“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 2/6.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.”,

which has been published under the **‘VIXRA’ Mathematics subheading:- *‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 2/6.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$+j\begin{bmatrix}\hat{x}_3(x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2) - \\ \hat{y}_3(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + \\ x_3(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + \\ y_3(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1)\end{bmatrix}$$

$$+k\begin{bmatrix}\hat{y}_3(x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2) + \\ \hat{x}_3(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) - \\ y_3(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + \\ x_3(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1)\end{bmatrix}$$

$$= \begin{bmatrix}x_3x_1x_2 - x_3y_1y_2 - x_3\hat{x}_1\hat{x}_2 - x_3\hat{y}_1\hat{y}_2 - y_3x_1y_2 - y_3x_2y_1 - y_3\hat{x}_1y_2 + \\ y_3\hat{x}_2\hat{y}_1 - \hat{x}_3\hat{x}_1x_2 + \hat{x}_3y_1\hat{y}_2 - \hat{x}_3\hat{x}_1x_2 - \hat{x}_3\hat{y}_1y_2 - \hat{y}_3x_1\hat{y}_2 - \hat{y}_3\hat{x}_2y_1 + \hat{y}_3\hat{x}_1y_2 - \hat{y}_3x_2\hat{y}_1\end{bmatrix}$$

$$+i\begin{bmatrix}y_3x_1x_2 - y_3y_1y_2 - y_3\hat{x}_1\hat{x}_2 - y_3\hat{y}_1\hat{y}_2 + \\ x_3x_1y_2 + x_3x_2y_1 + x_3\hat{x}_1\hat{y}_2 - x_3\hat{x}_2\hat{y}_1 + \\ \hat{y}_3x_1\hat{x}_2 - \hat{y}_3y_1\hat{y}_2 + \hat{y}_3\hat{x}_1x_2 + \hat{y}_3\hat{y}_1y_2 - \\ \hat{x}_3x_1\hat{y}_2 - \hat{x}_3\hat{x}_2y_1 + \hat{x}_3\hat{x}_1y_2 - \hat{x}_3x_2\hat{y}_1\end{bmatrix}$$

$$+j\begin{bmatrix}\hat{x}_3x_1x_2 - \hat{x}_3y_1y_2 - \hat{x}_3\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{y}_1\hat{y}_2 - \\ \hat{y}_3x_1y_2 - \hat{y}_3x_2y_1 - \hat{y}_3\hat{x}_1\hat{y}_2 + \hat{y}_3\hat{x}_2\hat{y}_1 + \\ x_3x_1\hat{x}_2 - x_3y_1\hat{y}_2 + x_3\hat{x}_1x_2 + x_3\hat{y}_1y_2 + \\ y_3x_1\hat{y}_2 + y_3\hat{x}_2y_1 - y_3\hat{x}_1y_2 + y_3x_2\hat{y}_1\end{bmatrix}$$

$$+k\begin{bmatrix}\hat{y}_3x_1x_2 - \hat{y}_3y_1y_2 - \hat{y}_3\hat{x}_1\hat{x}_2 - \hat{y}_3\hat{y}_1\hat{y}_2 + \\ \hat{x}_3x_1y_2 + \hat{x}_3x_2y_1 + \hat{x}_3\hat{x}_1\hat{y}_2 - \hat{x}_3\hat{x}_2\hat{y}_1 - \\ y_3x_1\hat{x}_2 + y_3y_1\hat{y}_2 - y_3\hat{x}_1x_2 - y_2\hat{y}_1y_2 + \\ x_3x_1\hat{y}_2 + x_3\hat{x}_2y_1 - x_3\hat{x}_1y_2 + x_3x_2\hat{y}_1\end{bmatrix}$$

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$= q_1(q_2 q_3)$, as required. Q.E.D.

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Theorem TII-3

Let there exist three quaternion numbers, q_1, q_2, q_3. Hence, for any such quaternions, we obtain the following distributive laws :-

(i) $q_1(q_2 + q_3) = q_1 q_2 + q_1 q_3$,

(ii) $(q_2 + q_3) q_1 = q_2 q_1 + q_3 q_1$.

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PROOF :-

As previously indicated, we let

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$

$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$$

$$q_3 = x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3.$$

In order to prove that the distributive law (i) holds for all quaternions, we henceforth note that, by virtue of Definitions DII-1 and DII-2 respectively,

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$$q_1(q_2+q_3) = (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)\begin{bmatrix} x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2 + \\ x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3 \end{bmatrix}$$

$$= (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(x_2 + x_3 + i(y_2+y_3) + j(\hat{x}_2+\hat{x}_3) + k(\hat{y}_2+\hat{y}_3))$$

$$\begin{aligned} = \; & x_1(x_2+x_3) - y_1(y_2+y_3) - \hat{x}_1(\hat{x}_2+\hat{x}_3) - \hat{y}_1(\hat{y}_2+\hat{y}_3) + \\ & i[x_1(y_2+y_3) + (x_2+x_3)y_1 + \hat{x}_1(\hat{y}_2+\hat{y}_3) - (\hat{x}_2+\hat{x}_3)\hat{y}_1] + \\ & j[x_1(\hat{x}_2+\hat{x}_3) - y_1(\hat{y}_2+\hat{y}_3) + \hat{x}_1(\hat{x}_2+\hat{x}_3) + \hat{y}_1(y_2+y_3)] + \\ & k[x_1(\hat{y}_2+\hat{y}_3) + (\hat{x}_2+\hat{x}_3)y_1 - \hat{x}_1(y_2+y_3) + (x_2+x_3)\hat{y}_1] \quad . \end{aligned}$$

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Furthermore, we deduce that

$$\begin{aligned} q_1q_2 + q_1q_3 = \; & (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2) + \\ & (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3) \end{aligned}$$

$$\begin{aligned} = \; & x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + \\ & j(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + k(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1) + \\ & x_1x_3 - y_1y_3 - \hat{x}_1\hat{x}_3 - \hat{y}_1\hat{y}_3 + i(x_1y_3 + x_3y_1 + \hat{x}_1\hat{y}_3 - \hat{x}_3\hat{y}_1) + \\ & j(x_1\hat{x}_3 - y_1\hat{y}_3 + \hat{x}_1x_3 + \hat{y}_1y_3) + k(x_1\hat{y}_3 + \hat{x}_3y_1 - \hat{x}_1y_3 + x_3\hat{y}_1) \end{aligned}$$

$$\begin{aligned} = \; & x_1(x_2+x_3) - y_1(y_2+y_3) - \hat{x}_1(\hat{x}_2+\hat{x}_3) - \hat{y}_1(\hat{y}_2+\hat{y}_3) + \\ & i[x_1(y_2+y_3) + (x_2+x_3)y_1 + \hat{x}_1(\hat{y}_2+\hat{y}_3) - (\hat{x}_2+\hat{x}_3)\hat{y}_1] + \\ & j[x_1(\hat{x}_2+\hat{x}_3) - y_1(\hat{y}_2+\hat{y}_3) + \hat{x}_1(x_2+x_3) + \hat{y}_1(y_2+y_3)] + \\ & k[x_1(\hat{y}_2+\hat{y}_3) + (\hat{x}_2+\hat{x}_3)y_1 - \hat{x}_1(y_2+y_3) + (x_2+x_3)\hat{y}_1] \end{aligned}$$

$$= q_1(q_2+q_3), \text{ as required.}$$

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Similarly, by employing the hitherto mentioned definitions, we may likewise prove the validity of the distributive law (ii), namely –

$(q_2 + q_3)\, q_1 = q_2 q_1 + q_3 q_1$, as required. Q.E.D.

As a natural consequence of Eqs. (2–5) and (2–6), we further perceive than any quaternion, q, raised to an integral power, $n \in \mathbb{N}$, the set of natural numbers, is accordingly denoted by the defining equation :–

$$q^n = \underbrace{q \cdot q \cdots\cdots\cdots q}_{(n \text{ times})}, \quad \left\{ \begin{array}{l} \forall n \in \mathbb{N} \\ \forall q \in \mathbb{H} \end{array} \right\} \qquad (2-7),$$

being completely analogous with the exponential properties of both real and complex numbers via the equations :–

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$$z^n = \underbrace{z \cdot z \cdots\cdots\cdots z}_{(n \text{ times})}, \quad \left\{ \begin{array}{l} \forall n \in \mathbb{N} \\ \forall z \in \mathbb{C} \end{array} \right\} \qquad (2-8)$$

AND

$$a^n = \underbrace{a . a \cdots\cdots\cdots a}_{(n \text{ times})}, \quad \left\{ \begin{array}{l} \forall n \in \mathbb{N} \\ \forall a \in \mathbb{R} \end{array} \right\} \qquad (2-9).$$

Finally, we remark on a more general note that it is indeed instructive to compare the hitherto described properties of quaternion hypercomplex

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numbers with those same properties thus pertaining to real and complex numbers, whereupon the following table has been drawn up for that very purpose :-

Description of Property	Concomitant Response of Variables
COMMUTATIVITY OF ADDITION	$a_1 + a_2 = a_2 + a_1$, $\forall a_1, a_2 \in \mathbb{R}$. $z_1 + z_2 = z_2 + z_1$, $\forall z_1, z_2 \in \mathbb{C}$. $q_1 + q_2 = q_2 + q_1$, $\forall q_1, q_2 \in \mathbb{H}$.
ASSOCIATIVITY OF ADDITION	$a_1 + (a_2 + a_3) = (a_1 + a_2) + a_3$, $\forall a_1, a_2, a_3 \in \mathbb{R}$. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, $\forall z_1, z_2, z_3 \in \mathbb{C}$. $q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3$, $\forall q_1, q_2, q_3 \in \mathbb{H}$.
COMMUTATIVITY OF MULTIPLICATION	$a_1 a_2 = a_2 a_1$, $\forall a_1, a_2 \in \mathbb{R}$. $z_1 z_2 = z_2 z_1$, $\forall z_1, z_2 \in \mathbb{C}$. $q_1 q_2 \neq q_2 q_1$, $\forall q_1, q_2 \in \mathbb{H}$.
DISTRIBUTIVE LAW	$a_1 (a_2 + a_3) = (a_2 + a_3) a_1$, $\forall a_1, a_2, a_3 \in \mathbb{R}$. $z_1 (z_2 + z_3) = (z_2 + z_3) z_1$, $\forall z_1, z_2, z_3 \in \mathbb{C}$. $q_1 (q_2 + q_3) \neq (q_2 + q_3) q_1$, $\forall q_1, q_2, q_3 \in \mathbb{H}$. continued overleaf

Description of Property	Concomitant Response of Variables
ASSOCIATIVITY OF MULTIPLICATION	$a_1(a_2 a_3) = (a_1 a_2) a_3 \,,\ \forall a_1, a_2, a_3 \in \mathbb{R}$. $z_1(z_2 z_3) = (z_1 z_2) z_3 \,,\ \forall z_1, z_2, z_3 \in \mathbb{C}$. $q_1(q_2 q_3) = (q_1 q_2) q_3 \,,\ \forall q_1, q_2, q_3 \in \mathbb{H}$.
EXPONENTIAL LAW	$a^n = \underbrace{a \cdot a \ldots\ldots\ldots a}_{(n \text{ times})}, \left\{\begin{matrix} \forall n \in \mathbb{N} \\ \forall a \in \mathbb{R} \end{matrix}\right\}$. $z^n = \underbrace{z \cdot z \ldots\ldots\ldots z}_{(n \text{ times})}, \left\{\begin{matrix} \forall n \in \mathbb{N} \\ \forall z \in \mathbb{C} \end{matrix}\right\}$. $q^n = \underbrace{q \cdot q \ldots\ldots\ldots q}_{(n \text{ times})}, \left\{\begin{matrix} \forall n \in \mathbb{N} \\ \forall q \in \mathbb{H} \end{matrix}\right\}$.

Table II (1)

2. The Modulus and the Quaternion Conjugate.

Bearing in mind the definitions and theorems previously enunciated with respect to the modulus and conjugate of a complex number, we can now extend these concepts to quaternion hypercomplex numbers by means of two definitions and a theorem which are analogously derived as follows :-

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Definition DII – 3

The *modulus* or *absolute value* of a quaternion hypercomplex number,

$$q = x + iy + j\hat{x} + k\hat{y} \quad ,$$

is defined by

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$$|q| = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2} \; .$$

Definition DII – 4

Let there exist quaternion hypercomplex numbers, $\bar{q}_\lambda$, $\bar{q}_\rho$; such that

$$\bar{q}_\lambda \cdot q = |q|^2 \quad \text{and} \quad q \cdot \bar{q}_\rho = |q|^2 \; .$$

We accordingly define $\bar{q}_\lambda$ and $\bar{q}_\rho$ as the *laevo – conjugate* and *dextro – conjugate* of q respectively.[+]

Theorem TII – 4

The laevo – and dextro – conjugates of any quaternion hypercomplex number are, in fact, equal to one another, that is to say

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$\bar{q}_\lambda = \bar{q}_\rho = \bar{q} \Rightarrow q\bar{q} = \bar{q}q = |q|^2,$

wherein the entity, $\bar{q}$, is henceforth referred to as the quaternion hypercomplex conjugate.

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PROOF :-

By virtue of Definition DII-4, we initially stated that

$\bar{q}_\lambda \cdot q = |q|^2$ and $q \cdot \bar{q}_\rho = |q|^2$.

† Here we note that the Latin prefixes 'laevo-' and 'dextro-' mean 'left-hand' and 'right-hand' respectively.

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Henceforth, we may further write

$(\bar{q}_\lambda \cdot q)\,\bar{q}_\rho = |q|^2 \bar{q}_\rho$ and $\bar{q}_\lambda (q \cdot \bar{q}_\rho) = \bar{q}_\lambda |q|^2$

$\therefore\ \bar{q}_\lambda (q \cdot \bar{q}_\rho) = |q|^2 \bar{q}_\rho$ and $(\bar{q}_\lambda \cdot q)\,\bar{q}_\rho = \bar{q}_\lambda |q|^2$

$\therefore\ \bar{q}_\lambda |q|^2 = |q|^2 \bar{q}_\rho$ and $|q|^2 \bar{q}_\rho = \bar{q}_\lambda |q|^2,$

whereupon cancellation of the commutative common factor, $|q|^2 \in \mathbb{R}$,† thus yields

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$$\bar{q}_\lambda = \bar{q}_\rho, \text{ as required.}$$

In summary, we perceive that

$$\bar{q}_\lambda = \bar{q}_\rho = \bar{q},$$

the quaternion hypercomplex conjugate, thus implying that

$$q\bar{q} = \bar{q}q = |q|^2. \quad \underline{\underline{Q.E.D.}}$$

† The fact that $|q|^2$ and $1/|q|^2$ commute when multiplied by any other quaternion may be easily verified from Definition DII-2.

Additional properties of quaternion conjugates and moduli will likewise be elucidated in Parts 4 and 6 respectively of this section.

3. The Inverse and the Quotient.

The basic definitions for both the *inverse* and the *quotient* of a

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quaternion hypercomplex number follow immediately from those pertaining to real and complex numbers. However, due to the general lack of commutativity thus characterised by quaternion products, we shall soon perceive in due course that the criteria, previously established with respect to the inverses and quotients of real and complex numbers, cannot be wholly applied to their quaternion hypercomplex counterparts, as is evident from the following definitions and theorems to be stated and verified henceforth :-

Definition DII-5

The inverse of a quaternion hypercomplex number, q, is denoted by

$\frac{1}{q} = q^{-1}$, $\forall q \neq 0$, such that

$q \cdot q^{-1} = q^{-1} \cdot q = 1$, the identity element of multiplication.

Theorem TII-5

Let q be any quaternion hypercomplex number. Hence, it may be shown that its corresponding inverse,

$q^{-1} = \frac{1}{q} = \bar{q}/|q|^2$, $\forall q \neq 0$.

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PROOF :-

In accordance with Theorem TII - 4, we firstly note that

$q \cdot \bar{q} = \bar{q} \cdot q = |q|^2.$

Consequently, from Definition DII - 5, it likewise follows that

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$$q \cdot q^{-1} = 1 \Rightarrow \bar{q} \cdot q \cdot q^{-1} = \bar{q} \cdot 1 \Rightarrow |q|^2 q^{-1} = \bar{q}$$

$\therefore q^{-1} = \bar{q}/|q|^2$, as required. Q.E.D.

Theorem TII - 6

Let there exist two quaternion hypercomplex numbers, q_1 and q_2. Henceforth, it may be proven that, corresponding to every quaternion product, $q_1 q_2$, there exists an inverse,

$$(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}, \quad \forall q_1, q_2 \neq 0.$$

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PROOF ;-

From Definition DII - 5, we immediately deduce that

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$$q_1 q_2 (q_1 q_2)^{-1} = 1.$$

Hence, it futhermore follows that

$$q_1^{-1} q_1 q_2 (q_1 q_2)^{-1} = q_1^{-1} \cdot 1$$

$$\therefore 1 \cdot q_2 (q_1 q_2)^{-1} = q_1^{-1} \cdot 1$$

$$\therefore q_2 (q_1 q_2)^{-1} = q_1^{-1}$$

$$\therefore q_2^{-1} \cdot q_2 (q_1 q_2)^{-1} = q_2^{-1} \cdot q_1^{-1}$$

$$\therefore 1 \cdot (q_1 q_2)^{-1} = q_2^{-1} \cdot q_1^{-1}$$

$\therefore (q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$, as required. Q.E.D.

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Definition DII-6

The quotient of any two quaternion hypercomplex numbers, q_1 and q_2, is accordingly denoted by

$$\frac{q_1}{q_2} = \begin{cases} q_1 \left(\frac{1}{q_2}\right) \\ \left(\frac{1}{q_2}\right) q_1 \end{cases} = \begin{cases} q_1 q_2^{-1} \\ q_2^{-1} q_1 \end{cases}, \ \forall q_2 \neq 0.$$

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Theorem TII-7

The quaternion hypercomplex quotient may be explicitly expressed as

$$q_1/q_2 = \begin{cases} \bar{q}_2 q_1/|q_2|^2 \,, \\ q_1\bar{q}_2/|q_2|^2 \end{cases}$$

where $q_1\bar{q}_2 \neq \bar{q}_2 q_1$, $\forall q_1, q_2 \in \mathbb{H}$, such that

$$q_1\bar{q}_2/|q_2|^2 \neq \bar{q}_2 q_1/|q_2|^2, \quad \forall q_2 \neq 0.$$

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PROOF :-

By virtue of Theorem TII-5, we firstly note that the inverse,

$$1/q = \bar{q}/|q|^2,$$

whereupon we may rewrite the above stated quotient as

$$q_1/q_2 = \begin{cases} q_1(1/q_2) \\ (1/q_2)\, q_1 \end{cases} = \begin{cases} q_1 q_2^{-1} \\ q_2^{-1} q_1 \end{cases}$$

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$$= \begin{cases} q_1(\bar{q}_2/|q_2|^2) \\ (\bar{q}_2/|q_2|^2)\,q_1 \end{cases}$$

$$= \begin{cases} q_1\bar{q}_2/|q_2|^2 \\ \bar{q}_2 q_1/|q_2|^2 \text{, as required.} \end{cases}$$

Finally, in view of Definition DII-2, we are once again reminded that the multiplication of the quaternions, q_1 and $\bar{q}_2$, is generally not commutative, $\forall q_1, \bar{q}_2 \in \mathbb{H}$, that is to say

$$q_1\bar{q}_2 \neq \bar{q}_2 q_1 \Longrightarrow \underline{q_1\bar{q}_2/|q_2|^2 \neq \bar{q}_2 q_1/|q_2|^2,\ \forall q_2 \neq 0.} \quad \underline{\underline{\text{Q.E.D.}}}$$

From Definition DII-5 and Theorem TII-5, we further surmise that the formulae arising from the above stated definition and theorem are completely analogous with

$$1/z = z^{-1},\ \forall z \neq 0, \text{ such that } zz^{-1} = z^{-1}z = 1 \qquad (2\text{-}10)$$

AND

$$z^{-1} = \bar{z}/|z|^2,\ \forall z \neq 0 \qquad (2\text{-}11),$$

being none other than the familiar complex analogues of the inverse and its subsequent rationalisation. But here the similarities end, as is evident in the case of the quaternion hypercomplex quotient, q_1/q_2, which is not uniquely defined but instead takes on two distinct values. The reader will no doubt appreciate the real significance of this particular phenomenon in relation to the fundamental role played by the difference quotient as a prerequisite for the calculus of quaternion hypercomplex functions (cf. Section IV).

In the meantime, however, we will proceed to summarise the behavioural

properties of both the inverses and quotients of real, complex and quaternion hypercomplex numbers respectively, by means of Table II (2) provided below:-

TO BE CONTINUED.

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Description of Property	Concomitant Response of Variables
INVERSES	$a^{-1} = \frac{1}{a}, \forall a \in \mathbb{R} - \{0\}.$ $z^{-1} = \frac{1}{z} = \bar{z}/\lvert z\rvert^2, \forall z \in \mathbb{C} - \{0\}.$ $q^{-1} = \frac{1}{q} = \bar{q}/\lvert q\rvert^2, \forall q \in \mathbb{H} - \{0\}.$
INVERSE PRODUCTS	$(a_1a_2)^{-1} = a_1^{-1}a_2^{-1}$ $= a_2^{-1}a_1^{-1}, \forall a_1, a_2 \in \mathbb{R} - \{0\}.$ $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$ $= z_2^{-1}z_1^{-1}, \forall z_1, z_2 \in \mathbb{C} - \{0\}.$ $(q_1q_2)^{-1} = q_2^{-1}q_1^{-1}$ $\neq q_1^{-1}q_2^{-1}, \forall q_1, q_2 \in \mathbb{H} - \{0\}.$
QUOTIENTS	$a_1/a_2 = a_1a_2^{-1} = a_2^{-1}a_1, \forall a_1 \in \mathbb{R} \;\&\; \forall a_2 \in \mathbb{R} - \{0\}.$ $z_1/z_2 = z_1z_2^{-1} = z_2^{-1}z_1, \forall z_1 \in \mathbb{C} \;\&\; \forall z_2 \in \mathbb{C} - \{0\}.$ $q_1/q_2 = \begin{cases} q_1q_2^{-1}, \\ q_2^{-1}q_1 \end{cases}$ where $q_1q_2^{-1} \neq q_2^{-1}q_1, \forall q_1 \in \mathbb{H} \;\&\; \forall q_2 \in \mathbb{H} - \{0\}.$

Table II (2)

4. Properties of the Quaternion Conjugate.

Having formally established an algebraic relationship between the quaternion hypercomplex conjugate, $\bar{q}$, its generic co-factor, q, as well as

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the modulus thereof, $|q|$, by way of Theorem TII−4, we now wish to

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investigate further properties thus pertaining to this particular entity. Henceforth, we shall accordingly enunciate two additional theorems and another definition for that very purpose.

Theorem TII − 8

Let there exist a quaternion hypercomplex number,

$q = x + iy + j\hat{x} + k\hat{y}, \ \forall x, y, \hat{x}, \hat{y} \in \mathbb{R}.$

Hence, it may be shown that its corresponding conjugate,

$\bar{q} = x - iy - j\hat{x} - k\hat{y}.$

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PROOF :-

From Definition DII − 2, it follows that

$(x + iy + j\hat{x} + k\hat{y})(x - iy - j\hat{x} - k\hat{y})$

$= x \cdot x - y(-y) - \hat{x}(-\hat{x}) - \hat{y}(-\hat{y}) + i[x(-y) + xy - \hat{x}\hat{y} - (-\hat{x})\hat{y}] +$
$j[x(-\hat{x}) - y(-\hat{y}) + x\hat{x} + \hat{y}(-y)] + k[x(-\hat{y}) + (-\hat{x})y - \hat{x}(-y) + x\hat{y}]$

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$$= x^2 + y^2 + \hat{x}^2 + \hat{y}^2 + i(-xy + xy - \hat{x}\hat{y} + \hat{x}\hat{y}) +$$
$$j(-x\hat{x} + y\hat{y} + x\hat{x} - y\hat{y}) + k(-x\hat{y} - \hat{x}y + \hat{x}y + x\hat{y})$$

$$= x^2 + y^2 + \hat{x}^2 + \hat{y}^2 + i \cdot 0 + j \cdot 0 + k \cdot 0$$

$$= x^2 + y^2 + \hat{x}^2 + \hat{y}^2.$$

Similarly, by virtue of the same definition, we likewise deduce that

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$$(x - iy - j\hat{x} - k\hat{y})(x + iy + j\hat{x} + k\hat{y}) = x^2 + y^2 + \hat{x}^2 + \hat{y}^2.$$

However, in terms of Definition DII-3, we also perceive that the square of the modulus, $|q|$, that is to say,

$$|q|^2 = x^2 + y^2 + \hat{x}^2 + \hat{y}^2$$

and since we had initially stated that

$$q = x + iy + j\hat{x} + k\hat{y},$$

we accordingly obtain

$$q(x - iy - j\hat{x} - k\hat{y}) = |q|^2 \text{ and } (x - iy - j\hat{x} - k\hat{y})q = |q|^2 \text{ respectively.}$$

Furthermore, in accordance with Theorem TII-4, we have already ascertained that

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$$q\bar{q} = \bar{q}q = |q|^2,$$

whence it naturally follows that the quaternion hypercomplex conjugate,

$\bar{q} = x - iy - j\hat{x} - k\hat{y}$, as anticipated. Q.E.D.

Theorem TII-9

Let there exist two quaternion hypercomplex numbers,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$
$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2.$$

Henceforth, it may be proven that the following properties, thus pertaining to their corresponding conjugates, apply, $\forall x_1, \ldots\ldots\ldots, \hat{y}_2 \in \mathbb{R}$:-

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(i) $\overline{q_1 + q_2} = \bar{q}_1 + \bar{q}_2$,

(ii) $\overline{q_1 - q_2} = \bar{q}_1 - \bar{q}_2$,

(iii) $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$

(iv) $\overline{q_2 q_1} = \bar{q}_1 \bar{q}_2$.

* * *

PROOF :-

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(i) By virtue of Definition DII-1, we have

$$q_1 + q_2 = x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2),$$

whence it immediately follows that the quaternion hypercomplex conjugate <u>sum</u>,

$$\overline{q_1 + q_2} = x_1 + x_2 - i(y_1 + y_2) - j(\hat{x}_1 + \hat{x}_2) - k(\hat{y}_1 + \hat{y}_2).$$

Now since we initially defined the existence of quaternion hypercomplex numbers, q_1 and q_2, such that

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$
$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$$

then their corresponding conjugates are accordingly given by

$$\bar{q}_1 = x_1 - iy_1 - j\hat{x}_1 - k\hat{y}_1,$$

$$\bar{q}_2 = x_2 - iy_2 - j\hat{x}_2 - k\hat{y}_2,$$

thereby implying that

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$$\bar{q}_1 + \bar{q}_2 = x_1 - iy_1 - j\hat{x}_1 - k\hat{y}_1 + (x_2 - iy_2 - j\hat{x}_2 - k\hat{y}_2)$$

$$= x_1 + x_2 - i(y_1 + y_2) - j(\hat{x}_1 + \hat{x}_2) - k(\hat{y}_1 + \hat{y}_2)$$

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$$= \overline{q_1 + q_2}\text{, as required.} \quad \underline{\underline{\text{Q.E.D.}}}$$

(ii) Similarly, by virtue of Definition DII-1, we have

$$q_1 - q_2 = x_1 - x_2 + i(y_1 - y_2) + j(\hat{x}_1 - \hat{x}_2) + k(\hat{y}_1 - \hat{y}_2),$$

whence it immediately follows that the quaternion hypercomplex conjugate <u>difference</u>,

$$\overline{q_1 - q_2} = x_1 - x_2 - i(y_1 - y_2) - j(\hat{x}_1 - \hat{x}_2) - k(\hat{y}_1 - \hat{y}_2).$$

Furthermore, we deduce that

$$\begin{aligned}
\bar{q}_1 - \bar{q}_2 &= x_1 - iy_1 - j\hat{x}_1 - k\hat{y}_1 - (x_2 - iy_2 - j\hat{x}_2 - k\hat{y}_2) \\
&= x_1 - x_2 + i(y_2 - y_1) + j(\hat{x}_2 - \hat{x}_1) + k(\hat{y}_2 - \hat{y}_1) \\
&= \overline{q_1 - q_2}\text{, as required.} \quad \underline{\text{Q.E.D.}}
\end{aligned}$$

(iii) From Definition DII-2, we obtain

$$\begin{aligned}
q_1 q_2 = {} & x_1 x_2 - y_1 y_2 - \hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{y}_2 + i(x_1 y_2 + x_2 y_1 + \hat{x}_1 \hat{y}_2 - \hat{x}_2 \hat{y}_1) + \\
& j(x_1 \hat{x}_2 - y_1 \hat{y}_2 + \hat{x}_1 x_2 + \hat{y}_1 y_2) + k(x_1 \hat{y}_2 + \hat{x}_2 y_1 - \hat{x}_1 y_2 + x_2 \hat{y}_1),
\end{aligned}$$

whence it follows that the conjugate of this <u>product</u>,

$$\begin{aligned}
\overline{q_1 q_2} = {} & x_1 x_2 - y_1 y_2 - \hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{y}_2 - i(x_1 y_2 + x_2 y_1 + \hat{x}_1 \hat{y}_2 - \hat{x}_2 \hat{y}_1) - \\
& j(x_1 \hat{x}_2 - y_1 \hat{y}_2 + \hat{x}_1 x_2 + \hat{y}_1 y_2) - k(x_1 \hat{y}_2 + \hat{x}_2 y_1 - \hat{x}_1 y_2 + x_2 \hat{y}_1).
\end{aligned}$$

Similarly, in accordance with the above definition, we also deduce that

$$\bar{q}_2\bar{q}_1 = (x_2 - iy_2 - j\hat{x}_2 - k\hat{y}_2)(x_1 - iy_1 - j\hat{x}_1 - k\hat{y}_1)$$

$$= x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 - i(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) - j(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) - k(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1)$$

$= \overline{q_1q_2}$, as required. Q.E.D.

(iv) From Definition DII-2, we obtain

$$q_2q_1 = x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_2y_1 + x_1y_2 + \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2) + j(\hat{x}_1x_2 - \hat{y}_1y_2 + x_1\hat{x}_2 + y_1\hat{y}_2) + k(x_2\hat{y}_1 + \hat{x}_1y_2 - \hat{x}_2y_1 + x_1\hat{y}_2),$$

whence it follows that the conjugate of this product,

$$\overline{q_2q_1} = x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 - i(x_2y_1 + x_1y_2 + \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2) - j(\hat{x}_1x_2 - \hat{y}_1y_2 + x_1\hat{x}_2 + y_1\hat{y}_2) - k(x_2\hat{y}_1 + \hat{x}_1y_2 - \hat{x}_2y_1 + x_1\hat{y}_2).$$

Similarly, in accordance with the above definition, we also deduce that

$$\bar{q}_1\bar{q}_2 = (x_1 - iy_1 - j\hat{x}_1 - k\hat{y}_1)(x_2 - iy_2 - j\hat{x}_2 - k\hat{y}_2)$$

$$= x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 - i(x_2y_1 + x_1y_2 + \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2) - j(\hat{x}_1x_2 - \hat{y}_1y_2 + x_1\hat{x}_2 + y_1\hat{y}_2) - k(x_2\hat{y}_1 + \hat{x}_1y_2 - \hat{x}_2y_1 + x_1\hat{y}_2)$$

$= \overline{q_2q_1}$, as required. Q.E.D.

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<u>Definition DII−7</u>

Let there exist a quaternion hypercomplex number,

$$q = x + iy + j\hat{x} + k\hat{y}.$$

Hence, we accordingly observe the following properties of its corresponding

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conjugate, $\bar{q}$:−

(i) $$\bar{\bar{q}} = \overline{x - iy - j\hat{x} - k\hat{y}}$$

$$= \overline{x + i(-y) + j(-\hat{x}) + k(-\hat{y})}$$

$$= x + iy + j\hat{x} + k\hat{y}$$

$$= q,$$

(ii) $$|q| = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}$$

$$= \sqrt{x^2 + (-y)^2 + (-\hat{x})^2 + (-\hat{y})^2}$$

$$= |\bar{q}|.$$

Once again, we take the opportunity to review the material presented

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in both this part and in the preceding Part 2 of this section by means of an appropriate table provided below, wherein any valid comparisons arising from the respective properties of complex and quaternion hypercomplex conjugates have been clearly indicated as such :—

Description of Property	Concomitant Response of Variables
RESOLUTION OF CONJUGATES	$\bar{z} = x - iy, \ \forall x, y \in \mathbb{R}.$ $\bar{q} = x - iy - j\hat{x} - k\hat{y}, \ \forall x, y, \hat{x}, \hat{y} \in \mathbb{R}.$
CORRELATION OF CONJUGATES WITH MODULI	$\bar{\bar{z}} = z, \ \forall z \in \mathbb{C}.$ $\bar{\bar{q}} = q, \ \forall q \in \mathbb{H}.$ $z \cdot \bar{z} = \bar{z} \cdot z = \lvert z \rvert^2, \ \forall z \in \mathbb{C}.$ $q \cdot \bar{q} = \bar{q} \cdot q = \lvert q \rvert^2, \ \forall q \in \mathbb{H}.$ $\lvert \bar{z} \rvert = \lvert z \rvert, \ \forall z \in \mathbb{C}.$ $\lvert \bar{q} \rvert = \lvert q \rvert, \ \forall q \in \mathbb{H}.$ continued overleaf

Description of Property	Concomitant Response of Variables
CONJUGATE SUMS	$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2, \forall z_1, z_2 \in \mathbb{C}.$ $\overline{q_1+q_2} = \bar{q}_1+\bar{q}_2, \forall q_1, q_2 \in \mathbb{H}.$
CONJUGATE DIFFERENCES	$\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2, \forall z_1, z_2 \in \mathbb{C}.$ $\overline{q_1-q_2} = \bar{q}_1-\bar{q}_2, \forall q_1, q_2 \in \mathbb{H}.$
CONJUGATE PRODUCTS	$\overline{z_1 z_2} = \bar{z}_1\bar{z}_2 = \bar{z}_2\bar{z}_1 = \overline{z_2 z_1}, \forall z_1, z_2 \in \mathbb{C}.$ $\overline{q_1 q_2} = \bar{q}_2\bar{q}_1 \neq \bar{q}_1\bar{q}_2 = \overline{q_2 q_1}, \forall q_1, q_2 \in \mathbb{H}.$

Table II(3)

5. Definition and Properties of the Real and Imaginary Parts of Quaternions.

From complex variable analysis, we perceive that any complex number, $z = x+iy$, has real and imaginary parts respectively defined as

$$Re(z) = x, \quad Im(z) = y \qquad (2-12).$$

Here, it is only natural that one should analogously extend these same concepts to quaternions, bearing in mind the relative simplicity of their algebraic structure which readily permits the application of such notions thereupon.

Definition DII – 8

Let there exist a quaternion hypercomplex number,

$$q = x + iy + j\hat{x} + k\hat{y} .$$

We accordingly define the following characteristic properties of q :–

(i) the real part of q,

$$Re(q) = x ,$$

(ii) the imaginary part, with respect to i, of q,

$$Im_i(q) = y ,$$

(iii) the imaginary part, with respect to j, of q,

$$Im_j(q) = \hat{x} ,$$

(iv) the imaginary part, with respect to k, of q,

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$\mathcal{I}m_k(q) = \hat{y}.$

Furthermore, it naturally follows that

$$\left.\begin{array}{l} \mathcal{R}\ell(q) \\ \mathcal{I}m_i(q) \\ \mathcal{I}m_j(q) \\ \mathcal{I}m_k(q) \end{array}\right\} \le |q| = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}$$

$$= \sqrt{[\mathcal{R}\ell(q)]^2 + [\mathcal{I}m_i(q)]^2 + [\mathcal{I}m_j(q)]^2 + [\mathcal{I}m_k(q)]^2}\ .$$

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<u>Theorem TII-10</u>

Let there exist a quaternion,

$$q = x + iy + j\hat{x} + k\hat{y} \Longrightarrow \bar{q} = x - iy - j\hat{x} - k\hat{y}.$$

Hence, it may be proven that the following properties hold for all quaternions ;—

(i) $\mathcal{R}\ell(q) = \dfrac{q + \bar{q}}{2}$,

(ii) $\mathcal{I}m_i(q) = \dfrac{i\bar{q} - qi}{2} = \dfrac{\bar{q}i - iq}{2}$,

(iii) $\mathcal{I}m_j(q) = \dfrac{j\bar{q} - qj}{2} = \dfrac{\bar{q}j - jq}{2}$,

(iv) $\mathcal{I}m_k(q) = \dfrac{k\bar{q} - qk}{2} = \dfrac{\bar{q}k - kq}{2}$.

* * *

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PROOF :-

From previous definitions, we know that

$$q = x + iy + j\hat{x} + k\hat{y},$$
$$\bar{q} = x - iy - j\hat{x} - k\hat{y},$$
$$Rl(q) = x,\ Im_i(q) = y,\ Im_j(q) = \hat{x},\ Im_k(q) = \hat{y}.$$

(i) Subsequently, we deduce that

$$q + \bar{q} = x + iy + j\hat{x} + k\hat{y} + x - iy - j\hat{x} - k\hat{y}$$

$$= 2x + i(y - y) + j(\hat{x} - \hat{x}) + k(\hat{y} - \hat{y})$$

$$= 2x + i \cdot 0 + j \cdot 0 + k \cdot 0$$

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$$= 2x$$

$$= 2 Rl(q)$$

$\therefore Rl(q) = \frac{q + \bar{q}}{2}$, as required. Q.E.D.

(ii) By making 'i' a co-factor of both q and $\bar{q}$ and hence applying the product rules specified in Definition DI-1, we deduce that

$$-iq = -ix + y - ij\hat{x} - ik\hat{y} = -ix + y - k\hat{x} + j\hat{y},$$

$$\bar{q}i = ix + y - ji\hat{x} - ki\hat{y} = ix + y + k\hat{x} - j\hat{y}$$

$$\therefore -iq + \bar{q}i = 2y = 2\mathcal{I}m_i(q)$$

$$\therefore \bar{q}i - iq = 2\mathcal{I}m_i(q)$$

$$\therefore \mathcal{I}m_i(q) = \frac{\bar{q}i - iq}{2}$$

and similarly

$$-qi = -ix + y - ji\hat{x} - ki\hat{y} = -ix + y + k\hat{x} - j\hat{y}$$

$$i\bar{q} = ix + y - ij\hat{x} - ik\hat{y} = ix + y - k\hat{x} + j\hat{y},$$

$$\therefore -qi + i\bar{q} = 2y = 2\mathcal{I}m_i(q)$$

$$\therefore i\bar{q} - qi = 2\mathcal{I}m_i(q)$$

$$\therefore \mathcal{I}m_i(q) = \frac{i\bar{q} - qi}{2} = \frac{\bar{q}i - iq}{2}$$, as required. Q.E.D.

(iii) The required proof is completely analogous with the preceding part (ii) of this theorem, insofar as we substitute 'j' for 'i' as a co-factor of both q and $\bar{q}$ and hence apply the product rules specified in Definition DI-1.

Q.E.D.

(iv) The required proof is completely analogous with the preceding part (ii) of this theorem, insofar as we substitute 'k' for 'i' as a co-factor of both q and $\bar{q}$ and hence apply the product rules specified in Definition DI − 1.

Q.E.D.

Given the content of Definition DII − 8 and Theorem TII − 10, it is only appropriate that we likewise tabulate the behavioural properties of the real and imaginary parts of quaternions and hence compare them directly with the real and imaginary parts of complex numbers by means of Table II (4) designated below :-

Description of Property	Concomitant Response of Variables
RESOLUTION OF REAL AND IMAGINARY PARTS	$\left.\begin{array}{l} \mathcal{R}\ell(z) = x \\ \mathcal{I}m(z) = y \end{array}\right\}, \forall x, y \in \mathbb{R} \ \& \ \forall z \in \mathbb{C}.$ $\left.\begin{array}{l} \mathcal{R}\ell(q) = x \\ \mathcal{I}m_i(q) = y \\ \mathcal{I}m_j(q) = \hat{x} \\ \mathcal{I}m_k(q) = \hat{y} \end{array}\right\}, \forall x, y, \hat{x}, \hat{y} \in \mathbb{R} \ \& \ \forall q \in \mathbb{H}.$
CORRELATION OF REAL AND IMAGINARY PARTS WITH CONJUGATES AND MODULI	$\left.\begin{array}{l} \mathcal{R}\ell(z) \\ \mathcal{I}m(z) \end{array}\right\} \leq \lvert z \rvert, \ \left.\begin{array}{l} \mathcal{R}\ell(q) \\ \mathcal{I}m_i(q) \\ \mathcal{I}m_j(q) \\ \mathcal{I}m_k(q) \end{array}\right\} \leq \lvert q \rvert.$ continued overleaf

Description of Property	Concomitant Response of Variables
CORRELATION OF REAL AND IMAGINARY PARTS WITH CONJUGATES AND MODULI	$\left.\begin{aligned} Re(z) &= \frac{z+\bar{z}}{2} \\ Im(z) &= \frac{i(\bar{z}-z)}{2} \end{aligned}\right\}, \forall z \in \mathbb{C}.$ $\left.\begin{aligned} Re(q) &= \frac{q+\bar{q}}{2} \\ Im_i(q) &= \frac{i\bar{q}-qi}{2} \\ &= \frac{\bar{q}i-iq}{2} \\ Im_j(q) &= \frac{j\bar{q}-qj}{2} \\ &= \frac{\bar{q}j-jq}{2} \\ Im_k(q) &= \frac{k\bar{q}-qk}{2} \\ &= \frac{\bar{q}k-kq}{2} \end{aligned}\right\}, \forall q \in \mathbb{H}.$

Table II (4)

TO BE CONTINUED.

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6. Moduli Products and the Triangle Inequality.

We conclude our formal discussion of the algebra of quaternions with an analysis of the properties of moduli products and the triangle inequality, for which the material presented in Parts 2, 4 and 5 of this section will accordingly serve as a suitable basis. Moreover, in keeping with established practice, we shall also draw the reader's attention to the comparable behaviour of various properties thus pertaining to the moduli products and triangle inequalities of complex numbers, $z \in \mathbb{C}$, as well

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as the fundamental definitions for modulus in terms of both the complex and quaternion hypercomplex number systems.

In the meantime, however, we will henceforth state and likewise verify three more theorems for the purposes of our ensuing analysis :-

Theorem TII - 11

Let there exist two quaternions, q_1 and q_2. Here, it may be proven that the modulus of the quaternion hypercomplex product, $q_1 q_2$, is given by

$$|q_1 q_2| = |q_1||q_2| .$$

* * *

PROOF :-

From Theorem TII - 4, we observe that the square of the modulus

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product,

$$|q_1 q_2|^2 = q_1 q_2 \overline{q_1 q_2}$$

$$= q_1 q_2 \overline{q_2}\,\overline{q_1}$$

$$= q_1 (q_2 \overline{q_2}) \overline{q_1}$$

$$= q_1 |q_2|^2 \overline{q_1}$$

$$= q_1 \overline{q_1} |q_2|^2, \text{ since } |q_2|^2 \in \mathbb{R},$$

$$= |q_1|^2 |q_2|^2 .$$

Since the moduli of quaternions are always nonnegative, we accordingly deduce that

$$\sqrt{|q_1 q_2|^2} = \sqrt{|q_1|^2 |q_2|^2}$$

$$\therefore \sqrt{|q_1 q_2|^2} = \sqrt{(|q_1| . |q_2|)^2}, \text{ since both } |q_1|, |q_2| \in \mathbb{R},$$

$$\therefore |q_1 q_2| = |q_1||q_2|, \text{ as required.} \quad \underline{\text{Q.E.D.}}$$

Theorem TII – 12

Let there exist two quaternions, q_1 and q_2. Hence, it may be shown that the <u>modulus of the sum</u> of these particular numbers,

$$|q_1 + q_2| \leqslant |q_1| + |q_2| .$$

This result shall otherwise be referred to as the <u>triangle inequality</u>.

* * *

<u>PROOF</u> :-

Firstly, in accordance with Theorem TII-4, we observe that

$$|q_1 + q_2|^2 = (q_1 + q_2)\overline{(q_1 + q_2)} ,$$

whereupon, by virtue of Theorems TII-9, TII-10 and TII-11 as well as Definitions DII-7 and DII-8, it also follows that

$$\begin{aligned}
(q_1 + q_2)\overline{(q_1 + q_2)} &= (q_1 + q_2)(\overline{q_1} + \overline{q_2}) \\
&= q_1(\overline{q_1} + \overline{q_2}) + q_2(\overline{q_1} + \overline{q_2}) \\
&= q_1\overline{q_1} + q_1\overline{q_2} + q_2\overline{q_1} + q_2\overline{q_2} \\
&= |q_1|^2 + q_1\overline{q_2} + q_2\overline{q_1} + |q_2|^2 \\
&= |q_1|^2 + q_1\overline{q_2} + \overline{\overline{q_2}}\,\overline{q_1} + |q_2|^2
\end{aligned}$$

$$= |q_1|^2 + q_1\overline{q_2} + \overline{q_1\overline{q_2}} + |q_2|^2$$

$$= |q_1|^2 + 2\mathcal{Re}(q_1\bar{q}_2) + |q_2|^2$$

$$\therefore |q_1+q_2|^2 = |q_1|^2 + 2\mathcal{Re}(q_1\bar{q}_2) + |q_2|^2 .$$

However, since

$$\mathcal{Re}(q_1\bar{q}_2) \leqslant |q_1\bar{q}_2| = |q_1||\bar{q}_2| = |q_1||q_2| ,$$

then we likewise perceive that

$$\begin{aligned} |q_1+q_2|^2 &= |q_1|^2 + 2\mathcal{Re}(q_1\bar{q}_2) + |q_2|^2 \\ &\leqslant |q_1|^2 + 2|q_1||q_2| + |q_2|^2 \\ &= (|q_1|+|q_2|)^2 . \end{aligned}$$

Insofar as the moduli of quaternion hypercomplex numbers are always nonnegative, we finally deduce that

$$\sqrt{|q_1+q_2|^2} \leqslant \sqrt{(|q_1|+|q_2|)^2}$$

$\therefore |q_1+q_2| \leqslant |q_1| + |q_2|$, as required. <u>Q.E.D.</u>

<u>Theorem TII – 13</u>

Let there exist two quaternion hypercomplex numbers, q_1 and q_2. Accordingly, the <u>moduli</u> of the respective <u>sum</u> and <u>difference</u> of these quaternions possess the following properties :–

<u>(i)</u> $||q_1| - |q_2|| \leqslant |q_1+q_2|$,

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(ii) $||q_1| - |q_2|| \leqslant |q_1 - q_2|$.

* * *

PROOF :-

(i) We firstly consider the modulus, $|q_1|$, and accordingly write

$$|q_1| = |(q_1 + q_2) + (-q_2)| .$$

Now, by virtue of Theorem TII-12, we further obtain

$$\begin{aligned} |q_1| &= |(q_1 + q_2) + (-q_2)| \\ &\leqslant |q_1 + q_2| + |-q_2| \\ &= |q_1 + q_2| + |q_2| , \end{aligned}$$

whence it immediately follows that

$$|q_1| - |q_2| \leqslant |q_1 + q_2| .$$

Clearly,

$$|q_1| - |q_2| = ||q_1| - |q_2|| , \text{ if and only if } |q_1| \geqslant |q_2| ,$$

and thus

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$$||q_1|-|q_2|| \leqslant |q_1+q_2|, \text{ whenever } |q_1| \geqslant |q_2|.$$

Supposing, on the other hand, we now require that

$$|q_2| > |q_1|.$$

Once again, we likewise deduce that the modulus,

****

$$\begin{aligned} |q_2| &= |(q_2+q_1)+(-q_1)| \\ &\leqslant |q_2+q_1|+|-q_1| \\ &= |q_2+q_1|+|q_1|, \end{aligned}$$

whence it immediately follows that

$$|q_2|-|q_1| \leqslant |q_1+q_2|.$$

Furthermore, it is evident that

$$|q_2|-|q_1| = ||q_2|-|q_1||, \text{ whenever } |q_2| > |q_1|,$$

and since

$$||q_2|-|q_1|| = |-(|q_1|-|q_2|)| = |-1|\,||q_1|-|q_2|| = ||q_1|-|q_2||,$$

we finally conclude that

****

$$||q_1|-|q_2|| = ||q_2|-|q_1|| = |q_2|-|q_1| \leqslant |q_1+q_2| \Longrightarrow$$

$||q_1|-|q_2|| \leqslant |q_1+q_2|$, wherever $|q_2| > |q_1|$.

Hence, we have demonstrated that the inequality,

$$||q_1|-|q_2|| \leqslant |q_1+q_2|,$$

is valid $\forall\, |q_1|, |q_2| \in \mathbb{R}$, as required. <u>Q.E.D.</u>

<u>(ii)</u> We firstly consider the modulus, $|q_1|$, and accordingly write

$$|q_1| = |(q_1-q_2)+q_2|.$$

Now, by virtue of Theorem TII-13, we further obtain

$$|q_1| = |(q_1-q_2)+q_2|$$
$$\leqslant |q_1-q_2| + |q_2|,$$

whence it immediately follows that

$$|q_1|-|q_2| \leqslant |q_1-q_2|.$$

Clearly,

$|q_1|-|q_2| = ||q_1|-|q_2||$, if and only if, $|q_1| \geqslant |q_2|$, and thus

$$||q_1| - |q_2|| \leq |q_1 - q_2|, \text{ whenever } |q_1| \geq |q_2|.$$

Supposing, on the other hand, we now require that

$$|q_2| > |q_1|.$$

Once again, we likewise deduce that the modulus,

$$\begin{aligned} |q_2| &= |(q_2 - q_1) + q_1| \\ &\leq |q_2 - q_1| + |q_1| \\ &= |-(q_1 - q_2)| + |q_1| \\ &= |-1||q_1 - q_2| + |q_1| \\ &= |q_1 - q_2| + |q_1|, \end{aligned}$$

To be continued via the author's next submission, namely -

"An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 3/6."

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