Relativistic Quantum Theory of Atoms and Gravitation from 3 Postulates

Author : Jérémy KERNEIS, Brest (France)
jeremy.kerneis.11@eigsi.fr

Abstract

We use three postulates **P1**, **P2a/b** and **P3** :

P1 : $E = H = \gamma m_0 c^2 - k/r$, defines the Hamiltonian for central potential problems (which can be adapted to other potentials)
P2a : $k = Zq^2/(4\pi\epsilon_0)$ / **P2b** : $k = GM\gamma m_0$ define the electromagnetic/gravitationnal potentials
P3 : $\Psi = e^{iS/\hbar}$ defines the wavefunction, with S relativistic action, deduced from **P1**

Combining **P1** and **P2a** with "Sommerfeld's quantum rules" corresponds to the original quantum theory of Hydrogen, which produces the correct relativistic energy levels of atoms (Sommerfeld's and Dirac's theories of matter produces the same energy levels, and Schrodinger's theory produces the approximation of those energy levels). **P3** can be found in Schrodinger's famous paper introducing his equation, **P3** being his first assumption (a second assumption, suppressed here, is required to deduce his equation). **P3** implies that Ψ is solution of both Schrodinger's and Klein-Gordon's equations in the non interacting case ($k = 0$ in **P1**) while, in the interacting case ($k \neq 0$), it implies "Sommerfeld's quantum rules" : **P1**, **P2a**, and **P3** then produce the correct relativistic energy levels of atoms. We check that the required degeneracy is justified by pure deduction, without any other assumption (Schrodinger's theory only justifies one half of the degeneracy).
We observe that the introduction of an interaction in **P1** ($k = 0 \to k \neq 0$) is equivalent to a modification of the metric inside Ψ in **P3**, such that the equation of motion of a system can be deduced with two different methods, with or without the metric. Replacing the electromagnetic potential **P2a** by the suggested gravitationnal potential **P2b**, the equation of motion (deduced with and without the metric) is equivalent to the equation of motion of General Relativity in the low field approximation (with accuracy 10^{-6} at the surface of the Sun). We have no coordinate singularity inside the metric. Other motions can be obtained by modifying **P2b**, the theory is adaptable.
First of all, we discuss classical Kepler problems (Newtonian motion of the Earth around the Sun), explain the link between Kelpler law of periods (1619) and Plank's law (1900) and observe the links between all historical models of atoms (Bohr, Sommerfeld, Pauli, Schrodinger, Dirac, Fock). This being done, we introduce **P1**, **P2a/b**, and **P3** to then describe electromagnetism and gravitation in the same formalism.

January 26, 2017

I – New results in classical physics

We first defines the quantities in **P1** and **P2** : E [J] is the energy, m_0 [kg] the mass of the orbiting particle/planet, c [m/s] the speed of light, q [C] the unit proton charge ($-q$ is the electron charge), Z [-] the number of protons in an atom (we will fix $Z = 1$ for simplicity, corresponding to Hydrogen), ϵ_0 [F/m] the vacuum permittivity , G [$m^3/kg/s^2$] Newton's constant, M [kg] the mass of the central object, $\gamma = 1/\sqrt{1 - v^2/c^2}$ the relativistic factor.

In this part, the potential k/r can refer to the electromagnetic ($k = q^2/(4\pi\epsilon_0) = \alpha\hbar c$) or classical gravitationnal potential ($k = GMm_0$). α [-] is the fine structure constant, and $\hbar$ [J.s] is the reduced Planck constant. Equations (1) to (4) are well known results of classical physics and can be found in any student book. With $\epsilon = E - m_0c^2, \gamma m_0c^2 \approx m_0c^2 + 1/2m_0v^2 = m_0c^2 + p^2/(2m_0)$, the classical energy $\epsilon \leq 0$ is given by :

$$\epsilon = \frac{p^2}{2m_0} - \frac{k}{r} = \frac{(\vec{p}.\vec{r})^2}{2m_0r^2} + \frac{(\vec{p} \wedge \vec{r})^2}{2m_0r^2} - \frac{k}{r} = \frac{p_r^2}{2m_0} + \frac{L^2}{2m_0r^2} - \frac{k}{r} = \frac{m_0\dot{r}^2}{2} + \frac{L^2}{2m_0r^2} - \frac{k}{r} \quad (1)$$

With $L = m_0r^2\frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{m_0r^2d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$\epsilon = \frac{L^2}{2m_0}(\frac{du}{d\phi})^2 + \frac{L^2}{2m_0}u^2 - ku \Leftrightarrow 0 = \frac{du}{d\phi}(\frac{L^2}{m_0}\frac{d^2u}{d\phi^2} + \frac{L^2}{m_0}u - k) \Leftrightarrow 0 = \frac{d^2u}{d\phi^2} + u = \frac{m_0k}{L^2} \quad (2)$$

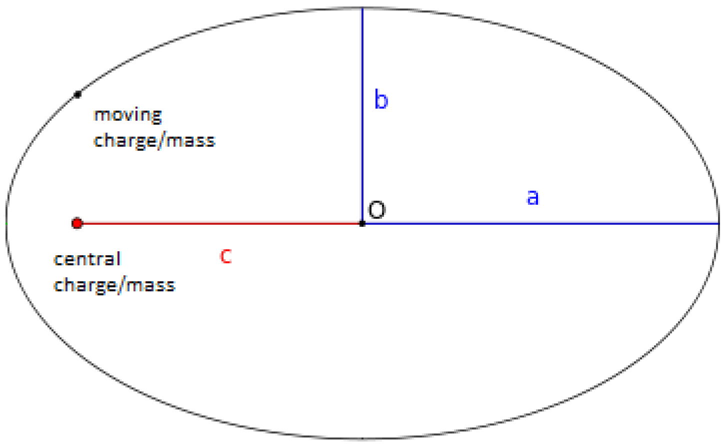
The solution, with u proportionnal to the potential k/r, takes the form

$$u = 1/r = \frac{m_0k}{L^2}(1 + ecos(\phi - \phi_0)) \ with \ e = \sqrt{1 + \frac{2\epsilon L^2}{k^2m_0}} \quad (3)$$

e (the eccentricity) and ϕ_0 are constants of integration, and $r(\phi)$ is given by

$$r = \frac{L^2}{m_0k + \sqrt{m_0^2k^2 + 2m_0\epsilon L^2}cos(\phi - \phi_0)} = \frac{L^2/(m_0k)}{1 + ecos(\phi - \phi_0)} = \frac{l}{1 + ecos(\phi - \phi_0)} \quad (4)$$

l is called "semi latus rectum". It is well known that the solution is an ellipse, described by 3 parameters a, b, c where $a^2 = b^2 + c^2$, $e = c/a$ and $l = b^2/a$).



$$a = \frac{m_0k}{2m_0|\epsilon|} = \frac{J}{\sqrt{2m_0|\epsilon|}}; \ b = \frac{L}{\sqrt{2m_0|\epsilon|}}; \ c = \frac{\sqrt{m_0^2k^2/(2m_0|\epsilon|) - L^2}}{\sqrt{2m_0|\epsilon|}} = \frac{K}{\sqrt{2m_0|\epsilon|}} \quad (5)$$

Here K, which is proportionnal to the eccentricity e, is the norm of the well known Laplace-Runge-Lenz-Pauli vector, the second converved angular momentum of Kepler Problems :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(\vec{p}\wedge\vec{L} - m_0 k\frac{\vec{r}}{r}) \quad , \quad \epsilon \; classical \; energy < 0 \tag{6}$$

since (with $\dot{\vec{L}} = 0$ and $\vec{r}\wedge\vec{v}\wedge\vec{r} = r^2\vec{v} - (\vec{v}.\vec{r})\vec{r}$, and $\vec{v}.\vec{r} = \dot{x}x + \dot{y}y + \dot{z}z = \dot{r}r$ which can be easily checked from the right to the left) :

$$\frac{d(\vec{p}\wedge\vec{L})}{dt} = \dot{\vec{p}}\wedge\vec{L} = \frac{k\vec{r}}{r^3}\wedge(m_0\vec{v}\wedge\vec{r}) = \frac{m_0 k}{r^3}(r^2\vec{v} - (\vec{v}.\vec{r})\vec{r}) = \frac{m_0 k}{r^3}(r^2\vec{v} - (\dot{r}r)\vec{r}) = \frac{d(m_0 k\vec{r}/r)}{dt} \tag{7}$$

Equation (7) shows that $\vec{K}$ is conserved, which is well known. We can now give new results (equations (9), (10) and (11)), the conserved Runge-Lenz vector can be rewritten :

$$\vec{K} = \frac{1}{\sqrt{-2m_0\epsilon}}(m_0^2\vec{v}\wedge\vec{r}\wedge\vec{v} - m_0 k\frac{\vec{r}}{r}) = \frac{1}{\sqrt{-2m_0\epsilon}}[(m_0^2 v^2 - m_0\frac{k}{r})\vec{r} - m_0^2(\vec{v}.\vec{r})\vec{v}] \tag{8}$$

$$\vec{K} = (\frac{m_0^2 v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})\vec{r} - (\frac{\vec{p}.\vec{r}}{\sqrt{-2m_0\epsilon}})\vec{p} = (p_w)\vec{r} - (w)\vec{p} \tag{9}$$

where we can check that

$$\dot{w} = \frac{\dot{\vec{p}}.\vec{r} + \vec{p}.\dot{\vec{r}}}{\sqrt{-2m_0\epsilon}} = \frac{m_0 v^2 - \frac{k}{r}}{\sqrt{-2m_0\epsilon}} = \frac{p_w}{m_0} \tag{10}$$

We can then define 6 rotations and a total angular momentum J such that

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} \qquad \vec{K} = \begin{bmatrix} xp_w - wp_x \\ yp_w - wp_y \\ zp_w - wp_z \end{bmatrix} = \vec{r}p_w - w\vec{p} \tag{11}$$

$$\vec{J} = \vec{L} + \vec{K} \; ; \; J^2 = K^2 + L^2 \; (\vec{K}.\vec{L} = 0 \; can \; be \; easily \; checked \; from \; equations \; (6) \; or \; (11)) \tag{12}$$

The 6 independant previous rotations defines the SO(4) symmetry. The two angular momenta have opposite parity. In 1926, Pauli [Pauli, 1926] used this definition of equation (12) for J to deduce the non relativistic energy levels of Hydrogen, without solving for the wave function. In 1935, Fock [Fock, 1935], studying Schrodinger's Hydrogen [Schrodinger, 1926] in momentum space, observed that, with an $1/r$ potential (SO(3) symmetry) he could describe an intrinsic SO(4) in the model. On this subject we suggest Torres del Castillo [Torres del Castillo, 2007]. Concerning SO(4), we now give a new result involving p_w and $\vec{J}$:

$$r^2P^2 = r^2[p_x^2 + p_y^2 + p_z^2 + p_w^2] = r^2[p^2 + (\frac{m_0^2 v^2 - m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] \tag{13}$$

$$r^2P^2 = r^2[p^2 + (\frac{m_0^2 v^2 - 2m_0\frac{k}{r} + m_0\frac{k}{r}}{\sqrt{-2m_0\epsilon}})^2] = r^2[p^2 + (-\sqrt{-2m_0\epsilon} + \frac{m_0 k/r}{\sqrt{-2m_0\epsilon}})^2] \tag{14}$$

$$r^2P^2 = r^2[p^2 - 2m_0\epsilon - 2m_0 k/r + J^2/r^2) = J^2 \tag{15}$$

This shows that J combines both SO(3) and SO(4) symmetries. From this (and the definition of angular momentum L) we easily deduce

$$J^2 = r^2(p^2 + p_w^2) \; ; \; L^2 = r^2(p^2 - p_r^2) \; ; \; K^2 = r^2(p_w^2 + p_r^2) \tag{16}$$

We now recall Kepler's third law of periods for Planetary motion (left hand side of equation (17) below), and observe that it can be rewritten in a new form (using equation (5)):

$$T^2 = \frac{4\pi^2}{GM}a^3 = \frac{4\pi^2}{GM}(a)(a^2) = \frac{4\pi^2}{GM}(\frac{m_0^2 GM}{2m_0|\epsilon|})(\frac{J^2}{2m_0|\epsilon|}) \Leftrightarrow |\epsilon|T = \pi J \; (new \; form) \tag{17}$$

Equations (15), (16) and (17) are new results for classical physics. Evidently, with h as Planck constant (and $\hbar = h/(2\pi)$), fixing J equal to one unit of angular momentum, $J = \hbar$, and introducing the frequency $\nu = 1/T$ gives $|\epsilon| = h\nu/2$. In particular, when the motion around the central object is a circle (corresponding to Bohr's model of Hydrogen [Bohr, 1913]), the electromagnetic interaction energy $V(r) = -k/r = E_i$ is constant, and $|E_i| = 2|\epsilon| = h\nu$: we recognize here Planck's law [Planck, 1900] for the electromagnetic field, hidden in Kepler's third law [Kepler, 1619]. Considering non circular orbits, and calling $< E_i >$ the time average value of the potential, we can write, (using equation (1) and the definition of L below equation (1)) :

$$T < E_i >= T(\frac{1}{T}\int_0^T \frac{-k}{r}\mathrm{d}t) = \int_0^T \epsilon - \frac{m_0\dot{r}^2}{2} - \frac{L^2}{2m_0r^2}\mathrm{d}t = \epsilon T - \int_0^T \frac{m_0\dot{r}^2}{2} + \frac{L}{2}\dot{\phi}\mathrm{d}t \tag{18}$$

With equation (17) and a simple change of the integration variables we have

$$T < E_i >= -\pi J - \int_{r(t=0)}^{r(T)} \frac{p_r}{2}\mathrm{d}r - \int_{\phi(t=0)}^{\phi(T)} \frac{L}{2}\mathrm{d}\phi = -\pi J - \oint \frac{p_r}{2}\mathrm{d}r - \pi L = -\pi J - [\pi(J-L)] - \pi L = -2\pi J \tag{19}$$

The term $[\pi(J-L)]$ as result of the integral over $\mathrm{d}r$ is justified in part II. The analogy between Kepler's third law and Planck's law remains valid for non circular orbits, while equation (19) looks like to a saturation of Heisenberg's relation $\Delta t\Delta E \approx h$ for $J = \hbar$. This is our last new result for classical physics.

II – Sommerfeld′s model of atoms :

We reproduce here Sommerfeld's book [Sommerfeld, 1916], nothing is new except equation (31) and maybe equations (24) and (25). Starting with **P1** : $E = H = \gamma m_0c^2 - k/r$ which, in polar coordinates (cylindrical coordinates with $z = 0$), becomes

$$(E + \frac{k}{r})^2 = p^2c^2 + m_0^2c^4 \Leftrightarrow E^2 - m_0^2c^4 = p^2c^2 - \frac{2Ek}{r} - \frac{k^2}{r^2} = p_r^2c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \tag{20}$$

$$E^2 - m_0^2c^4 = (\gamma m_0\dot{r})^2c^2 + \frac{L'^2c^2}{r^2} - \frac{2Ek}{r} \tag{21}$$

$L'^2 = L^2 - (k/c)^2 = L^2 - (\alpha\hbar)^2$ will be important. With $L = \vec{p} \wedge \vec{r} = \gamma m_0r^2\frac{d\phi}{dt} \Leftrightarrow \dot{r} = \frac{dr}{dt} = \frac{Ldr}{\gamma m_0r^2d\phi}$, and fixing $u = 1/r \Leftrightarrow \frac{du}{d\phi} = -\frac{dr/d\phi}{r^2}$ the previous equation can be rewritten and derivated according to :

$$E^2 - m_0^2c^4 = L^2c^2(\frac{du}{d\phi})^2 + L'^2c^2u^2 - 2Eku \tag{22}$$

$$0 = \frac{du}{d\phi}(2L^2c^2\frac{d^2u}{d\phi^2} + 2L'^2c^2u - 2Ek) \Leftrightarrow \frac{d^2u}{d\phi^2} + u(1 - (\frac{k}{Lc})^2) = \frac{Ek}{L^2c^2} \tag{23}$$

The solution takes the form $u = 1/r = \frac{Ek}{L^2c^2}(1 + ecos(\Gamma\phi - \phi_0))$ with $e = \sqrt{1 + \frac{(E^2-m_0^2c^4)L'^2c^2}{E^2k^2}}$ and $\Gamma^2 = 1-(\frac{k}{Lc})^2$. The Γ factor produces a shift of the perihelion, as illustrated by Sommerfeld :

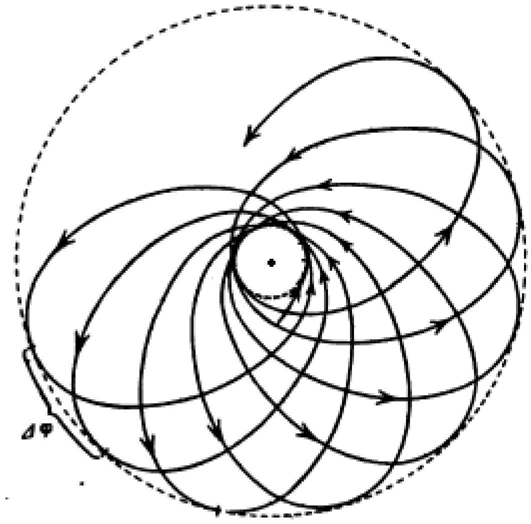


Figure 1 : Perihelion's shift for 8 loops (from [Sommerfeld, 1916])

The 3 parameters of the ellipse are now given by (with $E < m_0c^2$) :

$$a = \frac{Ek}{m_0^2c^4 - E^2} = \frac{Jc}{\sqrt{m_0^2c^4 - E^2}} \; ; \; b = \frac{c\sqrt{L^2 - (k/c)^2}}{\sqrt{m_0^2c^4 - E^2}} = \frac{L'c}{\sqrt{m_0c^2 - E^2}} \tag{24}$$

$$c = \frac{\sqrt{(Ek/c)^2/(m_0^2c^4 - E^2) - c^2(L^2 - (k/c)^2)}}{\sqrt{m_0^2c^4 - E^2}} = \frac{Kc}{\sqrt{m_0^2c^4 - E^2}} \; ; \; J^2 = L'^2 + K^2 \tag{25}$$

In classical physics $r(t)$ and $\phi(t)$ are cyclic functions of time with period T. In the relativistic domain, there is a precession of the perihelion, such that $r(t)$ and $\phi(t)$ have different periods, T_r and T_ϕ. Sommerfeld's (postulated) quantum rules [Sommerfeld, 1916], with n_ϕ and n_r integers, are

$$\int_t^{t+T_r} (\gamma m_0 \dot{r}^2)\mathrm{d}t \int_{r(t)}^{r(t+T_r)} p_r \mathrm{d}r = \oint p_r \mathrm{d}r = n_r h \tag{26}$$

$$\int_t^{t+T_\phi} L\dot{\phi}\mathrm{d}t = \int_0^{2\pi} L\mathrm{d}\phi = \oint L\mathrm{d}\phi = \; 2\pi L = n_\phi h \;\; (\Rightarrow L' = \hbar\sqrt{n_\phi^2 - \alpha^2}) \tag{27}$$

Sommerfeld gave two methods to compute equation (26). We reproduce them is Annex. The final result is (with $E < m_0c^2$ for $V < 0$):

$$\oint p_r \mathrm{d}r = 2\pi(J - L') = n_r h \Leftrightarrow J = \frac{Ek/c}{\sqrt{m_0^2c^4 - E^2}} = \frac{E\alpha\hbar}{\sqrt{m_0^2c^4 - E^2}} = (n_r + \sqrt{n_\phi^2 - \alpha^2})\hbar \tag{28}$$

This can be rewritten

$$E^2\alpha^2 = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2(m_0^2c^4 - E^2) \Leftrightarrow E^2(\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2) = (n_r + \sqrt{n_\phi^2 - \alpha^2})^2 m_0^2c^4 \tag{29}$$

$$E^2 = \frac{m_0^2c^4}{\frac{\alpha^2 + (n_r + \sqrt{n_\phi^2 - \alpha^2})^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} = \frac{m_0^2c^4}{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}} \Leftrightarrow E = \frac{m_0c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}}} \tag{30}$$

This last equation is called "fine structure of Hydrogen". In Dirac's theory, n_ϕ is replaced by $j + 1/2$ with $j = n_\phi \pm 1/2$ (see later). The energy levels are then the same : it was the great triumph of Dirac's theory that it reproduced Sommerfeld's energy levels.

With equations (28) and (25), it is interesting to rewrite the energy in a new form

$$E = \frac{m_0c^2}{\sqrt{1+\frac{(\alpha\hbar)^2}{J^2}}} = m_0c^2\frac{\sqrt{J^2}}{\sqrt{J^2+(\alpha\hbar)^2}} = m_0c^2\sqrt{\frac{J^2+(\alpha\hbar)^2-(\alpha\hbar)^2}{J^2+(\alpha\hbar)^2}} = m_0c^2\sqrt{1-\frac{(\alpha\hbar)^2}{K^2+L^2}} \tag{31}$$

III – A 7$^{\text{th}}$ model of matter

Inspired by Bohr, Sommerfeld, Schrodinger, Pauli, Dirac and Fock, we now suggest a new model of matter. Since our definition of the quantum wave function is based on the action, we first establish some new results. Our first postulate is assumed to be written in spherical coordinates r, θ, φ (with $r = \sqrt{x^2+y^2+z^2}$). We understand this coordinate system as being natural. In this coordinates system, the periods $T_r \neq T_\phi$ (used in polar coordinates) are now $T_r \neq T_\theta = T_\varphi$ $(= T_\phi)$. We start from

$$H = \gamma m_0c^2 - \frac{k}{r} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - \frac{k}{r} = \frac{m_0c^2(1-v^2/c^2)+v^2}{\sqrt{1-v^2/c^2}} - \frac{k}{r} \tag{32}$$

$$H = \frac{m_0v^2}{\sqrt{1-v^2/c^2}} + m_0c^2\sqrt{1-v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} + m_0c^2\sqrt{1-v^2/c^2} - \frac{k}{r} = \vec{p}.\vec{v} - \mathcal{L} \tag{33}$$

Here $\mathcal{L} = -m_0c^2\sqrt{1-v^2/c^2}+\frac{k}{r}$ is the usual Lagrangian. It can be found in the famous "Landau and Lifschitz" [Landau, Ed 1975]. Its properties are well known, and the action S is then given by :

$$S = \int \mathcal{L}\mathrm{d}t = \int \vec{p}.\vec{v} - H\mathrm{d}t = -Ht + \int \vec{p}.\vec{v}\mathrm{d}t \tag{34}$$

In spherical coordinates, with

$$\vec{r} = (r,0,0)\ ;\ \dot{\vec{r}} = \vec{v} = (\dot{r}, r\dot{\theta}, rsin\theta\dot{\varphi})\ ;\ \vec{L} = (r,0,0)\wedge\gamma m_0(\dot{r}, r\dot{\theta}, rsin\theta\dot{\varphi}) = \gamma m_0(0, r^2\dot{\theta}, r^2sin\theta\dot{\varphi}) \tag{35}$$

The Lagrangian is explicitly

$$\mathcal{L} = -m_0c\sqrt{c^2-v^2} + \frac{k}{r} = -m_0c\sqrt{c^2-\dot{r}^2-r^2\dot{\theta}^2-r^2sin^2\theta\dot{\varphi}^2} + \frac{k}{r} \tag{36}$$

The action is, according to equation (34) (and writing $L_\theta = \gamma m_0r^2\dot{\theta}, L_\varphi = \gamma m_0r^2sin^2\theta\dot{\varphi}$)

$$S = -Ht + \int \gamma m_0(\dot{r}^2 + r^2\dot{\theta}^2 + r^2sin^2\theta\dot{\varphi}^2)\mathrm{d}t = -Ht + \int \gamma m_0\dot{r}^2 + L_\theta\dot{\theta} + L_\varphi\dot{\varphi}\ \mathrm{d}t \tag{37}$$

with a change of variables we now obtain the desired expression for the action $S = S(t, r, \theta, \varphi)$

$$S = -Ht + \int \gamma m_0\dot{r}\mathrm{d}r + \int L_\theta\mathrm{d}\theta + \int L_\varphi\mathrm{d}\varphi = -Ht + \int p_r(r)\mathrm{d}r + \int L_\theta(\theta)\mathrm{d}\theta + L_\varphi\varphi \tag{38}$$

L_φ is constant since $\mathcal{L}$ does not depend on φ. From the right hand side of above we deduce $H + \frac{\partial S}{\partial t} = 0$, which is the usual Hamilton-Jacobi equation, here extended to the relativistic domain. From equations (36) and (38) we deduce

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial \mathcal{L}}{\partial \dot{r}}\ ;\ L_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial S}{\partial \theta}\ ;\ L_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial S}{\partial \varphi} \tag{39}$$

$p_r(r)$ is given by equations (20-21). From (35) and the definition of L_θ, L_φ, the squared norm of the angular momentum is given by

$$L^2 = L_\theta^2 + \frac{{L_\varphi}^2}{sin^2\theta} \Leftrightarrow L_\theta\dot{\theta} + L_\varphi\dot{\varphi} = \frac{L_\theta^2}{\gamma m_0 r^2} + \frac{L_\varphi^2}{\gamma m_0 r^2 sin^2\theta} = \frac{L^2}{\gamma m_0 r^2} = L\dot{\phi} \quad (40)$$

The right hand side of above gives the relation between our spherical coordinates system and the polar coordinates system (or cylindrical coordinates system with $z = 0$) used by Sommerfeld. We now use our third postulate : **P3** : $\Psi = e^{iS/\hbar}$ and observe that, according to (38) Ψ is a seperable function $\Psi(r,\theta,\phi,t) = \Re(r)\Theta(\theta)\Phi(\varphi)e^{-iHt/\hbar}$. Since $r(t)$, $\theta(t)$ and $\varphi(t)$ are cyclic variables $\Re(r)$, $\Theta(\theta)$ and $\Phi(\varphi)$ are cyclic functions :

$$\Phi(\varphi(t)) = \Phi(\varphi(t+T_\varphi)) \Leftrightarrow e^{i(\int_{\varphi(t_0)}^{\varphi(t)} L_\varphi \mathrm{d}\varphi)/\hbar} = e^{i(\int_{\varphi(t_0)}^{\varphi(t+T_\varphi)} L_\varphi \mathrm{d}\varphi)/\hbar} \quad (41)$$

or

$$1 = e^{i2\pi n_\varphi} = e^{i(\int_{\varphi(t)}^{\varphi(t+T_\varphi)} L_\varphi \mathrm{d}\varphi)/\hbar} \Leftrightarrow \oint L_\varphi \mathrm{d}\varphi = n_\varphi h \quad (42)$$

and similarily

$$\Theta(\theta(t)) = \Theta(\theta(t+T_\theta)) \Leftrightarrow e^{i(\int_{\theta(t_0)}^{\theta(t)} L_\theta \mathrm{d}\theta)/\hbar} = e^{i(\int_{\theta(t_0)}^{\theta(t+T_\theta)} L_\theta \mathrm{d}\theta)/\hbar} \quad (43)$$

According to equation (40) we can now write

$$1 = e^{i2\pi n_\theta} = e^{i(\int_{\theta(t)}^{\theta(t+T_\theta)} L_\theta \mathrm{d}\theta)/\hbar} \Leftrightarrow \oint L_\theta \mathrm{d}\theta = n_\theta h = \oint L\dot{\phi} - L_\varphi\dot{\varphi}\mathrm{d}t = (n_\phi - n_\varphi)h \quad (44)$$

The right hand side of above can be found in Sommerfeld's book.

$$\Re(r(t)) = \Re(r(t+T_r)) \Leftrightarrow e^{i(\int_{r(t_0)}^{r(t)} p_r(r)\mathrm{d}r)/\hbar} = e^{i(\int_{r(t_0)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \quad (45)$$

or

$$1 = e^{i2\pi n_r} = e^{i(\int_{r(t)}^{r(t+T_r)} p_r(r)\mathrm{d}r)/\hbar} \Leftrightarrow \oint p_r \mathrm{d}r = n_r h \quad (46)$$

We recognize here "Sommerfeld's quantum rules" producing the required energy levels. In equation (46), the integral over dr is made from aphelion to perihelion and from perihelion to aphelion, symmetrically, while the integral over the angles coordinates is made in one direction. Indeed, $-\varphi$ and $-\theta$ are possible values, while $-r$ is forbidden, especially in the action $S(t,r,\theta,\varphi)$ given in equation (38). We then restrict our quantum number : $n_r > 0$ but make no such constraint on $n_\varphi, n_\theta, n_\phi$. We observe that (for $x,y,z \neq 0$):

$$\vec{L} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} = \begin{bmatrix} yz(p_z/z - p_y/y) \\ zx(p_x/x - p_z/z) \\ xy(p_y/y - p_x/x) \end{bmatrix} \quad (47)$$

$L_z = L_\varphi \neq 0 \Rightarrow p_y/y \neq p_x/x \Rightarrow L^2 > L_\varphi^2$ and $|n_\phi| > |n_\varphi|$. This last relation can be deduced from equation (40)(left hand side) and the definition of L_θ too. Clearly, equation (23) and its solution are only defined for $L \neq 0 \Leftrightarrow |n_\phi| \geq 1$. In the classical limit (with $L' \to L,\ n_\phi^2 - \alpha^2 \to n_\phi^2$), the energy levels (30), will produces Schrodinger's energy levels :

$$E = \frac{m_0c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{n_\phi^2 - \alpha^2})^2}}} \approx m_0c^2 - \frac{\alpha^2 m_0 c^2}{2(n_r + |n_\phi|)} \to \epsilon = -\frac{\alpha^2 m_0 c^2}{2(n_r + |n_\phi|)} = -\frac{\alpha^2 m_0 c^2}{2n_s}\ ;\ n_s = n_r + |n_\phi| \quad (48)$$

Here n_s is Schrodinger's main quantum number. For a given value of $|n_\phi|$ there are two possible values of n_ϕ and $2|n_\phi| - 1$ possible values of n_φ. This point justifies that our degeneracy is twice Schrodinger's degeneracy (see Table 1) since in his model $n_\phi < 0$ is forbidden, making the wavefunction divergent. On the contrary, n_φ can be either positive or negative in both models. The fact that half of the observed degeneracy was missing in Schrodinger's theory was called "duplexity phenomena" by Dirac [Dirac, 1928]. There is no missing degeneracy in the present model since the degeneracy is, in the classical limit, $2n_s^2$:

Table 1 : Degeneracy for Hydrogen, in the classical limit

n_s	n_r	n_ϕ	n_φ	Degeneracy
1	0	±1	0	$2 = 2n_s^2$
2	1	±1	0	$2+6 = 8 = 2n_s^2$
	0	±2	$0, \pm1$	
	2	±1	0	
3	1	±2	$0, \pm1$	$2+6+10 = 18 = 2n_s^2$
	0	±3	$0, \pm1, \pm2$	

In general, the degeneracy is $2\sum_{n_\phi=1}^{n_\phi=n_s} 2|n_\phi| - 1 = 4\frac{n_s(n_s+1)}{2} - 2n_s = 2n_s^2$

Considering a non interacting system, in cartesian coordinates, the action S, according to equation (33) will take the form

$$S(x,y,z,t) = -Ht + \int p_x v_x + p_y v_y + p_z v_z \; dt = -Ht + p_x(x-x_0) + p_y(y-y_0) + p_z(z-z_0) \quad (49)$$

Then Ψ will take the form

$$\Psi = e^{iS/\hbar} = e^{i(-Ht+p_x(x-x_0)+p_y(y-y_0)+p_z(z-z_0))/\hbar} = e^{i(-wt+k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0))} \quad (50)$$

which is the fundamental solution of both Klein-Gordon equation (with $H^2 = p^2c^2 + m_0^2c^4$) and Schrodinger equation (with $H = \frac{p^2}{2m}$) :

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = -\hbar^2c^2(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}) + m_0^2c^4\Psi \quad (Klein-Gordon) \quad (51)$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m_0}(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}) = H_s\Psi \quad (Schrodinger) \quad (52)$$

H_s is Schrodinger's hamiltonian for a free particle. The general solution of the previous equations takes the form (fixing for simplicity $x_0 = y_0 = z_0 = 0$ in (50)) :

$$\Psi' = \int\int\int\int \psi(E, p_x, p_y, p_z)e^{i(-Ht+p_x x+p_y y)+p_z z)/\hbar}dE\; dp_x\; dp_y\; dp_z \;; \quad (53)$$

with

$$\int\int\int\int |\psi|^2 dE\; dp_x\; dp_y\; dp_z \; < \infty \quad (54)$$

where we recognize the Fourier transform, in the previous equations.
Then, in this theory, a free particle obeys a wave-equation : free particles exhibit a wave-like behavior. The previous equations being linear, if Ψ_1 and Ψ_2 are solutions, their sum is a solution. With equations (38),(39) and (40), equation (20) can be written :

$$E^2 - m_0^2c^4 = p_r^2c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} = (\frac{\partial S}{\partial r})^2c^2 + \frac{(\frac{\partial S}{\partial\phi})^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \quad (55)$$

With $\Psi = e^{iS/\hbar}$ we deduce ($\Leftrightarrow S = -i\hbar ln(\Psi)$, $\frac{\partial S}{\partial r} = -i\hbar\frac{\partial\Psi}{\Psi\partial r}$, $\frac{\partial S}{\partial\phi} = -i\hbar\frac{\partial\Psi}{\Psi\partial\phi}$)

$$(E^2 - m_0^2c^4)\Psi^2 = -\hbar^2(\frac{\partial\Psi}{\partial r})^2c^2 + \frac{-\hbar^2(\frac{\partial\Psi}{\partial\phi})^2c^2 - k^2\Psi^2}{r^2} - \frac{2Ek}{r}\Psi^2 \quad (56)$$

The classical limit of this equation can be found in Schrodinger's paper [Schrodinger, 1926], expressed in cartesian coordinates, just before deducing his well knwon equation, by an ad hoc assumption. The above equation is not linear, superposed states are not allowed for Hydrogen. The difference is induced by the integrals in equation (38) (right hand side) while there is no integral in the free particle case, see equation (49)(right hand side). Without the integral (=without interaction), the wave equation is obeyed since :

$$-\hbar^2(\frac{\partial\Psi}{\partial x})^2 = -\hbar^2(\frac{\partial^2\Psi}{\partial x^2})\Psi = p_x^2\Psi^2 \Leftrightarrow p_x^2\Psi = -\hbar^2(\frac{\partial^2\Psi}{\partial x^2}) \tag{57}$$

The New York Times [New York Times, 2002] produces an analysis from J. A. Wheeler concerning the superposition principle : " In the other is a "great smoky dragon," which is how Dr. Wheeler refers sometimes to one of the supreme mysteries of nature. That is the ability, according to the quantum mechanic laws that govern subatomic affairs, of a particle like an electron to exist in a murky state of possibility – to be anywhere, everywhere or nowhere at all – until clicked into substantiality by a laboratory detector or an eyeball." In the present theory, the interaction (or observation by interaction) is incompatible with superposed solutions of the wave function Ψ, because the theory is not linear.

The general solution Ψ' of equation (56) will be given by placing the action (38) instead of the action (49) in equation (53)

IV – Comparison with Schrodinger′s & Dirac′s theories

1) Both Dirac's and Schrodinger's theories are linear : they admit, in the interacting case, unobserved superposed states (Schrodinger's cat paradox). Such unobserved states are not allowed in the present theory.

2) As previously discussed, in Schrodinger's theory, half of the degeneracy is missing and the theory is not relativistic. Our relativistic theory has the correct degeneracy.

3) In a letter to Heisenberg written in 1936, 4 years after the experimental discovery of antimatter, Pauli described Dirac's theory as follow : "If, instead of Dirac's equation, one assumes as a basis the old scalar Klein-Gordon relativistic equation, it possesses the following properties: the charge density may be either positive or negative and the energy density is always ≥ 0, it can never be negative. This is exactly the opposite situation as in Dirac's theory and exactly what one wants to have. [...] Still am I happy to beat against my old enemy, the Dirac theory of the spinning electron". We don't have this kind of problems in our theory.

4) In Schrodinger's theory, when solving the Hydrogen atom, the radial part of the wave function, $\Re(r) = \Re(\rho)$, obey the following equation (see the well known book from D.J Griffith, chapter 3 on the Hydrogen atom [D.J. Griffith, 1995]) with $\kappa = \sqrt{2m_0|\epsilon|}/\hbar$; $\rho = \kappa r$; $\rho_0 = \frac{m_0 q^2}{2\pi\epsilon_0\hbar^2\kappa}$

$$\frac{d^2\Re}{d\rho^2} = [1 - \frac{\rho_0}{\rho} + \frac{n_\phi(n_\phi+1)}{\rho^2}]\Re \ with\ solution \rightarrow \Re = \rho^{n_\phi+1}e^{-\rho}\sum_{j=0} a_j\rho^j \tag{58}$$

In Schrodinger's and Dirac's theories, n_ϕ is usually called l (and $l = 0$ is a possible value). By a well known analysis firstly given by Schrodinger, Griffith shows that the integrability of the wave function requires that the sum in equation (58) is finite : there must exist a maximum value for j, called j_{max}, such that

$$2(j_{max} + n_\phi + 1) = \rho_0 \tag{59}$$

This last equation produces the energy levels for ϵ given in equation (48) (with j_{max} instead of n_r, $n_\phi = 0$ being allowed by Schrodinger's theory). A similar process exist in Dirac's theory. Both Dirac's and Schrodinger's theories then require that there exist an integer N such that $j_{max} < N$ since $j_{max} = \infty$ would suppress the integrability of the solution. Such maximum number, N, must then exist in both theories, but its value is not given by any theories. On the contrary, no such constraint exist in our theory.

5) The solution of both Schrodinger's and Dirac's equations decay exponentially when a central potential is added (like Hydrogen), as it can be checked in the solution of equation (58) : then the wave function is clearly not localized, it is defined for any point of space, arbitrarily far away from the nucleus. When the energy level of an electron is modified, for all values of r, θ, φ (including arbitrarily large values of r) the wave function must be modified instantaneously (to keep the integrability and a continuous solution) : such a mechanism is not explained by their theories. We did not interpret the present theory but we don't have this problem since the values of the parameter r inside the wave function are limited by the values of the aphelion and the perihelion of Sommerfeld's model. Only the integral over dr could have a physical meaning in fact, depending on the interpretation.

6) Dirac's solution for the Hydrogen atom can be found on the website of the University of Califnornia San Diego by example, the weblink is given in references [USCD]. The energy levels take the form (we keep notations)

$$E = \frac{m_0 c^2}{\sqrt{1 + \frac{\alpha^2}{(n_r + \sqrt{(j+1/2)^2 - \alpha^2})^2}}} \quad ; \quad l\pm = j \pm 1/2 \ ; \ l \ integer \tag{60}$$

With those notations, fixing $l = 0$ and $j = -1/2$ makes the energy imaginary. This unphysical state is often suppressed (or at least hidden) by changing notations. Considering a free particle by fixing $\alpha = 0$, in both Dirac's and Schrodinger's energy levels (equations (60) and (48)), gives the rest energy of the electron ($E = m_0 c^2$ or $\epsilon = 0$) while the non interacting Hamiltonian (see equation (52) for Schrodinger's Hamiltonian), in both theories, correspond to a moving particle with non vanishing momentum. We found no explanation in literature to justify that supressing the interaction in the energy levels (fixing $\alpha = 0$), gives the energy of a particle at rest in both Dirac's and Schrodinger's theories, while their Hamiltonian (=energy) have momentum terms. In our theory, the fact that $E = m_0 c^2$ when $\alpha = 0$ is easily justified by the fact that our quantum rules are based on the periodicity of the system : if a non interacting system is periodic, it is at rest. Attached to the reference [USCD] we give a weblink to a criticism of Dirac equation when applied to Hydrogen : by example, the solution being a spinor, the author claims that some components of the spinor correspond to unobserved states. Many other elements are discussed.

7) No one has been able to conciliates Dirac's relativistic theory (with matrices and spinors) with a relativistic theory of gravitation producing a satisfying description of matter. Our relativistic quantum theory of matter is described by exactly the same process we will use for gravitation (see later). It justifies the relativitic energy levels of atoms and their degeneracy.

V – Metric and wavefunction

We now investigate the connection between the quantum principle of minimal coupling (see [Dirac, 1928]) and the wave function, and observe the modification "*non interacting system* $\rightarrow$ *interacting system*" with $V = -k/r$:

$$E = \gamma m_0 c^2 \rightarrow E - V = \gamma m_0 c^2 \Leftrightarrow E = \gamma m_0 c^2 + V \tag{61}$$

$$E\Psi = \hbar \frac{\partial}{\partial t}\Psi \rightarrow (E - V)\Psi = E(1 - \frac{V}{E})\Psi = \hbar \frac{\partial}{\partial t'}\Psi = \hbar \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\Psi \tag{62}$$

then

$$\frac{\partial t}{\partial t'} = (1 - \frac{V}{E}) \Leftrightarrow t' = \frac{t}{1 - \frac{V}{E}} = t(\frac{E}{E - V}) = t(1 + \frac{V}{E - V}) = t(1 + \frac{V}{\gamma m_0 c^2}) \tag{63}$$

We deduce that the metric corresponding to the minimal coupling, for a central potential, is then :

$$c^2 dt^2 (1 - \frac{k}{\gamma m_0 c^2 r})^2 = dx^2 + dy^2 + dz^2 + ds^2 \tag{64}$$

Or, in polar coordinates with $dz = 0$,

$$c^2 dt^2 (1 - \frac{k}{\gamma m_0 c^2 r})^2 = dr^2 + r^2 d\phi^2 + ds^2 \tag{65}$$

Precisely, from this metric, we now deduce the equation of motion (22) (which will show the equivalence between minimal coupling and the above metric). This is done from equations (66) to (74) and is a new result. With :

$$E = \gamma m_0 c^2 - \frac{k}{r} \Leftrightarrow \gamma = (E + \frac{k}{r})/(m_0 c^2) \quad (\gamma^{-2} = 1 - \frac{dr^2}{c^2 dt^2} - \frac{r^2 d\phi^2}{c^2 dt^2}) \tag{66}$$

using le left hand side of the previous equation and the definition of L :

$$L = \gamma m_0 r^2 \frac{d\phi}{dt} \Leftrightarrow d\phi^2 = (\frac{L}{\gamma m_0 r^2})^2 dt^2 = (\frac{L}{(E + \frac{k}{r})r^2})^2 c^2 dt^2 \tag{67}$$

$$\frac{d\phi^2}{dt^2} = (\frac{L}{\gamma m_0 r^2})^2 = (\frac{Lc^2}{(E + \frac{k}{r})r^2})^2 \tag{68}$$

from the metric (65) and equation (66)

$$\frac{ds^2}{c^2 dt^2} = (1 - \frac{k}{\gamma m_0 c^2 r})^2 - 1 + \frac{1}{\gamma^2} \tag{69}$$

using the metric (65) and the previous equation,

$$dr^2 = [(1 - \frac{k}{\gamma m_0 c^2 r})^2 - \frac{r^2 d\phi^2}{c^2 dt^2} - \frac{ds^2}{c^2 dt^2}]c^2 dt^2 = [-\frac{r^2 d\phi^2}{c^2 dt^2} + 1 - \frac{1}{\gamma^2}]c^2 dt^2 \tag{70}$$

using equations (68) and (66), we rewrite equation (70) (right hand side)

$$dr^2 = [-(\frac{(Lc)^2}{(E + \frac{k}{r})r})^2 + 1 - (\frac{m_0 c^2}{E + \frac{k}{r}})^2]c^2 dt^2 = [\frac{-(Lc)^2 + (E + \frac{k}{r})^2 r^2 - m_0^2 c^4 r^2}{(E + \frac{k}{r})^2 r^2}]c^2 dt^2 \tag{71}$$

using the previous equation and (67)

$$\frac{dr^2}{d\phi^2} = \frac{-L^2 c^2 r^2 + (E + \frac{k}{r})^2 r^4 - m_0^2 c^4 r^4}{L^2 c^2} \tag{72}$$

$$L^2 c^2 (\frac{1}{r^4}\frac{dr^2}{d\phi^2}) = -\frac{L^2 c^2}{r^2} + E^2 + 2E\frac{k}{r} + \frac{k^2}{r^2} - m_0 c^4 \tag{73}$$

With $u = 1/r, (\frac{du}{d\phi})^2 = (-\frac{dr}{r^2 d\phi})^2 = (\frac{1}{r^4}\frac{dr^2}{d\phi^2})$ and $L'^2 = L^2 - (k/c)^2$ this equation becomes

$$E^2 - m_0 c^4 = L^2 c^2 (\frac{du^2}{d\phi^2}) + L'^2 c^2 u^2 - 2Eku \tag{74}$$

We recognize equation (22) deduced from the metric (65).

VI – A new relativistic theory of gravitation

We now simply repeat the analysis of the previous part, for gravitation. Fixing, for gravitation, inertial mass $\leftrightarrow$ gravitationnal mass, (or energy $\gamma m_0 c^2$ (mass+kinetic) $\leftrightarrow$ gravitationnal charge) we obtain the postulate **P2b** :

$$V = -\frac{GM\gamma m_0}{r} = -\frac{GM\gamma m_0 c^2}{c^2 r} \tag{75}$$

producing, according to equations (61) to (65), the metric :

$$c^2 dt^2 (1 - \frac{GM}{c^2 r})^2 = dr^2 + r^2 d\phi^2 + ds^2 \tag{76}$$

From equation (77) to equation (92), we will now compute the equation of the motion corresponding to this potential, and compare it with the equation of the motion of general relativity given in equation (96). We write, as prelimaries :

$$E - V = \gamma m_0 c^2 \Leftrightarrow E + \frac{GM\gamma m_0 c^2}{c^2} = \gamma m_0 c^2 \tag{77}$$

$$E = \gamma m_0 c^2 (1 - \frac{GM}{c^2 r}) \Leftrightarrow \gamma = \frac{E}{m_0 c^2 (1 - \frac{GM}{c^2 r})} \tag{78}$$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = 1 - (\frac{dr}{dct})^2 - r^2 (\frac{d\phi}{dct})^2 \tag{79}$$

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{p} \wedge \vec{r})}{dt} = \frac{d\vec{p}}{dt} \wedge \vec{r} + \frac{d\vec{r}}{dt} \wedge \vec{p} = \vec{F} \wedge \vec{r} + \vec{v} \wedge \gamma m_0 \vec{v} = 0 \tag{80}$$

using (78) and the definition of $(\vec{p} \wedge \vec{r})$

$$L = \gamma m_0 r^2 \frac{d\phi}{dt} = \frac{E}{c^2 (1 - \frac{GM}{c^2 r})} r^2 \frac{d\phi}{dt} \tag{81}$$

$$d\phi = \frac{Lc^2}{E} (1 - \frac{GM}{c^2 r}) \frac{1}{r^2} dt \Leftrightarrow r^2 \frac{d\phi}{dt} = \frac{Lc^2}{E} (1 - \frac{GM}{c^2 r}) \tag{82}$$

And we now start, in polar coordinates (spherical coordinates with $\theta = \pi/2$) :

$$ds^2 = (1 - \frac{GM}{c^2 r})^2 c^2 dt^2 - dr^2 - r^2 d\phi^2 \Leftrightarrow \frac{ds^2}{c^2 dt^2} = (1 - \frac{GM}{c^2 r})^2 - 1 + 1 - (\frac{dr^2}{c^2 dt^2}) - r^2 (\frac{d\phi^2}{c^2 dt^2}) \tag{83}$$

and using (79) and (78),

$$\frac{ds^2}{c^2 dt^2} = (1 - \frac{GM}{c^2 r})^2 - 1 + (\frac{m_o c^2}{E})^2 (1 - \frac{GM}{c^2 r})^2 \tag{84}$$

Using the metric (76)

$$dr^2 = [(1 - \frac{GM}{c^2 r})^2 - \frac{ds^2}{c^2 dt^2} - r^2 \frac{d\phi^2}{c^2 dt^2}] c^2 dt^2 \tag{85}$$

$$dr^2 = [(1 - \frac{GM}{c^2 r})^2 - \frac{ds^2}{c^2 dt^2} - \frac{1}{r^2 c^2} (r^2 \frac{d\phi}{dt})^2] c^2 dt^2 \tag{86}$$

using now (84) and (82)

$$dr^2 = [(1-\frac{GM}{c^2r})^2+1-(1-\frac{GM}{c^2r})^2-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2-\frac{1}{r^2c^2}(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2 \quad (87)$$

$$dr^2 = [1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2-\frac{1}{r^2c^2}(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2 \quad (88)$$

using (82),

$$\frac{dr^2}{d\phi^2} = \frac{[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2-\frac{1}{r^2c^2}(\frac{Lc}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2}{[(\frac{Lc^2}{E})^2(1-\frac{GM}{c^2r})^2\frac{1}{r^4c^2}]c^2dt^2} \quad (89)$$

$$\frac{dr^2}{d\phi^2} = r^4c^2(\frac{E}{Lc^2})^2\frac{1}{(1-\frac{GM}{c^2r})^2}-r^4c^2(\frac{m_0}{L})^2-r^2 \quad (90)$$

$$\frac{dr^2}{r^4d\phi^2} = (\frac{E}{Lc})^2\frac{1}{(1-\frac{GM}{c^2r})^2}-c^2(\frac{m_0}{L})^2-\frac{1}{r^2} \quad (91)$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{r^2d\phi})^2 = (\frac{E}{Lc})^2-(1-\frac{GM}{c^2r})^2((\frac{m_0c}{L})^2+\frac{1}{r^2}) \quad (92)$$

Changing variables with

$$u=\frac{1}{r} \quad ; \quad (\frac{du}{d\phi})^2 = (-\frac{\frac{dr}{d\phi}}{r^2})^2 = (\frac{dr}{r^2d\phi})^2 \quad (93)$$

gives

$$(1-\frac{GMu}{c^2})^2(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2-(1-\frac{GMu}{c^2})^2((\frac{m_0c}{L})^2+u^2) \quad (94)$$

and taking the low field approximation : $(1-\frac{GMu}{c^2})^2 \rightarrow (1-\frac{2GMu}{c^2})$, produces

$$(1-\frac{2GMu}{c^2})(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2-(1-\frac{2GMu}{c^2})((\frac{m_0c}{L})^2+u^2) \quad (95)$$

This equation can be compared with the equation of motion in general relativity (See C Magnan reproducing J.A. Wheeler and S. Weinberg for an example [Magnan, 2007]):

$$(\frac{du}{d\phi})^2 = (\frac{E}{Lc})^2-(1-\frac{2GMu}{c^2})((\frac{m_0c}{L})^2+u^2) \quad (96)$$

where we have the 3 same roots for $du/d\phi = 0$ and then the same aphelion and perihelion for orbiting particles/planets. The two equations differ by an extra term $(-\frac{2Gmu}{c^2} = -\frac{2Gm}{c^2r})$ in equation (95) such that the behavior of a test particle (including a photon with $m_0 = 0$) will be similar if this term is negligible when compared to unity. Table 2 recalls well known experiments which validated the theory of general relativity.

Table 2 : Experimental accuracy for measurements in general relativity

Reference	Measurement	Accuracy (exp)	$\frac{2GM}{c^2r}$
Hafele -Keating (1972)	Gravitationnal time dilation on Earth	$\approx 10^{-1}$	$\approx 10^{-9}$
Pound-Rebka (1959)	Gravitationnal redshift on Earth	$\approx 10^{-1}$	$\approx 10^{-9}$
Vessot et al. (1980)	Gravitationnal redshift on Earth	$\approx 10^{-4}$	$\approx 10^{-9}$
Shapiro (1968)	Gravitationnal time delay induced by Sun	$\approx 10^{-1}$	$\approx 10^{-6}$
Clemence (1947)	Mercury perihelion	$\approx 10^{-1}$	$\approx 10^{-8}$
TMET (1973)	Light Deflection by Sun (during eclipse)	$\approx 10^{-1}$	$\approx 10^{-6}$

TMET refers to "Texas Mauritanian Eclipse Team".

In each case, the extra term $\frac{2GM}{c^2r}$ is negligible when compared to unity or experimental uncertainties. Evidently, is was justified to take the low field approximation from equation (94) to (95).
In the cassical limit ($E^2-m_0^2c^4 \to 2m_0c^2\epsilon < 0$) it is known that equation (96) can be rewritten,

$$(\frac{du}{d\phi})^2 = \frac{2m_0\epsilon}{L^2}+\frac{2GMm_0^2}{L^2}-u^2+\frac{2GMu^3}{c^2} \Leftrightarrow \epsilon = \frac{L^2}{2m_0}(\frac{du}{d\phi})^2+\frac{L^2}{2m_0}u^2-GMm_0u-\frac{GML^2u^3}{m_0c^2} \quad (97)$$

We recognize here equation (2) (left hand side) with an extra term, producing a shift of the perihelion.

Studying the electromagnetic potential, we observed that the equation of motion could be deduced with or without the metric. We now show that our definition of the gravitationnal interaction is sufficient to deduce the equation of motion (92), without the use of the metric (76). This is done fom equation (98) to equation (114). From

$$E-V=\gamma m_0c^2 \Leftrightarrow E+\frac{GM\gamma m_0c^2}{c^2} = \gamma m_0c^2 \Leftrightarrow E=\gamma m_0c^2(1-\frac{GM}{c^2r}) \quad (98)$$

we deduce :

$$(\frac{E}{\gamma m_0c^2})^2 = (1-\frac{v^2}{c^2})(\frac{E}{m_0c^2})^2 = (\frac{E}{m_0c^2})^2[1-(\frac{dr^2}{c^2dt^2})-r^2(\frac{d\phi^2}{c^2dt^2})] = (1-\frac{GM}{c^2r})^2 \quad (99)$$

from which we deduce

$$r^2(\frac{d\phi^2}{c^2dt^2}) = 1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2-(\frac{dr^2}{c^2dt^2}) \quad (100)$$

Then with (98) and the definition of $|\vec{L}| = |\vec{p}\wedge\vec{r}| = |\gamma m_0r^2(\frac{d\phi}{dt})|$

$$r^2\frac{d\phi}{dt} = \frac{L}{\gamma m_0} = \frac{Lc^2}{E}(1-\frac{GM}{c^2r}) \Leftrightarrow r^2(\frac{d\phi^2}{c^2dt^2}) = \frac{1}{r^2c^2}(r^4\frac{d\phi^2}{dt^2}) = (\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2 \quad (101)$$

we rewrite (99)

$$[1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(\frac{dr^2}{c^2dt^2})](\frac{E}{m_0c^2})^2 = (1-\frac{GM}{c^2r})^2 \quad (102)$$

$$[1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(\frac{dr^2}{c^2dt^2})] = (1-\frac{GM}{c^2r})^2(\frac{m_0c^2}{E})^2 \quad (103)$$

$$1-(\frac{Lc}{rE})^2(1-\frac{GM}{c^2r})^2-(1-\frac{GM}{c^2r})(\frac{m_0c^2}{E})^2 = (\frac{dr^2}{c^2dt^2}) \quad (104)$$

$$[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]c^2dt^2 = dr^2 \quad (105)$$

rewriting (100)

$$r^2d\phi^2 = [1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]c^2dt^2-dr^2 \quad (106)$$

dividing (105) by (106) gives

$$(\frac{rd\phi}{dr})^2 = -1+\frac{[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]} \quad (107)$$

$$(\frac{rd\phi}{dr})^2 = \frac{[-1+(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]+[1-(\frac{m_0c^2}{E})^2(1-\frac{GM}{c^2r})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]]} \quad (108)$$

$$(\frac{rd\phi}{dr})^2 = \frac{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]}{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]} \tag{109}$$

$$(\frac{dr}{rd\phi})^2 = \frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2+(\frac{m_0c^2}{E})^2]}{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]} \tag{110}$$

$$(\frac{dr}{rd\phi})^2 = \frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{m_0c^2}{E})^2]}{[(1-\frac{GM}{c^2r})^2[(\frac{Lc}{rE})^2]} - 1 \tag{111}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2 = \frac{[1-(1-\frac{GM}{c^2r})^2[(\frac{m_0c^2}{E})^2]}{[(\frac{Lc}{rE})^2]} - (1-\frac{GM}{c^2r})^2 \tag{112}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2 = [(\frac{rE}{Lc})^2 - (1-\frac{GM}{c^2r})^2[(\frac{m_0cr}{L})^2] - (1-\frac{GM}{c^2r})^2 \tag{113}$$

$$(1-\frac{GM}{c^2r})^2(\frac{dr}{rd\phi})^2 = [(\frac{rE}{Lc})^2 - (1-\frac{GM}{c^2r})^2[(\frac{m_0cr}{L})^2+1] \tag{114}$$

We recognize (92), deduced without the metric (76).

VII – Comparison with general relativity

We previsouly compared general relativity and the present theory of gravitation in the low field approximation, corresponding to the only available experimental data (see Table 2 in the previous section). Here we compare the theories in the black hole limit. Our point of view should be clarified : the present study, based on a similar description of gravitation and electromagnetism (only those two forces for central potentials are described here), does not include the running of coupling constants observed in particles physics (and existing for electromagnetic, strong and weak forces). Here we have no reason to believe that this running of the constants does not apply to gravitation. Under this consideration, discussing black holes in the present theory is certainly extending the theory outside its scope. Nevertheless, we believe our theory has some advantages, when compared to general relativity, in the black hole limit.

1) We first reproduce the results of general relativity, taken from [Blau, 2016], Y. Choquet Bruhat [Choquet Bruhat, 2015] and E. Scornet [Scornet, 2010]. We recall Schwarzchild metric (all cited authors use the (- +++) signature) :

$$ds^2 = -(1-\frac{2GM}{c^2r})c^2dt^2 + \frac{dr^2}{1-\frac{2GM}{c^2r}} + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{115}$$

This metric has a singular behavior in the limit $r \to 2GM/c^2$ (the radial part becomes divergent). Following the above mentionned authors, we introduce a retarded time $v = ct + r + \frac{2GM}{c^2}\ ln(\frac{rc^2}{2GM} - 1)$ such that the metric can be written :

$$ds^2 = -(1-\frac{2GM}{c^2r})dv^2 + 2drdv + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{116}$$

All cited authors agree that equation (115) is singular for $r \to 2GM/c^2$ but they emphasize that this singular behavior is only apparent, since equation (116) is not singular in the same limit. We would like to comment this point of view by computing dv :

$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial r}dr = cdt + dr + 2GM/c^2[\frac{c^2/(2GM)}{\frac{rc^2}{2GM}-1}]dr = cdt + dr[1+\frac{1}{\frac{rc^2}{2GM}-1}] = cdt + \frac{dr}{1-\frac{2GM}{c^2r}} \tag{117}$$

Since dv is singular in the limit $r \to 2GM/c^2$, we conclude that ds^2 remains singular in this limit, for both equations (115) and (116). In the case $r < 2GM/c^2$, the time and radial parts of the metric change signs in equation (115), the physical interpretation of such behavior remains speculative (when discussed). This simply suggest, in our opinion, that strong gravitationnal fields ($r \approx 2GM/c^2$) are outside the scope of general relativity, exactly the same way that particles with velocity $v \approx c$ are outside the scope of the classical theory (the gravitationnal field at the horizon of a black hole has never been measured). The advantage of our theory is that there is no such singularity in our metric, and the metric it self was not required.

2) The central singularity ($r \to 0$) is commented as follow by E. Scornet : "From the physical point of view, this means there exist particles which can't escape the black hole and which stop to exist after a finite time (because they reach the [central] singularity which does not belong to space-time)". Y. Choquet-Bruhat gives the following comment on the central singularity : "The idea now is rather to consider that, near this signularity, the gravitationnal field is so strong that it becomes a quantum field or string field". The interesting point is that the same singularity ($r \to 0$), has been naturally supressed in our description of electromagnetism : the lowest energy level in equation (30) correspond to $n_r = 0 \Leftrightarrow p_r = 0 (\Leftrightarrow K = 0)$ and $n_\phi = 1 \Leftrightarrow L = \hbar$. Equation (20) can then be rewritten (with $k = \alpha\hbar c$) :

$$E^2 - m_0^2c^4 = \frac{\hbar^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \Leftrightarrow 0 = (1-\alpha^2)\hbar^2 - 2E\alpha\hbar cr + [m_0^2c^4 - E^2]r^2 \tag{118}$$

with solution

$$r = \frac{E\alpha\hbar c \pm \sqrt{(E\alpha\hbar c)^2 - (1-\alpha^2)\hbar^2(m_0^2c^4 - E^2)}}{m_0^2c^4 - E^2} = \alpha\hbar c[\frac{E \pm \sqrt{E^2 - m_0^2c^4(1-\alpha^2)}}{m_0^2c^4 - E^2}] \tag{119}$$

According to equation (31), the square root vanishes (it could be deduced from the fact that the motion is a circle too), and replacing E by its value gives

$$r = \frac{E\alpha\hbar c}{m_0^2c^4 - E^2} = \frac{m_0c^2\sqrt{1-\alpha^2}\ \alpha\hbar c}{(1-(1-\alpha^2))m_0^2c^4} = \frac{\sqrt{1-\alpha^2}\ \hbar}{\alpha m_0 c} \approx \frac{\hbar}{\alpha m_0 c} \quad (Bohr\ radius) \tag{120}$$

which is the minimal value for r. The same analysis would be much more complicated for gravitation since the gravitationnal charge is not constant, but the quantum rules of our theory would give similarily a minimal value for the parameter r, suppressing the problem of the central singularity.

3)Looking at equation (98), we see that the energy is negative for gravitation if $1 - GM/(c^2r) < 0$, for any positive kinetic energy of a particle. Assuming that our theory can be applied for such gravitationnal fields, negative energy regions of our theory have radii which are half the radii of black holes in general relativity.

Conclusion :

In part I, we examined classical Kepler's problem, gave some new links involving Planck's law, Kepler's third law, and Heiseinberg uncertainties relation. We established some geometrical properties of the three involved angular momenta.
In part II to part IV, we recalled Sommerfeld's model of matter, gave a new model of atoms justifying their relativistic energy levels, their degeneracy, and their wave like behavior. We emphazied on the superposition paradox and examined the differences with Schrodinger's and Dirac's theories. All historical models of atoms (Bohr, Sommerfeld, Pauli, Schrodinger, Dirac, Fock) were discussed.

In part V, using the quantum wave function, we linked the defnition of energy with the definition of a metric (space-time): it was then possible to deduce the equation of motion, with or without the metric, for both electromagnetism and gravitation.
In part VI and VII, we gave a new relativistic theory of gravitation, which we compared to general relativity, from the experimental point of view (the observed differences were of order 10^{-6} to 10^{-9}, much less than the experimental accuracy) and in the black hole limit.
Everything came from 3 very simple postulates **P1**, **P2a/b** and **P3**, only **P2b** being new, since **P3** has been previously suggested by Schrodinger, while **P1** and **P2a** are usual statement of special relativity.
Gravitation and electromagnetism were described in a strictly similar manner.
Other energy levels of motions can be obtained by modifying the interactions in **P2a/b** (which will modify **P1** and **P3**) or the $\gamma m_0 c^2$ term : the present theory of matter, space and time is adaptable.

References

Max Planck, 1900
”Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum”
Verhandlungen der Deutschen Physikalischen Gesellschaft
http : //grundpraktikum.physik.uni − saarland.de/scripts/Planck$_1$*.pdf*

Johannes Kepler, 1619
”Harmonices Mundi [The Harmony of the World], book 5 chapter 3, p189”

Niels Bohr, 1913
”On the Constitution of Atoms and Molecules”
Philosophical Magazine 26 (1913), p.1-24
http : //www.bibnum.education.fr/physique/physique − quantique/de − la − constitution − des − atomes − et − des − molecules

Erwin Schrodinger, 1926
”Quantisierung als Eigenwertproblem”
Ann. Phys. 79, 361 (1926)

Paul Adrien Maurice Dirac, 1928
”The quantum theory of the electron”
Royal Society Publishing
http : //rspa.royalsocietypublishing.org/content/117/778/610

Wolfgang Pauli Jr, 1926
”ber das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik.
Zeitschrift fr Physik 36.5 (1926), p. 336363

Arnold Sommerfeld, 1916
”Atom Bau und Spektrallinien”

Vladimir Fock, 1935
”On the theory of the Hydrogen atom”
Z. Physik 98, 145

Hafele, J. C.; Keating, R. E. (July 14, 1972).
”Around-the-World Atomic Clocks: Observed Relativistic Time Gains”.
Science. 177 (4044): 168170

Pound, R. V.; Rebka Jr. G. A. (November 1, 1959).
”Gravitational Red-Shift in Nuclear Resonance”.
Physical Review Letters. 3 (9): 439441

Vessot, R. F. C.; M. W. Levine, E. M. Mattison, E. L. Blomberg, T. E. Hoffman, G. U. Nystrom, B. F. Farrel, R. Decher, P. B. Eby, C. R. Baugher, J. W. Watts, D. L. Teuber and F. D. Wills (December 29, 1980).
”Test of Relativistic Gravitation with a Space-Borne Hydrogen Maser”.
Physical Review Letters. 45 (26): 20812084

Irwin I. Shapiro; Gordon H. Pettengill; Michael E. Ash; Melvin L. Stone; et al. (1968).
”Fourth Test of General Relativity: Preliminary Results”.

Physical Review Letters. 20 (22): 12651269.

Clemence, G. M. (1947).
"The Relativity Effect in Planetary Motions".
Reviews of Modern Physics. 19 (4): 36136

Texas Mauritanian Eclipse Team,
Gravitational deflection of-light: solar eclipse of 30 June 1973 I. Description of procedures and final results
Astronomical Journal, v.81, p.452

Landau L, Lifschitz
"The Classical Theory of Fields"
Volume 2 in Course of Theoretical Physics, (Fourth Edition)

New York Times, March 12, 2002, by Dennis Overbye
"Peering Through the Gates of Time"
http://www.nytimes.com/2002/03/12/science/peering-through-the-gates-of-time.html

GF Torres del Castillo, 2007
"The hydrogen atom via the four-dimensional spherical harmonics"
Revista Mexicana de Fisica 53(5) 407-414

David J. Griffith, 1995
"Introduction to quantum mechanic"
Library of Congress, Cataloging in publication data

USCD-University of California San Diego
"Solution of the Dirac Equation for Hydrogen"
http : //quantummechanics.ucsd.edu/ph130a/130$_n$otes/node501.html
Weblink for a critcism of Dirac Hydrogen : *http : //www7b.biglobe.ne.jp/ kcy05t/dirachy.html*

Christian Magnan, 2007
"Precession of Mercury Perihelion" (French Website)
https : //lacosmo.com/PrecessionMercure/index.html

Erwan Scornet, 2010
"Relativite generale et trou noir spherique"
http : //www.normalesup.org/ scornet/Memoire − L3.pdf

Matthias Blau, 2016
"Lecture Notes on General Relativity"
http : //www.blau.itp.unibe.ch/newlecturesGR.pdf

Yvonne Choquet-Bruhat, Thibault Damour, 2015
Introduction to General Relativity, Black Holes, and Cosmology
Oxford University Press

Annex : Sommerfeld's integral

We reproduce here [Sommerfeld, 1916] to compute equation (2). Comments are suppressed.

$$p_r = \gamma m_0 \dot{r} = \gamma m_0 \frac{dr}{d\phi}\dot{\phi} = \frac{L}{r^2}\frac{dr}{d\phi} \; ; \; dr = \frac{dr}{d\phi}d\phi \; ; \; p_r dr = L(\frac{dr}{rd\phi})^2 d\phi \tag{121}$$

We have seen under equation (23) that (we fix $\phi_0 = 0$) that $u = 1/r$ takes the form

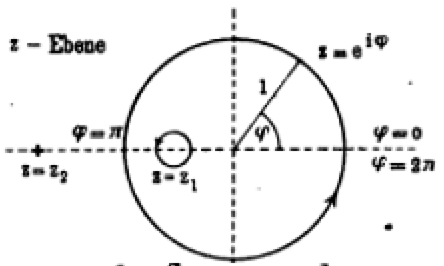
$$u = A(1 + ecos(\Gamma\phi) \Rightarrow \frac{dr}{rd\phi} = \frac{e\Gamma sin(\Gamma\phi)}{1 + ecos(\Gamma\phi)} \tag{122}$$

From this result we deduce, with $\varphi = \Gamma\phi$

$$\oint p_r dr = L\int_0^{2\pi}(\frac{e\Gamma sin(\Gamma\phi)}{1 + ecos(\Gamma\phi)})^2 d\phi = L\Gamma e^2 \int_0^{2\pi}(\frac{sin(\varphi)}{1 + ecos(\varphi)})^2 d\varphi \tag{123}$$

We recall $L\Gamma = L'$. Sommerfeld uses ϵ for e, γ for Γ :

Fig. 100.



$$(1) \quad J_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{1 + \varepsilon\cos\varphi}$$

$$z = e^{i\varphi}$$

$$J_1 = \frac{1}{2\pi i}\oint\frac{dz}{z\left[1 + \frac{\varepsilon}{2}\left(z + \frac{1}{z}\right)\right]} = \frac{1}{\pi i \varepsilon}\oint\frac{dz}{z^2 + 2\eta z + 1}$$

$$\eta = \frac{1}{\varepsilon}; \qquad \begin{cases} z_1 = -\eta + \sqrt{\eta^2 - 1} \; \ldots\ldots -z_1 < 1 \\ z_2 = -\eta - \sqrt{\eta^2 - 1} \; \ldots\ldots -z_2 > 1 \end{cases}$$

$$J_1 = \frac{1}{\pi i \varepsilon}\oint\frac{dz}{(z - z_1)(z - z_2)}$$

$$= \frac{1}{\pi i \varepsilon (z_1 - z_2)}\left(\oint\frac{dz}{z - z_1} - \oint\frac{dz}{z - z_2}\right)$$

$$J_1 = \frac{2\pi i}{\pi i \varepsilon (z_1 - z_2)} \qquad z_1 - z_2 = 2\sqrt{\eta^2 - 1} = \frac{2}{\varepsilon}\sqrt{1 - \varepsilon^2},$$

$$J_1 = \frac{1}{\sqrt{1 - \varepsilon^2}}$$

$$J_2 = \frac{\varepsilon^2}{2\pi}\int_0^{2\pi}\frac{\sin^2\varphi}{(1 + \varepsilon\cos\varphi)^2}d\varphi = \frac{\varepsilon}{2\pi}\frac{\sin\varphi}{1 + \varepsilon\cos\varphi}\Bigg|_0^{2\pi} - \frac{\varepsilon}{2\pi}\int_0^{2\pi}\frac{\cos\varphi}{1 + \varepsilon\cos\varphi}$$

$$J_2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1 + \varepsilon\cos\varphi} - 1\right)d\varphi = J_1 - 1 = \frac{1}{\sqrt{1 - \varepsilon^2}} - 1.$$

with $e = K/J (= c/a)$ we deduce from equation (123) and the above result :

$$\oint p_r dr = 2\pi L'(\frac{1}{\sqrt{1 - e^2}} - 1) = 2\pi L'(\frac{J}{\sqrt{J^2 - K^2}} - 1) = 2\pi L'(\frac{J}{L'} - 1) = 2\pi(J - L') \tag{124}$$

In the classical limit $L' \to L$. The second method starts from the equation (20)

$$E^2 - m_0^2c^4 = p_r^2c^2 + \frac{L^2c^2 - k^2}{r^2} - \frac{2Ek}{r} \Leftrightarrow p_r = \sqrt{[(\frac{E}{c})^2 - m_0^2c^2] + 2\frac{Ek/c^2}{r} - \frac{L'^2}{r^2}} = \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} \tag{125}$$

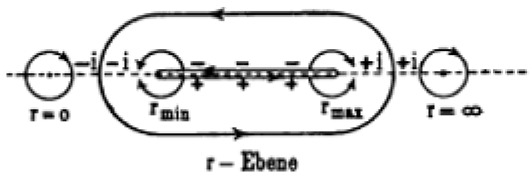
From which we deduce,

$$\oint p_r dr = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} dr \tag{126}$$

This integral must be performed from the aphelion to the perihelion (r_{min} and r_{max}), Sommerfeld computed this integral over the complex plane by means of the residue theorem.

$$J_s = \oint \sqrt{A + 2\frac{B}{r} + \frac{C}{r^2}} dr$$

Fig. 101.



$$r = 0 \Rightarrow \sqrt{C} \int \frac{dr}{r}\left(1 + \frac{B}{C} r + \cdots\right) \Rightarrow -2\pi i \sqrt{C}$$

$$s = \frac{1}{r}, \quad dr = -\frac{ds}{s^2} \qquad J_s = -\int \sqrt{A + 2Bs + Cs^2} \frac{ds}{s^2}$$

$$= -\sqrt{A} \int \left(1 + \frac{B}{A} s + \cdots\right) \frac{ds}{s^2}.$$

$$\Rightarrow + 2\pi i \frac{B}{\sqrt{A}}.$$

$$\Rightarrow J_s = -2\pi i \left(\sqrt{C} - \frac{B}{\sqrt{A}}\right)$$

We deduce,

$$\oint p_r dr = -2\pi i(iL' - iJ) = 2\pi(J - L') \ \ (or\ 2\pi(J - L)\ in\ classical\ limit) \tag{127}$$