“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 4/6.***”

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## I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“***An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.***”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

## II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 4/6.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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Theorem TIII - 4

Let '$f \circ g$' be any composite quaternion hypercomplex function. Now, if '$f$' is continuous at $g(q_0)$ and '$g$' is continuous at $q_0$, then '$f \circ g$' is likewise continuous at $q_0$.

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PROOF:-

We firstly note that Definition DIII - 11 specifies, inter alia, that the composite function,

$$(f \circ g)(q) = f(g(q)).$$

Furthermore, if '$f$' is continuous at $g(q_0)$, then, by virtue of Definition DIII - 13, we have

$$\lim_{g(q) \to g(q_0)} [f(g(q))] = f(g(q_0)),$$

thereby implying the existence of real numbers, $\gamma, \epsilon > 0$, such that

$$|f(g(q)) - f(g(q_0))| < \epsilon, \text{ whenever } |g(q) - g(q_0)| < \gamma.$$

Similarly, if '$g$' is continuous at $q_0$, then, by virtue of the same definition, we have

$$\lim_{q \to q_0} [g(q)] = g(q_0),$$

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thereby implying the existence of real numbers, $\gamma, \delta > 0$, such that

$|g(q) - g(q_0)| < \gamma$, whenever $|q - q_0| < \delta$.

Clearly, the simultaneity of the above stated inequalities further yields

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$|f(g(q)) - f(g(q_0))| < \epsilon$, whenever $|q - q_0| < \delta$,

which are precisely the conditions deemed necessary for the existence of the limit,

$\lim_{q \to q_0} [f(g(q))] = f(g(q_0))$,

and hence it automatically follows that the composite function, "$f \circ g$," is also continuous at $q_0$. Q.E.D.

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From the context of Definition DIII-13 and Theorem TIII-4, it is evident that any former restrictions on the existence of $f(q_0)$ as the limit of $f(q)$ at the point, $q = q_0$, thus characterized by the inequality,

$0 < |q - q_0| < \delta$ (3-8),

are effectively removed by instead admitting the inequality,

$|q - q_0| < \delta$ (3-9),

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which now guarantees the existence of every functional value, $f(q_0)$, as the limit at that particular point. As a matter of fact, this important, albeit subtle, distinction is crucial to the continuity of all such quaternion hypercomplex functions and, moreover, it provides a theoretical basis for the material covered in both the final part of Section III and also all of Section IV following immediately thereafter.

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## 5. Definition for an Arc embedded in q-Space and the Continuity of Quaternion Hypercomplex Functions restricted to such Arcs.

The concept of an arc is well understood with regard to the analysis of real and complex variable functions. Refering in particular to the latter named branch of mathematics, it is instructive to note that Churchill et al. (cf. Reference B2, Section VII) have formally introduced this notion as a prerequisite to the development of both the definite and indefinite integral for a complex variable function and, in so doing, have subsequently found it necessary to invoke the concomitant properties of continuity as a means of elucidating this very idea.

Allied to the foregoing, our extended notion of an arc, $C$, defined on the domain of definition of any given quaternion hypercomplex function, '$f$', is likewise presented within the context of two separate definitions, wherein the hitherto mentioned domain shall, for the purposes of these definitions, be denoted more simply as a q-space.

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Definition DIII − 14

An arc, C, is any set of points denoted by

$q = (x, y, \hat{x}, \hat{y})$

and thus embedded in quaternion hypercomplex q – space such that

$x = x(t),\ y = y(t),\ \hat{x} = \hat{x}(t),\ \hat{y} = \hat{y}(t),\quad \forall t \in [a, b],$

wherever $x(t), y(t), \hat{x}(t)$ and $\hat{y}(t)$ are continuous functions of the real parameter 't'.

Here, it is convenient to algebraically describe the individual points of C, with respect to the closed interval,

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$[a, b] = \{t : a \leq t \leq b\},$

by the equation –

$q = q(t) = x(t) + i\,y(t) + j\,\hat{x}(t) + k\,\hat{y}(t),\quad \forall t \in [a, b].$

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Definition DIII − 15

A simple, or Jordan, arc, C, is one which does not intersect itself, that is to say,

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$q(t_1) \neq q(t_2)$, whenever $t_1 \neq t_2$, $\forall t_1, t_2 \in [a, b]$.

Conversely, an arc, C, becomes a simple, or Jordan, closed curve, if and only if

$q(a) = q(b)$.

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The above stated definitions further induce us to derive two more theorems which respectively establish

(a) the conditions for the continuity of a quaternion hypercomplex function, restricted to such arcs, in terms of the continuity of its constituent real and imaginary parts,

(b) the continuity of the sums, products and quotients of quaternion hypercomplex functions, similarly restricted,

all of which shall henceforth be elucidated as follows :-

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Theorem TIII - 5

Let there exist a quaternion hypercomplex function,

$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})$.

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Consequently, it may be shown that the limit of such a function, thus defined on an arc, $C$, embedded in $q$-space,

$$\lim_{t\to t_0}[f(q(t))] = u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + i\,v_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + j\,u_2(x(t_0), y(t_0), \hat{x}(t_0), y(t_0)) + k\,v_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0))$$

also exists and hence may be rewritten as

$$\lim_{t\to t_0}[f^*(t)] = u_1^*(t_0) + i\,v_1^*(t_0) + j\,u_2^*(t_0) + k\,v_2^*(t_0)\ ,$$

provided that

(i) the function, $f(q(t))$, is continuous at $q(t_0)$, thus implying that the functions, $u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)), \ldots\ldots, v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$, are also continuous at $t_0$

AND

(ii) the functions, $x(t)$, $y(t)$, $\hat{x}(t)$ and $\hat{y}(t)$, are continuous at $t_0$, thus implying that the function, $q(t)$, is also continuous at $t_0$.

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PROOF :-

In accordance with Definition DIII-14, we firstly note that

$$x = x(t),\ y = y(t),\ \hat{x} = \hat{x}(t),\ \hat{y} = \hat{y}(t)\ ,\ \forall t \in [a,b]\ ,$$

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$$\Longrightarrow \begin{cases} q = q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \\ q(t_0) = x(t_0) + iy(t_0) + j\hat{x}(t_0) + k\hat{y}(t_0) \end{cases} .$$

Now, by substituting the above stated equations into the defining equation for $f(q)$, we further obtain

$$f(q) = f(q(t)) = u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + iv_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + \\ ju_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) + kv_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$$

and hence

$$f(q(t_0)) = u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + iv_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + \\ ju_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + kv_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) .$$

Since we have already stipulated that

(i) the function, $f(q(t))$, is continuous at $q(t_0)$

AND

(ii) the functions, $x(t), y(t), \hat{x}(t)$ and $\hat{y}(t)$, are continuous at $t_0$,

then, by virtue of Definition DIII-13, it follows that

(i) $\lim_{q(t) \to q(t_0)} \left[ f(q(t)) \right] = f(q(t_0))$,

whereupon there exist real numbers, $\gamma, \epsilon > 0$, such that

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$$|f(q(t)) - f(q(t_0))| < \epsilon, \text{ whenever } |q(t) - q(t_0)| < \gamma$$

$$\therefore \left.\begin{array}{l} |u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) - u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0))| \\ \vdots \\ |v_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) - v_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0))| \end{array}\right\} < \epsilon,$$

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$$\text{whenever } \left.\begin{array}{l} |x(t) - x(t_0)| \\ |y(t) - y(t_0)| \\ |\hat{x}(t) - \hat{x}(t_0)| \\ |\hat{y}(t) - \hat{y}(t_0)| \end{array}\right\} < \gamma$$

and similarly, from real variable analysis,

(ii) $\lim_{t \to t_0} [x(t)] = x(t_0)$, $\lim_{t \to t_0} [y(t)] = y(t_0)$,

$\lim_{t \to t_0} [\hat{x}(t)] = \hat{x}(t_0)$, $\lim_{t \to t_0} [\hat{y}(t)] = \hat{y}(t_0)$,

whereupon there exist real numbers, $\gamma, \delta > 0$, such that

$$\left.\begin{array}{l} |x(t) - x(t_0)| \\ |y(t) - y(t_0)| \\ |\hat{x}(t) - \hat{x}(t_0)| \\ |\hat{y}(t) - \hat{y}(t_0)| \end{array}\right\} < \gamma, \text{ whenever } |t - t_0| < \delta.$$

Clearly, the simultaneity of the hitherto mentioned inequalities furthermore provides the very conditions deemed necessary for the existence of the limits,

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$$\lim_{t \to t_0} [f(q(t))] = f(q(t_0))$$

$$= u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + i\, v_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + j\, u_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) + k\, v_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) \,,$$

$$\lim_{t \to t_0} [u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] = u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) \,,$$

$$\vdots$$

$$\lim_{t \to t_0} [v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] = v_2(x(t_0), y(t_0), \hat{x}(t_0), y(t_0)) \,,$$

$$\lim_{t \to t_0} [q(t)] = q(t_0) \,.$$

Finally, by writing

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$$f(q(t)) = f^*(t) \,,$$

$$u_1(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) = u_1^*(t_0) \,,$$

$$\vdots$$

$$v_2(x(t_0), y(t_0), \hat{x}(t_0), \hat{y}(t_0)) = v_2^*(t_0) \,,$$

we likewise perceive that

$$\lim_{t \to t_0} [f^*(t)] = u_1^*(t_0) + i\, v_1^*(t_0) + j\, u_2^*(t_0) + k\, v_2^*(t_0) \,,$$

as required. Q.E.D.

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*Theorem TIII – 6*

Let there exist two continuous quaternion hypercomplex functions, thus defined on '$t$' and hence denoted respectively as $f^*(t)$ and $F^*(t)$, such that

$$\lim_{t \to t_0}[f^*(t)] = f^*(t_0) \quad \text{and} \quad \lim_{t \to t_0}[F^*(t)] = F^*(t_0) \ .$$

Henceforth, the following algebraic properties with regard to the continuity of these functions may be established :–

(i) $\lim_{t \to t_0}[f^*(t) + F^*(t)] = f^*(t_0) + F^*(t_0)$ ,

(ii) $\lim_{t \to t_0}[f^*(t)F^*(t)] = f^*(t_0)F^*(t_0)$ ,

(iii) $\lim_{t \to t_0}[F^*(t)f^*(t)] = F^*(t_0)f^*(t_0)$ ,

(iv) $\lim_{t \to t_0}[f^*(t)/F^*(t)] = f^*(t_0)/F^*(t_0)$ , $(F^*(t_0) \neq 0)$ .

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PROOF :–

We firstly postulate the existence of two quaternion hypercomplex functions, $f(q(t))$ and $F(q(t))$, defined with respect to an arc, $C$, in $q$–space and which are both continuous at $q(t_0)$, such that

$$\lim_{q(t) \to q(t_0)}[f(q(t))] = f(q(t_0)) \quad \text{and} \quad \lim_{q(t) \to q(t_0)}[F(q(t))] = F(q(t_0)) \ .$$

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Now, in an analogous manner to Theorem TIII−2, it likewise follows that

(i) $\lim_{q(t)\to q(t_0)} [f(q(t)) + F(q(t))] = f(q(t_0)) + F(q(t_0))$ ,

(ii) $\lim_{q(t)\to q(t_0)} [f(q(t))F(q(t))] = f(q(t_0))F(q(t_0))$ ,

(iii) $\lim_{q(t)\to q(t_0)} [F(q(t))f(q(t))] = F(q(t_0))f(q(t_0))$ ,

(iv) $\lim_{q(t)\to q(t_0)} [f(q(t))/F(q(t))] = f(q(t_0))/F(q(t_0))$ , $(F(q(t_0)) \neq 0)$ ,

bearing in mind the conditions for continuity previously specified in Theorem TIII−5 as well as in theorems pertaining to the continuity of real variable functions (cf. Appendix A1), whereupon it is evident that the above stated equations may also be rewritten respectively as −

(i) $\lim_{t\to t_0} [f(q(t)) + F(q(t))] = f(q(t_0)) + F(q(t_0))$ ,

(ii) $\lim_{t\to t_0} [f(q(t))F(q(t))] = f(q(t_0))F(q(t_0))$ ,

(iii) $\lim_{t\to t_0} [F(q(t))f(q(t))] = F(q(t_0))f(q(t_0))$ ,

(iv) $\lim_{t\to t_0} [f(q(t))/F(q(t))] = f(q(t_0))/F(q(t_0))$ , $(F(q(t_0)) \neq 0)$ .

Finally, by writing

$$f(q(t)) = f^*(t) \;\&\; F(q(t)) = F^*(t) \Longrightarrow f(q(t_0)) = f^*(t_0) \;\&\; F(q(t_0)) = F^*(t_0) ,$$

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we henceforth obtain —

(i) $\lim_{t \to t_0} [f^*(t) + F^*(t)] = f^*(t_0) + F^*(t_0)$ ,

(ii) $\lim_{t \to t_0} [f^*(t) F^*(t)] = f^*(t_0) F^*(t_0)$ ,

(iii) $\lim_{t \to t_0} [F^*(t) f^*(t)] = F^*(t_0) f^*(t_0)$ ,

(iv) $\lim_{t \to t_0} [f^*(t)/F^*(t)] = f^*(t_0)/F^*(t_0)$ , $(F^*(t_0) \neq 0)$,

as required. Q.E.D.

Once again, we remark that the material covered in this final part of Section III will attain a much greater significance from the reader's viewpoint, as we proceed to analyse the fundamental principles underlying the differentiation and integration of quaternion hypercomplex functions — a topic to which we subsequently address ourselves in Section IV of this dissertation.

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## IV. Elementary Principles of Differentiation and Integration applied to Quaternion Hypercomplex Functions

Before venturing to investigate the fundamental principles underlying the differential and integral calculus of quaternion hypercomplex functions, we will assume that these same principles as applied to the calculus of real and complex variable functions are already well understood. Indeed, by way of formal introduction, the reader will instantly recall from real variable analysis that the first derivative of a function, $f(x)$, is given by

$$\frac{d}{dx}[f(x)] = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = f'(x) \qquad (4-1),$$

provided that this limit (ie. derivative) actually exists. Similarly, in relation to complex variable analysis, we likewise define the first derivative of a function, $f(z)$, as

$$\frac{d}{dz}[f(z)] = \lim_{\delta z \to 0}\left[\frac{f(z+\delta z)-f(z)}{\delta z}\right] = f'(z) \qquad (4-2),$$

provided that such a limit (ie. derivative) also exists.

Furthermore, we must specify the conditions which are both necessary and sufficient in order to guarantee the existence of the above stated derivatives. Broadly speaking, the differentiability of any real function, thus typified by Eq. (4-1), is dependant upon two factors, namely –

(a) the formation of a suitable difference quotient involving such functions

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AND

(b) the existence of a limiting value for this difference quotient as $h \to 0$,

whereas, in the case of Eq. (4-2), we surmise that the differentiability of any complex function is wholly dependant upon its analyticity over a given region (domain) of the complex $z$-plane, for which Churchill et al.

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(cf. Reference B2, Section VII) have subsequently provided a very detailed account.

However, with regard to the differentiability of quaternion hypercomplex functions, we will discover in ~~due course~~ that this particular class of functions is not generally analytic in its behaviour. As suggested previously, the ultimate reason for this lies in the fact that the difference quotients of any two quaternion hypercomplex functions are not uniquely defined but instead take on two distinct values. Henceforth, we are compelled to substantially modify our existing notions on the calculus of functions in order to meet the specific requirements of our ensuing analysis and, needless to say, the remainder of this section is devoted to just such an endeavour.

1. The Difference Quotient and the First Derivative of a Quaternion Hypercomplex Function restricted to an Arc embedded in $q$-Space.

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From Eqs. (4–1) and (4–2), it is evident that there exists 'a priori' difference quotients,

$$\frac{f(x+h)-f(x)}{h} \in \mathbb{R}\,, \quad \frac{f(z+\delta z)-f(z)}{\delta z} \in \mathbb{C}\,,$$

which respectively pre-suppose the existence of <u>first derivatives</u> for a large number of real and complex variable functions.

This being the case, we are naturally encouraged to introduce the concept of a <u>difference quotient</u> into our study of the properties of quaternion hyper-complex functions, whereupon we accordingly state the following definition:–

<u>Definition DIV–1</u>

The <u>difference quotient</u> corresponding to any quaternion hypercomplex function, $f(q)$, is defined as –



$$\frac{\delta f(q)}{\delta q} = \frac{f(q+\delta q)-f(q)}{\delta q} = \begin{cases} [\,f(q+\delta q)-f(q)\,]\,\overline{\delta q}\,/\,|\delta q|^2\,, \\ \overline{\delta q}\,[\,f(q+\delta q)-f(q)\,]\,/\,|\delta q|^2 \end{cases}$$

where $\delta(f(q))$ and $\delta q$ are designated to be the constituent <u>increments</u> of the entities, $f(q)$ and $q$, defined respectively therein.

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The above definition, in conjunction with Eqs. (4–1) and (4–2), further tends to suggest that one may presumably express the <u>first derivative of any quaternion hypercomplex function, $f(q)$,</u> as

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$$\frac{d}{dq}[f(q)] = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q)-f(q)}{\delta q}\right] \qquad (4-3),$$

provided that this limit (ie. derivative) exists. To test the validity of our hypothesis, we now consider the following examples of functions to which Eq. (4-3) can be directly applied :-

(i) The constant function, $f(q) = k$, where the constant, $k \in \mathbb{H}$.

Given that $f(q) = k$ is a constant in $\mathbb{H}$, the set of all quaternions, we likewise perceive that

$$f(q+\delta q) = k$$

and hence the corresponding difference quotient,

$$\frac{\delta(f(q))}{\delta q} = \frac{f(q+\delta q)-f(q)}{\delta q} = \begin{cases} [f(q+\delta q) - f(q)]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\,[f(q+\delta q) - f(q)]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} 0\cdot\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\cdot 0/|\delta q|^2 \end{cases}$$

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$$= \begin{cases} 0 \\ 0 \end{cases}$$

$$= 0 .$$

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Finally, the required first derivative,

$$\frac{d}{dq}[f(q)] = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right]$$

$$= \lim_{\delta q \to 0}[0]$$

$$= 0, \text{ as anticipated.}$$

(ii) <u>The linear function, $f(q) = k_1 q + k_2$, where the constants, $k_1 k_2 \in \mathbb{H}$.</u>

Given that $f(q) = k_1 q + k_2$ is a <u>linear</u> function with $k_1, k_2 \in \mathbb{H}$, the set of all quaternions, we likewise perceive that

$$f(q+\delta q) = k_1(q+\delta q) + k_2$$
$$= k_1 q + k_1 \delta q + k_2$$

and hence the corresponding difference quotient,

$$\frac{\delta f(q)}{\delta q} = \frac{f(q+\delta q) - f(q)}{\delta q} = \begin{cases} [f(q+\delta q) - f(q)]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}[f(q+\delta q) - f(q)]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} [k_1 q + k_1 \delta q + k_2 - (k_1 q + k_2)]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}[k_1 q + k_1 \delta q + k_2 - (k_1 q + k_2)]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} k_1 \delta q\, \overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\, k_1 \delta q/|\delta q|^2 \end{cases}$$

$$= \begin{cases} k_1 \\ \overline{\delta q}\, k_1\, \delta q / |\delta q|^2 , \text{ since } \delta q\, \overline{\delta q} = |\delta q|^2 . \end{cases}$$

Accordingly, we wish to evaluate the derivative, $\frac{d}{dq}(k_1 q + k_2)$, in terms of Eq. (4–3). However, we immediately recognise that there arises a problem in this regard since the entity,

$$\lim_{\delta q \to 0} [\overline{\delta q}\, k_1\, \delta q / |\delta q|^2] = \lim_{\delta q \to 0} [(\overline{\delta q}/|\delta q|)\, k_1\, (\delta q/|\delta q|)] \qquad (4\text{–}4),$$

cannot be explicitly defined whilst the limits, $\lim_{\delta q \to 0}[\overline{\delta q}/|\delta q|]$ and $\lim_{\delta q \to 0}[\delta q/|\delta q|]$, remained undefined, whereas the constituent limits,

$$\lim_{\delta q \to 0}[\delta q] = \lim_{\delta q \to 0}[\overline{\delta q}] = \lim_{\delta q \to 0}[|\delta q|] = 0 \qquad (4\text{–}5).$$

Here it should be noted that we can easily verify the existence of the limit,

$$\lim_{\delta q \to 0}[\overline{\delta q}] = 0,$$

by virtue of the simultaneous occurrence of the inequalities,

$$|\overline{\delta q}| < \epsilon \text{ and } 0 < |\delta q| < \delta,$$

which thus conform to the precepts of Definition DIII – 12. From Definition DII – 7, we also perceive that, for any quaternion, $q$, the corresponding modulus,

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$|q| = |\bar{q}|$,

whence it naturally follows that

$|\delta q| = |\overline{\delta q}|$.

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Finally, we obtain

$|\delta q| < \epsilon$, whenever $0 < |\delta q| < \delta$,

which are precisely the conditions deemed necessary for the existence of the limit,

$\lim_{\delta q \to 0} [\delta q] = 0$, as originally stated.

On the other hand, if we were to specify that the constant, $k_1 \in \mathbb{R}$, the set of all real numbers, then we may write

$$\overline{\delta q}\, k_1\, \delta q / |\delta q|^2 = k_1\, \overline{\delta q}\, \delta q / |\delta q|^2 = k_1 |\delta q|^2 / |\delta q|^2 = k_1 \qquad (4\text{-}6),$$

bearing in mind the commutativity of any real number with all such quaternion products. In the circumstances, the required first derivative of the linear function, $f(q) = k_1 q + k_2$, $\forall k_1 \in \mathbb{R}$ & $\forall k_2 \in \mathbb{H}$, is given by

$$\frac{d}{dq}[f(q)] = \lim_{\delta q \to 0} \left[ \frac{f(q+\delta q) - f(q)}{\delta q} \right]$$

$$= \lim_{\delta q \to 0} \left[ \begin{cases} k_1 \\ k_1 \end{cases} \right]$$

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$$= \lim_{\delta q \to 0} [k_1]$$

$$= k_1 \text{, as anticipated.}$$

(iii) <u>The polynomial function, $f(q) = \sum_{l=0}^{n} k_l q^l$, where the constants, $k_l \in \mathbb{H}$ and $n \in \mathbb{N}$, the set of natural numbers.</u>

Given that $f(q) = \sum_{l=0}^{n} k_l q^l$ is a <u>polynomial</u> function with $k_l \in \mathbb{H}$ and $n \in \mathbb{N}$, the set of natural numbers, we likewise perceive that

$$f(q+\delta q) = \sum_{l=0}^{n} k_l (q+\delta q)^l$$



and hence the corresponding difference quotient,

$$\frac{\delta f(q)}{\delta q} = \frac{f(q+\delta q) - f(q)}{\delta q} = \begin{cases} [f(q+\delta q) - f(q)]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\,[f(q+\delta q) - f(q)]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} [\sum_{l=0}^{n} k_l (q+\delta q)^l - \sum_{l=0}^{n} k_l q^l]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\,[\sum_{l=0}^{n} k_l (q+\delta q)^l - \sum_{l=0}^{n} k_l q^l]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} [\sum_{l=0}^{n} k_l ((q+\delta q)^l - q^l)]\,\overline{\delta q}/|\delta q|^2 \\ \overline{\delta q}\,[\sum_{l=0}^{n} k_l ((q+\delta q)^l - q^l)]/|\delta q|^2 \end{cases}$$

$$= \begin{cases} \sum_{l=0}^{n} [k_l((q+\delta q)^l - q^l)\overline{\delta q}/|\delta q|^2] \\ \sum_{l=0}^{n} [\overline{\delta q} k_l((q+\delta q)^l - q^l)/|\delta q|^2] \end{cases} .$$

Subsequently, we observe that the above stated difference quotient is really a sum of n-monomial terms, designated respectively as

$$\frac{\delta(k_l q^l)}{\delta q} = \begin{cases} k_l((q+\delta q)^l - q^l)\overline{\delta q}/|\delta q|^2 \ , \forall l \in \{0,1,\ldots\ldots\ldots,n\} \\ \overline{\delta q} k_l((q+\delta q)^l - q^l)/|\delta q|^2 \qquad (4-7) , \end{cases}$$

which therefore facilitates our study of the concomitant difference properties of any given polynomial quaternion hypercomplex function.

Furthermore, by expanding the increment,

$$(q+\delta q)^l - q^l = (q+\delta q)^{l-1}(q+\delta q) - q^{l-1} q$$

$$= (q+\delta q)^{l-1}(q+\delta q) - (q+\delta q)^{l-1} q + (q+\delta q)^{l-1} q + q^{l-1} q$$

$$= (q+\delta q)^{l-1}(q+\delta q - q) + [(q+\delta q)^{l-1} - q^{l-1}] q$$

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$$= (q+\delta q)^{l-1}\delta q + [(q+\delta q)^{l-1} - q^{l-1}] q ,$$

and hence substituting this expansion into Eq. (4-7), we thus obtain

$$\frac{\delta(k_\ell q^\ell)}{\delta_q} = \begin{cases} \dfrac{k_\ell\left((q+\delta_q)^{\ell-1}\delta_q + [(q+\delta_q)^{\ell-1} - q^{\ell-1}]q\right)\bar{\delta_q}}{|\delta_q|^2} \\[2ex] \dfrac{\bar{\delta_q}\, k_\ell\left((q+\delta_q)^{\ell-1}\delta_q + [(q+\delta_q)^{\ell-1} - q^{\ell-1}]q\right)}{|\delta_q|^2} \end{cases}$$

$$= \begin{cases} \dfrac{k_\ell(q+\delta_q)^{\ell-1}\delta_q\bar{\delta_q} + k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q\bar{\delta_q}}{|\delta_q|^2} \\[2ex] \dfrac{\bar{\delta_q}k_\ell(q+\delta_q)^{\ell-1}\delta_q + \bar{\delta_q}k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q}{|\delta_q|^2} \end{cases}$$

$$= \begin{cases} \dfrac{k_\ell(q+\delta_q)^{\ell-1}|\delta_q|^2}{|\delta_q|^2} + \dfrac{k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q\bar{\delta_q}}{|\delta_q|^2} \\[2ex] \dfrac{\bar{\delta_q}k_\ell(q+\delta_q)^{\ell-1}\delta_q}{|\delta_q|^2} + \dfrac{\bar{\delta_q}k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q}{|\delta_q|^2} \end{cases}$$

$$= \begin{cases} k_\ell(q+\delta_q)^{\ell-1} + \dfrac{k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q\bar{\delta_q}}{|\delta_q|^2} \\[2ex] \dfrac{\bar{\delta_q}k_\ell(q+\delta_q)^{\ell-1}\delta_q}{|\delta_q|^2} + \dfrac{\bar{\delta_q}k_\ell[(q+\delta_q)^{\ell-1} - q^{\ell-1}]q}{|\delta_q|^2} \end{cases} \qquad (4\text{–}8).$$

We now wish to evaluate the derivative, $\frac{d}{dq}(k_\ell q^\ell)$, in terms of Eq. (4–3). However, since the quaternion products,

$q\bar{\delta_q} \neq \bar{\delta_q}q$, $\bar{\delta_q}k_\ell \neq k_\ell\bar{\delta_q}$, $\bar{\delta_q}k_\ell(q+\delta_q)^{\ell-1} \neq k_\ell(q+\delta_q)^{\ell-1}\bar{\delta_q}$, etc.,

$\forall q, \bar{\delta_q}, k_\ell \in \mathbb{H}$, it likewise follows that the real term denominator, $|\delta q|^2$, does not cancel with respect to each of the numerators contained in the quaternion quotients,

$$\frac{k_\ell[(q+\delta q)^{\ell-1}-q^{\ell-1}]q\bar{\delta q}}{|\delta q|^2}\ ,\ \frac{\bar{\delta q}k_\ell(q+\delta q)^{\ell-1}\delta q}{|\delta q|^2}\quad \text{and}$$

$$\frac{\bar{\delta q}k_\ell[(q+\delta q)^{\ell-1}-q^{\ell-1}]q}{|\delta q|^2}\ ,\ \forall q, \bar{\delta q}, k_\ell \in \mathbb{H},$$

whereupon we invariably deduce that the corresponding limits of these quotients, as $\delta q \to 0$, are not explicitly defined for the same reasons outlined in the preceding example (ii) and hence our hypothetical derivative,

$$\frac{d}{dq}(k_\ell q^\ell) = \lim_{\delta q \to 0}\left[\frac{\delta(k_\ell q^\ell)}{\delta q}\right] \qquad (4-9),$$

cannot be similarly evaluated as a function of '$q$'.

As a direct consequence of the above mentioned factors, we finally conclude that the polynomial quaternion hypercomplex function,

$$f(q) = \sum_{k=0}^{n} k_\ell q^\ell\ ,\ \forall k_\ell \in \mathbb{H} \text{ and } \forall n \in \mathbb{N},$$

is also not generally differentiable with respect to every '$q$' in $\mathbb{H}$.

* * *

In retrospect, we surmise that, apart from the special cases of

(a) the constant function, $f(q) = k \in \mathbb{H}$,
(b) the linear function, $f(q) = k_1 q + k_2$, $\forall k_1 \in \mathbb{R}$ and $\forall k_2 \in \mathbb{H}$,

there now exists a substantial body of quaternion hypercomplex functions which are not differentiable in '$q$' with respect to their maximum domain of definition (i.e. $q$-space). Admittedly, the resultant inadequacies of both Definition D IV-1 and Eq. (4-3) might very well cause one to consider any further attempts at differentiating quaternion hypercomplex functions along the lines suggested thus far as being a completely futile exercise. However,



such a judgement in the author's opinion is rather short sighted, since it will be shown that we can still apply the concept of a first derivative to many such functions, provided that we restrict our arguments to a given arc, $C$, thus embedded in $q$-space, as opposed to utilising any combination of continuous regions located therein.

The reader will no doubt recall that Definition D III-14 accordingly provides us with a suitable explanation for the above stated notion of an arc, $C$, and this in turn will play a crucial role in the formulation of the first derivative of a quaternion hypercomplex function thus restricted to such arcs. Subsequently, we require that the existence of this particular derivative shall be wholly dependant upon a set of three new provisos, namely —

(i) the existence of a difference quotient of a quaternion hypercomplex function, $f(q)$, hitherto restricted to an arc, $C$, embedded in $q$-space, which we denote as

$$\left[\frac{\delta f(q)}{\delta q}\right]_C ,$$

(ii) the existence of a parametric first derivative, with respect to '$t$', of a quaternion hypercomplex function, $\hat{h}(q)$, hitherto restricted to an arc, $C$, embedded in $q$-space, which we denote as

$$\frac{d}{dt}\left[\hat{h}(q(t))\right],$$

(iii) the existence of a differential operator, $\left[\frac{d}{dq}\right]_C$,

for which the following definitions should suffice in relation thereto—

Definition DIV-2

The difference quotient of a quaternion hypercomplex function, $f(q)$, thus restricted to an arc, $C$, embedded in $q$-space, is defined as



$$\left[\frac{\delta f(q)}{\delta q}\right]_C = \left(\frac{f(q+\delta q)-f(q)}{h}\right) \Big/ \frac{\delta q}{h}$$

$$= \begin{cases} \dfrac{\left[\dfrac{f(q+\delta q)-f(q)}{h}\right]\overline{\dfrac{\delta q}{h}}}{\left|\dfrac{\delta q}{h}\right|^2}, \\ \dfrac{\overline{\dfrac{\delta q}{h}}\left[\dfrac{f(q+\delta q)-f(q)}{h}\right]}{\left|\dfrac{\delta q}{h}\right|^2} \end{cases}$$

where the entities,

$q = q(t)$,

$$\left.\begin{array}{l} f(q) = f(q(t)), \\ \delta q = q(t+h) - q(t), \\ f(q+\delta q) = f(q(t+h)), \end{array}\right\} (h \in \mathbb{R}),$$

with respect to both sides of this equation.

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Definition DIV − 3

Let there exist a quaternion hypercomplex function, $\hat{h}(q)$, which is henceforth restricted to an arc, $C$, embedded in $q$ − space.

Consequently, the parametric first derivative, with respect to '$t$', of such a function is defined as

$$\frac{d}{dt}[\hat{h}(q(t))] = \lim_{h \to 0}\left[\frac{\hat{h}(q(t+h)) - \hat{h}(q(t))}{h}\right],$$

provided that this derivative exists.

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Definition DIV − 4

Let there exist a differential operator, $\left[\frac{d}{dq}\right]_C$, such that the first derivative, with respect to '$q$', of any quaternion hypercomplex function, $f(q)$, thus



restricted to an arc, $C$, embedded in $q$-space, is accordingly given by

$$\left[\frac{d}{dq}\right]_C(f(q)) = \lim_{h \to 0}\left[\frac{\delta f(q)}{\delta q}\right]_C = \lim_{h \to 0}\left[\left(\frac{f(q+\delta q) - f(q)}{h}\right)\Big/\frac{\delta q}{h}\right],$$

provided that this derivative exists.

Given the content of the preceding Definitions DIV−2, 3 and 4, we now take the opportunity to elucidate various points mentioned therein, which are as follows :−

(a) Definition DIV−2 is simply a modification of Definition DIV−1, insofar as the difference quotient, originally postulated with respect to the latter named definition, has been appropriately rewritten after having both its numerator and denominator divided through by the real term, $h$.

(b) Definitions DIV−3 and 4 formally introduce the notion of a first derivative for quaternion hypercomplex functions as a limit of the corresponding difference quotients thereof. In the case of Definition DIV−4, the reader will also note the usage of the notation for the differential operator, $\left[\frac{d}{dq}\right]_C$, thereby symbolising the process of differentiation applied to this particular class of functions hitherto restricted to any given arc, $C$. The author has found it necessary to introduce this notation in order to avoid any possible confusion with the differential operator, $\frac{d}{dq}$, which both characterises Eq. (4−3) and is furthermore a contributing factor to its inherent deficiencies as a defining equation for the first derivative, with respect to '$q$', of any quaternion hypercomplex function, $f(q)$. In summary, we shall henceforth refer to the operator, $\left[\frac{d}{dq}\right]_C$, as the first order differential operator restricted to an arc, $C$, embedded in $q$−space.

(c) Whilst Definitions DIV−3 and 4 postulate the existence of the above mentioned first derivatives, neither of them, on the other hand, sets out a systematic procedure for evaluating such derivatives or, for that matter, specifies the conditions necessary for their very existence. We therefore address ourselves to this problem by both stating and verifying the next theorem which is essentially the culmination of the hitherto described definitions as well as the provisos that each of these definitions likewise entails.

## Theorem TIV-1

Let the difference quotient of a quaternion hypercomplex function, $f(q)$, thus restricted to an arc, $C$, embedded in $q$-space, be defined in accordance with Definition DIV-2.

Furthermore, by defining the first derivative of such a function in accordance with Definition DIV-4, it may therefore be clearly established that this derivative is likewise expressed as

$$\left[\frac{d}{dq}\right]_C (f(q)) = \lim_{h\to 0}\left[\left(\frac{f(q+\delta q)-f(q)}{h}\right)\Big/\frac{\delta q}{h}\right]$$

$$= \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\cdot\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\ \\ \dfrac{\overline{\frac{d}{dt}[q(t)]}\cdot\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases},$$

provided that the corresponding real and imaginary parts of the constituent functions, $f(q(t))$ and $q(t)$, are differentiable in '$t$'.

*   *   *

PROOF:-

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We initially postulate the existence of quaternion hypercomplex functions, $f(q(T))$ and $q(T)$, whose corresponding difference quotients, each possessing a real denominator, $T-t$, may be respectively defined as

$$\frac{f(q(T))-f(q(t))}{T-t} \text{ and } \frac{q(T)-q(t)}{T-t}, \ \forall t, T \in \mathbb{R},$$

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where
$$\begin{aligned} f(q(T)) = f^*(T) &= u_1^*(T) + i v_1^*(T) + j u_2^*(T) + k v_2^*(T) \\ &= u_1(x(T), y(T), \hat{x}(T), \hat{y}(T)) + \dots\dots\dots\dots + \\ &\quad k v_2(x(T), y(T), \hat{x}(T), \hat{y}(T)) \quad , \end{aligned}$$

$$q(T) = x(T) + i y(T) + j\hat{x}(T) + k\hat{y}(T),$$

$f(q(t))$ and $q(t)$ are analogously defined, $\forall t \in \mathbb{R}$.

Furthermore, by setting

$$T = t + h,$$

it likewise follows that

$$\begin{aligned} \lim_{T\to t}\left[\frac{f(q(T))-f(q(t))}{T-t}\right] &= \lim_{t+h\to t}\left[\frac{f(q(t+h))-f(q(t))}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{f(q(t+h))-f(q(t))}{h}\right] \\ &= \frac{d}{dt}\left[f(q(t))\right]^{+} \\ &= \frac{d}{dt}[u_1^*(t)] + i\frac{d}{dt}[v_1^*(t)] + j\frac{d}{dt}[u_2^*(t)] + k\frac{d}{dt}[v_2^*(t)] \end{aligned}$$

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AND

$$\lim_{T \to t}\left[\frac{q(T) - q(t)}{T - t}\right] = \lim_{t+h \to t}\left[\frac{q(t+h) - q(t)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{q(t+h) - q(t)}{h}\right]$$

$$= \frac{d}{dt}[q(t)]^{+}$$

$$= \frac{d}{dt}[x(t)] + i\frac{d}{dt}[y(t)] + j\frac{d}{dt}[\hat{x}(t)] + k\frac{d}{dt}[\hat{y}(t)],$$

$^{+}$ cf. the provisions of Definition DIV-3.



provided that

$$\lim_{T - t}\left[\frac{u_1^*(T) - u_1^*(t)}{T - t}\right] = \frac{d}{dt}[u_1^*(t)], \quad \lim_{T \to t}\left[\frac{x(T) - x(t)}{T - t}\right] = \frac{d}{dt}[x(t)],$$

$$\lim_{T \to t}\left[\frac{v_1^*(T) - v_1^*(t)}{T - t}\right] = \frac{d}{dt}[v_1^*(t)], \quad \lim_{T \to t}\left[\frac{y(T) - y(t)}{T - t}\right] = \frac{d}{dt}[y(t)],$$

$$\lim_{T \to t}\left[\frac{u_2^*(T) - u_2^*(t)}{T - t}\right] = \frac{d}{dt}[u_2^*(t)], \quad \lim_{T \to t}\left[\frac{\hat{x}(T) - \hat{x}(t)}{T - t}\right] = \frac{d}{dt}[\hat{x}(t)],$$

$$\lim_{T - t}\left[\frac{v_2^*(T) - v_2^*(t)}{T - t}\right] = \frac{d}{dt}[v_2^*(t)], \quad \lim_{T \to t}\left[\frac{\hat{y}(T) - \hat{y}(t)}{T - t}\right] = \frac{d}{dt}[\hat{y}(t)],$$

being the corresponding real and imaginary parts of the above stated limits (ie. derivatives) and thus satisfying the criteria prescribed in Theorem TIII-5.

Finally, in accordance with Theorem TIII-6, we deduce that

$$\lim_{T\to t}\left[\frac{f(q(T))-f(q(t))}{T-t}\Big/\frac{q(T)-q(t)}{T-t}\right] = \frac{d}{dt}[f(q(t))]\Big/\frac{d}{dt}[q(t)]$$

$$\therefore \lim_{h\to 0}\left[\frac{f(q(t+h))-f(q(t))}{h}\Big/\frac{q(t+h)-q(t)}{h}\right] = \frac{d}{dt}[f(q(t))]\Big/\frac{d}{dt}[q(t)]$$

$$\therefore \lim_{h\to 0}\left[\left(\frac{f(q+\delta q)-f(q)}{h}\right)\Big/\frac{\delta q}{h}\right] = \frac{d}{dt}[f(q(t))]\Big/\frac{d}{dt}[q(t)]$$

and hence the derivative, by virtue of Definition DIV-4,

$$\left[\frac{d}{dq}\right]_c(f(q)) = \frac{\frac{d}{dt}[f(q(t))]}{\frac{d}{dt}[q(t)]}$$

$$= \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\cdot\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\cdot\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases}$$

, as required. <u>Q.E.D.</u>

From the proof of Theorem TIV-1, it is evident that the existence of the first derivative, $\left[\frac{d}{dq}\right]_c(f(q))$, is wholly dependant upon the



existence of its constituent derivatives, $\frac{d}{dt}[q(t)]$ and $\frac{d}{dt}[f(q(t))]$, whose respective real and imaginary parts are denoted by

$$Re\left(\frac{d}{dt}[q(t)]\right) = \frac{d}{dt}[x(t)]\ ,\quad Re\left(\frac{d}{dt}[f(q(t))]\right) = \frac{d}{dt}[u_1^*(t)]\ ,$$

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$$Im_i\left(\frac{d}{dt}[q(t)]\right) = \frac{d}{dt}[y(t)]\,, \quad Im_i\left(\frac{d}{dt}[f(q(t))]\right) = \frac{d}{dt}[v_1^*(t)]\,,$$

$$Im_j\left(\frac{d}{dt}[q(t)]\right) = \frac{d}{dt}[\hat{x}(t)]\,, \quad Im_j\left(\frac{d}{dt}[f(q(t))]\right) = \frac{d}{dt}[u_2^*(t)]\,,$$

$$Im_k\left(\frac{d}{dt}[q(t)]\right) = \frac{d}{dt}[\hat{y}(t)]\,, \quad Im_k\left(\frac{d}{dt}[f(q(t))]\right) = \frac{d}{dt}[v_2^*(t)] \qquad (4-10).$$

Ultimately, we can determine the algebraic structure of this derivative after having evaluated all of its component derivatives thus contained in Eq. (4-10) by means of various well established techniques arising from the differential calculus of real variable functions.

In finalising our discussion of this particular topic, the author accordingly wishes to draw the reader's attention towards the following characteristic properties of the function, $f(q)$, whenever the quaternion,

$$q = x + iy,\ x + j\hat{x},\ x + k\hat{y} \qquad (4-11),$$

respectively :-

(a) From Definition DIII-9, we ascertain that any quaternion hypercomplex function, $f(q)$, may be defined as

$$f(q) = f(x + iy + j\hat{x} + k\hat{y})$$

$$= u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})\,,$$

$$\forall x, y, \hat{x}, \hat{y},\ u_1(x,y,\hat{x},\hat{y}), \ldots\ldots\ldots\ldots, v_2(x,y,\hat{x},\hat{y}) \in \mathbb{R}.$$

Substitution of Eq. (4-11) into this equation henceforth yields –

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$$f(x+iy) = u_1(x,y,0,0) + i\,v_1(x,y,0,0) + j\,u_2(x,y,0,0) + k\,v_2(x,y,0,0)$$

$$= u_1(x,y,0,0) + i\,v_1(x,y,0,0), \text{ with } u_2(x,y,0,0) = v_2(x,y,0,0) = 0,$$

$$f(x+j\hat{x}) = u_1(x,0,\hat{x},0) + i\,v_1(x,0,\hat{x},0) + j\,u_2(x,0,\hat{x},0) + k\,v_2(x,0,\hat{x},0)$$

$$= u_1(x,0,\hat{x},0) + j\,u_2(x,0,\hat{x},0), \text{ with } v_1(x,0,\hat{x},0) = v_2(x,0,\hat{x},0) = 0,$$

$$f(x+k\hat{y}) = u_1(x,0,0,\hat{y}) + i\,v_1(x,0,0,\hat{y}) + j\,u_2(x,0,0,\hat{y}) + k\,v_2(x,0,0,\hat{y})$$

$$= u_1(x,0,0,\hat{y}) + k\,v_2(x,0,0,\hat{y}), \text{ with } v_1(x,0,0,\hat{y}) = u_2(x,0,0,\hat{y}) = 0.$$

Note that each of these functions, being special cases of $f(q)$, is analogous with the complex variable function,

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y) \qquad (4-12),$$

$\forall\, x, y, u(x,y), v(x,y) \in \mathbb{R}$, whereupon the complex number, $i = (0,1)$, is <u>isomorphic</u> with the quaternions, $i$, $j$ and $k$, in this connection. As a direct consequence of the above stated factors, it therefore follows that, for the sub-set of arcs, $C$, embedded in $q$-space and thus denoted by the equations,

$$q(t) = \begin{cases} x(t) + iy(t) \\ x(t) + j\hat{x}(t) \\ x(t) + k\hat{y}(t) \end{cases} \qquad (4-13),$$

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the corresponding first derivatives, $\frac{d}{dt}[f(q(t))]$ and $\overline{\frac{d}{dt}[q(t)]}$, respectively form commutative products, being a characteristic property of both real and complex variable functions.

In the circumstances, we likewise perceive that the first derivative,

$$\left[\frac{d}{dq}\right]_c (f(q)) = \frac{\frac{d}{dt}[f(q(t))]}{\frac{d}{dt}[q(t)]}$$

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$$= \frac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2} = \frac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \qquad (4-14),$$

$\forall q(t) = x(t) + iy(t),\ x(t) + j\hat{x}(t),\ x(t) + k\hat{y}(t)$,

as anticipated.

(b) From complex variable analysis, the reader will no doubt recall that ~~any~~ function,

$$f(z) = u(x,y) + iv(x,y),$$

is differentiable in 'z', if and only if it satisfies

(i) the Cauchy – Riemann equations, namely –

$$\left.\begin{aligned} \frac{\partial}{\partial x}[u(x,y)] &= \frac{\partial}{\partial y}[v(x,y)] \\ \frac{\partial}{\partial y}[v(x,y)] &= -\frac{\partial}{\partial x}[v(x,y)] \end{aligned}\right\} \qquad (4-15)$$

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AND

(ii) the general condition for continuity within a neighbourhood, via the equation —

$$\frac{d}{dz}[f(z)] = \frac{\partial}{\partial x}[u(x,y)] + i\frac{\partial}{\partial x}[v(x,y)] \qquad (4-16),$$

which, in conjunction with Eq. (4-15), accordingly guarantees the analyticity of such functions over any pre-determined region of the z-plane.

Consequently, in an analogous manner to Eqs. (4-15) and (4-16), we foresee that the first derivative,

$$\left[\frac{d}{dq}\right]_C (f(q)) = \frac{d}{dq}[f(q)] \qquad (4-17),$$

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$$\forall q = x + iy,\ x + j\hat{x},\ x + k\hat{y},$$

is a special case of Eq. (4-14), if and only if any one of the following conditions simultaneously applies :–

$$\left.\begin{aligned} \frac{\partial}{\partial x}[u_1(x,y,0,0)] &= \frac{\partial}{\partial y}[v_1(x,y,0,0)] \\ \frac{\partial}{\partial y}[u_1(x,y,0,0)] &= -\frac{\partial}{\partial x}[v_1(x,y,0,0)] \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} \frac{d}{dq}[f(q)] = \frac{\partial}{\partial x}[u_1(x,y,0,0)] \\ + \\ i\frac{\partial}{\partial x}[v_1(x,y,0,0)], \end{aligned}\right.$$

$$\forall q = x + iy, \qquad (4-18)$$

OR

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$$\left.\begin{aligned}\frac{\partial}{\partial x}[u_1(x,0,\hat{x},0)] &= \frac{\partial}{\partial \hat{x}}[u_2(x,0,\hat{x},0)]\\ \frac{\partial}{\partial \hat{x}}[u_1(x,0,\hat{x},0)] &= -\frac{\partial}{\partial x}[u_2(x,0,\hat{x},0)]\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned}\frac{d}{dq}[f(q)] = \frac{\partial}{\partial x}[u_1(x,0,\hat{x},0)] \\ + \\ j\frac{\partial}{\partial x}[u_2(x,0,\hat{x},0)]\end{aligned}\right. ,$$

$$\forall q = x + j\hat{x}, \qquad (4-19)$$

OR

$$\left.\begin{aligned}\frac{\partial}{\partial x}[u_1(x,0,0,\hat{y})] &= \frac{\partial}{\partial \hat{y}}[v_2(x,0,0,\hat{y})]\\ \frac{\partial}{\partial \hat{y}}[u_1(x,0,0,\hat{y})] &= -\frac{\partial}{\partial x}[v_2(x,0,0,y)]\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned}\frac{d}{dq}[f(q)] = \frac{\partial}{\partial x}[u_1(x,0,0,\hat{y})] \\ + \\ k\frac{\partial}{\partial x}[v_2(x,0,0,\hat{y})]\end{aligned}\right. ,$$

$$\forall q = x + k\hat{y} \qquad (4-20).$$

Indeed, these are the only circumstances which will allow us to transcend the usual restrictions previously imposed upon us via the intermediary of an arc, $C$, embedded in $q$−space, with regard to the differentiation of quaternion hypercomplex functions in general.



## 2. Smooth Arcs and Contours.

As we have already seen from the preceding Part 1 of this section, the role of an arc, $C$, embedded in $q$−space, is central to our development of the differential calculus for quaternion hypercomplex functions and this in itself assuredly warrants closer examination. To be more specific, we will consider the notion of the <u>smoothness</u> of such arcs, bearing in mind that this same issue has been both examined and satisfactorily resolved within the content of complex variable analysis.

Consequently, we shall enunciate two more definitions to that effect which, to all intents and purposes, may be legitimately looked upon as a logical extension of their better known complex variable counterparts :–

## Definition DIV – 5

An arc, C, represented by the equation –

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \,, \; \forall t \in [a, b] \,,$$

is called smooth, provided that

(i) $q'(t) = \frac{d}{dt}[q(t)]$ exists,

(ii) $q'(t)$ is continuous, $\forall t \in (a, b)$, and $q'(t) \neq 0$ with respect to any value of '$t$' contained in this interval.

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## Definition DIV – 6

A contour, or piecewise smooth arc, C, is an arc consisting of a finite number of smooth arcs joined end to end. When the initial and final values of $q(t)$ are the same, a contour, C, is subsequently called a simple closed contour.

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Referring in particular to Definition DIV-5, we firstly observe that the existence of the derivative, $q'(t) = \frac{d}{dt}[q(t)]$, was previously established with respect to Theorem TIV-1 and hence the real derivatives,

$$\frac{d}{dt}[x(t)] = x'(t), \quad \frac{d}{dt}[\hat{x}(t)] = \hat{x}'(t),$$

$$\frac{d}{dt}[y(t)] = y'(t), \quad \frac{d}{dt}[\hat{y}(t)] = \hat{y}'(t) \qquad (4-21),$$

also exist.

From real variable analysis, it can be easily proven that any function, $f(x)$, which is differentiable in '$x$', is therefore continuous and since the real variable functions, $x(t)$, $y(t)$, $\hat{x}(t)$ and $\hat{y}(t)$, are differentiable in '$t$', they are also continuous, thereby implying that the quaternion hypercomplex function, $q(t)$, is likewise continuous. Furthermore, the functions, $x(t)$, $y(t)$, $\hat{x}(t)$ and $\hat{y}(t)$, by virtue of their differentiability, can be expressed respectively as Taylor series expansions, via the formulae —

$$\left.\begin{aligned} x(t) &= \sum_{n=0}^{\infty}\left(\frac{x^{(n)}(t_0)}{n!}\right)(t-t_0)^n \\ y(t) &= \sum_{n=0}^{\infty}\left(\frac{y^{(n)}(t_0)}{n!}\right)(t-t_0)^n \\ \hat{x}(t) &= \sum_{n=0}^{\infty}\left(\frac{\hat{x}^{(n)}(t_0)}{n!}\right)(t-t_0)^n \\ \hat{y}(t) &= \sum_{n=0}^{\infty}\left(\frac{\hat{y}^{(n)}(t_0)}{n!}\right)(t-t_0)^n \end{aligned}\right\}, \ \forall t_0 \in (a,b) \qquad (4-22),$$

whereupon their corresponding derivatives,

$$\left.\begin{aligned}
\frac{d}{dt}[x(t)] &= x'(t) = \sum_{n=1}^{\infty}\left(\frac{x^{(n)}(t_0)}{(n-1)!}\right)(t-t_0)^{n-1}\\
\frac{d}{dt}[y(t)] &= y'(t) = \sum_{n=1}^{\infty}\left(\frac{y^{(n)}(t_0)}{(n-1)!}\right)(t-t_0)^{n-1}\\
\frac{d}{dt}[\hat{x}(t)] &= \hat{x}'(t) = \sum_{n=1}^{\infty}\left(\frac{\hat{x}^{(n)}(t_0)}{(n-1)!}\right)(t-t_0)^{n-1}\\
\frac{d}{dt}[\hat{y}(t)] &= \hat{y}'(t) = \sum_{n=1}^{\infty}\left(\frac{\hat{y}^{(n)}(t_0)}{(n-1)!}\right)(t-t_0)^{n-1}
\end{aligned}\right\},\ \forall t_0 \in (a,b) \qquad (4-23),$$

To be continued via the author's next submission, namely -

***"An Introduction to Functions of a Quaternion Hypercomplex Variable -  PART 5/6."***

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