fashion (cf. Fig. (3-2) illustrated below).

Can we likewise represent quaternion hypercomplex numbers as ordered quadruplets in an analogous manner to the Argand diagram? Quite clearly, we cannot do so in view of the fact that, as previously stated, the set of all quaternions, $\mathbb{H}$, is defined as a four-dimensional vector space over $\mathbb{R}$. However, in spite of the obvious visual handicap, we may still resolve this fundamental problem of representation by introducing the concept of a neighbourhood, for which we accordingly provide the following definition :-

Definition DIII-2

Any quaternion hypercomplex number, $q \in S$, a subset of quaternions contained in $\mathbb{H}$, is said to lie within the δ-neighbourhood, or more simply neighbourhood, of a fixed quaternion, q_0, if and only if there exists the inequality,

TO BE CONTINUED.

…./[10]

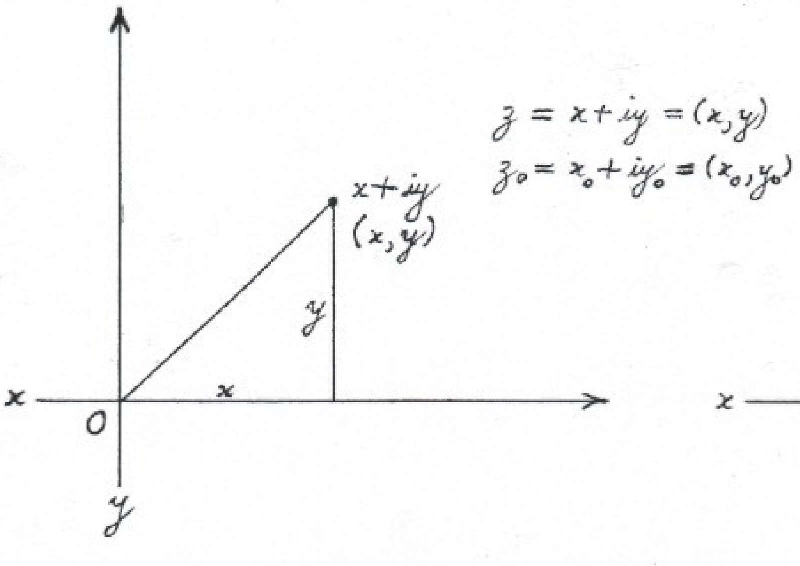


Figure (3–1)

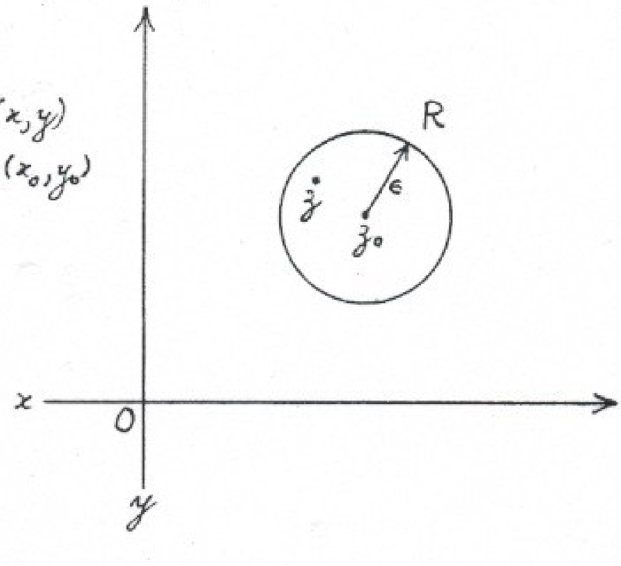


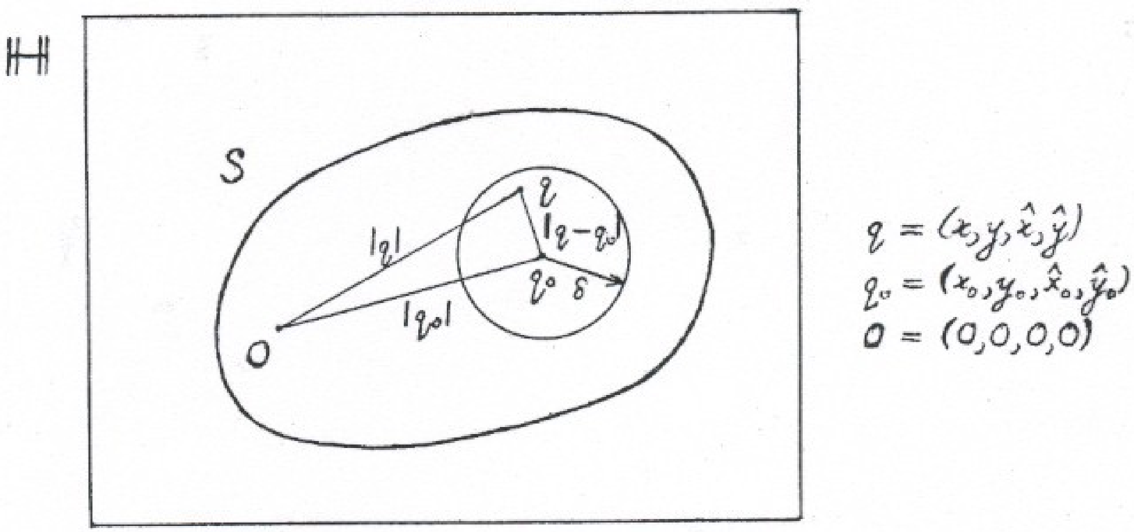
Figure (3–2)

$|q - q_0| < \delta$, where 'δ' is some positive four-dimensional radius embedded in S.

The meaning of this definition becomes much clearer from Fig. (3–3), illustrated below, wherein it should be noted that we have subsequently adopted a schematic convention reminiscent of Venn diagrams for the purposes of enhancing its intended representation. In so doing, we are therefore admitting that the modulus, $|q|$, of any quaternion, q, may be geometrically interpreted as the length of a directed segment (vector) embedded in four-dimensional real space, even though we cannot physically represent it as such on a four-dimensional co-ordinate diagram. Nevertheless, we have established a direct analogy between Definition DIII – 2 and its more familiar counterpart thus pertaining to the neighbourhood conditions for complex variables in $\mathbb{C}$.

Henceforth, we shall finalize our discussion of this particular topic by defining the notions of

(a) <u>interior point</u>, (b) <u>exterior point</u>, (c) <u>boundary point</u>, (d) <u>open set</u>, (e) <u>closed set</u>, (f) <u>connected set</u>, (g) <u>domain</u>, (h) <u>region</u>, (i) <u>bounded set</u>, (j) <u>unbounded set</u>,



<u>Figure (3−3)</u>

as follows :−

<u>Definition DIII − 3</u>

A point, q_0, is said to be an <u>interior point</u> of a subset of quaternions, S, whenever there is some neighbourhood of q_0 which contains only points of

S; otherwise it is called an exterior point of S.

Definition DIII – 4

If a point, q_0, does not conform to the criteria specified in the above definition, it is called a boundary point of S. Furthermore, the totality of all such boundary points is called the boundary of S.

Definition DIII – 5

A set, S, is open if it contains none of its boundary points. On the other hand, a set, S, is closed if it contains all of its boundary points.

Definition DIII – 6

An open set, S, is connected if each pair of points therein can be joined by a polygonal path, consisting of a finite number of linear segments joined end to end, which lies entirely within S. An open set which is connected is called a domain. It should also be noted that any neighborhood is accordingly classified as a domain and, furthermore, any domain combined with none, some or all of its boundary points is referred to as a region.

Definition DIII-7

A set, S, is bounded if every point of S lies within some closed surface,

$$|q| = R > 0 ;$$

otherwise it is unbounded.

Referring in particular to Definitions DIII-3 to DIII-7, we remark that the terms 'point' and 'quaternion hypercomplex number' are both synonymous and therefore interchangeable, being a convention which readily follows from the treatments given by Churchill et al. (cf. Reference B2, Section VII) of such notions, inter alia, with respect to complex variable analysis. Indeed, the reader may further wish to verify this fact by perusing pgs. 18 and 19 of the above mentioned reference.

2. Functions of a Quaternion Hypercomplex Variable and Some Algebraic Properties thereof.

Because the notion of a function and its subsequent algebraic properties are already well established in terms of both real and complex variable analysis, the logical extension of these ideas into the realm of quaternions will therefore present no real difficulty. Whilst we are generally mindful of the overall lack of commutativity thus manifested by the multiplication of quatenians, we can nevertheless proceed immediately to elucidate this basic concept by means of four new definitions which are accordingly stated as follows :—

Definition DIII — 8

Let S be any subset of quaternion hypercomplex numbers contained in $\mathbb{H}$. A function 'f', defined on S, is a rule which accordingly assigns to each $q \in S$ a quaternion hypercomplex number 'w', that is to say

$$w = f(q) .$$

Furthermore, the set, S, is otherwise referred to as the domain of definition of 'f' and 'w' is likewise an element of the co-domain of 'f'.

Definition DIII — 9

Let there exist a quaternion hypercomplex number,

$$q = x + iy + j\hat{x} + k\hat{y} .$$

Consequently, we define a function of any quaternion hypercomplex variable, 'f', such that

$$w = f(q) = f(x + iy + j\hat{x} + k\hat{y})$$

$$= u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

whereupon $x, \ldots, \hat{y}, u_1(x,y,\hat{x},\hat{y}), \ldots, v_2(x,y,\hat{x},\hat{y}) \in \mathbb{R}$.

Definition DIII − 10

Let 'f' be a mapping or transformation of any quaternion hypercomplex variable. Consequently, we define

(i) the image of a point 'q' in the domain of definition, S, as the point '$w = f(q)$' and the set of images of all points in a subset, T, which is contained in S, is called the image of T,

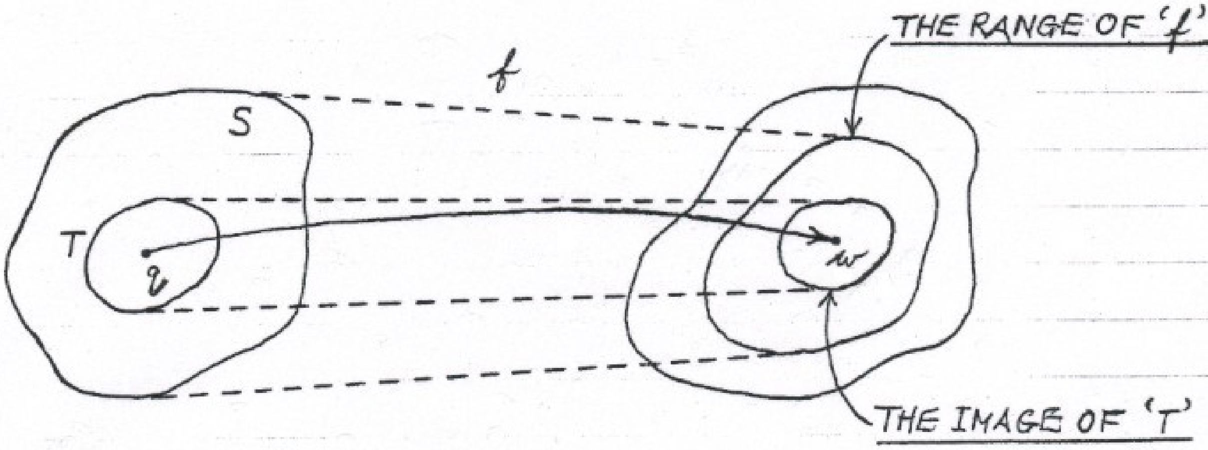
(ii) the image of the entire domain of definition, S, as the range of 'f', thus contained in the co − domain of 'f',

(iii) the inverse image of a point, '$w = f(q)$', as the set of all points, '$q = f^{-1}(w)$', in the domain of definition of 'f' which have 'w' as their image. The inverse image of a point may contain just one point, many points or none at all. The last consideration only arises whenever 'w' is not in the range of 'f'.

We graphically illustrate this definition by means of Fig. (3−4) provided below.

Definition DIII – 11

Let 'f' and 'g' be two functions of a quaternion hypercomplex variable defined on a common domain, S. Henceforth, the following algebraic operations, defined with respect to these functions, shall apply —



DOMAIN OF 'f' CO-DOMAIN OF 'f'

Figure (3–4)

(i) $(f+K)(q) = f(q) + K$, where K denotes any quaternion hypercomplex constant,

(ii) $(|f|)(q) = |f(q)|$,

(iii) $(f^n)(q) = (f(q))^n$, where $n \in \mathbb{Z}$, the set of integers,

(iv) $(f \pm g)(q) = f(q) \pm g(q)$,

(v) $(Kf)(q) = Kf(q)$,

(vi) $(fg)(q) = f(q)g(q)$,

(vii) $(f/g)(q) = f(q)/g(q), \ \forall g(q) \neq 0$,

(viii) $(f \circ g)(q) = f(g(q))$,

(ix) $(id \circ f)(q) = id(f(q))$

$= f(q)$, where 'id' denotes the identity function,

(x) $(f^{-1} \circ f)(q) = (f \circ f^{-1})(q)$

$= id(q)$

$= q$, provided that 'f' is 'one-to-one' and 'onto'.

Bearing in mind the provisions of Definitions DIII-8 to DIII-11 incl., we now make the following comments and observations :-

(1) In referring to the subset, S, of quaternions as the domain of definition of 'f', we shall henceforth for the remainder of this text designate this subset to be the maximum domain of definition, in other words the set, $\mathbb{H}$, of all quaternions, unless any restriction to the contrary is clearly indicated.

(2) Since, as previously stated, we cannot physically represent individual quaternions and, for that matter, whole regions thereof as points on a four-dimensional co-ordinate diagram, we are likewise compelled to represent any mappings (transformations) from the domain to the co-domain of 'f' by means of function diagrams thus typified by Fig. (3-4).

(3) With respect to part (vii) of Definition DIII-11, we further deduce that the quotient function, $(f/g)(q)$, of any two functions, 'f' and 'g', defined on a common domain is <u>not uniquely determined</u> but instead takes on <u>two distinct values</u>, namely

$$(f/g)(q) = f(q)/g(q) = \begin{cases} f(q)\overline{g(q)}/|g(q)|^2 , \ \forall g(q) \neq 0 \\ \overline{g(q)}f(q)/|g(q)|^2 \end{cases} \quad (3-7).$$

Indeed, this assertion can be readily ascertained from Theorem TII-7 and, should this otherwise be seen to contradict the basic notion of a function being a uniquely determined entity, then we are instantly reminded of the fact that in complex variable analysis there arise many examples of functions which are <u>multi-valued</u>, eg. the function,

$$f(z) = z^{1/2} \quad (3-8),$$

which takes on two distinct values throughout its entire domain, except when $z = 0$. Consequently, we may justifiably claim that we are simply following a well established precedent and hence extending this same precedent into the realm of quaternion hypercomplex functions.

In the circumstances, we shall henceforth agree that, wherever the term 'function' is used in this context, we specifically mean a <u>single valued</u> function, unless a given example to the contrary is properly defined as such.

TO BE CONTINUED.

..../[19]

3. The Limit of a Quaternion Hypercomplex Function and Some Algebraic Properties thereof.

Having formulated a suitable definition for the quaternion hypercomplex function and its subsequent algebraic properties, we now wish to investigate a number of properties governing the limits of these functions. To do

this, we will firstly enunciate a sound working definition for the limit of a quaternion hypercomplex function, followed by three theorems in connection with

(i) the relationship between a limit and its constituent real and imaginary parts,

(ii) the limits of the sums, products and quotients of quaternion hypercomplex functions,

(iii) the limit of the modulus of a quaternion hypercomplex function.

As a matter of fact, the procedure we are about to employ is essentially the same as that employed to investigate parallel properties of the limits of complex functions, whereupon the hitherto described definition may be enunciated accordingly :-

Definition DIII – 12

Let there exist a quaternion hypercomplex function,

$$w = f(q),$$

such that its corresponding limit,

$$\lim_{q \to q_0} [f(q)] = w_0 ,$$

exists if and only if there exist real numbers, $\delta, \epsilon > 0$, which simultaneously give rise to the pair of inequalities,

$$0 < |q - q_0| < \delta \quad \text{and} \quad |f(q) - w_0| < \epsilon .$$

The above definition is graphically illustrated in Fig. (3–5) set out below.

Definition DIII–12 therefore provides us with a rigorous method of verifying the existence or non-existence of a limit of a quaternion hypercomplex function and can be applied to relatively simple functions. By the same token, the reader will no doubt also appreciate that the δ, ϵ – technique becomes very tedious and cumbersome when applied to more elaborate quaternion hypercomplex functions. Indeed, one can well imagine the difficulties that would have otherwise been encountered if this same technique were to be analogously applied to any function, $f(z)$, in $\mathbb{C}$ of comparable complexity (cf. Reference B2, Section VII).

Hence, it is for this reason that we choose to dispense with any further discussion of the δ-ε technique by way of specific, albeit simple, illustrations thereof; we will instead be seeking an alternative approach in order to overcome our basic objection to the 'practical' limitations inherent in this particular algebraic method. Assuredly, we may in part overcome these difficulties by firstly deriving our next theorem which correlates the limiting value of a quaternion hypercomplex function with its constituent real and imaginary parts :-

Theorem TIII-1

Suppose that there exists a function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

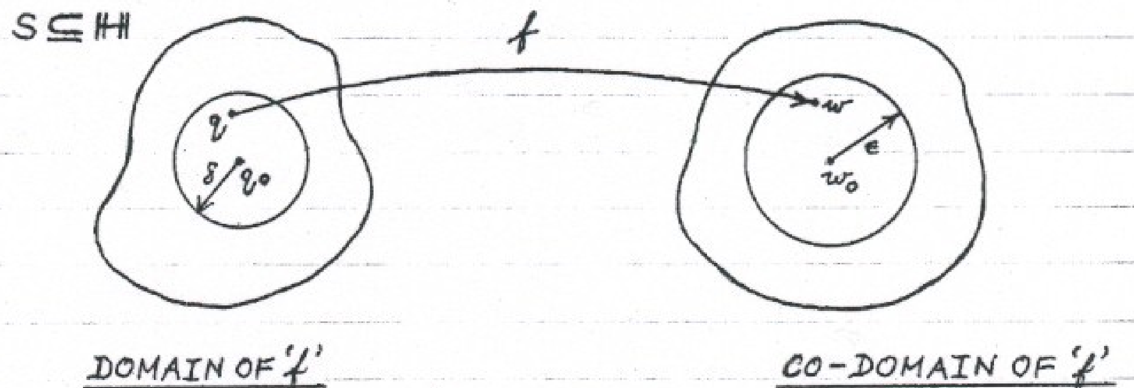


Figure (3–5)

as well as fixed points,

$$q_0 = x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0 ,$$

$$w_0 = u_{10} + iv_{10} + ju_{20} + kv_{20} ,$$

then the limit,

$$\lim_{q \to q_0} [f(q)] = w_0 ,$$

exists if and only if the real variable limits,

(i) $$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [u_1(x,y,\hat{x},\hat{y})] = u_{10} ,$$

(ii) $$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [v_1(x,y,\hat{x},\hat{y})] = v_{10} ,$$

(iii) $$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,y_0)} [u_2(x,y,\hat{x},\hat{y})] = u_{20} ,$$

(iv) $$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [v_2(x,y,\hat{x},\hat{y})] = v_{20} ,$$

likewise exist.

* * *

<u>PROOF:–</u>

In accordance with Definition DIII–12, we note that

$$\lim_{q \to q_0} [f(q)] = w_0$$

exists, if and only if there likewise exist real numbers, $\delta, \epsilon > 0$, such that

$$0 < |q - q_0| < \delta \Rightarrow |f(q) - w_0| < \epsilon .$$

By substituting the previously established defining equations for q, $f(q)$, q_0, w_0 respectively into the above stated inequalities, we may now write

$$0 < |x + iy + j\hat{x} + k\hat{y} - (x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0)| < \delta \Rightarrow$$

$$\left| \begin{array}{l} u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y}) \\ -(u_{10} + iv_{10} + ju_{20} + kv_{20}) \end{array} \right| < \epsilon$$

which may be further simplified to

$$0 < |(x - x_0) + i(y - y_0) + j(\hat{x} - \hat{x}_0) + k(\hat{y} - \hat{y}_0)| < \delta \Rightarrow$$

$$\left| \begin{array}{l} (u_1(x,y,\hat{x},\hat{y}) - u_{10}) + i(v_1(x,y,\hat{x},\hat{y}) - v_{10}) + \\ j(u_2(x,y,\hat{x},\hat{y}) - u_{20}) + k(v_2(x,y,\hat{x},\hat{y}) - v_{20}) \end{array} \right| < \epsilon$$

and hence

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (y-\hat{y}_0)^2} < \delta \Rightarrow$$

$$\sqrt{\begin{array}{l} (u_1(x,y,\hat{x},\hat{y}) - u_{10})^2 + (v_1(x,y,\hat{x},\hat{y}) - v_{10})^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_{20})^2 + (v_2(x,y,\hat{x},\hat{y}) - v_{20})^2 \end{array}} < \epsilon .$$

Since the inequalities,

$$\left.\begin{aligned}
|u_1(x,y,\hat{x},\hat{y}) - u_{10}| &= \sqrt{(u_1(x,y,\hat{x},\hat{y}) - u_{10})^2} \\
|v_1(x,y,\hat{x},\hat{y}) - v_{10}| &= \sqrt{(v_1(x,y,\hat{x},\hat{y}) - v_{10})^2} \\
|u_2(x,y,\hat{x},\hat{y}) - u_{20}| &= \sqrt{(u_2(x,y,\hat{x},\hat{y}) - u_{20})^2} \\
|v_2(x,y,\hat{x},\hat{y}) - v_{20}| &= \sqrt{(v_2(x,y,\hat{x},\hat{y}) - v_{20})^2}
\end{aligned}\right\} \leqslant \begin{bmatrix} (u_1(x,y,\hat{x},\hat{y}) - u_{10})^2 + \\ (v_1(x,y,\hat{x},\hat{y}) - v_{10})^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_{20})^2 + \\ (v_2(x,y,\hat{x},\hat{y}) - v_{20})^2 \end{bmatrix}^{\frac{1}{2}},$$

it likewise follows that

$$|u_1(x,y,\hat{x},\hat{y}) - u_{10}| < \epsilon, \; |v_1(x,y,\hat{x},\hat{y}) - v_{10}| < \epsilon,$$

$$|u_2(x,y,\hat{x},\hat{y}) - u_{20}| < \epsilon, \; |v_2(x,y,\hat{x},\hat{y}) - v_{20}| < \epsilon,$$

whenever

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2} < \delta,$$

which from real variable analysis are precisely the conditions deemed necessary for the existence of the limits (i), (ii), (iii) and (iv), thus denoted in the preamble to this proof.

Conversely, for each positive number 'ϵ', let there exist four positive numbers, δ_1, δ_2, δ_3 and δ_4, such that

$$|u_1(x,y,\hat{x},\hat{y}) - u_{10}| < \tfrac{\epsilon}{4}, \; |v_1(x,y,\hat{x},\hat{y}) - v_{10}| < \tfrac{\epsilon}{4},$$

$$|u_2(x,y,\hat{x},\hat{y}) - u_{20}| < \tfrac{\epsilon}{4}, \; |v_2(x,y,\hat{x},\hat{y}) - v_{20}| < \tfrac{\epsilon}{4},$$

wherever

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2} < \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}.$$

Furthermore, since

$$\sqrt{\begin{matrix}(u_1(x,y,\hat{x},\hat{y})-u_{10})^2+(v_1(x,y,\hat{x},\hat{y})-v_{10})^2+ \\ (u_2(x,y,\hat{x},\hat{y})-u_{20})^2+(v_2(x,y,\hat{x},\hat{y})-v_{20})^2\end{matrix}}$$

$$= \sqrt{\begin{matrix}|u_1(x,y,\hat{x},\hat{y})-u_{10}|^2+|v_1(x,y,\hat{x},\hat{y})-v_{10}|^2+ \\ |u_2(x,y,\hat{x},\hat{y})-u_{20}|^2+|v_2(x,y,\hat{x},\hat{y})-v_{20}|^2\end{matrix}}$$

$$\leq \left[\begin{matrix}|u_1(x,y,\hat{x},\hat{y})-u_{10}|^2+2|u_1(x,y,\hat{x},\hat{y})-u_{10}||v_1(x,y,\hat{x},\hat{y})-v_{10}|+ \\ |v_1(x,y,\hat{x},\hat{y})-v_{10}|^2+2(|u_1(x,y,\hat{x},\hat{y})-u_{10}|+|v_1(x,y,\hat{x},\hat{y})-v_{10}|)\times \\ (|u_2(x,y,\hat{x},\hat{y})-u_{20}|+|v_2(x,y,\hat{x},\hat{y})-v_{20}|)+|u_2(x,y,\hat{x},\hat{y})-u_{20}|^2+ \\ 2|u_2(x,y,\hat{x},\hat{y})-u_{20}||v_2(x,y,\hat{x},\hat{y})-v_{20}|+|v_2(x,y,\hat{x},\hat{y})-v_{20}|^2\end{matrix}\right]^{\frac{1}{2}}$$

$$= \sqrt{\left(\begin{matrix}|u_1(x,y,\hat{x},\hat{y})-u_{10}|+|v_1(x,y,\hat{x},\hat{y})-v_{10}|+ \\ |u_2(x,y,\hat{x},\hat{y})-u_{20}|+|v_2(x,y,\hat{x},\hat{y})-v_{20}|\end{matrix}\right)^2}$$

$$= |u_1(x,y,\hat{x},\hat{y})-u_{10}|+|v_1(x,y,\hat{x},\hat{y})-v_{10}|+|u_2(x,y,\hat{x},\hat{y})-u_{20}|+$$
$$|v_2(x,y,\hat{x},\hat{y})-v_{20}|$$

$$< \epsilon,$$

it therefore follows that

$$\sqrt{\begin{array}{l}(u_1(x,y,\hat{x},\hat{y})-u_{10})^2+(v_1(x,y,\hat{x},\hat{y})-v_{10})^2+\\(u_2(x,y,\hat{x},\hat{y})-u_{20})^2+(v_2(x,y,\hat{x},\hat{y})-v_{20})^2\end{array}}<\epsilon,$$

wherever $0<\sqrt{(x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2}<\delta$

and hence

$$\left|\begin{array}{l}(u_1(x,y,\hat{x},\hat{y})-u_{10})+i(v_1(x,y,\hat{x},\hat{y})-v_{10})+\\j(u_2(x,y,\hat{x},\hat{y})-u_{20})+k(v_2(x,y,\hat{x},\hat{y})-v_{20})\end{array}\right|<\epsilon,$$

wherever $0<|(x-x_0)+i(y-y_0)+j(\hat{x}-\hat{x}_0)+k(\hat{y}-\hat{y}_0)|<\delta$

$$\therefore\left|\begin{array}{l}u_1(x,y,\hat{x},\hat{y})+iv_1(x,y,\hat{x},\hat{y})+ju_2(x,y,\hat{x},\hat{y})+kv_2(x,y,\hat{x},\hat{y})\\-(u_{10}+iv_{10}+ju_{20}+kv_{20})\end{array}\right|<\epsilon,$$

wherever $0<|x+iy+j\hat{x}+k\hat{y}-(x_0+iy_0+j\hat{x}_0+k\hat{y}_0)|<\delta,$

$\therefore |f(q)-w_0|<\epsilon,$ wherever $0<|q-q_0|<\delta,$

-64-

which, in accordance with Definition DIII-12, are precisely the conditions deemed necessary for the existence of the limit,

$\lim_{q\to q_0}[f(q)]=w_0,$ as required. Q.E.D.

Once again, the author wishes to emphasise that this theorem does not in any way diminish the validity of Definition DIII-12. On the contrary, it merely serves to expedite the task of verifying the existence, or otherwise, of the limits of quaternion hypercomplex functions by explicitly demonstrating the dependence of such limits upon their constituent real and imaginary parts. In so doing, the potential usefulness of Theorem TIII-1 should become ever more apparent to the reader as we proceed to derive our next theorem, which we state as follows :-

Theorem TIII-2

Let there exist two quaternion hypercomplex functions, $f(q)$ and $F(q)$, whose respective limits are denoted by

$\lim_{q \to q_0}[f(q)] = w_0$ and $\lim_{q \to q_0}[F(q)] = W_0$. Hence, it may be proven that

(i) $\lim_{q \to q_0}[f(q) + F(q)] = w_0 + W_0$,

(ii) $\lim_{q \to q_0}[f(q)F(q)] = w_0 W_0$,

(iii) $\lim_{q \to q_0}[F(q)f(q)] = W_0 w_0$,

(iv) $\lim_{q \to q_0}[f(q)/F(q)] = w_0/W_0 \quad (W_0 \neq 0)$.

* * *

PROOF :-

Let

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

$$F(q) = U_1(x,y,\hat{x},\hat{y}) + i V_1(x,y,\hat{x},\hat{y}) + j U_2(x,y,\hat{x},\hat{y}) + k V_2(x,y,\hat{x},\hat{y}),$$

$$q_0 = x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0,$$

$$w_0 = u_{10} + i v_{10} + j u_{20} + k v_{20},$$

$$W_0 = U_{10} + i V_{10} + j U_{20} + k V_{20},$$

as previously indicated.

(i) Clearly, by virtue of Definition DII-1, we initially obtain

$$\begin{aligned} f(q) + F(q) = {} & u_1(x,y,\hat{x},\hat{y}) + U_1(x,y,\hat{x},\hat{y}) + i(v_1(x,y,\hat{x},\hat{y}) + V_1(x,y,\hat{x},\hat{y})) + \\ & j(u_2(x,y,\hat{x},\hat{y}) + U_2(x,y,\hat{x},\hat{y})) + k(v_2(x,y,\hat{x},\hat{y}) + V_2(x,y,\hat{x},\hat{y})) \end{aligned}$$

as well as

$$w_0 + W_0 = u_{10} + U_{10} + i(v_{10} + V_{10}) + j(u_{20} + U_{20}) + k(v_{20} + V_{20}).$$

Upon rewriting the algebraic sums, $f(q) + F(q)$ and $w_0 + W_0$, respectively as

$$f(q) + F(q) = A_1(x,y,\hat{x},\hat{y}) + i A_2(x,y,\hat{x},\hat{y}) + j A_3(x,y,\hat{x},\hat{y}) + k A_4(x,y,\hat{x},\hat{y}),$$

$$w_0 + W_0 = A_{10} + i A_{20} + j A_{30} + k A_{40},$$

we further deduce that, by equating the corresponding real and imaginary parts thereof,

$$A_1(x,y,\hat{x},\hat{y}) = u_1(x,y,\hat{x},\hat{y}) + U_1(x,y,\hat{x},\hat{y}),$$

$$A_2(x,y,\hat{x},\hat{y}) = v_1(x,y,\hat{x},\hat{y}) + V_1(x,y,\hat{x},\hat{y}),$$

$$A_3(x,y,\hat{x},\hat{y}) = u_2(x,y,\hat{x},\hat{y}) + U_2(x,y,\hat{x},\hat{y}),$$

$$A_4(x,y,\hat{x},\hat{y}) = v_2(x,y,\hat{x},\hat{y}) + V_2(x,y,\hat{x},\hat{y}),$$

$$A_{10} = u_{10} + U_{10},$$

$$A_{20} = v_{10} + V_{10},$$

$$A_{30} = u_{20} + U_{20},$$

$$A_{40} = v_{20} + V_{20}.$$

Now, in accordance with Theorem TIII-1, it follows that

$$\lim_{q \to q_0} [f(q) + F(q)] = A_{10} + iA_{20} + jA_{30} + kA_{40},$$

since the definitions and theorems pertaining to the limits of real variable functions (cf. Appendix A1) respectively yield –

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [A_1(x,y,\hat{x},\hat{y})] = \lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [u_1(x,y,\hat{x},\hat{y}) + U_1(x,y,\hat{x},\hat{y})]$$

$$= u_{10} + U_{10} = A_{10},$$

$$\lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[A_2(x,y,\hat{x},\hat{y})\right] = \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[v_1(x,y,\hat{x},\hat{y}) + V_1(x,y,\hat{x},\hat{y})\right]$$

$$= v_{10} + V_{10} = A_{20},$$

-67-

$$\lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[A_3(x,y,\hat{x},\hat{y})\right] = \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[u_2(x,y,\hat{x},\hat{y}) + U_2(x,y,\hat{x},\hat{y})\right]$$

$$= u_{20} + U_{20} = A_{30},$$

$$\lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[A_4(x,y,\hat{x},\hat{y})\right] = \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[v_2(x,y,\hat{x},\hat{y}) + V_2(x,y,\hat{x},\hat{y})\right]$$

$$= v_{20} + V_{20} = A_{40},$$

bearing in mind that, by virtue of the above mentioned theorems,

$$\lim_{q\to q_0}\left[f(q)\right] = u_{10} + iv_{10} + ju_{20} + kv_{20}$$

$$\Longrightarrow \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[u_1(x,y,\hat{x},\hat{y})\right] = u_{10}, \quad \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[v_1(x,y,\hat{x},\hat{y})\right] = v_{10},$$

$$\lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[u_2(x,y,\hat{x},\hat{y})\right] = u_{20}, \quad \lim_{\substack{(x,y,\hat{x},\hat{y})\to\\(x_0,y_0,\hat{x}_0,\hat{y}_0)}}\left[v_2(x,y,\hat{x},\hat{y})\right] = v_{20}$$

AND

$$\lim_{q \to q_0} [F(q)] = U_{10} + iV_{10} + jU_{20} + kV_{20}$$

$$\Longrightarrow \lim_{\substack{(x,y,\hat{x},\hat{y}) \to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} [U_1(x,y,\hat{x},\hat{y})] = U_{10}, \quad \lim_{\substack{(x,y,\hat{x},\hat{y}) \to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} [V_1(x,y,\hat{x},\hat{y})] = V_{10},$$

$$\lim_{\substack{(x,y,\hat{x},\hat{y}) \to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} [U_2(x,y,\hat{x},\hat{y})] = U_{20}, \quad \lim_{\substack{(x,y,\hat{x},\hat{y}) \to \\ (x_0,y_0,\hat{x}_0,\hat{y}_0)}} [V_2(x,y,\hat{x},\hat{y})] = V_{20}.$$

Finally, in view of the fact that we had previously defined

$$w_0 + W_0 = A_{10} + iA_{20} + jA_{30} + kA_{40},$$

we further obtain

$$\lim_{q \to q_0} [f(q) + F(q)] = A_{10} + iA_{20} + jA_{30} + kA_{40} = w_0 + W_0,$$

as required. <u>Q.E.D.</u>

(ii) By virtue of Definitions DII-2 and DIII-11, we initially obtain

$$f(q)F(q) = [u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y})] \times [U_1(x,y,\hat{x},\hat{y}) + iV_1(x,y,\hat{x},\hat{y}) + jU_2(x,y,\hat{x},\hat{y}) + kV_2(x,y,\hat{x},\hat{y})]$$

$$\begin{aligned} = &(u_1U_1 - v_1V_1 - u_2U_2 - v_2V_2)(x,y,\hat{x},\hat{y}) + \\ &i(u_1V_1 + U_1v_1 + u_2V_2 - U_2v_2)(x,y,\hat{x},\hat{y}) + \\ &j(u_1U_2 - v_1V_2 + u_2U_1 + v_2V_1)(x,y,\hat{x},\hat{y}) + \\ &k(u_1V_2 + U_2v_1 - u_2V_1 + U_1v_2)(x,y,\hat{x},\hat{y}), \end{aligned}$$

as well as

$$\begin{aligned} w_0 W_0 = {} & u_{10} U_{10} - v_{10} V_{10} - u_{20} U_{20} - v_{20} V_{20} + \\ & i(u_{10} V_{10} + U_{10} v_{10} + u_{20} V_{20} - U_{20} v_{20}) + \\ & j(u_{10} U_{20} - v_{10} V_{20} + u_{20} U_{10} + v_{20} V_{10}) + \\ & k(u_{10} V_{20} + U_{20} v_{10} - u_{20} V_{10} + U_{10} v_{20}) \quad . \end{aligned}$$

Upon rewriting the algebraic products, $f(q)F(q)$ and $w_0 W_0$, respectively as

$$f(q)F(q) = A_1(x,y,\hat{x},\hat{y}) + i A_2(x,y,\hat{x},\hat{y}) + j A_3(x,y,\hat{x},\hat{y}) + k A_4(x,y,\hat{x},\hat{y}) ,$$

$$w_0 W_0 = A_{10} + i A_{20} + j A_{30} + k A_{40} ,$$

we further deduce that, by equating the corresponding real and imaginary parts thereof,

$$A_1(x,y,\hat{x},\hat{y}) = (u_1 U_1 - v_1 V_1 - u_2 U_2 - v_2 V_2)(x,y,\hat{x},\hat{y}) ,$$

$$A_2(x,y,\hat{x},\hat{y}) = (u_1 V_1 + U_1 v_1 + u_2 V_2 - U_2 v_2)(x,y,\hat{x},\hat{y}) ,$$

$$A_3(x,y,\hat{x},\hat{y}) = (u_1 U_2 - v_1 V_2 + u_2 U_1 + v_2 V_1)(x,y,\hat{x},\hat{y}) ,$$

$$A_4(x,y,\hat{x},\hat{y}) = (u_1 V_2 + U_2 v_1 - u_2 V_1 + U_1 v_2)(x,y,\hat{x},\hat{y}) ,$$

$$A_{10} = u_{10} U_{10} - v_{10} V_{10} - u_{20} U_{20} - v_{20} V_{20} ,$$

$$A_{20} = u_{10} V_{10} + U_{10} v_{10} + u_{20} V_{20} - U_{20} v_{20} ,$$

****

$$A_{30} = u_{10}U_{20} - v_{10}V_{20} + u_{20}U_{10} + v_{20}V_{10},$$

$$A_{40} = u_{10}V_{20} + U_{20}v_{10} - u_{20}V_{10} + U_{10}v_{20}.$$

Now, in accordance with Theorem TIII-1, it follows that

$$\lim_{q \to q_0} [f(q)F(q)] = A_{10} + iA_{20} + jA_{30} + kA_{40},$$

since the definitions and theorems pertaining to the limits of real variable functions (cf. Appendix A1) respectively yield —

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [A_1(x,y,\hat{x},\hat{y})] = u_{10}U_{10} - v_{10}V_{10} - u_{20}U_{20} - v_{20}V_{20} = A_{10},$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [A_2(x,y,\hat{x},\hat{y})] = u_{10}V_{10} + U_{10}v_{10} + u_{20}V_{20} - U_{20}v_{20} = A_{20},$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [A_3(x,y,\hat{x},\hat{y})] = u_{10}U_{20} - v_{10}V_{20} + u_{20}U_{10} + v_{20}V_{10} = A_{30},$$

$$\lim_{(x,y,\hat{x},\hat{y}) \to (x_0,y_0,\hat{x}_0,\hat{y}_0)} [A_4(x,y,\hat{x},\hat{y})] = u_{10}V_{20} + U_{20}v_{10} - u_{20}V_{10} + U_{10}v_{20} = A_{40},$$

bearing in mind the conditions, previously specified in part (i) of this theorem,

****
****
****

for the existence of the limits,

****

$$\lim_{q \to q_0}[f(q)] = u_{10} + iv_{10} + ju_{20} + kv_{20}$$

AND

$$\lim_{q \to q_0}[F(q)] = U_{10} + iV_{10} + jU_{20} + kV_{20} .$$

Finally, in view of the fact that we had previously defined

$$w_0 W_0 = A_{10} + iA_{20} + jA_{30} + kA_{40} ,$$

we further obtain

$$\lim_{q \to q_0}[f(q)F(q)] = A_{10} + iA_{20} + jA_{30} + kA_{40} = w_0 W_0 \text{, as required. } \underline{\underline{Q.E.D.}}$$

(iii) The proof required for this part of the theorem is completely analogous with part (ii) thereof, insofar as the corresponding real and imaginary parts of the functions, $f(q)$ and $F(q)$, may be readily transposed with respect to the product functions, $f(q)F(q)$ and $F(q)f(q)$. <u>Q.E.D.</u>

(iv) By virtue of Theorem TII − 7, we initially ascertain that the quotient function,

$$f(q)/F(q) = \begin{cases} \overline{F(q)}f(q)/|F(q)|^2 \\ f(q)\overline{F(q)}/|F(q)|^2 \end{cases} ,$$

where

$$f(q)\overline{F(q)} \neq \overline{F(q)}f(q) \Rightarrow$$

****

$$f(q)\overline{F(q)}/|F(q)|^2 \neq \overline{F(q)}f(q)/|F(q)|^2, \ \forall F(q) \neq 0.$$

****

****

****

Furthermore, we deduce that

(a) $\lim_{q \to q_0} [f(q)] = w_0$

AND

(b) since

$$\overline{F(q)}/|F(q)|^2 = \frac{U_1(x,y,\hat{x},\hat{y}) - iV_1(x,y,\hat{x},\hat{y}) - jU_2(x,y,\hat{x},\hat{y}) - kV_2(x,y,\hat{x},\hat{y})}{(U_1^2+V_1^2+U_2^2+V_2^2)(x,y,\hat{x},\hat{y})},$$

then, in accordance with Theorem TIII-1 and the definitions and theorems pertaining to limits of real variable functions (cf. Appendix A1), it follows that

$$\begin{aligned} \lim_{q \to q_0} [\overline{F(q)}/|F(q)|^2] &= (U_{10}/(U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2)) - \\ &\quad i(V_{10}/(U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2)) - \\ &\quad j(U_{20}/(U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2)) - \\ &\quad k(V_{20}/(U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2)) \qquad + \\ &= (U_{10} - iV_{10} - jU_{20} - kV_{20})/(U_{10}^2+V_{10}^2+U_{20}^2+V_{20}^2) \\ &= \overline{W_0}/|W_0|^2. \end{aligned}$$

****

****

Finally, by virtue of the preceding part (ii) of this theorem, we likewise perceive that

$$\lim_{q\to q_0}[f(q)/F(q)] = \lim_{q\to q_0}\begin{cases}(\overline{F(q)}f(q))/|F(q)|^2\\ (f(q)\overline{F(q)})/|F(q)|^2\end{cases}$$

† A formal justification for this statement is provided in Appendix A2.

****

****

****

****

-72-

$$= \begin{cases}\lim_{q\to q_0}[(\overline{F(q)}f(q))/|F(q)|^2]\\ \lim_{q\to q_0}[(f(q)\overline{F(q)})/|F(q)|^2]\end{cases}$$

$$= \begin{cases}\lim_{q\to q_0}[(\overline{F(q)}/|F(q)|^2)f(q)]\\ \lim_{q\to q_0}[f(q)(\overline{F(q)}/|F(q)|^2)]\end{cases}$$

$$= \begin{cases}(\overline{W}_0 w_0)/|W_0|^2\\ (w_0\overline{W}_0)/|W_0|^2\end{cases} = \begin{cases}(\overline{W}_0/|W_0|^2)w_0\\ w_0(\overline{W}_0/|W_0|^2)\end{cases} \dagger$$

$= w_0/W_0$, as required. <u>Q.E.D.</u>

****

****

Invariably, our analytical procedure for evaluating the limits of quaternion hypercomplex functions is greatly enhanced by the introduction of Theorem TIII-2, since this particular theorem fulfills exactly the same requirements as does its more familiar counterpart with respect to the <u>sums, products and quotients</u> of functions thus encountered in complex variable analysis.

We therefore conclude our formal discussion of this topic by enunciating and hence proving our last theorem therein which deals explicitly with the limits of the <u>moduli</u> of quaternion hypercomplex functions :-

<u>Theorem TIII - 3</u>

Let there exist a quaternion hypercomplex function, $f(q)$. Hence, the corresponding limit of this function,

$$\lim_{q \to q_0} [f(q)] = w_0 ,$$

† Here, we note that $1/|W_0|^2 \in \mathbb{R}$ and hence readily commutes with any quaternion product.

****

further implies the existence of the limit,

$$\lim_{q \to q_0} [|f(q)|] = |w_0| .$$

* * *

<u>PROOF</u>:-

****

****

From Definition DIII-12, we ascertain that the limit,

$$\lim_{q \to q_0}[f(q)] = w_0,$$

exists if and only if there exist real numbers, $\delta, \epsilon > 0$, such that

$$|f(q) - w_0| < \epsilon, \text{ whenever } 0 < |q - q_0| < \delta.$$

However, in accordance with part (ii) of Theorem TII-13, we also deduce that

$$||f(q)| - |w_0|| \leqslant |f(q) - w_0|$$

and hence we obtain

$$||f(q)| - |w_0|| < \epsilon, \text{ whenever } 0 < |q - q_0| < \delta,$$

which are precisely the conditions deemed necessary for the existence of the limit,

$$\lim_{q \to q_0}[|f(q)|] = |w_0|.$$ <u>Q.E.D.</u>

Needless to say, the reader will realise that the above stated theorem is simply a logical extension of those theorems, which pertain to the limits of the <u>moduli</u> of both real and complex variable functions, and hence no further explanation as to its relevance in the study of quaternion hypercomplex

****

****

functions should be required.

4: Conditions for the Continuity of Quaternion Hypercomplex Functions; the Continuity of the Composites of such Functions.

The behavioural properties, previously outlined in Part 3 of this section, further induce us to consider the notion of continuity as applied to quaternion hypercomplex functions. Since this notion has already been defined and elucidated with respect to both real and complex variable functions, it should therefore be extended to include quaternion hypercomplex functions as well. Now, in order to fulfill this expectation, we henceforth introduce our next definition and theorem to that effect as follows :-

Definition DIII - 13

A quaternion hypercomplex function 'f' is continuous at a point, q_0, if and only if the following conditions are satisfied :-

(i) $\lim_{q \to q_0} [f(q)]$ exists,

(ii) $f(q_0)$ is defined,

(iii) $\lim_{q \to q_0} [f(q)] = f(q_0)$.

****

We further note that statement (iii) automatically implies statements (i) and (ii), as well as the existence of real numbers, $\delta, \epsilon > 0$, such that we obtain

$$|f(q) - f(q_0)| < \epsilon, \text{ whenever } |q - q_0| < \delta.$$

To be continued via the author's next submission, namely -

"An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 4/6."

* * * * * * * *

* * * * * * * *

[THIS PAGE HAS BEEN INTENTIONALLY LEFT BLANK.]