EXISTENCE OF TRAVELING WAVES IN THE FRACTIONAL BURGERS EQUATION

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Abstract. We construct traveling waves in the Burgers equation with the fractional laplacian $(D^2)^\alpha$, for $\alpha \in (1/2, 1)$. This is done with first constructing odd solutions u_ε of $uu' = K_{\varepsilon_1} * u - k_{\varepsilon_1} u + \varepsilon_2 u''$, $u(-\infty) = u_c > 0$, with $K_{\varepsilon_1} * u - k_{\varepsilon_1} u$ nonsingular, and then passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$, where we get $K_{\varepsilon_1} * u_\varepsilon - k_{\varepsilon_1} u_\varepsilon \to (D^2)^\alpha u_0$ pointwise from an operator splitting trick estimates, that we discovered and used in a simpler way earlier.

1. Introduction

We study the equation

$$u_t + uu_x - (\partial_{xx})^\alpha u = 0, \tag{1.1}$$

in which the fractional power of the laplacian in one dimension for $\alpha \in (0, 1]$ can be represented as

$$(D^2)^\alpha u(x) = -\frac{1}{2\Gamma(-2\alpha)\cos(\pi\alpha)} \text{ p.v.} \int_R \frac{u(y) - u(x)}{|x-y|^{1+2\alpha}} dy, \tag{1.2}$$

where p.v. denotes the Cauchy principal value. $(D^2)^\alpha$ is also a pseudo-differential operator of symbol $-|\xi|^{2\alpha}$:

$$(D^2)^\alpha u = \mathcal{F}^{-1}(-|\xi|^{2\alpha}(\mathcal{F}u)) \ \forall u \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz class.

The simplest Cauchy problem for the classical Burgers equation

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = (1 - H(x))u_- + H(x)u_+, \end{cases} \tag{1.3}$$

where H is the Heaviside function, has two types of solutions. If $u_- > u_+$, the shock wave $u(x,t) = (1 - H(x - st))u_- + H(x - st)u_+$, with $s = \frac{u_- + u_+}{2}$, is the unique weak solution of (1.3). If $u_- < u_+$, the rarefaction wave

$$u(x,t) = \begin{cases} u_-, & x < u_- t, \\ x/t, & u_- t \le x \le u_+ t, \\ u_+, & x > u_+ t, \end{cases} \tag{1.4}$$

is the unique weak solution of (1.3) with the entropy condition.

Here we are interested in the first type of solutions of (1.1), i.e., traveling waves $U(x - st)$, such that $U(-\infty) = u_-$ and $U(\infty) = u_+$, where $u_- > u_+$. U satisfies

$$U'(U - s) = (D^2)^\alpha U.$$

Date: November 25, 2014.

If we let $u = U - s$, then

$$uu' = (D^2)^\alpha u. \tag{1.5}$$

Integrating (1.5) over R shows that $u_c = u(-\infty) = -u(\infty)$, thus $u_c = \frac{1}{2}(u_- - u_+)$ and $s = \frac{1}{2}(u_- + u_+)$ (the Rankine-Hugoniot condition). These solutions are expected to be globally stable, i.e., the solution of the Cauchy problem with initial value having tails asymptotic to u_- and u_+, $u_- > u_+$, should converge to a translate of the traveling wave asymptotic to u_- and u_+.

The only result in the literature in this direction is the formal nonexistence of smooth traveling waves of (1.1) in the case $\alpha \in (0, 1/2]$ in [5, Proposition 5.1]. Solutions of the Cauchy problem with initial value having tails asymptotic to u_- and u_+, $u_- < u_+$, were shown to converge to the rarefaction wave (1.4) in the case $\alpha \in (1/2, 1)$ [11], to a self-similar solution in the case $\alpha = 1/2$ [4] and to the solution of the linear part of (1.1) with $(1 - H(x))u_- + H(x)u_+$ initial condition in the case $\alpha \in (0, 1/2)$ [4]. Solutions of the Cauchy problem were shown to always remain smooth in the case $\alpha \in (1/2, 1)$ [10] and $\alpha = 1/2$ [12], and to possibly become discontinuous in the case $\alpha \in (0, 1/2)$ [3]. A weak such solution was shown to eventually become smooth for α a little less than $1/2$ [6]. Some other papers on the subject are [1, 2].

Existence of traveling waves was a longstanding problem for the nonlocal Burgers equation

$$u_t + uu_x - K * u + u = 0, \tag{1.6}$$

where K is nonsingular. It was solved in [8] and in more generality in [7]. There the traveling wave can be a shock wave, i.e., discontinuous, if u_c is large enough. Traveling waves of (1.6) are the starting point of our construction, which uses the idea from [9] of getting traveling waves of (1.1) from an appropriate limit. In [9] we constructed traveling waves of

$$u_t - (\partial_{xx})^\alpha + f(u) = 0,$$

where f is bistable, with passing to the limit of traveling wave solutions of

$$u_t - b_\alpha(J_\varepsilon * u - j_\varepsilon u) + f(u) = 0, \tag{1.7}$$

where $b_\alpha = -\frac{1}{2\Gamma(-2\alpha)\cos(\pi\alpha)}$,

$$J_\varepsilon(x) = \begin{cases} \frac{1}{|x|^{1+2\alpha}}, & |x| \geq \varepsilon, \\ \frac{1}{\varepsilon^{1+2\alpha}}, & |x| < \varepsilon, \end{cases} \tag{1.8}$$

and $j_\varepsilon = \int_R J_\varepsilon = (\frac{1}{\alpha} + 2)\frac{1}{\varepsilon^{2\alpha}}$, so that formally $b_\alpha(J_\varepsilon * u - j_\varepsilon u) \to (D^2)^\alpha u$. Traveling waves of (1.7) are guaranteed to be smooth (not discontinuous), if j_ε is large enough. This should also be the case for (1.6) if the nonlocal operator is as in (1.7), however, it is not known how to show it. Since members of the limiting sequence need to be smooth, we overcome this difficulty by constructing first from (1.6) odd solutions of

$$uu' = b_\alpha(K_{\varepsilon_1} * u - k_{\varepsilon_1} u) + \varepsilon_2 u'', \tag{1.9}$$

with $u_c = u(-\infty)$ and $K_{\varepsilon_1} = J_\varepsilon$ in (1.8). If $\varepsilon_2 > 0$ is appropriately chosen, we can then pass to the limit $\varepsilon_1, \varepsilon_2 \to 0$ to obtain in Section 2

Theorem 1.1. *Let $\alpha \in (1/2, 1)$. There exists an odd and smooth solution of (1.5) such that $u(-\infty) = u_c$ and $u' < 0$.*

It should be noted that getting the needed estimates for this passage to the limit is harder than in [9], even though we use the same operator splitting trick.

2. Existence

In [8] we showed

Proposition 1. *Let* $\int_R K = 1$, K *even*, $K \geq 0$, $K \in W^{1,1}(R)$, K *nonincreasing on* $(0, \infty)$, $K(y) = o(\frac{1}{y^4})$ *as* $y \to \infty$. *There exists an odd solution* u *of*

$$uu' = K * u - u, \tag{2.1}$$

such that $u' < 0$ *and* $u(-\infty) = u_c$. *Moreover, if* $u_c > 2\int_R |x|K(x)dx$, u *is discontinuous at* 0.

It follows that there is such a solution u_δ of the equation

$$uu' = b_\alpha(K_{\varepsilon_1,\delta} * u - k_{\varepsilon_1,\delta}u) + \frac{1}{\delta^2}(L_\delta * u - u), \tag{2.2}$$

where $K_{\varepsilon_1,\delta} \nearrow K_{\varepsilon_1}$ and $\frac{1}{\delta^2}(L_\delta * \phi - \phi) \to \varepsilon_2\phi''$ for smooth enough ϕ as $\delta \to 0$. Here $L_\delta(x) = \frac{1}{\delta}L(\frac{x}{\delta})$, $L \geq 0$, $\int_R L = 1$ and $\varepsilon_2 = \frac{1}{2}\int_R x^2 L(x)dx$. Since each u_δ is monotone, from Helly's Theorem there is a subsequence of u_δ, denoted again by u_δ, such that $u_\delta \to u_0$ as $\delta \to 0$. We need to show that u_0 satisfies (1.9) and that $u_0(-\infty) = u_c$. For the first we use weak formulation, for the second strong. Let $S \geq 0$ be such that $\int_R S = 1$, $S \in W^{2,1}(R)$ and $v_\delta = S * u_\delta$. Apply S to (2.2) and integrate from $-\infty$ to x:

$$\frac{1}{2}u_c^2 - S * (\frac{1}{2}u_\delta^2) = \int_{-\infty}^x \big[b_\alpha(K_{\varepsilon_1,\delta} * v_\delta - k_{\varepsilon_1,\delta}v_\delta) + \frac{1}{\delta^2}(L_\delta * v_\delta - v_\delta)\big].$$

Passing to the limit $\delta \to 0$ and integrating from 0 to x we get

$$\int_0^x (\frac{1}{2}u_c^2 - S * (\frac{1}{2}u_0^2)) = \int_0^x \int_R y b_\alpha K_{\varepsilon_1}(y) \int_0^1 v_0(s + ty)dtdyds + \varepsilon_2 v_0(x), \tag{2.3}$$

where $v_0 = S * u_0$. This is from

$$\begin{aligned}\int_{-\infty}^x \big[b_\alpha(K_{\varepsilon_1,\delta} * v_\delta - k_{\varepsilon_1,\delta}v_\delta) &= \lim_{r\to-\infty}\int_r^x \int_R y b_\alpha K_{\varepsilon_1,\delta}(y) \int_0^1 v_\delta'(s + ty)dtdyds \\ &= \int_R y b_\alpha K_{\varepsilon_1,\delta}(y) \int_0^1 [v_\delta(x + ty) - u_c]dtdy \\ &\to \int_R y b_\alpha K_{\varepsilon_1}(y) \int_0^1 v_0(x + ty)dtdy \text{ as } \delta \to 0.\end{aligned}$$

where we used Fubini's Theorem and Dominated Convergence twice, and

$$\int_{-\infty}^x \frac{1}{\delta^2}(L_\delta * v_\delta - v_\delta) = \varepsilon_2 v_\delta' + \int_R L(y) \int_0^y (y - t)[v_\delta'(x + \delta t) - v_\delta'(x)]dtdy,$$

where $v_\delta' \to v_0'$ from $\int_R |S'| < \infty$ with Dominated Convergence and

$$\begin{aligned}\big|\int_R L(y) \int_0^y (y - t)[v_\delta'(x + \delta t) - v_\delta'(x)]dtdy\big| &\leq \delta \max|v_\delta''| \int_R L(y)\big|\int_0^y (y - t)tdt\big|dy \\ &\to 0 \text{ as } \delta \to 0,\end{aligned}$$

from $\int_R |S''| < \infty$ and an addition assumption that $\int_R |y^3|L(y)dy < \infty$.

It is clear that $u_0 \not\equiv 0$. We would now like to take $S_\varepsilon(x) = \frac{1}{\varepsilon}S(\frac{x}{\varepsilon})$ in (2.3) and pass to the limit $\varepsilon \to 0$. However, we do not know if u_0 is continuous, and if it is not at x_{dc}, $v_0(x_{dc}) \to \frac{1}{2}(u_0(x_{dc}-) + u_0(x_{dc}+))$ as $\varepsilon \to 0$. Before we return to (2.3), we use weak formulation in particular to show that u_0 is continuous.

Multiplying (2.2) by $\phi \in C_0^\infty$, integrating over R and passing to the limit $\delta \to 0$, we get

$$\int_R \big[\frac{1}{2}u_0^2\phi' + b_\alpha(K_{\varepsilon_1} * u_0 - k_{\varepsilon_1}u_0)\phi + \varepsilon_2 u_0\phi''\big] = 0. \tag{2.4}$$

For any finite a, b, with integration in (2.4) over (a,b) we get

$$\int_a^b f\phi'' = 0,$$

where

$$f(x) = -\int_0^x \frac{1}{2}u_0^2 + \int_0^x ds\int_0^s b_\alpha(K_{\varepsilon_1} * u_0 - k_{\varepsilon_1}u_0) + \varepsilon_2 u_0(x).$$

It is standard that

$$f(x) = c_1 + c_2 x \text{ a.e.}, \tag{2.5}$$

where c_1, c_2 satisfy the system $F_1(b) = 0$, $F_2(b) = 0$, with

$$F_1(x) = \int_a^x (f(s) - c_1 - c_2 s)ds,\ F_2(x) = \int_a^x F_1.$$

If x_{dc} is a point of discontinuity of u_0, then let a, b be such that $x_{dc} \in (a,b)$. There is sequence of points tending to x_{dc} from the left, and another tending to x_{dc} from the right, at which (2.5) is satisfied. Since in (2.5) only u_0 is potentially not continuous, we pass to the limit getting

$$u_0(x_{dc}-) = u_0(x_{dc}+),$$

where we used that u_0 is monotone. We can now differentiate (2.5) twice to get that u_0 is a solution of (1.9).

With S_ε we pass to the limit $\varepsilon \to 0$ in (2.3) and differentiate it to get

$$\frac{1}{2}u_c^2 - \frac{1}{2}u_0^2(x) = \int_R yb_\alpha K_{\varepsilon_1}(y)\int_0^1 u_0(x+ty)dtdy + \varepsilon_2 u_0'(x).$$

On the other hand, we integrate (1.9) with u_0 from $-\infty$ to x to get

$$\frac{1}{2}u_0^2(-\infty) - \frac{1}{2}u_0^2(x) = \int_R yb_\alpha K_{\varepsilon_1}(y)\int_0^1 u_0(x+ty)dtdy + \varepsilon_2 u_0'(x). \tag{2.6}$$

Thus $u_0(-\infty) = u_c$.

In passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$, if we can show that the three first derivatives of the solution $u_{\varepsilon_1,\varepsilon_2}$ of (1.9) are uniformly bounded, then from Arzelà-Ascoli Theorem there is a subsequence of $u_{\varepsilon_1,\varepsilon_2}$, also denoted by $u_{\varepsilon_1,\varepsilon_2}$, such that $u_{\varepsilon_1,\varepsilon_2} \to u_0$ as $\varepsilon_1, \varepsilon_2 \to 0$ pointwise on R and

$$b_\alpha(K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1}u_{\varepsilon_1,\varepsilon_2}) + \varepsilon_2 u''_{\varepsilon_1,\varepsilon_2} \to (D^2)^\alpha u_0 \text{ as } \varepsilon_1, \varepsilon_2 \to 0$$

pointwise on R, shown e.g., in [9], so that u_0 satisfies (1.5). The idea is to split

$$b_\alpha K_{\varepsilon_1} = P_{\varepsilon_1} + R_{\varepsilon_1},$$

with $R_{\varepsilon_1} \geq 0$, $P_{\varepsilon_1} \in W^{1,1}(R)$ and $p_{\varepsilon_1} = \int_R P_{\varepsilon_1} = 2|\min_{x\in R} u'_{\varepsilon_1,\varepsilon_2}(x)|$, namely

$$P_{\varepsilon_1}(x) = \begin{cases} b_\alpha K_{\varepsilon_1}(x), & x \in R\backslash[-e,e], \\ b_\alpha K_{\varepsilon_1}(-e), & x \in (e,e), \end{cases} \tag{2.7}$$

where $e = \frac{1}{p_{\varepsilon_1}^{1/2\alpha}}(2+\frac{1}{\alpha})^{1/2\alpha}$. Note that the min is attained, since $u'_{\varepsilon_1,\varepsilon_2}(x) \to 0$ as $x \to -\infty$ from (2.6). Let $r_{\varepsilon_1} = \int_R R_{\varepsilon_1}$. After differentiating (1.9), at the min we get

$$-\frac{p_{\varepsilon_1}^2}{4} = P'_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} + R_{\varepsilon_1} * u'_{\varepsilon_1,\varepsilon_2} - r_{\varepsilon_1} u'_{\varepsilon_1,\varepsilon_2} + \varepsilon_2 u'''_{\varepsilon_1,\varepsilon_2} \geq P'_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2}.$$

With (2.7) this becomes

$$\frac{p_{\varepsilon_1}^2}{4} \leq \frac{2u_c}{\left(2+\frac{1}{\alpha}\right)^{\frac{1+2\alpha}{2\alpha}}} p_{\varepsilon_1}^{\frac{1+2\alpha}{2\alpha}}.$$

Since $\alpha > 1/2$, p_{ε_1} is bounded. We need to show though that such a splitting exists. Note that here we are adjusting it to the solution, whereas in [9] the splitting in (1.7) was adjusting to the nonlinearity, i.e., we had $J_\varepsilon = K + S_\varepsilon$ with $b_\alpha k + f' > 0$. We show that $|\min_{x\in R} u'_{\varepsilon_1,\varepsilon_2}(x)|$ is of order lower than $b_\alpha k_{\varepsilon_1}$. From (2.6) we get

$$\varepsilon_2|u'| \leq \frac{1}{2}u_c^2 + u_c b_\alpha k_{\varepsilon_1}^{1-\frac{1}{2\alpha}} \Big[\frac{2(2\alpha-1)}{\left(2+\frac{1}{\alpha}\right)^{1-\frac{1}{2\alpha}}} + \left(2+\frac{1}{\alpha}\right)^{\frac{1}{\alpha}-1}\Big].$$

It now suffices to take $\varepsilon_2 = \frac{1}{k_{\varepsilon_1}^\beta}$, where $\beta < \frac{1}{2\alpha}$.

To estimate $\max_{x\in R} u''_{\varepsilon_1,\varepsilon_2}$, first note that this max is attained, since $u''_{\varepsilon_1,\varepsilon_2} \to 0$ as $x \to -\infty$ from (1.9) and $u'_{\varepsilon_1,\varepsilon_2}(x) \to 0$ as $x \to -\infty$. Using another splitting

$$b_\alpha K_{\varepsilon_1} = P + R_{\varepsilon_1}, \tag{2.8}$$

after differentiating (1.9) twice, at the max we get

$$(p + 3u'_{\varepsilon_1,\varepsilon_2})u''_{\varepsilon_1,\varepsilon_2} \leq P' * u'_{\varepsilon_1,\varepsilon_2}.$$

Since $|u'_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded, so is $|u''_{\varepsilon_1,\varepsilon_2}|$ after taking P such that $p + 3u'_{\varepsilon_1,\varepsilon_2} > 0$.

To estimate $\max_{x\in R} u'''_{\varepsilon_1,\varepsilon_2}$, use another splitting as in (2.8). After differentiating (1.9) three times, at the max we get

$$(p + 4u'_{\varepsilon_1,\varepsilon_2})u'''_{\varepsilon_1,\varepsilon_2} = K' * u''_{\varepsilon_1,\varepsilon_2},$$

so as before $|u'''_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded.

To show that $u_0(-\infty) = u_c$, we argue as before. Integrate (1.9) from $-\infty$ to x, pass to the limit $\varepsilon_1, \varepsilon_2 \to 0$, then integrate (1.5) from $-\infty$ to x and compare the two. The only difficulty is in showing that

$$\int_{-\infty}^x (K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1} u_{\varepsilon_1,\varepsilon_2}) \to \int_{-\infty}^x (D^2)^\alpha u_0 \text{ as } \varepsilon_1 \to 0.$$

To manage the singularity it is standard that we consider separately integration on $R\backslash(-1,1)$ and $(-1,1)$, i.e.,

$$\int_{-\infty}^x (K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1} u_{\varepsilon_1,\varepsilon_2}) = I_1 + I_2,$$

where

$$I_1 = \lim_{r \to -\infty} \int_r^x \int_{R\setminus(-1,1)} yK_{\varepsilon_1}(y) \int_0^1 u'_{\varepsilon_1,\varepsilon_2}(s+ty)dtdyds$$
$$= \int_{R\setminus(-1,1)} yK_{\varepsilon_1}(y) \int_0^1 u_{\varepsilon_1,\varepsilon_2}(x+ty)dtdy$$

and

$$I_2 = \lim_{r \to -\infty} \int_r^x \int_{-1}^1 y^2 K_{\varepsilon_1}(y) \int_0^1 (1-t)u''_{\varepsilon_1,\varepsilon_2}(s+ty)dtdyds$$
$$= \int_{-1}^1 y^2 K_{\varepsilon_1}(y) \int_0^1 (1-t)u'_{\varepsilon_1,\varepsilon_2}(x+ty)dtdy.$$

Passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$ in I_1 and I_2 and doing the same integrations in $\int_{-\infty}^x (D^2)^\alpha u_0$ we get

$$I_1 + I_2 \to \int_{-\infty}^x (D^2)^\alpha u_0.$$

Note that we can show that I_1 is finite only for $\alpha > 1/2$.

To show that $u'_0 < 0$, differentiate (1.5). If $u'_0(x_{\max}) = 0$ at a point $x_{\max}$, then also $u''_0(x_{\max}) = 0$. On the other hand, $((D^2)^\alpha u_0)(x_{\max}) < 0$, a contradiction. To justify $((D^2)^\alpha u_0)' = (D^2)^\alpha u'_0$, it suffices that $|u'''_0|$ is bounded, which is after additionally showing that $|u''''_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded, as above.

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