Neutrosophic set and Logic

Mumtaz Ali

Department of Mathematics, Quaid-i-azam University, Islamabad, 45320, Pakistan

Florentin Smarandache

University of New Mexico, 705 Gurley Ave., Gallup, New Mexico 87301, USA

Luige Vladareanu

Institute of Solid Mechanics, Bucharest, Romania

Abstract

Neutrosophic sets and Logic plays a significant role in approximation theory. It is a generalization of fuzzy sets and intuitionistic fuzzy set. Neutrosophic set is based on the neutrosophic philosophy in which every idea Z, has opposite denoted as anti(Z) and its neutral which is denoted as neut(Z). This is the main feature of neutrosophic sets and logic. This chapter is about the basic concepts of neutrosophic sets as well as some of their hybrid structures. This chapter starts with the introduction of fuzzy sets and intuitionistic fuzzy sets respectively. The notions of neutrosophic set are defined and studied their basic properties in this chapter. Then we studied neutrosophic crisp sets and their associated properties and notions. Moreover, interval valued neutrosophic sets are studied with some of their properties. Finally, we presented some applications of neutrosophic sets in the real world problems.

1. Introduction

The data in real life problems like engineering, social, economic, computer, decision making, medical diagnosis etc. are often uncertain and imprecise. This type of data is not necessarily crisp, precise and deterministic nature because of their fuzziness and vagueness. To handle this kind of data, Zadeh introduced fuzzy set sets (1965). Several types of approaches have been proposed which is based on fuzzy sets such as interval valued fuzzy sets (1986), intuitionistic fuzzy sets (1986), and so on. Researchers throughout the world have been successfully applied fuzzy sets in several areas like signal processing, knowledge representation, decision making, stock markets, pattern recognition, control, data mining, artificial intelligence etc.

Atanassov (1986), observed that there is some kind of uncertainty in the data which is not handled by fuzzy sets. Therefore, intuitionistic fuzzy sets were proposed by Atanassov in (1986), which became the generalization of fuzzy sets by inserting the non-membership degree to fuzzy sets. An intuitionistic fuzzy set has a membership function as well as a non-membership function. Intuitionistic fuzzy sets define more beautifully the fuzzy objects of the real world. A huge amount of research study has been conducted on intuitionistic fuzzy sets from different aspects. Intuitionistic fuzzy sets have been successfully applied in several fields such as modeling imprecision, decision making problems, pattern recognition, economics, computational intelligence, medical diagnosis and so on.

Smarandache in (1995), coined the theory of neutrosophic sets and logic under the neutrosophy which is a new branch of philosophy that study the origin, nature, and scope of neutralities as well as their interactions with ideational spectra. A neutrosophic set can be characterized by a truth membership function T, an indeterminacy membership function I and falsity membership function F. Neutrosophic set is the generalization of fuzzy sets (1965), intuitionistic fuzzy sets (1986), paraconsistent set (1995) etc. Neutrosophic sets can treat uncertain, inconsistent, incomplete, indeterminate and false information. The neutrosophic sets and their related set theoretic operators need to be specified from scientific or engineering point of view. Indeterminacy are quantified explicitly in neutrosophic sets and T, I, and F operators are complementally independent which is very significant in several applications such as information fusion, physics, computer, networking, decision making, information theory etc.

In this chapter, we present the notions of neutrosophic sets and logic. In section 1, we presented a brief introduction. In section 2, we studied neutrosophic sets with some of their basic properties. In the next section 3, the hybrid structure neutrosophic crisp sets and their associated properties and notions have been studied. In section 4, interval valued neutrosophic sets have been studied. Section 5 is about to study some practical life applications of neutrosophic sets.

2. Neutrosophic set, Similarity Measures, Neutrosophic Norms

In this section the notions of neutrosophic sets, some similarity measures of neutrosophic sets, neutrosophic norms respectively.

2.1. Neutrosophic Set

In this subsection the neutrosophic set is presented with their basic properties and notions with illustrative examples.

Definition 2.1.1: Let X be a universe of discourse and a neutrosophic set A on X is defined as

$$A=\{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle,\ x\in X\}$$

where $T,\ I,\ F\ :\ X\to]^{-}0,1^{+}[$ and $^{-}0\le T_A(x)+I_A(x)+F_A(x)\le 3^{+}$.

From philosophical point of view, neutrosophic set takes the value in the interval $[0,1]$, because it is difficult to use neutrosophic set with value from real standard or non-standard subsets of $]^{-}0,1^{+}[$ in real life application like scientific and engineering problems.

Definition 2.1.2: A neutrosophic set A is contained in another neutrosophic set B, if

$$T_A(x)\le T_B(x),\ I_A(x)\le I_A(x),\ F_A(x)\ge F_B(x)\ \text{ for all } x\in X\ .$$

Definition 2.1.3: An element x of U is called significant with respect to neutrosophic set A of U if the degree of truth-membership or falsity-membership or indeterminancy-membership value, i.e.,$T_{A(x)}$ or $F_{A(x)}$or $I_{A(x)}\le 0.5$.

Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity-membership all cannot be significant. We define an intuitionistic neutrosophic set by A = {< x: $T_{A(x)}$,$I_{A(x)}$,$F_{A(x)}$ >,x ∈ U},where min { $T_{A(x)}$, $F_{A(x)}$ } ≤ 0.5, min { $T_{A(x)}$, $I_{A(x)}$ } ≤ 0.5, min { $F_{A(x)}$, $I_{A(x)}$ } ≤ 0.5, for all x ∈ U, with the condition $0\le T_{A(x)}+I_{A(x)}+F_{A(x)}\le 2$.

As an illustration, let us consider the following example.

Example 2.1.4: Assume that the universe of discourse U={x_1,x_2,x_3},where x_1 characterizes the capability, x_2 characterizes the trustworthiness and x_3 indicates the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in [0,1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components viz. the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is an intuitionistic neutrosophic set (IN S) of U, such that,

$$A = \{< x_1,0.3,0.5,0.4 >,< x_2,0.4,0.2,0.6 >,< x_3,0.7,0.3,0.5 >\},$$

where the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.4 etc.

Definition 2.1.5: Let X is a space of points (objects) with generic elements in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $\mathrm{T_A(x)}$, an indeterminacy membership function $\mathrm{I_A(x)}$, and a falsity membership function $\mathrm{F_A(x)}$ if the functions $\mathrm{T_A(x), I_A(x), F_A(x)}$ are singletons subintervals/subsets in the real standard $[0,1]$, i.e. $\mathrm{T_A(x): X \to [0,1], I_A(x): X \to [0,1], F_A(x): X \to [0,1]}$. Then a simplification of the neutrosophic set A is denoted by $\mathrm{A} = \{< x: \mathrm{T_A(x), I_A(x), F_A(x)} >, x \in X\}$.

Definition 2.1.6: Let X is a space of points (objects) with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function$F_A(x)$, for each point$x \in X$, $T_A(x), I_A(x), F_A(x) \in [0,1]$. Therefore, a SVNS A can be written as $A_{SVNS} = \{< x:\ T_A(x), I_A(x), F_A(x) >, x \in X\}$.

For two SVNS, $A_{SVNS} = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\}$ and $B_{SVNS} = \{< x: T_B(x), I_B(x), F_B(x) >, x \in X\}$, the following expressions are defined in [2] as follows:

$A_{NS} \subseteq B_{NS}$ if and only if $T_A(x) \le T_B(x), I_A(x) \ge I_B(x), F_A(x) \ge F_B(x)$. $A_{NS} = B_{NS}$ if and only if $T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$. $A^c =< x, F_A(x), 1 - I_A(x), T_A(x) >$

For convenience, a SVNS A is denoted by $A =< T_A(x), I_A(x), F_A(x) >$ for any x$\in X$; for two SVNSs A and B. Then,

$$(1)\ A \cup B =< \max\big(T_A(x), T_B(x)\big), min\big(I_A(x), I_B(x)\big), min\big(T_A(x), T_B(x)\big) >$$

$$(2)\ A \cap B =< min\big(T_A(x), T_B(x)\big), max\big(I_A(x), I_B(x)\big), max\big(T_A(x), T_B(x)\big) >$$

2.2 Jaccard, Dice and Cosine similarity Measures of Neutrosophic Sets

The vector similarity measure is one of the most important techniques to measure the similarity between objects. In this subsection, the Jaccard, Dice and Cosine similarity measures between two vectors have been studied.

Definition 2.2.1: Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be the two vectors of length n where all the coordinates are positive. The Jaccard index of these two vectors is defined as

$$J(X,Y) = \frac{X.Y}{\|X\|_2{}^2 + \|Y\|_2{}^2 + X.Y} = \frac{\sum_{i=1}^n x_i.y_i}{\sum_{i=1}^n x_i{}^2 + \sum_{i=1}^n y_i{}^2 - \sum_{i=1}^n x_i.y_i},$$

where $X.Y = \sum_{i=1}^n x_i.y_i$ is the inner product of the vectors $X\ and\ Y$.

Definition 2.2.2: The Dice similarity measure is defined as

$$J(X,Y)=\frac{2X.Y}{\|X\|_2{}^2+\|Y\|_2{}^2}=\frac{2\sum_{i=1}^{n}x_i.y_i}{\sum_{i=1}^{n}x_i{}^2+\sum_{i=1}^{n}y_i{}^2}.$$

Cosine formula is defined as the inner product of these two vectors divided by the product of their lengths. This is the cosine of the angle between the vectors.

Definition 2.2.3: The cosine similarity measure is defined as

$$C(X,Y)=\frac{X.Y}{\|X\|_2{}^2.\|Y\|_2{}^2}=\frac{\sum_{i=1}^{n}x_i.y_i}{\sum_{i=1}^{n}x_i{}^2.\sum_{i=1}^{n}y_i{}^2}.$$

It is obvious that the Jaccard, Dice and cosine similarity measures satisfy the following properties

$$(\mathrm{P}_1)\ 0\le \mathrm{J(X,Y)},\mathrm{D(X,Y)},\mathrm{C(X,Y)}\le 1,$$

$$(\mathrm{P}_2)\ \mathrm{J(X,Y)}=\mathrm{J(Y,X)},\mathrm{D(X,Y)}=\mathrm{D(Y,X)}\text{ and }\mathrm{C(X,Y)}=\mathrm{C(Y,X)},$$

$$(\mathrm{P}_3)\ \mathrm{J(X,Y)}=1,\ \mathrm{D(X,Y)}=1\text{ and }\mathrm{C(X,Y)}=1\text{ if }\mathrm{X}=\mathrm{Y}.$$

i.e., $x_i=y_i(i=1,2,\dots,n)$for every $x_i\in X$ and $y_i\in Y$. Also Jaccard, Dice, cosine weighted similarity measures between two SNSs A and B as discussed in [6] are

$$WJ(A,B)=\sum_{i=1}^{n}w_i\frac{\begin{array}{c}T_A(x_i)T_B(x_i)\\+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i)\end{array}}{\begin{array}{c}(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\\+(T_B(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2\\-T_A(x_i)T_B(x_i)-T_B(x_i)T_C(x_i)-T_C(x_i)T_A(x_i)\end{array}},$$

$$WD(A,B)=\sum_{i=1}^{n}w_i\frac{2\begin{pmatrix}T_A(x_i)T_B(x_i)\\+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i)\end{pmatrix}}{\begin{array}{c}(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\\+(T_B(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2\end{array}},$$

$$WC(A,B)=\sum_{i=1}^{n}w_i\frac{\begin{pmatrix}T_A(x_i)T_B(x_i)\\+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i)\end{pmatrix}}{\begin{array}{c}\sqrt{(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2}\\\sqrt{(T_B(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2+(T_A(x_i))^2}\end{array}}.$$

2.3. Neutrosophic N-norms and Neutrosophic N-conorms

In this subsection 2.3, neutrosophic norms and their related properties discussed by the authors.

Definition 2.3.1: $N_n:\ \left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)^2\to\left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)$ where $N_n(x(T_1,I_1,F_1),\ y(T_2,I_2,F_2))\ =\ (N_nT(x,y),\ N_nI(x,y),\ N_nF(x,y)),$ $N_nT(.,.),\ N_nI(.,.),\ N_nF(.,.)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components. N_n have to satisfy, for any $x,\ y,\ z$ in the neutrosophic logic/set M of the universe of discourse U , the following axioms:

a) Boundary Conditions: $N_n(x,0) = 0,\ N_n(x,1) = x.$

b) Commutativity: $N_n(x, y) = N_n(y, x).$

c) Monotonicity: If $x \le y,$ *then* $N_n(x, z) \le N_n(y, z).$

d) Associativity: $N_n(N_n(x, y), z) = N_n(x, N_n(y, z)).$

N_n represent the and operator in neutrosophic logic, and respectively the intersection operator in neutrosophic set theory.

Example 2.3.2: A general example of N-norm would be this.
Let $x(T_1, I_1, F_1)$ and $y(T_1, I_1, F_1)$ be in the neutrosophic set/logic M. Then: $N_n(x, y) = (T_1 \wedge T_2,\ I_1 \vee I_2,\ F_1 \vee F_2).$

Definition 2.3.3: $N_c : (\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,)^2 \to (\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,)$, where $N_c(x(T_1,I_1,F_1),\ y(T_2,I_2,F_2)) = (N_cT(x,y),\ N_cI(x,y),\ N_cF(x,y)),$
$N_cT(.,.),\ N_cI(.,.),\ N_cF(.,.)$ are the truth /membership, indeterminacy, and respectively falsehood /non-membership components. N_c have to satisfy, for any x, y, z in the neutrosophic logic/set M of the universe of discourse U , the following axioms:

a) Boundary Conditions: $N_c(x,0) = x,\ N_c(x,1) = 1.$

b) Commutativity: $N_c(x, y) = N_c(y, x).$

c) Monotonicity: If $x \le y,$ *then* $N_c(x, z) \le N_c(y, z).$

d) Associativity: $N_c(N_c(x, y), z) = N_c(x, N_c(y, z)).$

Example 2.3.4: A general example of N-conorm would be this. Let $x(T_1, I_1, F_1)$ and $y(T_1, I_1, F_1)$ be in the neutrosophic set/logic M. Then: $N_c(x, y) = (T_1 \vee T_2,\ I_1 \wedge I_2,\ F_1 \wedge F_2)$ where the "/\" operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the "\/" operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the above N-conorms axioms). For example, /\ can be the Algebraic Product T-norm/N-norm, so T1/\T2 = T1·T2; and \/ can be the Algebraic Product T-conorm/N-conorm, so T1\/T2 = T1+T2-T1·T2. Or /\ can be any T-norm/N-norm, and \/ any T-conorm/N-conorm from the above and below; for example the easiest way would be to consider the min for crisp components (or inf for subset components) and respectively max for crisp components (or sup for subset components).

Theorem 2.3.5: For any s-norm s(x, y) and for all $\alpha \ge 1$, we get the following s-norms and t-norms:

1. $S_\alpha^s(x,y) = \sqrt[\alpha]{s(x^\alpha, y^\alpha)}$,
2. $T_\alpha^s(x,y) = 1 - \sqrt[\alpha]{s((1-x)^\alpha, (1-y)^\alpha)}$.

Theorem 2.3.6: For any t-norm t(x, y) and for all $\alpha > 1$, we get the following t-norms and s-norms:

1. $T_\alpha^t(x,y) = \sqrt[\alpha]{t(x^\alpha, y^\alpha)}$,
2. $S_\alpha^t(x,y) = 1 - \sqrt[\alpha]{t((1-x)^\alpha, (1-y)^\alpha)}$.

Theorem 2.3.7: Let $f,\ g:[0,1]\to[0,1]$ be bijective functions such that $f(0)=0$, $f(1)=1$, $g(0)=1$ and $g(1)=0$. For any s-norm $s(x,y)$ we get the following s-norm and t-norm:

1. $S_f^s(x,y)=f^{-1}\left[s\left(f(x),f(y)\right)\right]$,
2. $T_g^s(x,y)=g^{-1}\left[s\left(g(x),g(y)\right)\right]$.

Corollary 2.3.8: Let $f(x)=\sin\frac{\pi}{2}x$ and $g(x)=\cos\frac{\pi}{2}x$ then

1. $S_{\sin}^s(x,y)=\frac{2}{\pi}\sin^{-1}s\left(\sin\frac{\pi}{2}x,\sin\frac{\pi}{2}y\right)$ is an s-norm
2. $T_{\cos}^s(x,y)=\frac{2}{\pi}\cos^{-1}s\left(\cos\frac{\pi}{2}x,\cos\frac{\pi}{2}y\right)$ is a t-norm

Theorem 2.3.9: Let $f,\ g:[0,1]\to[0,1]$ be bijective functions such that $f(0)=0$, $f(1)=1$, $g(0)=1$ and $g(1)=0$. For any t-norm $t(x,y)$ we get the following t-norm and s-norm:

1. $T_f^t(x,y)=f^{-1}\left[t\left(f(x),f(y)\right)\right]$,
2. $S_g^t(x,y)=g^{-1}\left[t\left(g(x),g(y)\right)\right]$

Corollary 2.3.10: Let f(x)= $\sin\frac{\pi}{2}x$ and g(x)= $\cos\frac{\pi}{2}x$ then

1. $T_{\sin}^t(x,y)=\frac{2}{\pi}\sin^{-1}\left(t\left(\sin\frac{\pi}{2}x,\sin\frac{\pi}{2}y\right)\right)$ is a t-norm
2. $S_{\cos}^t(x,y)=\frac{2}{\pi}\cos^{-1}\left(t\left(\cos\frac{\pi}{2}x,\cos\frac{\pi}{2}y\right)\right)$ is an s-norm.

Definition 2.3.11: $T_n:\ \left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)^2\to\left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)$ where
$T_n\left(x(T,I,F),\ y(T,I,F)\right)=\left(t(x_T,y_T),\ s(x_I,y_I),\ s(x_F,y_F)\right)$,
$t(x_T,y_T),\ s(x_I,y_I),\ s(x_F,y_F)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components and *s* and *t* are the fuzzy s-norm and fuzzy t-norm respectively. T_n have to satisfy, for any $x,\ y,\ z$ in the neutrosophic logic/set M of the universe of discourse U, the following axioms:

a) Boundary Conditions: $T_n(x,0)=0,\ T_n(x,1)=x.$

b) Commutativity: $T_n(x,\ y)=T_n(y,\ x).$

c) Monotonicity: If $x\le y,\ then\ T_n(x,\ z)\le T_n(y,\ z).$

d) Associativity: $T_n\left(T_n(x,\ y),\ z\right)=T_n\left(x,\ T_n(y,\ z)\right).$

Definition 2.3.12: $S_n:\ \left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)^2\to\left(\,]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\,\right)$ where
$S_n\left(x(T,I,F),\ y(T,I,F)\right)=\left(s(x_T,y_T),\ t(x_I,y_I),\ t(x_F,y_F)\right)$,
$s(x_T,y_T),\ t(x_I,y_I),\ t(x_F,y_F)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components and *s* and *t* are the fuzzy s-norm and fuzzy t-norm respectively. S_n have to satisfy, for any $x,\ y,\ z$ in the neutrosophic logic/set M of the universe of discourse U, the following axioms:

a) Boundary Conditions: $S_n(x,0)=x,\ S_n(x,1)=1.$

b) Commutativity: $S_n(x,\ y)=S_n(y,\ x).$

c) Monotonicity: If $x\le y,\ then\ S_n(x,\ z)\le S_n(y,\ z).$

d) Associativity: $S_n\left(S_n(x,\ y),\ z\right)=S_n\left(x,\ S_n(y,\ z)\right).$

From now we use the following notation for N-norm and N-conorm respectively $T_n-(x,y)$ and $S_n-(x,y)$.

We will use the following border $0(0,1,1)$ and $1(1,0,0)$.

Theorem 2.3.13: *For any* $S_n-(x,y)$ *and for all* $\alpha \geq 1$, by *using any fuzzy union s-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

1. $\underset{s}{S_n^\alpha}(x,y)=\left\langle \begin{array}{l} \sqrt[\alpha]{s(x_T{}^\alpha, y_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_I)^\alpha,(1-y_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_F)^\alpha,(1-y_F)^\alpha)} \end{array} \right\rangle$ and 2. $\underset{s}{T_n^\alpha}(x,y)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{s\left(\left(1-x_T{}^\alpha\right),\left(1-y_T{}^\alpha\right)\right)}, \\ \sqrt[\alpha]{s\left(x_I{}^\alpha, y_I{}^\alpha\right)}, \\ \sqrt[\alpha]{s\left(x_F{}^\alpha, y_F{}^\alpha\right)} \end{array} \right\rangle$,

where *s* any s-norm (fuzzy union).

Proof. 1.

Axiom 1.

$$\underset{s}{S_n^\alpha}(0,x)=\left\langle \begin{array}{l} \sqrt[\alpha]{s(0^\alpha, x_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-1)^\alpha,(1-x_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-1)^\alpha,(1-x_F)^\alpha)} \end{array} \right\rangle = x(x_T, x_I, x_F).$$

$$\underset{s}{S_n^\alpha}(1,x)=\left\langle \begin{array}{l} \sqrt[\alpha]{s(1^\alpha, x_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-0)^\alpha,(1-x_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-0)^\alpha,(1-x_F)^\alpha)} \end{array} \right\rangle = 1(1,0,0)$$

Axiom 2.

$$\underset{s}{S_n^\alpha}(x,y)=\left\langle \begin{array}{l} \sqrt[\alpha]{s(x_T{}^\alpha, y_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_I)^\alpha,(1-y_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_F)^\alpha,(1-y_F)^\alpha)} \end{array} \right\rangle$$
$$=\left\langle \begin{array}{l} \sqrt[\alpha]{s(y_T{}^\alpha, x_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-y_I)^\alpha,(1-x_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-y_F)^\alpha,(1-x_F)^\alpha)} \end{array} \right\rangle.$$
$$=\underset{s}{S_n^\alpha}(y,x)$$

Axiom 3. Let $x(x_1,\ x_2,\ x_3)\leq y(y_1,\ \mathrm{y}_2,\ \mathrm{y}_3)$ then $x_1\leq y_1,\ x_2\geq \mathrm{y}_2,\ x_3\geq \mathrm{y}_3$ and $s\left(x_1^\alpha, z_1^\alpha\right)\geq s\left(y_1^\alpha, z_1^\alpha\right)$ which implies

$$\sqrt[\alpha]{s\left(x_1^\alpha, z_1^\alpha\right)}\geq \sqrt[\alpha]{s\left(y_2^\alpha, z_2^\alpha\right)}. \qquad (1)$$

Also we have $(1-x_2)^\alpha \leq (1-y_2)^\alpha$ then

$s\left(\left(1-x_2\right)^\alpha,\left(1-z_2\right)^\alpha\right)\leq s\left(\left(1-y_2\right)^\alpha,\left(1-z_2\right)^\alpha\right)$, which implies that

$$1-\sqrt[\alpha]{s\left(\left(1-x_2\right)^\alpha,\left(1-z_2\right)^\alpha\right)}\geq 1-\sqrt[\alpha]{s\left(\left(1-y_2\right)^\alpha,\left(1-z_2\right)^\alpha\right)} \qquad (2)$$

And we have $(1-x_3)^\alpha \leq (1-y_3)^\alpha$ then

$s\left(\left(1-x_3\right)^\alpha,\left(1-z_3\right)^\alpha\right)\leq s\left(\left(1-y_3\right)^\alpha,\left(1-z_3\right)^\alpha\right)$, which implies that

$$1-\sqrt[\alpha]{s\left(\left(1-x_3\right)^{\alpha},\left(1-z_3\right)^{\alpha}\right)}\geq 1-\sqrt[\alpha]{s\left(\left(1-y_3\right)^{\alpha},\left(1-z_3\right)^{\alpha}\right)} \qquad (3)$$

From (1), (2) and (3) we have $\underset{s}{S_n^{\alpha}}(x,z)\geq \underset{s}{S_n^{\alpha}}(y,z)$.

Axiom 4.

$$\underset{s}{S_n^{\alpha}}\left(\underset{s}{S_n^{\alpha}}(x,y),z\right)=S_n^{\alpha}\left(\left\langle\begin{array}{l}\sqrt[\alpha]{s(x_T{}^{\alpha},y_T{}^{\alpha})},\\ 1-\sqrt[\alpha]{s((1-x_I)^{\alpha},(1-y_I)^{\alpha})},\\ 1-\sqrt[\alpha]{s((1-x_F)^{\alpha},(1-y_F)^{\alpha})}\end{array}\right\rangle,z\right)$$

$$=\left\langle\begin{array}{l}\sqrt[\alpha]{s\left(\sqrt[\alpha]{s\left(x_T{}^{\alpha},y_T{}^{\alpha}\right)}^{\alpha},z_T{}^{\alpha}\right)},\\ 1-\sqrt[\alpha]{s\left(\left(1-\left(1-\sqrt[\alpha]{s\left(\left(1-x_I\right)^{\alpha},\left(1-y_I\right)^{\alpha}\right)}\right)\right)^{\alpha},\left(1-z_I\right)^{\alpha}\right)},\\ 1-\sqrt[\alpha]{s\left(\left(1-\left(1-\sqrt[\alpha]{s\left(\left(1-x_F\right)^{\alpha},\left(1-y_F\right)^{\alpha}\right)}\right)\right)^{\alpha},\left(1-z_F\right)^{\alpha}\right)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\sqrt[\alpha]{s\left(s\left(x_T{}^{\alpha},y_T{}^{\alpha}\right),z_T{}^{\alpha}\right)},\\ 1-\sqrt[\alpha]{s\left(s\left(\left(1-x_I{}^{\alpha}\right),\left(1-y_I{}^{\alpha}\right)\right),\left(1-z_I{}^{\alpha}\right)\right)},\\ 1-\sqrt[\alpha]{s\left(s\left(\left(1-x_F{}^{\alpha}\right),\left(1-y_F{}^{\alpha}\right)\right),\left(1-z_F{}^{\alpha}\right)\right)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\sqrt[\alpha]{s\left(x_T{}^{\alpha},s\left(y_T{}^{\alpha},z_T{}^{\alpha}\right)\right)},\\ 1-\sqrt[\alpha]{s\left(\left(1-x_I{}^{\alpha}\right),s\left(\left(1-y_I{}^{\alpha}\right),\left(1-z_I{}^{\alpha}\right)\right)\right)},\\ 1-\sqrt[\alpha]{s\left(\left(1-x_F{}^{\alpha}\right),s\left(\left(1-y_F{}^{\alpha}\right),\left(1-z_F{}^{\alpha}\right)\right)\right)}\end{array}\right\rangle$$

$$=S_n^{\alpha}\left(x,\underset{s}{S_n^{\alpha}}(y,z)\right)$$

Therefore $\underset{s}{S_n^{\alpha}}(x,y)$ is an N-conorm.

2. The proof is similar to Proof 1. □

Theorem 2.3.14: *For any* $T_n-(x,y)$ *and for all* $\alpha\geq 1$, *by using any fuzzy intersection t-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

1. $\underset{t}{S_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{t\left(\left(1-x_T{}^{\alpha}\right),\left(1-y_T{}^{\alpha}\right)\right)},\\ \sqrt[\alpha]{t\left(x_I{}^{\alpha},y_I{}^{\alpha}\right)},\\ \sqrt[\alpha]{t\left(x_F{}^{\alpha},y_F{}^{\alpha}\right)}\end{array}\right\rangle$ and 2. $\underset{t}{T_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{t\left(x_T{}^{\alpha},y_T{}^{\alpha}\right)},\\ 1-\sqrt[\alpha]{t\left(\left(1-x_I{}^{\alpha}\right),\left(1-y_I{}^{\alpha}\right)\right)},\\ 1-\sqrt[\alpha]{t\left(\left(1-x_F{}^{\alpha}\right),\left(1-y_F{}^{\alpha}\right)\right)}\end{array}\right\rangle$,

where *t* any t-norm (fuzzy intersection).

Proof. The proof is similar to Proof of theorem 3.3.

By these theorems we can generate infinitely many *N-norms* and *N-conorms* by using two bijective functions with certain conditions.

Theorem 2.3.15: *Let* $f,\ g:[0,1]\to[0,1]$ *be bijective functions such that* $f(0)=0$, $f(1)=1$, $g(0)=1$ *and* $g(1)=0$. *For any* $S_n-(x,y)$ *and by using any fuzzy union s-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

1. $S^s_{n\atop f,g}(x,y)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f(x_T),f(y_T)\right)\right], \\ g^{-1}\left[s\left(g(x_I),g(y_I)\right)\right], \\ g^{-1}\left[s\left(g(x_F),g(y_F)\right)\right]\end{array}\right\rangle$, and 2. $T^s_{n\atop f,g}(x,y)=\left\langle\begin{array}{l} g^{-1}\left[s\left(g(x_T),g(y_T)\right)\right], \\ f^{-1}\left[s\left(f(x_I),f(y_I)\right)\right], \\ f^{-1}\left[s\left(f(x_F),f(y_F)\right)\right]\end{array}\right\rangle$.

Proof. 1.

Axiom 1.

$$S^s_{n\atop f,g}(x,0)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f(x_T),f(0)\right)\right], \\ g^{-1}\left[s\left(g(x_I),g(1)\right)\right], \\ g^{-1}\left[s\left(g(x_F),g(1)\right)\right]\end{array}\right\rangle=x.$$

$$S^s_{n\atop f,g}(x,1)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f(x_T),f(1)\right)\right], \\ g^{-1}\left[s\left(g(x_I),g(0)\right)\right], \\ g^{-1}\left[s\left(g(x_F),g(0)\right)\right]\end{array}\right\rangle=1(1,0,0).$$

Axiom 2.

$$S^s_{n\atop f,g}(x,y)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f(x_T),f(y_T)\right)\right], \\ g^{-1}\left[s\left(g(x_I),g(y_I)\right)\right], \\ g^{-1}\left[s\left(g(x_F),g(y_F)\right)\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left[s\left(f(y_T),f(x_T)\right)\right], \\ g^{-1}\left[s\left(g(y_I),g(x_I)\right)\right], \\ g^{-1}\left[s\left(g(y_F),g(x_F)\right)\right]\end{array}\right\rangle=S^s_{n\atop f,g}(y,x).$$

Axiom 3. Let $x\le y$. Since f is bijective on the interval $[0,1]$ and by Axiom s3 we have $s\left(f(x_T),f(z_T)\right)\le s\left(f(y_T),f(z_T)\right)$ then $f^{-1}\left[s\left(f(x_T),f(z_T)\right)\right]\le f^{-1}\left[s\left(f(y_T),f(z_T)\right)\right]$ (1)
Also since g is bijective on the interval $[0,1]$ and by Axiom t3 we have

$s\left(g\left(x_I\right),g\left(z_I\right)\right)\geq s\left(g\left(y_I\right),g\left(z_I\right)\right)$ then $g^{-1}\left[s\left(g\left(x_I\right),g\left(z_I\right)\right)\right]\geq g^{-1}\left[s\left(g\left(y_I\right),g\left(z_I\right)\right)\right]$ (2)

And

$s\left(g\left(x_F\right),g\left(z_F\right)\right)\geq s\left(g\left(y_F\right),g\left(z_F\right)\right)$ then $g^{-1}\left[s\left(g\left(x_F\right),g\left(z_F\right)\right)\right]\geq g^{-1}\left[s\left(g\left(y_F\right),g\left(z_F\right)\right)\right]$ (3)

From (1), (2) and (3) we have $\underset{f,g}{S_n^s}\left(x,z\right)\leq \underset{f,g}{S_n^s}\left(y,z\right)$

Axiom 4.

$$\underset{f,g}{S_n^s}\left(S_n^s\left(x,y\right),z\right)=\underset{f,g}{S_n^s}\left(\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(x_T\right),f\left(y_T\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_I\right),g\left(y_I\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_F\right),g\left(y_F\right)\right)\right]\end{array}\right\rangle,z\right)$$

$$=\left\langle\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left[s\left(f\left(x_T\right),f\left(y_T\right)\right)\right]\right),f\left(z_T\right)\right),\\ g^{-1}\left(g\left(g^{-1}\left[s\left(g\left(x_I\right),g\left(y_I\right)\right)\right]\right),g\left(z_I\right)\right),\\ g^{-1}\left(g\left(g^{-1}\left[s\left(g\left(x_F\right),g\left(y_F\right)\right)\right]\right),g\left(z_F\right)\right)\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left(s\left(f\left(x_T\right),f\left(y_T\right)\right),f\left(z_T\right)\right),\\ g^{-1}\left(s\left(g\left(x_I\right),g\left(y_I\right)\right),g\left(z_I\right)\right),\\ g^{-1}\left(s\left(g\left(x_F\right),g\left(y_F\right)\right),g\left(z_F\right)\right)\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left(f\left(x_T\right),s\left(f\left(y_T\right),f\left(z_T\right)\right)\right),\\ g^{-1}\left(g\left(x_I\right),s\left(g\left(y_I\right),g\left(z_I\right)\right)\right),\\ g^{-1}\left(g\left(x_F\right),s\left(g\left(y_F\right),g\left(z_F\right)\right)\right)\end{array}\right\rangle$$

$$=S_n^s\left(x,\underset{f,g}{S_n^s}\left(y,z\right)\right).$$

Therefore $\underset{f,g}{S_n^s}\left(x,y\right)$ is an $S_n-\left(x,y\right)$ □

2. The Proof is similar to Proof 1.

Corollary 2.3.16: *Let* $f(x)=\sin\frac{\pi}{2}x$ *and* $g(x)=\cos\frac{\pi}{2}x$ *then*

$$1.\quad \underset{\sin,\cos}{S_n^s}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1} s\left(\sin\frac{\pi}{2}x_T,\sin\frac{\pi}{2}y_T\right), \\ \frac{2}{\pi}\cos^{-1} s\left(\cos\frac{\pi}{2}x_I,\cos\frac{\pi}{2}y_I\right), \\ \frac{2}{\pi}\cos^{-1} s\left(\cos\frac{\pi}{2}x_F,\cos\frac{\pi}{2}y_F\right) \end{array}\right\rangle \text{ is an } S_n-(x,y)$$

$$2.\quad \underset{\sin,\cos}{T_n^s}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1} s\left(\cos\frac{\pi}{2}x_T,\cos\frac{\pi}{2}y_T\right), \\ \frac{2}{\pi}\sin^{-1} s\left(\sin\frac{\pi}{2}x_I,\sin\frac{\pi}{2}y_I\right), \\ \frac{2}{\pi}\sin^{-1} s\left(\sin\frac{\pi}{2}x_F,\sin\frac{\pi}{2}y_F\right) \end{array}\right\rangle \text{ is a } T_n-(x,y)$$

Theorem 2.3.17: *Let* f, $g:[0,1]\rightarrow[0,1]$ *be bijective functions such that* $f(0)=0$, $f(1)=1$, $g(0)=1$ *and* $g(1)=0$. *For any* $T_n-(x,y)$ *and by using any fuzzy intersection t-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

$$1.\quad \underset{f,g}{S_n^t}(x,y)=\left\langle \begin{array}{l} g^{-1}\left[t\left(g(x_T),g(y_T)\right)\right], \\ f^{-1}\left[t\left(f(x_I),f(y_I)\right)\right], \\ f^{-1}\left[t\left(f(x_F),f(y_F)\right)\right] \end{array}\right\rangle, \text{ and } \quad 2.\quad \underset{f,g}{T_n^t}(x,y)=\left\langle \begin{array}{l} f^{-1}\left[t\left(f(x_T),f(y_T)\right)\right], \\ g^{-1}\left[t\left(g(x_I),g(y_I)\right)\right], \\ g^{-1}\left[t\left(g(x_F),g(y_F)\right)\right] \end{array}\right\rangle.$$

Proof. The proof is similar to Proof of theorem 4.1. □

Corollary 2.3.18: *Let* $f(x)=\sin\frac{\pi}{2}x$ *and* $g(x)=\cos\frac{\pi}{2}x$ *then*

$$1.\quad \underset{\sin,\cos}{S_n^t}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1} t\left(\cos\frac{\pi}{2}x_T,\cos\frac{\pi}{2}y_T\right), \\ \frac{2}{\pi}\sin^{-1} t\left(\sin\frac{\pi}{2}x_I,\sin\frac{\pi}{2}y_I\right), \\ \frac{2}{\pi}\sin^{-1} t\left(\sin\frac{\pi}{2}x_F,\sin\frac{\pi}{2}y_F\right) \end{array}\right\rangle \text{ is an } S_n-(x,y)$$

$$2.\quad \underset{\sin,\cos}{T_n^t}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1} t\left(\sin\frac{\pi}{2}x_T,\sin\frac{\pi}{2}y_T\right), \\ \frac{2}{\pi}\cos^{-1} t\left(\cos\frac{\pi}{2}x_I,\cos\frac{\pi}{2}y_I\right), \\ \frac{2}{\pi}\cos^{-1} t\left(\cos\frac{\pi}{2}x_F,\cos\frac{\pi}{2}y_F\right) \end{array}\right\rangle \text{ is a } T_n-(x,y).$$

We now generate some new $S_n-(x,y)$ and $T_n-(x,y)$ from existing $S_n-(x,y)$ and $T_n-(x,y)$ using the Generating Theorems and the Bijective Generating Theorems.

Example 2.3.19: BOUNDED SUM GENERATING CLASSES
New $S_n-(x,y)$ and $T_n-(x,y)$ from the bounded sum *s*-norm.

$$\underset{s}{S_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} \sqrt[\alpha]{\min(x_T{}^{\alpha}+y_T{}^{\alpha},1)}, \\ 1-\sqrt[\alpha]{\min((1-x_I)^{\alpha}+(1-y_I)^{\alpha},1)}, \\ 1-\sqrt[\alpha]{\min((1-x_F)^{\alpha}+(1-y_F)^{\alpha},1)} \end{array} \right\rangle \text{ and } \underset{bs}{T_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{\min\left((1-x_T)^{\alpha}+(1-y_T)^{\alpha},1\right)}, \\ \sqrt[\alpha]{\min\left(x_I{}^{\alpha}+y_I{}^{\alpha},1\right)}, \\ \sqrt[\alpha]{\min\left(x_F{}^{\alpha}+y_F{}^{\alpha},1\right)} \end{array} \right\rangle.$$

$$\underset{\sin,\cos}{S_n^{bs}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right),1\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right),1\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right),1\right)\right) \end{array} \right\rangle \underset{\sin,\cos}{T_n^{bs}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right),1\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right),1\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right),1\right)\right) \end{array} \right\rangle.$$

Example 2.3.20: ALGEBRAIC SUM GENERATING CLASSES
New $S_n-(x,y)$ and $T_n-(x,y)$ from the algebraic sum *s*-norm.

$$\underset{as}{S_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} \sqrt[\alpha]{x_T{}^{\alpha}+y_T{}^{\alpha}-x_T{}^{\alpha}.y_T{}^{\alpha}}, \\ 1-\sqrt[\alpha]{(1-x_I)^{\alpha}+(1-y_I)^{\alpha}-(1-x_I)^{\alpha}.(1-y_I)^{\alpha}}, \\ 1-\sqrt[\alpha]{(1-x_F)^{\alpha}+(1-y_F)^{\alpha}-(1-x_F)^{\alpha}.(1-y_F)^{\alpha}} \end{array} \right\rangle, \quad \underset{as}{T_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{(1-x_T)^{\alpha}+(1-y_T)^{\alpha}-(1-x_T)^{\alpha}(1-y_T)^{\alpha}}, \\ \sqrt[\alpha]{x_I{}^{\alpha}+y_I{}^{\alpha}-x_I{}^{\alpha}.y_I{}^{\alpha}}, \\ \sqrt[\alpha]{x_F{}^{\alpha}+y_F{}^{\alpha}-x_F{}^{\alpha}.y_F{}^{\alpha}} \end{array} \right\rangle.$$

$$\underset{\sin,\cos}{S_n^{as}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)\right) \end{array} \right\rangle, \quad \underset{\sin,\cos}{T_n^{as}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)-\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)-\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)\right) \end{array} \right\rangle.$$

Example 2.3.21: EINSTEIN SUM GENERATING CLASSES
New $S_n-(x,y)$ and $T_n-(x,y)$ from the Einstein sum *s*-norm.

$$\underset{es}{S_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} \sqrt[\alpha]{\dfrac{x_T{}^{\alpha}+y_T{}^{\alpha}}{1+x_T{}^{\alpha}y_T{}^{\alpha}}}, \\ 1-\sqrt[\alpha]{\dfrac{(1-x_I)^{\alpha}+(1-y_I)^{\alpha}}{1+(1-x_I)^{\alpha}(1-y_I)^{\alpha}}}, \\ 1-\sqrt[\alpha]{\dfrac{(1-x_F)^{\alpha}+(1-y_F)^{\alpha}}{1+(1-x_F)^{\alpha}(1-y_F)^{\alpha}}} \end{array} \right\rangle, \quad \underset{es}{T_n^{\alpha}}(x,y)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{\dfrac{(1-x_T)^{\alpha}+(1-y_T)^{\alpha}}{1+(1-x_T)^{\alpha}(1-y_T)^{\alpha}}}, \\ \sqrt[\alpha]{\dfrac{x_I{}^{\alpha}+y_I{}^{\alpha}}{1+x_I{}^{\alpha}y_I{}^{\alpha}}}, \\ \sqrt[\alpha]{\dfrac{x_F{}^{\alpha}+y_F{}^{\alpha}}{1+x_F{}^{\alpha}y_F{}^{\alpha}}} \end{array} \right\rangle.$$

$$\underset{\sin,\cos}{S_n^{bs}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)}{1+\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)}\right),\\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)}{1+\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)}\right),\\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)}{1+\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)}\right)\end{array}\right\rangle,\quad \underset{\sin,\cos}{T_n^{bs}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)}{1+\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)}\right),\\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)}{1+\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)}\right),\\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)}{1+\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)}\right)\end{array}\right\rangle.$$

Example 2.3.22: BOUNDED PRODUCT GENERATING CLASSES
New $S_n-(x,y)$ and $T_n-(x,y)$ from the bounded product *t*-norm.

$$\underset{bp}{T_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{\max\left(x_T^{\ \alpha}+y_T^{\ \alpha}-1,0\right)},\\ 1-\sqrt[\alpha]{\max\left(\left(1-x_I\right)^{\alpha}+\left(1-y_I^{\ \alpha}\right)-1,0\right)},\\ 1-\sqrt[\alpha]{\max\left(\left(1-x_F\right)^{\alpha}+\left(1-y_F\right)^{\alpha}-1,0\right)}\end{array}\right\rangle,\quad \underset{bp}{S_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{\max\left(\left(1-x_T\right)^{\alpha}+\left(1-y_T\right)^{\alpha}-1,0\right)},\\ \sqrt[\alpha]{\max\left(x_I^{\ \alpha}+y_I^{\ \alpha}-1,0\right)},\\ \sqrt[\alpha]{\max\left(x_F^{\ \alpha}+y_F^{\ \alpha}-1,0\right)}\end{array}\right\rangle$$

$$\underset{\sin,\cos}{T_n^{bp}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-1,0\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-1,0\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-1,0\right)\right)\end{array}\right\rangle,\quad \underset{\sin,\cos}{S_n^{t}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-1,0\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_I+\sin\frac{\pi}{2}y_I\right)-1,0\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_F+\sin\frac{\pi}{2}y_F\right)-1,0\right)\right)\end{array}\right\rangle.$$

Example 2.3.23:EINSTEIN PRODUCT GENERATING CLASSES
New $S_n-(x,y)$ and $T_n-(x,y)$ from the Einstein product *t*-norm.

$$\underset{ep}{T_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{\frac{x_T^{\ \alpha}y_T^{\ \alpha}}{2-\left(x_T^{\ \alpha}+y_T^{\ \alpha}-x_T^{\ \alpha}y_T^{\ \alpha}\right)}},\\ 1-\sqrt[\alpha]{\frac{\left(1-x_I\right)^{\alpha}\left(1-y_I\right)^{\alpha}}{2-\left(\left(1-x_I\right)^{\alpha}+\left(1-y_I\right)^{\alpha}-\left(1-x_I\right)^{\alpha}\left(1-y_I\right)^{\alpha}\right)}},\\ 1-\sqrt[\alpha]{\frac{\left(1-x_F\right)^{\alpha}\left(1-y_F\right)^{\alpha}}{2-\left(\left(1-x_F\right)^{\alpha}+\left(1-y_F\right)^{\alpha}-\left(1-x_F\right)^{\alpha}\left(1-y_F\right)^{\alpha}\right)}}\end{array}\right\rangle,\quad \underset{ep}{S_n^{\alpha}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{\frac{\left(1-x_T\right)^{\alpha}\left(1-y_T\right)^{\alpha}}{2-\left(\left(1-x_T\right)^{\alpha}+\left(1-y_T\right)^{\alpha}-\left(1-x_T\right)^{\alpha}\left(1-y_T\right)^{\alpha}\right)}},\\ \sqrt[\alpha]{\frac{x_I^{\ \alpha}y_I^{\ \alpha}}{2-\left(x_I^{\ \alpha}+y_I^{\ \alpha}-x_I^{\ \alpha}y_I^{\ \alpha}\right)}},\\ \sqrt[\alpha]{\frac{x_F^{\ \alpha}y_F^{\ \alpha}}{2-\left(x_F^{\ \alpha}+y_F^{\ \alpha}-x_F^{\ \alpha}y_F^{\ \alpha}\right)}}\end{array}\right\rangle$$

$$\underset{\sin,\cos}{T_n^{ep}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)}{2-\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)\right)}\right), \\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)}{2-\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)\right)}\right), \\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)}{2-\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)\right)}\right) \end{array}\right\rangle, \quad \underset{\sin,\cos}{s_n^{ep}}(x,y)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)}{2-\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)\right)}\right), \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)}{2-\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)-\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)\right)}\right), \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)}{2-\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)-\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)\right)}\right) \end{array}\right\rangle$$

Note that for the *s*-norms max and drastic sum and *t*-norms min, algebraic product and drastic product we get the same norms.

3. Crisp Neutrosophic Set, Neutrosophic Crisp Neighborhood Systems, Neutrosophic Crisp Local Functions

In this section, crisp neutrosophic sets, neutrosophic crisp neighborhood systems and neutrosophic crisp local functions have been discussed.

3.1. Crisp Neutrosophic Set

In this subsection, the authors presented crisp neutrosophic sets and their related properties.

Definition 3.1.1 : Let X be a non-empty fixed set. A neutrosophic crisp set (NCS for short) A is an object having the form $A=\langle A_1, A_2, A_3\rangle$ where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

Definition 3.1.2 : Let X be a nonempty set and $p \in X$ Then the neutrosophic crisp point p_N defined by $p_N = \left\langle \{p\}, \phi, \{p\}^c \right\rangle$ is called a neutrosophic crisp point (NCP for short) in X, where NCP is a triple ({only one element in X}, the empty set,{the complement of the same element in X}).

Definition 3.1.3 : Let X be a non-empty set, and $p \in X$ a fixed element in X. Then the neutrosophic crisp set $p_{N_N} = \left\langle \phi, \{p\}, \{p\}^c \right\rangle$ is called "vanishing neutrosophic crisp point" (VNCP for short) in X, where VNCP is a triple (the empty set,{only one element in X},{the complement of the same element in X}).

Definition 3.1.4 : Let $p_N = \left\langle \{p\}, \phi, \{p\}^c \right\rangle$ be a NCP in X and $A=\left\langle A_1, A_2, A_3\right\rangle$ a neutrosophic crisp set in X.

(a) p_N is said to be contained in A ($p_N \in A$ for short) iff $p \in A_1$.

(b) Let p_{N_N} be a VNCP in X, and $A=\left\langle A_1, A_2, A_3\right\rangle$ a neutrosophic crisp set in X. Then p_{N_N} is said to be contained in A ($p_{N_N} \in A$ for short) iff $p \notin A_3$.

Definition 3.1.5 : Let X be non-empty set, and L a non–empty family of NCSs. We call L a neutrosophic crisp ideal (NCL for short) on X if

i. $A \in L$ and $B \subseteq A \Rightarrow B \in L$ [heredity],

ii. $A \in L$ and $B \in \mathrm{L} \Rightarrow A \vee B \in \mathrm{L}$ [Finite additivity].

A neutrosophic crisp ideal L is called a σ - neutrosophic crisp ideal if $\{M_j\}_{j \in N} \leq L$, implies $\underset{j \in J}{\cup} M_{j \in L}$ (countable additivity). The smallest and largest neutrosophic crisp ideals on a non-empty set X are $\{\phi_N\}$ and the NSs on X. Also, $NC\mathrm{L}_f$, NCL_c are denoting the neutrosophic crisp ideals (NCL for short) of neutrosophic subsets having finite and countable support of X respectively. Moreover, if A is a nonempty NS in X, then $\{B \in NCS : B \subseteq A\}$ is an NCL on X. This is called the principal NCL of all NCSs, denoted by NCL$\langle A \rangle$.

Proposition 3.1.6 : Let $\{L_j : j \in J\}$ be any non - empty family of neutrosophic crisp ideals on a set X. Then $\bigcap_{j \in J} L_j$ and $\bigcup_{j \in J} L_j$ are neutrosophic crisp ideals on X, where $\underset{j \in J}{\cap} L_j = \left\langle \underset{j \in J}{\cap} A_{j1}, \underset{j \in J}{\cap} A_{j2}, \underset{j \in J}{\cup} A_{j3} \right\rangle$ or $\underset{j \in J}{\cap} L_j = \left\langle \underset{j \in J}{\cap} A_{j1}, \underset{j \in J}{\cup} A_{j2}, \underset{j \in J}{\cup} A_{j3} \right\rangle$ and $\underset{j \in J}{\cup} L_j = \left\langle \underset{j \in J}{\cup} A_{j1}, \underset{j \in J}{\cup} A_{j2}, \underset{j \in J}{\cap} A_{j3} \right\rangle$ or $\underset{j \in J}{\cup} L_j = \left\langle \underset{j \in J}{\cup} A_{j1}, \underset{j \in J}{\cap} A_{j2}, \underset{j \in J}{\cap} A_{j3} \right\rangle$.

Remark 3.1.7 : The neutrosophic crisp ideal defined by the single neutrosophic set ϕ_N is the smallest element of the ordered set of all neutrosophic crisp ideals on X.

Proposition 3.1.8 : A neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$ in the neutrosophic crisp ideal L on X is a base of L iff every member of L is contained in A.

3.2. Neutrosophic Crisp Neighborhoods System

Definition 3.2.1: Let $A = \langle A_1, A_2, A_3 \rangle$, be a neutrosophic crisp set on a set X, then $p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle$, $p_1 \neq p_2 \neq p_3 \in X$ is called a neutrosophic crisp point

An NCP $p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle$, is said to be belong to a neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$, of X, denoted by $p \in A$, if may be defined by two types

i) **Type 1:** $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$

ii) **Type 2:** $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$

Theorem 3.2.2: Let $A = \langle\langle A_1, A_2, A_3 \rangle\rangle$, and $B = \langle\langle B_1, B_2, B_3 \rangle\rangle$, be neutrosophic crisp subsets of X. Then $A \subseteq B$ iff $p \in A$ implies $p \in B$ for any neutrosophic crisp point p in X.

Proof: Let $A \subseteq B$ and $p \in A$. Then two types

Type 1: $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$ or

Type 2: $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$. Thus $p \in B$. Conversely, take any x in X. Let $p_1 \in A_1$ and $p_2 \in A_2$ and $p_3 \in A_3$. Then p is a neutrosophic crisp point in X. and $p \in A$. By the hypothesis $p \in B$. Thus $p_1 \in B_1$, or Type 1: $\{p_1\} \subseteq B_1, \{p_2\} \subseteq B_2$ and $\{p_3\} \subseteq B_3$ or

Type 2: $\{p_1\} \subseteq B_1, \{p_2\} \supseteq B_2$ and $\{p_3\} \subseteq B_3$. Hence. $A \subseteq B$.

Theorem 3.2.3: Let $A = \langle A_1, A_2, A_3 \rangle$, be a neutrosophic crisp subset of X. Then $A = \cup\{p : p \in A\}$..

Proof: Since $\cup\{p : p \in A\}$. may be two types

Type 1: $\langle \cup\{p_1 : p_1 \in A_1\}, \cup\{p_2 : p_2 \in A_2\}, \cap\{p_3 : p_3 \in A_3\}\rangle$ or

Type 2: $\langle \cup\{p_1 : p_1 \in A_1\}, \cap\{p_2 : p_2 \in A_2\}, \cap\{p_3 : p_3 \in A_3\}\rangle$. Hence $A = \langle A_1, A_2, A_3 \rangle$

Proposition 3.2.4 : Let $\{A_j : j \in J\}$ is a family of NCSs in X. Then

(a_1) $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in \bigcap_{j \in J} A_j$ iff $p \in A_j$ for each $j \in J$.

(a_2) $p \in \bigcup_{j \in J} A_j$ iff $\exists j \in J$ such that $p \in A_j$.

Proposition 3.2.5 : Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ be two neutrosophic crisp sets in X. Then

a) $A \subseteq B$ iff for each p we have $p \in A \Leftrightarrow p \in B$ and for each p we have $p \in A \Rightarrow p \in B$.

b) $A = B$ iff for each p we have $p \in A \Rightarrow p \in B$ and for each p we have $p \in A \Leftrightarrow p \in B$.

Proposition3.2.6 : Let $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then

$A = \cup < \{p_1 : p_1 \in A_1\}, \{p_2 : p_2 \in A_2\}, \{p_3 : p_3 \in A_3\}$.

Definition 3.2.7 : Let $f : X \to Y$ be a function and p be a nutrosophic crisp point in X. Then the image of p under f, denoted by $f(p)$, is defined by $f(p) = \langle\{q_1\},\{q_2\},\{q_3\}\rangle$,where $q_1 = f(p_1), q_2 = f(p_2)$.

and $q_3 = f(p_3)$.

It is easy to see that $f(p)$ is indeed a NCP in Y, namely $f(p) = q$, where $q = f(p)$, and it is exactly the same meaning of the image of a NCP under the function f.

3.3. Neutrosophic Crisp Local functions

Here, the author discussed neutrosophic crisp local functions.

Definition 3.3.1: Let p be a neutrosophic crisp point of a neutrosophic crisp topological space (X, τ). A neutrosophic crisp neighbourhood (NCNBD for short) of a neutrosophic crisp point p if there is a neutrosophic crisp open set(NCOS for short) B in X such that $p \in B \subseteq A$.

Theorem 3.3.2: Let (X,τ) be a neutrosophic crisp topological space (NCTS for short) of X. Then the neutrosophic crisp set A of X is NCOS iff A is a NCNBD of p for every neutrosophic crisp set $p \in A$.

Proof: Let A be NCOS of X. Clearly A is a NCBD of any $p \in A$. Conversely, let $p \in A$. Since A is a NCBD of p, there is a NCOS B in X such that $p \in B \subseteq A$. So we have $A = \cup\{p : p \in A\} \subseteq \cup\{B : p \in A\} \subseteq A$ and hence $A = \cup\{B : p \in A\}$. Since each B is NCOS.

Definition 3.3.3: Let (X,τ) be a neutrosophic crisp topological spaces (NCTS for short) and L be neutrosophic crisp ideal (NCL, for short) on X. Let A be any NCS of X. Then the neutrosophic crisp local function $NCA^{*}(L,\tau)$ of A is the union of all neutrosophic crisp points(NCP, for short) $P = \langle\{p_1\},\{p_2\},\{p_3\}\rangle$, such that if $U \in N((p))$ and $NA^{*}(L,\tau) = \cup\{p \in X : A \wedge U \notin L$ for every U nbd of N(P)$\}$, $NCA^{*}(L,\tau)$ is called a neutrosophic crisp local function of A with respect to τ and L which it will be denoted by $NCA^{*}(L,\tau)$, or simply $NCA^{*}(\mathrm{L})$.

Example 3.3.4: One may easily verify that.

If L=$\{\phi_N\}$, then $NCA^{*}(L,\tau) = NCcl(A)$, for any neutrosophic crisp set $A \in NCSs$ on X.

If $L = \{\text{all NCSs on X}\}$ then $NCA^{*}(L,\tau) = \phi_N$, for any $A \in NCSs$ on X .

Theorem 3.3.5: Let (X,τ) be a NCTS and L_1, L_2 be two topological neutrosophic crisp ideals on X. Then for any neutrosophic crisp sets A, B of X. then the following statements are verified

i) $A \subseteq B \Rightarrow NCA^{*}(L,\tau) \subseteq NCB^{*}(L,\tau)$,

ii) $L_1 \subseteq L_2 \Rightarrow NCA^{*}(L_2,\tau) \subseteq NCA^{*}(L_1,\tau)$.

iii) $NCA^{*} = NCcl(A^{*}) \subseteq NCcl(A)$.

iv) $NCA^{**} \subseteq NCA^{*}$.

v) $NC(A \cup B)^{*} = NCA^{*} \cup NCB^{*}$.,

vi) $NC(A \cap B)^{*}(L) \subseteq NCA^{*}(L) \cap NCB^{*}(L)$.

vii) $\ell \in L \Rightarrow NC(A \cup \ell)^{*} = NCA^{*}$.

viii) $NCA^{*}(L,\tau)$ is neutrosophic crisp closed set .

Proof: Since $A \subseteq B$, let $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCA^{*}(L_1)$ then $A \cap U \notin L$ for every $U \in N(p)$. By hypothesis we get $B \cap U \notin L$, then $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NB^{*}(L_1)$.

i)Clearly. $L_1 \subseteq L_2$ implies $NCA^{*}(L_2,\tau) \subseteq NCA^{*}(L_1,\tau)$ as there may be other IFSs which belong to L_2 so that for GIFP $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCA^{*}(L_1)$ but P may not be contained in $NCA^{*}(L_2)$.

ii) Since $\{\phi_N\} \subseteq L$ for any NCL on X, therefore by (ii) and Example 3.1, $NCA^{*}(L) \subseteq NCA^{*}(\{O_N\}) = NCcl(A)$ for any NCS A on X. Suppose $P_1 = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCcl(A^{*}(L_1))$. So for every $U \in NC(P_1)$, $NC(A^{*}) \cap U \neq \phi_N$, there exists $P_2 = \langle\{q_1\},\{q_2\},\{q_3\}\rangle \in NCA^{*}(L_1) \cap U$ such that for every V NCNBD of $P_2 \in N(P_2)$, $A \cap U \notin L$. Since $U \wedge V \in N(p_2)$ then $A \cap (U \cap V) \notin L$ which leads to $A \wedge U \notin L$, for every $U \in N(P_1)$ therefore $P_1 \in NC(A^{*}(L\ ))$ and so $NCcl(NA^{*}) \subseteq NCA^{*}$ While, the other inclusion follows directly. Hence $NCA^{*} = NCcl(NCA^{*})$ .But the inequality $NCA^{*} \subseteq Ncl(NCA^{*})$.

iii) The inclusion $NCA^{*} \cup NCB^{*} \subseteq NC(A \cup B)^{*}$ follows directly by (i). To show the other implication, let $p \in NC(A \cup B)^{*}$ then for every $U \in NC(p)$, $(A \cup B) \cap U \notin L, i.e, (A \cap U) \cup (B \cap U) \notin L.$ then, we have two cases $A \cap U \notin L$ and $B \cap U \in L$ or the converse, this means that exist $U_1, U_2 \in N(P)$ such that $A \cap U_1 \notin L$, $B \cap U_1 \notin L$, $A \cap U_2 \notin L$ and $B \cap U_2 \notin L$. Then $A \cap (U_1 \cap U_2) \in L$ and $B \cap (U_1 \cap U_2) \in L$ this gives $(A \cup B) \cap (U_1 \cap U_2) \in L$, $U_1 \cap U_2 \in N(C(P))$ which contradicts the hypothesis. Hence the equality holds in various cases.

vi) By (iii), we have $NCA^{**} = NCcl(NCA^{*})^{*} \subseteq NCcl(NCA^{*}) = NCA^{*}$

Let (X, τ) be a NCTS and L be NCL on X . Let us define the neutrosophic crisp closure operator $NCcl^{*}(A) = A \cup NC(A^{*})$ for any NCS A of X. Clearly, let $NCcl^{*}(A)$ is a neutrosophic crisp operator. Let $NC\tau^{*}(L)$ be NCT generated by $NCcl^{*}$

.i.e $NC\tau^{*}(L) = \{A : NCcl^{*}(A^{c}) = A^{c}\}$ now $L = \{\phi_N\} \Rightarrow NCcl^{*}(A) = A \cup NCA^{*} = A \cup NCcl(A)$ for every neutrosophic crisp set A. So, $N\tau^{*}(\{\phi_N\}) = \tau$. Again $L = \{all\ \text{NCSs on X}\} \Longrightarrow NCcl^{*}(A) = A,$ because $NCA^{*} = \phi_N$, for every neutrosophic crisp set A so $NC\tau^{*}(L)$ is the neutrosophic crisp discrete topology on X. So we can conclude by Theorem 4.1.(ii). $NC\tau^{*}(\{\phi_N\}) = NC\tau^{*}(L)$ i.e. $NC\tau \subseteq NC\tau^{*}$, for any neutrosophic ideal L_1 on X. In particular, we have for two topological neutrosophic ideals L_1, and L_2 on X, $L_1 \subseteq L_2 \Rightarrow NC\tau^{*}(L_1) \subseteq NC\tau^{*}(L_2)$.

Theorem 3.3.6: Let τ_1, τ_2 be two neutrosophic crisp topologies on X. Then for any topological neutrosophic crisp ideal L on X, $\tau_1 \leq \tau_2$ implies $NA^{*}(L, \tau_2) \subseteq NA^{*}(L, \tau_1)$, for every A $\in L$ then $NC\tau^{*}{}_1 \subseteq NC\tau^{*}{}_2$

Proof: Clear. A basis $NC\beta(L, \tau)$ for $NC\tau^{*}(L)$ can be described as follows:

$NC\beta(L, \tau) = \{A - B : A \in \tau, B \in L\}$.

Then we have the following theorem.

Theorem 3.3.7: $NC\beta(L, \tau) = \{A - B : A \in \tau, B \in L\}$ Forms a basis for the generated NT of the NCT (X, τ) with topological neutrosophic crisp ideal L on X.

The relationship between $NC\tau$ and $NC\tau^{*}(L)$ established throughout the following result which have an immediately proof .

Theorem 3.3.8: Let τ_1, τ_2 be two neutrosophic crisp topologies on X. Then for any topological neutrosophic ideal L on X, $\tau_1 \subseteq \tau_2$ implies $NC\tau^{*}{}_1 \subseteq NC\tau^{*}{}_2$.

Theorem 3.3.9: Let (X, τ) be a NCTS and L_1, L_2 be two neutrosophic crisp ideals on X . Then for any neutrosophic crisp set A in X, we have

i) $NCA^{*}(L_1 \cup L_2, \tau) = NCA^{*}(L_1, NC\tau^{*}(L_1)) \wedge NCA^{*}(L_2, NC\tau^{*}(L_2))$ ii) $NC\tau^{*}(L_1 \cup L_2) = (NC\tau^{*}(L_1))^{*}(L_2) \wedge (NC\tau^{*}(L_2))^{*}(L_1)$

Proof : Let $p \notin (L_1 \cup L_2, \tau)$, this means that there exists $U \in NC(P)$ such that $A \cap U_p \in (L_1 \cup L_2)$ i.e. There exists $\ell_1 \in L_1$ and $\ell_2 \in L_2$ such that $A \cap U \in (\ell_1 \vee \ell_2)$ because of the heredity of L_1, and assuming $\ell_1 \wedge \ell_2 = O_N$ .Thus we have $(A \cap U) - \ell_1 = \ell_2$ and $(A \cap U_p) - \ell_2 = \ell_1$ therefore $(U - \ell_1) \cap A = \ell_2 \in L_2$ and $(U - \ell_2) \cap A = \ell_1 \in L_1$. Hence $p \notin NCA^{*}(L_2, NC\tau^{*}(L_1))$ or $P \notin NCA^{*}(L_1, NC\tau^{*}(L_2))$ because p must belong to either ℓ_1 or ℓ_2 but not to both. This gives $NCA^{*}(L_1 \cup L_2, \tau) \geq NCA^{*}(L_1, NC\tau^{*}(L_1)) \cap NCA^{*}(L_2, NC\tau^{*}(L_2))$ .To show the second inclusion, let us assume $P \notin NCA^{*}(L_1, NC\tau^{*}(L_2))$. This implies that there exist $U \in N(P)$ and $\ell_2 \in L_2$

such that $(U_p - \ell_2) \cap A \in L_1$. By the heredity of L_2, if we assume that $\ell_2 \subseteq A$ and define $\ell_1 = (U - \ell_2) \cap A$. Then we have $A \cap U \in (\ell_1 \cup \ell_2) \in L_1 \cup L_2$. Thus, $NCA^*(L_1 \cup L_2, \tau) \subseteq NCA^*(L_1, NC\tau^*(L_1)) \cap NCA^*(L_2, NC\tau^*(L_2))$ and similarly, we can get $NCA^*(L_1 \cup L_2, \tau) \subseteq NCA^*(L_2, \tau^*(L_1))$. This gives the other inclusion, which complete the proof.

Corollary 3.3.10: Let (X, τ) be a NCTS with topological neutrosophic crisp ideal L on X. Then

i) $NCA^*(L, \tau) = NCA^*(L, \tau^*)$ and $NC\tau^*(L) = NC(NC\tau^*(L))^*(L)$

ii) $NC\tau^*(L_1 \cup L_2) = (NC\tau^*(L_1)) \cup (NC\tau^*(L_2))$.

4 Interval Valued Neutrosophic Sets

In this section, we studied interval valued neutrosophic sets and their properties.

Definition 4.1: Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$.

Definition 4.2: For two IVNS, A_{INS} ={<x, [$T_A^L(x), T_A^U(x)$] , [$I_A^L(x), I_A^U(x)$],[$F_A^L(x), F_A^U(x)$]> | x $\in$ X }
And B_{INS} = {<x, ={<x, [$T_B^L(x), T_B^U(x)$] , [$I_B^L(x), I_B^U(x)$],[$F_B^L(x), F_B^U(x)$]> | x $\in$ X }>|x $\in$ X } the two relations are defined as follows:

(1) $A_{INS} \subseteq B_{INS}$ if and only if

$T_A^L(x) \leq T_B^L(x)$, $T_A^U(x) \leq T_B^U(x)$, $I_A^L(x) \geq I_B^L(x)$, $F_A^L(x) \geq F_B(x)$, $F_A^U(x) \geq F_B^U(x)$.

(2) $A_{INS} = B_{INS}$ if and only if ,

$T_A^L(x) = T_B^L(x)$, $T_A^U(x) = T_B^U(x)$, $I_A^L(x) = I_B^L(x)$, $I_A^U(x) = I_B^U(x)$, $F_A^L(x) = F_B^L(x)$, $F_A^U(x) = F_B^U(x)$

for any x $\in$ X.

Definition 4.3: Assume that there are two interval neutrosophic sets A and B in X ={ $x_1, x_2, \ldots, x_n$ } Based on the extension measure for fuzzy sets, a cosine similarity measure between interval valued neutrosophic sets A and B is proposed as follows:

$$C_N(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\Delta T_A(x_i)\Delta T_B(x_i) + \Delta I_A(x_i)\Delta I_B(x_i) + \Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2+(\Delta I_A(x_i))^2+(\Delta F_A(x_i))^2}\sqrt{(\Delta T_B(x_i))^2+(\Delta I_B(x_i))^2+(\Delta F_B(x_i))^2}}.$$

Where

$$\Delta T_A(x_i) = T_A^L(x_i) + T_A^U(x_i) \;,\; \Delta T_B(x_i) = T_B^L(x_i) + T_B^U(x_i)$$
$$\Delta I_A(x_i) = I_A^L(x_i) + I_A^U(x_i) \;,\; \Delta I_B(x_i) = I_B^L(x_i) + I_B^U(x_i)$$

$$\Delta F_A(x_i) = F_A^L(x_i) + F_A^U(x_i),\; \Delta F_B(x_i) = F_B^U(x_i) + F_B^U(x_i).$$

And

$$T_A^L(x_i) + T_A^U(x_i) \;,\; \Delta T_B(x_i) = T_B^L(x_i) + T_B^U(x_i)$$

$$\Delta I_A(x_i) = I_A^L(x_i) + I_A^U(x_i) \;,\; \Delta I_B(x_i) = I_B^L(x_i) + I_B^U(x_i)$$

$$\Delta F_A(x_i) = F_A^L(x_i) + F_A^U(x_i)\ ,\ \ \Delta F_B(x_i) = F_B^U(x_i) + F_B^U(x_i)\ .$$

Proposition 4.4: Let A and B be interval valued neutrosophic sets then

- **i.** **0** $\le C_N(A,B) \le$ **1**
- ii. $C_N(A,B) = C_N(B,A)$
- iii. $C_N(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$, $I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and $F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n.

Proof : (i) it is obvious that the proposition is true according to the cosine valued

(ii) it is obvious that the proposition is true.

(iii) When A =B, there are $T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$, $I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and $F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n, So there is $C_N(A,B) = 1$

If we consider the weights of each element x_i, a weighted cosine similarity measure between IVNSs A and B is given as follows:

$$C_{WN}(A,B) = \frac{1}{n}\sum_{i=1}^{n} w_i \frac{\Delta T_A(x_i)\Delta T_B(x_i) + \Delta I_A(x_i)\Delta I_B(x_i) + \Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2 + (\Delta I_A(x_i))^2 + (\Delta F_A(x_i))^2}\ \sqrt{(\Delta T_B(x_i))^2 + (\Delta I_B(x_i))^2 + (\Delta F_B(x_i))^2}}.$$

Where $w_i \in [0.1]$,i =1,2,…,n ,and $\sum_{i=1}^{n} w_i = 1$.

If we take $w_i = \frac{1}{n}$, i =1,2,…,n , then there is $C_{WN}(A,B) = C_N(A,B)$.

Definition 4.5: The weighted cosine similarity measure between two IVNSs A and B also satisfies the following properties:

- **i.** **0** $\le C_{WN}(A,B) \le$ **1**
- ii. $C_{WN}(A,B) = C_{WN}(B,A)$
- iii. $C_{WN}(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$. $I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and $F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n

Proposition 4.6: Let the distance measure of the angle as **d(A,B)= arcos** $C_N(A,B)$ **,**then it satisfies the following properties.

- **i.** d(A, B) $\ge$ 0, if **0** $\le C_N(A,B) \le$ **1**
- **ii.** d(A, B) = **arcos(1) = 0, if** $C_N(A,B)$ **= 1**
- **iii.** d(A, B) = d(B, A) if $C_N(A,B) = C_N(B,A)$
- **iv.** d(A, C) $\le$ d(A, B) + d(B, C) if A $\subseteq$ B $\subseteq$ C for any interval valued neutrosophic sets C.

Proof : obviously, d(A,B) satisfies the (i) – (iii). In the following , d(A,B) will be proved to satisfy the (iv).

For any C = { x_i }, A $\subseteq$ B $\subseteq$ C since Eq (7) is the sum of terms. Let us consider the distance measure of the angle between vectors:

d_i (A(x_i), B(x_i)) = **arcos(** C_N (A(x_i), B(x_i)**),**

d_i (B(x_i), C(x_i)) = **arcos(** C_N (B(x_i), C(x_i)**), and**

d_i (A(x_i), C(x_i)) = **arcos(** C_N (A(x_i), C(x_i)**),,** for i=1, 2, .., n, where

$$C_N(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_A(x_i)\Delta T_B(x_i)+\Delta I_A(x_i)\Delta I_B(x_i)+\Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2+(\Delta I_A(x_i))^2+(\Delta F_A(x_i))^2}\sqrt{(\Delta T_B(x_i))^2+(\Delta I_B(x_i))^2+(\Delta F_B(x_i))^2}}.$$

$$C_N(B,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_B(x_i)\Delta T_C(x_i)+\Delta I_B(x_i)\Delta I_C(x_i)+\Delta F_B(x_i)\Delta F_C(x_i)}{\sqrt{(\Delta T_B(x_i))^2+(\Delta I_B(x_i))^2+(\Delta F_B(x_i))^2}\sqrt{(\Delta T_C(x_i))^2+(\Delta I_C(x_i))^2+(\Delta F_C(x_i))^2}}.$$

$$C_N(A,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_A(x_i)\Delta T_C(x_i)+\Delta I_A(x_i)\Delta I_C(x_i)+\Delta F_A(x_i)\Delta F_C(x_i)}{\sqrt{(\Delta T_A(x_i))^2+(\Delta I_A(x_i))^2+(\Delta F_A(x_i))^2}\sqrt{(\Delta T_C(x_i))^2+(\Delta I_C(x_i))^2+(\Delta F_C(x_i))^2}}.$$

5. Applications of Neutrosophic Sets

In this section, the author gave some applications of neutrosophic sets in real life problems.

5.1. Multi-criteria group decision-making methods based on hybrid score-accuracy functions

In a multi-criteria group decision-making problem, let A= {A_1, A_2, …, A_m} be a set of alternatives and let C= {C_1, C_2, …, C_n} be a set of attributes. Then, the weights of decision makers and attributes are not assigned previously, where the information about the weights of the decision makers is completely unknown and the information about the weights of the attributes is incompletely known in the group decision-making problem. In such a case, we develop two methods based on the hybrid score-accuracy functions for multiple attribute group decision-making problems with unknown weights under single valued neutrosophic and interval neutrosophic environments.

Multi-criteria group decision-making method in single valued neutrosophic setting

In the group decision process under single valued neutrosophic environment, if a group of t decision makers or experts is required in the evaluation process, then the kth decision maker can provide the evaluation information of the alternative A_i (i= 1, 2, …, m) on the attribute C_j (j= 1, 2, …, n), which is represented by the form of a SVNS:

$A_i^k=\left\{\left\langle C_j, T_{A_i}^k(C_j), I_{A_i}^k(C_j), F_{A_i}^k(C_j)\right\rangle / C_j\in C\right\}$. Here, $0\le T_{A_i}^k(C_j)+I_{A_i}^k(C_j)+F_{A_i}^k(C_j)\le 3$,

$T_{A_i}^k(C_j)\in[0,1]$, $I_{A_i}^k(C_j)\in[0,1]$, $F_{A_i}^k(C_j)\in[0,1]$, for k = 1, 2, …., t, j=1, 2, …., n, i=1, 2, ….,m

For convenience, $a_{ij}^k=\left\langle T_{ij}^k, I_{ij}^k, F_{ij}^k\right\rangle$ is denoted as a SVNN in the SVNS. A_i^k (k= 1, 2, …t; i= 1, 2, …, m; j= 1, 2, …, n). Therefore, we can get the k-th single valued neutrosophic decision matrix $D^k=(A_{ij}^k)_{m\times n}$ (k= 1, 2, …, t).

Then, the group decision-making method is described as follows.

Step1: Calculate hybrid score-accuracy matrix

The hybrid score-accuracy matrix $Y^k = (Y_{ij}^k)_{m\times n}$ (k= 1, 2, …, t; i= 1, 2, …, m; j= 1, 2, …, n) is obtained from the decision matrix $D^k = (A_{ij}^k)_{m\times n}$ by the following formula:

$$Y_{ij}^k = \frac{1}{2}\alpha(1+T_{ij}^k - F_{ij}^k) + \frac{1}{3}(1-\alpha)(2+T_{ij}^k - I_{ij}^k - F_{ij}^k)$$

Step2: Calculate the average matrix

From the obtained hybrid score-accuracy matrices, the average matrix $Y^* = (Y_{ij}^*)_{m\times n}$ (k= 1, 2, …, t; i= 1, 2, …, m; j= 1, 2, …, n) is calculated by $Y_{ij}^* = \frac{1}{t}\sum_{k=1}^{t}(Y_{ij}^k)$.

The collective correlation coefficient between Y^k (k= 1, 2, …, t) and Y^* represents as follows:

$$e_k = \sum_{i=1}^{m} \frac{\sum_{j=1}^{n} Y_{ij}^k Y_{ij}^*}{\sqrt{\sum_{j=1}^{n}(Y_{ij}^k)^2}\sqrt{\sum_{j=1}^{n}(Y_{ij}^*)^2}}.$$

Step3: Determination decision maker's weights

In practical decision-making problems, the decision makers may have personal biases and some individuals may give unduly high or unduly low preference values with respect to their preferred or repugnant objects. In this case, we will assign very low weights to these false or biased opinions. Since the ‘‘mean value’’ is the ‘‘distributing center’’ of all elements in a set, the average matrix Y^* is the maximum compromise among all individual decisions of the group. In mean sense, a hybrid score-accuracy matrix Y^k is closer to the average one Y^*. Then, the preference value (hybrid score-accuracy value) of the k-th decision maker is closer to the average value and his/her evaluation is more reasonable and more important, thus the weight of the k-th decision maker is bigger. Hence, a weight model for decision makers can be defined as:

$$\lambda_k = \frac{e_k}{\sum_{k=1}^{t} e_k}, \text{ where } 0 \le \lambda_k \le 1,\ \sum_{k=1}^{t}\lambda_k = 1 \text{ for k=1, 2, ….,t.}$$

Step4: Calculate collective hybrid score-accuracy matrix

For the weight vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)^T$ of decision makers obtained from eqation.(6), we accumulate all individual hybrid score-accuracy matrices of $Y^k = (Y_{ij}^k)_{m\times n}$ (k= 1, 2, …, t; i= 1, 2, …, m; j= 1, 2, …, n) into a collective hybrid score-accuracy matrix $Y = (Y_{ij})_{m\times n}$ by the following formula:

$$Y_{ij} = \sum_{k=1}^{t}\lambda_k Y_{ij}^k$$

Step5: Weight model for attributes

For a specific decision problem, the weights of the attributes can be given in advance by a partially known subset corresponding to the weight information of the attributes, which is denoted by W. Reasonable weight values of the attributes should make the overall averaging value of all alternatives as large as possible because they can enhance the obvious differences and identification of various alternatives under the attributes to easily rank the alternatives. To determine the weight vector of the attributes Ye introduced the following optimization model:

$$\max W = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} W_j Y_{ij}$$

Subject to,

$$\sum_{j=1}^{n} W_j = 1$$

where $W_j > 0$

This is a linear programming problem, which can be easily solved to determine the weight vector of the attributes $W = (W_1, W_2, \ldots, W_n)^T$

Step6: Ranking alternatives

To rank alternatives, we can sum all values in each row of the collective hybrid score-accuracy matrix corresponding to the attribute weights by the overall weighted hybrid score-accuracy value of each alternative A_i (i= 1, 2, …, m):

$$M(A_i) = \sum_{j=1}^{n} W_j Y_{ij}$$

According to the overall hybrid score-accuracy values of $M(A_i)$ (i= 1, 2, …, m), we can rank alternatives A_i (i= 1, 2, …, m) in descending order and choose the best one.

Step7: End

5.2. Example of Teacher Recruitment Process

Suppose that a university is going to recruit in the post of an assistant professor for a particular subject.. After initial screening, five candidates (i.e. alternatives) A_1, A_2, A_3, A_4, A_5 remain for further evaluation. A committee of four decision makers or experts, D_1, D_2, D_3, D_4 has been formed to conduct the interview and select the most appropriate candidate. Eight criteria obtained from expert opinions, namely, academic performances (C_1), subject knowledge (C_2), teaching aptitude (C_3), research- experiences (C_4), leadership quality (C_5), personality (C_6), management capacity (C_7) and values (C_8) are considered for recruitment criteria. If four experts are required in the evaluation process, then the five possible alternatives A_i (i= 1, 2, 3, 4, 5) are evaluated by the form of SVNNs under the above eight attributes on the fuzzy concept "excellence". Thus the four single valued neutrosophic decision matrices can be obtained from the four experts and expressed, respectively, as follows:(see Table 1, 2, 3, 4).

Table1: Single valued neutrosophic decision matrix

$D_1=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	$\langle .8, .1, .1 \rangle$	$\langle .8, .1, .1 \rangle$	$\langle .7, .2, .1 \rangle$	$\langle .7, .2, .2 \rangle$	$\langle .7, .4, .1 \rangle$	$\langle .7, .4, .2 \rangle$	$\langle .7, .3, .1 \rangle$	$\langle .7, .4, .3 \rangle$
A_2	$\langle .8, .2, .2 \rangle$	$\langle .8, .2, .1 \rangle$	$\langle .7, .3, .2 \rangle$	$\langle .7, .3, .3 \rangle$	$\langle .7, .3, .2 \rangle$	$\langle .6, .4, .2 \rangle$	$\langle .7, .2, .2 \rangle$	$\langle .7, .3, .4 \rangle$
A_3	$\langle .8, .1, .2 \rangle$	$\langle .8, .3, .2 \rangle$	$\langle .7, .4, .3 \rangle$	$\langle .7, .3, .1 \rangle$	$\langle .7, .2, .3 \rangle$	$\langle .6, .3, .3 \rangle$	$\langle .7, .1, .3 \rangle$	$\langle .7, .3, .3 \rangle$
A_4	$\langle .8, .1, .0 \rangle$	$\langle .8, .2, .3 \rangle$	$\langle .7, .3, .4 \rangle$	$\langle .7, .1, .2 \rangle$	$\langle .7, .2, .2 \rangle$	$\langle .7, .2, .2 \rangle$	$\langle .7, .2, .3 \rangle$	$\langle .7, .2, .4 \rangle$
A_5	$\langle .8, .2, .2 \rangle$	$\langle .8, .3, .3 \rangle$	$\langle .7, .3, .4 \rangle$	$\langle .7, .2, .2 \rangle$	$\langle .7, .1, .3 \rangle$	$\langle .7, .1, .1 \rangle$	$\langle .7, .1, .2 \rangle$	$\langle .7, .1, .3 \rangle$

Table2: Single valued neutrosophic decision matrix

$D_2=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	$\langle .8, .2, .1\rangle$	$\langle .8, .1, .1\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .4, .2\rangle$	$\langle .7, .4, .2\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .3, .3\rangle$
A_2	$\langle .8, .2, .2\rangle$	$\langle .8, .2, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .2, .2\rangle$	$\langle .6, .4, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .4, .4\rangle$
A_3	$\langle .8, .2, .2\rangle$	$\langle .8, .3, .3\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .6, .3, .3\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .2, .3\rangle$
A_4	$\langle .8, .1, .0\rangle$	$\langle .8, .2, .2\rangle$	$\langle .7, .3, .4\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .4\rangle$
A_5	$\langle .8, .1, .2\rangle$	$\langle .8, .2, .3\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .3\rangle$

Table3: Single valued neutrosophic decision matrix

$D_3=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	$\langle .8, .1, .0\rangle$	$\langle .8, .1, .1\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .1\rangle$	$\langle .7, .3, .1\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .3, .3\rangle$
A_2	$\langle .8, .2, .1\rangle$	$\langle .8, .2, .1\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .6, .4, .4\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .3, .3\rangle$
A_3	$\langle .8, .2, .2\rangle$	$\langle .8, .2, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .6, .2, .3\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .3, .4\rangle$
A_4	$\langle .8, .1, .0\rangle$	$\langle .8, .2, .2\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .2, .3\rangle$
A_5	$\langle .8, .1, .2\rangle$	$\langle .8, .2, .3\rangle$	$\langle .7, .2, .4\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .1, .3\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .3\rangle$

Table4: Single valued neutrosophic decision matrix

$D_4=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	$\langle .8, .2, .1\rangle$	$\langle .8, .2, .1\rangle$	$\langle .7, .2, .1\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .3, .1\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .1\rangle$	$\langle .7, .4, .3\rangle$
A_2	$\langle .8, .2, .0\rangle$	$\langle .8, .2, .1\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .1, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .6, .4, .3\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .3, .3\rangle$
A_3	$\langle .8, .1, .2\rangle$	$\langle .8, .2, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .6, .3, .2\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .3, .3\rangle$
A_4	$\langle .8, .1, .0\rangle$	$\langle .8, .2, .3\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .2, .3\rangle$	$\langle .7, .2, .4\rangle$
A_5	$\langle .8, .2, .2\rangle$	$\langle .8, .3, .0\rangle$	$\langle .7, .3, .3\rangle$	$\langle .7, .2, .2\rangle$	$\langle .7, .1, .3\rangle$	$\langle .7, .1, .1\rangle$	$\langle .7, .1, .2\rangle$	$\langle .7, .2, .3\rangle$

Thus, we use the proposed method for single valued neutrosophic group decision-making to get the most suitable teacher. We take $\alpha = 0.5$ for demonstrating the computing procedure of the proposed method. For the above four decision matrices, the following hybrid score-accuracy matrices are obtained by equation(3):(see Table 5, 6, 7, 8)

Table5: Hybrid score accuracy matrix for D_1

$Y_1=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.7667	1.7167	1.6000	1.5167	1.5333	1.4500	1.5667	1.3667
A_2	1.6000	1.6833	1.4833	1.4000	1.4833	1.3667	1.5167	1.3167
A_3	1.6333	1.6500	1.3667	1.5667	1.4333	1.3167	1.4667	1.4000
A_4	1.8000	1.5167	1.3167	1.5500	1.5167	1.5167	1.4333	1.3500
A_5	1.6000	1.4833	1.3167	1.5167	1.4667	1.6333	1.5500	1.4667

Table6: Hybrid score accuracy matrix for D_2

$Y_2=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.6833	1.7167	1.5167	1.5500	1.4500	1.4500	1.4833	1.4000
A_2	1.6000	1.6000	1.4000	1.4000	1.5167	1.2833	1.4833	1.2833
A_3	1.6000	1.4833	1.4000	1.4833	1.4000	1.3167	1.4333	1.4333
A_4	1.8000	1.6000	1.3167	1.5500	1.4833	1.5167	1.4000	1.3167
A_5	1.6333	1.5167	1.4000	1.5167	1.4333	1.5500	1.5167	1.4333

Table7: Hybrid score accuracy matrix for D_3

$Y_3=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.8000	1.7167	1.5167	1.6000	1.5667	1.4833	1.4833	1.4000
A_2	1.6833	1.6833	1.4833	1.4333	1.4833	1.2000	1.4833	1.4000
A_3	1.6000	1.6000	1.4000	1.4833	1.5167	1.3500	1.4333	1.3167
A_4	1.8000	1.6000	1.4833	1.5500	1.5167	1.5167	1.4333	1.4333
A_5	1.6333	1.5167	1.3500	1.5500	1.4667	1.5500	1.5167	1.4333

Table8: Hybrid score accuracy matrix for D_4

$Y_4=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.6833	1.6833	1.6000	1.5167	1.5667	1.5167	1.6000	1.3667
A_2	1.7333	1.6833	1.4833	1.4667	1.4833	1.2833	1.5167	1.4000
A_3	1.6333	1.6000	1.4000	1.4833	1.5167	1.4000	1.4000	1.4000
A_4	1.8000	1.5167	1.4000	1.5500	1.5167	1.5167	1.4333	1.3500
A_5	1.6000	1.7333	1.4000	1.5167	1.4667	1.6333	1.5500	1.4333

From the above hybrid score-accuracy matrices, by using equation (4) we can yield the average matrix Y^*.(see Table 9)

Table 9: The average matrix

$Y^*=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.7208	1.7084	1.5584	1.5459	1.5292	1.4750	1.5333	1.3834
A_2	1.6417	1.6500	1.4625	1.4375	1.4917	1.2833	1.5000	1.3625
A_3	1.6167	1.5833	1.3917	1.5042	1.4792	1.3459	1.4333	1.3875
A_4	1.8000	1.5584	1.3792	1.5500	1.5084	1.5167	1.4250	1.3625
A_5	1.6167	1.5625	1.3667	1.3450	1.4584	1.5917	1.5334	1.4417

From the equations. (5) and (6), we determine the weights of the three decision makers as follows:
$\lambda_1 = 0.2505$ $\lambda_2 = 0.2510$ $\lambda_3 = 0.2491$ $\lambda_3 = 0.2494$

Hence, the hybrid score-accuracy values of the different decision makers' evaluations are aggregated[48] by equation (7) and the following collective hybrid score-accuracy matrix can be obtain as follows(see Table 10):

Table10: Collective hybrid score accuracy- matrix

$Y=$

.	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1.7209	1.7085	1.5584	1.5459	1.5292	1.4751	1.5334	1.3834
A_2	1.6417	1.6500	1.4624	1.4375	1.4918	1.2833	1.5000	1.3624
A_3	1.6168	1.5834	1.3917	1.5043	1.4792	1.3458	1.4332	1.3875
A_4	1.8001	1.5584	1.3793	1.5500	1.5085	1.5167	1.4250	1.3626
A_5	1.6167	1.5626	1.3667	1.3451	1.4584	1.5918	1.5334	1.4417

Assume that the information about attribute weights is incompletely known weight vectors,
$0.1 \le w_1 \le 0.2$, $0.1 \le w_2 \le 0.2$, $0.1 \le w_3 \le 0.2$, $0.1 \le w_4 \le 0.2$, $0.1 \le w_5 \le 0.2$, $0.1 \le w_6 \le 0.2$, $0.1 \le w_7 \le 0.2$, $0.1 \le w_8 \le 0.2$ given by the decision makers,

By using the linear programming model (8), we obtain the weight vector of the attributes as:

$W = [0.2, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1]^T$.

Wcan calculate the overall hybrid score-accuracy values $M(A_i)$ (i=1, 2, 3, 4, 5):

$M(A_1) = 1.58842$, $M(A_2) = 1.51208$, $M(A_3) = 1.49421$, $M(A_4) = 1.54591$, $M(A_5) = 1.50957$

According to the above values of $M(A_i)$ (i= 1, 2, 3, 4, 5), the ranking order of the alternatives is

$A_1 > A_4 > A_2 > A_5 > A_3$. Then, the alternative A_1 is the best teacher.

5.3. Application of Cosine Similarity Measure for Interval Valued Neutrosophic Numbers to Pattern Recognition

In order to demonstrate the application of the proposed cosine similarity measure for interval valued neutrosophic numbers to pattern recognition, we discuss the medical diagnosis problem as follows:
For example the patient reported temperature claiming that the patient has temperature between 0.5 and 0.7 severity /certainty, some how it is between 0.2 and 0.4 indeterminable if temperature is cause or the effect of his current disease. And it between 0.1 and 0.2 sure that temperature has no relation with his main disease.

This piece of information about one patient and one symptom may be written as:

(patient,Temperature) = <[0.5, 0.7], [0.2 ,0.4], [0.1, 0.2]>,

(patient, Headache) = < [0.2, 0.3], [0.3 ,0.5], [0.3, 0.6]>,

(patient, Cough) = <[0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>.

Then,

P = {< x_1, [0.5, 0.7], [0.2 ,0.4], [0.1, 0.2] >, < x_2 x_2, [0.2, 0.3], [0.3, 0.5], [0.3, 0.6] > ,< x_3, [0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>}.

And each diagnosis A_i (i=1, 2, 3) can also be represented by interval valued neutrosophic numbers with respect to all the symptoms as follows:

A_1= {< x_1, [0.5, 0.6], [0.2 ,0.3], [0.4, 0.5] >, < x_2, [0.2 , 0.6], [0.3 ,0.4], [0.6 , 0.7]>,< x_3, [0.1, 0.2], [0.3 ,0.6], [0.7, 0.8]>},

A_2= {< x_1, [0.4, 0.5], [0.3, 0.4], [0.5, 0.6] >, < x_2 , [0.3, 0.5], [0.4 ,0.6], [0.2, 0.4]> , < x_3, [0.3, 0.6], [0.1, 0.2], [0.5, 0.6]>}.

A_3= {< x_1, [0.6, 0.8], [0.4 ,0.5], [0.3, 0.4]>, < x_2 , [0.3, 0.7], [0.2, 0.3], [0.4, 0.7]> ,< x_3, [0.3, 0.5], [0.4, 0.7], [0.2, 0.6]>}.

Our aim is to classify the pattern P in one of the classes A_1, A_2, A_3 .According to the recognition principle of maximum degree of similarity measure between interval valued neutrosophic numbers, the process of diagnosis A_k to patient P is derived according to k = arg Max{ $C_N(A_i, P)$)}.

We can compute the cosine similarity between A_i (i=1, 2, 3) and P as follows;

$$C_N(A_1, P) = 0.8988,\ C_N(A_2, P) = 0.8560,\ C_N(A_3, P) = 0.9654$$

Then, we can assign the patient to diagnosis A_3 (Typoid) according to recognition of principal.

6. Conclusion

Neutrosophic set is a mathematical framework which handles uncertain, incomplete, inconsistent, false, indeterminate information. Several hybrid structures have been proposed which based on neutrosophic sets. In this chapter, the authors presented the study on neutrosophic sets which is a generalization of fuzzy sets and intuitionistic fuzzy sets. Some set theoretic operations and properties are studied in this chapter. The authors also studied the hybrid structures associated with neutrosophic sets such as neutrosophic crisp sets and their related properties. Further, interval valued neutrosophic sets have been presented under discussion in this chapter. At the end, some applications of neutrosophic sets presented to show the applicability of neutrosophic sets in the real life problems.

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KEY TERMS AND DEFINITIONS

Fuzzy Set: Let X be a non-empty collection of objects denoted by x. Then a fuzzy set A in X is a set of ordered pairs having the form $A=\{(x,\mu_A(x)):x\in X\}$, where the function $\mu_A:X\to[0,1]$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A. The interval M = $[0,1]$ is called membership space.

Interval Valued Fuzzy Set: Let D[0, 1] be the set of closed sub-intervals of the interval [0, 1]. An *interval-valued fuzzy set* in X, $X \neq \phi$ and Card(X) = n, is an expression A given by $A = \{(x, M_A(x)) : x \in X\}$, where $M_A : X \to D[0,1]$.

Intuitionistic Fuzzy Set: Let X be a non-empty set. Then an intuitionistic fuzzy set A is a set having the form A={(x, μ_A(x), γ_A(x)): x∈X} where the functions μ_A: X→[0,1] and γ_A: X→[0,1] represents the degree ofmembership and the degree of non-membership respectively of each element x∈X and $0 \leq \mu_A(x)+\gamma_A(x) \leq 1$ for each x∈X.

s-norm: The function $s:[0,1]\times[0,1]\to[0,1]$ is called an s-norm if it satisfies the following four axioms:
Axiom 1. $s(x,y) = s(y,x)$ (commutative condition).
Axiom 2. s(s(x, y), z) = s(x, s(y, z)) (associative condition).
Axiom 3. If $x_1 \geq x_2$ and $y_1 \geq y_2$, then $s(x_1, y_1) \geq s(x_2, y_2)$ (nondecreasing condition).
Axiom 4. s(1, 1) = 1, s(x, 0) = s(0, x) = x (boundary condition).

t-norm: The function $t:[0,1]\times[0,1]\to[0,1]$ is called a **t-norm** if it satisfies the following four axioms**:**
Axiom t1. t(x, y) = t(y, x) (commutative condition).
Axiom t2. t(t(x, y), z) = t(x, t(y, z)) (associative condition).
Axiom t3. If $x_1 \geq x_2$ and $y_1 \geq y_2$, then $t(x_1, y_1) \geq t(x_2, y_2)$ (nondecreasing condition).
Axiom t4 t(x, 1) = x (boundary condition).

Cosine Similarity Measure. Given two vectors of attributes, X= $(x_1, x_2, \ldots, x_n)$ and Y= $(y_1, y_2, \ldots, y_n)$, the cosine similarity, cosθ, is represented using a dot product and magnitude as

$$\textbf{Cos}\boldsymbol{\theta}\textbf{=} \frac{\sum_{i=}^{n} x_i y_i}{\sqrt{\sum_{i=}^{n} x_i^{\,2}}\sqrt{\sum_{i=}^{n} y_i^{\,2}}}$$

In vector space, a cosine similarity measure between two fuzzy set $\mu_A(x_i)$ and $\mu_B(x_i)$ defined as follows:

$$C_F(A,B) = \frac{\sum_{i=}^{n} \mu_A(x_i)\mu_B(x_i)}{\sqrt{\sum_{i=}^{n} \mu_A(x_i)^2}\sqrt{\sum_{i=}^{n} \mu_B(x_i)^2}}$$

The cosine of the angle between the vectors is within the values between 0 and 1.

In 2-D vector space, cosine similarity measure between IFS as follows:

$$C_{IFS}(A,B) = \frac{\sum_{i=}^{n} \mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\sum_{i=}^{n} \mu_A(x_i)^2 + \nu_A(x_i)^2}\sqrt{\sum_{i=}^{n} \mu_B(x_i)^2 + \nu_B(x_i)^2}}$$