Studies on finite semigroups, semigroup semirings, group semirings on semirings as distributive lattices

Ph D Dissertation

Mrs. Jayshree K.

University of Madras

India

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ABSTRACT OF THE THESIS

As the name suggests, a semigroup is a generalization of a group; because a semigroup need not in general have an element which has an inverse. The algebraic structure enjoyed by a semigroup is a non-empty set together with an associative closed binary operation. From the literature survey on semigroups it is clear that not significant amount of research has been done in which study is based on how far the finite semigroups satisfy the classical theorems for finite groups. To the best of one's knowledge such a study is absent. So here the problem of analyzing how far a finite semigroup relates itself to the properties of classical theorems enjoyed by finite group is carried out in this thesis This problem/study leads to the definition of several new and interesting concepts in semigroups. This study is new and innovative. Further, special elements like S-idempotents, S-units, S-zerodivisors, S-nilpotents and S-anti zero divisors are defined for semigroups for the first time. Conditions on these semigroups to contain these special elements is obtained. Here, the study of semigroup semirings and group semirings is carried out for the first time using only distributive finite lattices. As finite distributive lattices are nothing but two distinct idempotent semigroups connected by distributive laws

CONTENTS

CHAPTER ONE

AN INTRODUCTION TO BASIC CONCEPTS ON SEMIGROUPS, SEMIRINGS AND DISTRIBUTIVE LATTICES

The semigroup happens to be an algebraic structure with a single closed associative binary operation on a non-empty set. Semigroups have many interesting and important properties and applications. Moreover, the most generalized algebraic structures for groups are semigroups. The study of semigroups, most of the time is often carried out in a fashion similar to those for rings, like finding idempotents, zero divisors, regular elements and so on and so forth. But a study of this structure relating to groups is very meager. As semigroups are generalizations of groups, in this research work, those properties of groups which are satisfied or otherwise by semigroups are analysed and studied.

Semirings are nothing but semigroups on the same set with two binary operations + and × (or $\cup$ and $\cap$) related by the distributive operation. Hence the study of semirings has become mandatory. Here, semigroup semirings and group semirings are studied using distributive lattices as semirings. Finally the notion of Smarandache zero divisors (S-zero divisors), Smarandache idempotents (S-idempotents) and Smarandache nilpotents (S-nilpotents) can be referred from [98, 100].

1.1 SEMIGROUPS AND THEIR PROPERTIES

In this section, a brief recollection of semigroups and their properties are made. Semigroups are the generalization of groups; however, semigroups in general, as analyzed by researchers come under such topics like regular semigroups, inverse semigroups, homomorphism of semigroups etc. Most of the time, semigroups are defined in an abstract way by researchers as a result of which one is not in a position to visualize them.

Definition 1.1.1: *Let S be a non-empty set on which is defined a binary operation *. (S, *) is a semigroup if;*

> i) *for all a, b $\in$ S; a * b = c $\in$ S*
>
> ii) *a * (b * c) = (a * b) * c*

for all a, b, c $\in$ S.

If the number of elements in S is finite, S is called as a finite semigroup otherwise, S is an infinite semigroup.

Example 1.1.1: Let $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of integers modulo 10. Z_{10} under multiplication is a semigroup denoted by $\{Z_{10}, \times\}$.

Thus, throughout this thesis is $Z_n = \{0, 1, 2, \ldots, n - 1\}$ denotes the set of integers modulo n. The notation used in this thesis is $Z_n = \{0, g_1 = 1, g_2 = 2, \ldots, g_{n-1} = n - 1\}$ or $Z_n = \{0, 1, g_1 = 2\ g_2 = 3, \ldots, g_{n-2} = n - 1\}$.

Z_n under multiplication is a semigroup denoted by $\{Z_n, \times\}$.

Example 1.1.2: Let

$S(3)$ = {set of all mappings of the set (1, 2, 3) to (1 2 3)}

$$= \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\right.$$

$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&3\end{pmatrix},$$

$$\left.\begin{pmatrix}1&2&3\\3&2&3\end{pmatrix}\right\}$$

under the composition of mappings is a semigroup, with 3^3 elements in it.

This semigroup will be known as the symmetric semigroup of degree 3.

Throughout this thesis $S(n)$ = {set of all mappings of the set (1, 2, 3, …, n) to (1, 2, 3, …, n)}. $S(n)$ under the composition of maps will be known as the symmetric semigroup of degree n and $S(n)$ has n^n elements in it.

Definition 1.1.2: *A semigroup (S, *) which has an identity element e such that $e * s = s * e = S$ for all $s \in S$ will be known as the monoid.*

$S(n)$ is a monoid. $S = (Z_n, \times)$ is also a monoid as $1 \in Z_n$ acts as the identity of S: Finally the definition of subsemigroup and ideals are recalled from [20-21].

Definition 1.1.3: *Let (S, *) be a semigroup. If H is a proper subset of S such that (H, *) is a semigroup, then H is defined as the subsemigroup of S.*

If H is a subsemigroup and $s * h$ and $h * s \in H$ for all $s \in S$ and $h \in H$, then H is defined as an ideal of S.

Definition 1.1.4: *Let (S, *) be a semigroup. Let A be a proper subset of S such that (A, *) is a group, then S is defined as a Smarandache semigroup [99].*

Example 1.1.3: Let $S = (Z_{12}, \times)$ be a semigroup. $H = \{1, 11, \times\}$ is a subsemigroup of S. In fact H is a group under × so S is a Smarandache semigroup.

Let P = {0, 2, 4, 6, 8, 10} $\subseteq Z_{12}$, $\{P, \times\}$ is a subsemigroup as well as an ideal of S [99].

For every $s \in S$ and $p \in P$; $s \times p = p \times s \in P$.

Example 1.1.4: Let $S(4)$ be the symmetric semigroup of degree four under the operation of composition of mappings.

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\} \subseteq S(4)$$

is a group. So $S(4)$ is a Smarandache semigroup.

Now

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\} \subseteq S(4)$$

is subsemigroup of $S(4)$ which is also an ideal of $S(4)$.

Now let $\{S, \times\}$ be a semigroup with unit 1 or a monoid. If for $x \in S$ there exists a $y \in S$ such that $x \times y = y \times x = 1$ then x is defined as a unit in S.

If for some $a \in S$ there exist $a, b \in S$ such that $a \times b = 0 \in S$ then a is a zero divisor in S. If for some $t \in S \setminus \{0, 1\}$; $t \times t = t$ then t is defined as an idempotent of S.

Examples of this situation is given in the following from [99].

Example 1.1.5: Let $S = \{Z_{15}, \times\}$ be a semigroup of order 15. $X = 14 \in S$ is such that $x \times x = 1 \pmod{15}$ is a unit in S.

Let $y = 10 \in S$; $y \times y \equiv 10 \pmod{15}$ is an idempotent of S. Let $x = 3$ and $y = 5 \in S$; clearly $x \times y = y \times x = 0 \pmod{15}$ is a zero divisor in S.

Example 1.1.6: Let $S(5)$ be the symmetric semigroup of degree 5.

$$X = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \in S(5)$$

is an idempotent of $S(5)$.

$$y \circ y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

is the identity element of $S(5)$.

Example 1.1.7: Let $S = \{Z_{12}, \times\}$ be the semigroup of order 12.

Let $x = 6 \in S$, $x \times x = 0 \pmod{12}$ is a nilpotent element of S.

$y = 4$ and $z = 9 \in S$ are such that $4 \times 4 \equiv 4 \pmod{12}$ and $9 \times 9 \equiv 9 \pmod{12}$ are idempotents of S. Let $x = 2$ and $y = 6 \in S$. Clearly $x \times y \equiv 0 \pmod{12}$ is a zero divisor in S.

Now in the following section, some properties of groups and some classical theorems on finite group theory are recalled from [39, 56].

1.2 PROPERTIES OF GROUPS AND CLASSICAL THEOREMS ON FINITE GROUPS

In this section some basic properties of groups and the classical theorems on finite groups are recalled from [39, 56, 99].

It is a well-known fact that groups are the only algebraic structures with a single binary operation that is mathematically so perfect.

DEFINITION 1.2.1: *A non-empty set of elements G is said to form a group if in G there is defined a binary operation, called the product and denoted by '•' such that*

1. *$a, b \in G$ implies that $a \bullet b \in G$ (closed).*
2. *$a, b, c \in G$ implies $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ (associative law).*
3. *There exists an element $e \in G$ such that $a \bullet e = e \bullet a = a$ for all $a \in G$ (the existence of identity element in G).*
4. *For every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$ (the existence of inverse in G).*

DEFINITION 1.2.2: *A group G is said to be abelian (or commutative) if for every $a, b \in G$; $a \bullet b = b \bullet a$.*

A group, which is not abelian, is called naturally enough, non-abelian. Another natural characteristic of a group *G* is the number of elements it contains. The order of *G* and denote it by *o(G)*.

Here some interesting preliminary results about groups are recalled from [39, 56, 99].

THEOREM 1.2.1: *Let G be a group, then the identity element of G is unique.*

THEOREM 1.2.2: *If G is a group, then every* $a \in G$ *has a unique inverse in G.*

THEOREM 1.2.3: *Let G be a group; for every* $a \in G$, $(a^{-1})^{-1} = a$.

THEOREM 1.2.4: *Let G be a group. For* $a, b \in G$ $(a \bullet b)^{-1} = b^{-1} \bullet a^{-1}$.

DEFINITION 1.2.3: *A non-empty subset H of a group (G, *) is said to be a subgroup of G if, (H, *) is itself a group.*

The following remark is clear; if *H* is a subgroup of *G* and *K* is a subgroup of *H*, then *K* is a subgroup of *G*.

THEOREM 1.2.5: *A non-empty subset H of the group* $(G, \bullet)$ *is a subgroup of G if and only if*

1. $a, b \in H$ *implies that* $a \bullet b \in H$.
2. $a \in H$ *implies that* $a^{-1} \in H$.

THEOREM 1.2.6: *If H is a non-empty finite subset of group G and H is closed under multiplication, then H is a subgroup of G.*

DEFINITION 1.2.4: *Let G be a group. H a subgroup of G: for* $a,b \in G$, *a is congruent to b mod H, written as* $a \equiv b \pmod H$ *if* $ab^{-1} \in H$. *It is easily verified that the relation* $a \equiv b \pmod H$ *is an equivalence relation.*

DEFINITION 1.2.5: *If H is a subgroup of G,* $a \in G$, *then* $Ha = \{ha / h \in H\}$; *Ha is called a right coset of H in G.*

THEOREM 1.2.7: *For all* $a \in G$, $Ha = \{x \in G / a \equiv x \bmod H\}$.

DEFINITION 1.2.6: *If G is a group and $a \in G$, the order of a is the least positive integer m such that $a^m = e$.*

If no such integer exists than *a* is of infinite order. The notation *o(a)* for the order of *a* is used in this thesis.

DEFINITION 1.2.7: *A subgroup N of a group G is said to be a normal subgroup of G if for every $g \in G$ and $n \in N$, $g\,n\,g^{-1} \in N$.*

DEFINITION 1.2.8: *If $a, b \in G$, then b is said to be a conjugate of a in G if there exists an element $c \in G$ such that $b = c^{-1}ac$. This relation of conjugacy is denoted by $a \sim b$.*

THEOREM 1.2.8: *Conjugacy is an equivalence relation on G.*

DEFINITION 1.2.9: *Let G be a group. A and B subgroups of G, A and B are conjugate with each other if for some $g \in G$, $A = gBg^{-1}$.*

1.3 SEMILATTICES AND DISTRIBUTIVE LATTICES OF FINITE ORDER

In this section definition and properties of semilattices $(L, \cup)$ $(L, \cap)$ and distributive lattices are just recalled. It is to be noted that these semilattices are idempotent semigroups. Further the concept of distributive lattices is recalled from [14, 100].

Definition 1.3.1: *A relation R on a set A is called a partial order (relation) if R is reflexive, antisymmetric and transitive.*

The partial order (relation) is denoted by $\subseteq$ or $\leq$.

Definition 1.3.2: *A partial order relation $\leq$ on A called a total order if for each $a, b \in A$ either $a \leq b$ or $b \leq a$. $(A, \leq)$ is called a chain or a totally ordered set.*

Example 1.3.1: Let $A = \{1, 5, 8, 12, 16, 19\}$, $(A, \leq)$ is a total ordered set. Here $\leq$ is the usual "less than or equal to" relation.

Example 1.3.2: Let $X = \{a, b, c\}$. The power set of *X* denoted by $P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *P(X)* under the relation '$\subseteq$' inclusion of subsets or containment relation, is a partial order on *P(X)*.

It is important and interesting to note that a partially ordered set can be represented by Hasse Diagram [14, 100].

Hasse diagram of the poset (partially order set *A*) given in example 1.3.1 is as follows:

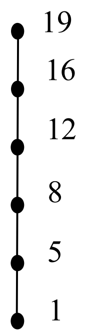


Figure 1.3.1

The Hasse diagram of the poset *P(X)* described in example 1.3.2 is as follows:

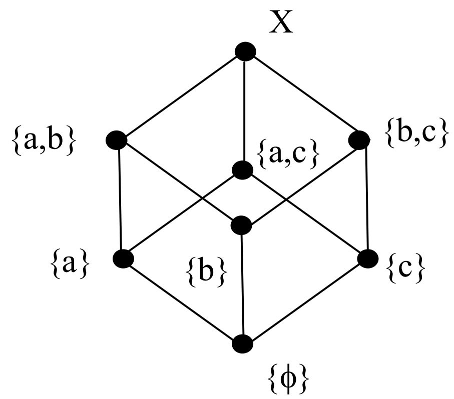


Figure 1.3.2

Definition 1.3.3: *A partially ordered set (poset) (L, ⊴) is called a semilattice if for every pair of elements x, y ∈ L, the sup (x, y) exist (or equivalently the inf (x, y) exist).*

It is to be noted that *sup(x, y)* is also denoted by '$x \cup y$' and *inf (x, y)* is denoted by '$x \cap y$' whenever it makes proper sense.

Now examples of this situation are given in the following:

Example 1.3.3: Let $P = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c\}\}$ be a poset under the operation $inf(x, y) = x \cap y$. The Hasse diagram associated with P is given in Figure 1.3.3.

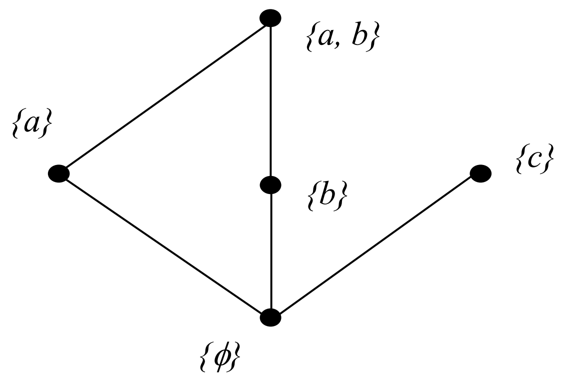


Figure 1.3.3

$$\{a\} \cap \{b\} = inf\{\{a\}, \{b\}\} = \{\phi\} \;\; inf\{\{b\}, \{c\}\} = \{b\} \cap \{c\} = \{\phi\},$$

$inf(\{a, b\}, \{a\}) = \{a, b\} \cap \{a\} = \{a\}$ and so on $\{P, \cap$ (or $inf(x, y)\}$ is a semilattice of order 5.

However $\{P, \cup\}$ or $sup(x, y)$ is not even closed for

$$sup\{\{b\}, \{c\}\} = \{b, c\} \notin P.$$

Example 1.3.4: Let $S = \{X = \{a, b, c\}, \{a\}, \{b], \{b, c\}, \{a, b\} \{a, c\}, sup\{(x, y)\} = x \cup y\}$ be the poset under *sup* operation. S is a semilattice of order six. The Hasse diagram related with S is given by the Figure 1.3.4 which is as follows:

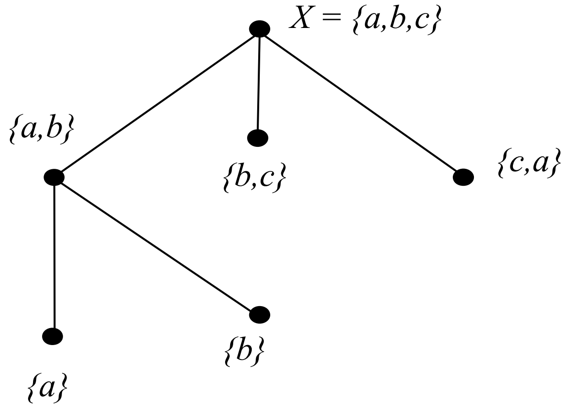


Figure 1.3.4

Now having seen examples of semilattices, the concept of lattice is introduced.

For more about these notions refer [14, 100].

Definition 1.3.4: *A poset (L, $\leq$) is called a lattice order if for every pair of elements x, y in L the sup(x, y) and inf (x, y) exist in L.*

This is illustrated by the following Hasse diagram.

Example 1.3.5: Let $L = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 1, \leq,$ *inf* and *sup*$\}$ be the lattice order given by the following Hasse diagram:

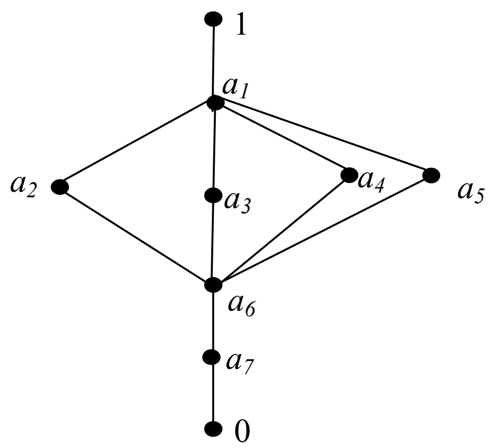


Figure 1.3.5

As in this thesis mainly algebraic operations on a lattice are used, the algebraic lattice is defined in the following:

Definition 1.3.5: *An algebraic lattice (L, $\cap$, $\cup$) is a non-empty set L with two closed binary operations $\cup$ (join) and $\cap$ (meet) (also called as union or sum and intersection or product for join and meet respectively) which satisfy the following conditions for all x, y, z $\in$ L*

L_1 $\quad x \cap y = y \cap x, \qquad x \cup y = y \cup x$

L_2 $\quad x \cap (y \cap z) = (x \cap y) \cap z, \qquad x \cup (y \cup z) = (x \cup y) \cup z$

$L_3 \quad x \cap (x \cup y) = x, \qquad x \cup (x \cap y) = x$

Two applications of L_3 *namely* $x \cap x = x \cap (x \cup (x \cap x)) = x$*; lead to the additional condition*

$$L_4 \quad x \cap x = x \qquad x \cup x = x;$$

the idempotent law.

It is important to note throughout this thesis 1 and 0 are in L and

$$1 \cap x = x, \quad 1 \cup x = 1,$$

$0 \cap x = 0$ *and* $0 \cup x = x$ *for all* $x \in L$.

1 is called the greatest element of L and 0 is defined as the least element of L.

The connection between the lattice order sets and algebraic lattices is as follows:

If $(L \leq)$ *is a lattice ordered set;* $x \cap y = \inf(x, y)$ *and* $x \cup y = \sup(x, y)$*, then* $(L, \cup, \cap)$ *is a algebraic lattice.*

If $(L, \cup, \cap)$ *is an algebraic lattice, then define* $x \leq y$ *if and only if* $x \cap y = x$ *(or* $x \leq y$ *if and only if* $x \cup y = y$*) then* $(L, \leq)$ *is a lattice ordered set.*

By order of L denoted by $|L|$ *or* $o(L)$ *it is meant that the number of distinct elements in L.*

Definition 1.3.6: *A chain lattice* C_n *is a totally ordered set* $0 < a_{n-2} < a_{n-1} < \ldots < a_1 < 1$*; where* $o((C_n)) = n$.

Next the concept of distributive lattices are recalled [14].

Definition 1.3.7: *A lattice L is called distributive if either of the following conditions hold good for all x, y, z in L.*

$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ *or* $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ *called the distributivity equations.*

Examples of distributive lattices. All chain lattices are distributive lattices.

Example 1.3.6: Let *L* be the lattice with the following Hasse diagram:

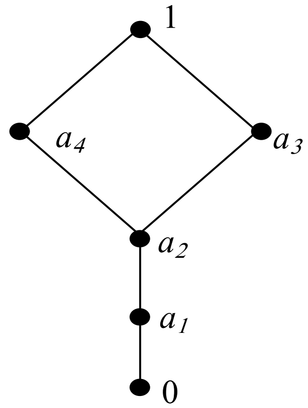


Figure 1.3.6

L is a distributive lattice. This lattice has no zero divisors as $a_i \cap a_j \neq 0$ for $a_i, a_j \in L \setminus \{0\}; 1 \leq i, j \leq 4$.

Definition 1.3.8: *Let L be a lattice with 0 and 1. L is said to be a complemented lattice if for each $x \in L$ there is at least one $y \in L$ such that $x \cap y = 0$ and $x \cup y = 1$. y is called a complement of x.*

Next the notion of Boolean algebra is recalled from [14].

Definition 1.3.9: *A complemented distributive lattice is called a Boolean algebra (or a Boolean lattice).*

Distributivity in a Boolean algebra guarantees the uniqueness of complements.

Example 1.3.7: Let B given by the following Hasse diagram is a Boolean algebra of order four.

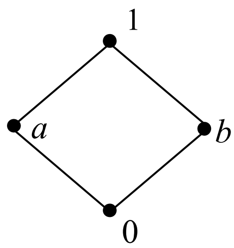


FIGURE 1.3.7

For more about Boolean algebras refer [14].

In fact if $X = \{x_1, x_2, \ldots, x_n\}$. $P(X)$ the power set of X is a Boolean algebra with 2^n elements in it [14].

Next the definition and properties of semirings are recalled.

1.4 SEMIRINGS AND THEIR PROPERTIES

In this section, semirings and their properties are recalled [61, 89, 100]. This is mainly carried out because one does not have many textbooks for semirings except in the book 'Handbook of Algebra' Vol. I, by Udo [89], which carries a section on semirings and semifields and more about semirings are given in [89, 100].

DEFINITION 1.4.1: *Let S be a non-empty set on which is defined two binary operations addition '+' and multiplication '•' satisfying the following conditions:*

1. *$(S, +)$ is a commutative monoid.*
2. *$(S, \bullet)$ is a semigroup.*
3. *$(a + b) \bullet c = a \bullet c + b \bullet c$ and $a \bullet (b + c) = a \bullet b + a \bullet c$ for all a, b, c in S.*

That is multiplication '•' distributes over the operation addition '+'. $(S, +, \bullet)$ is a semiring.

DEFINITION 1.4.2: *The semiring $(S, +, \bullet)$ is said to be a commutative semiring if the semigroup $(S, \bullet)$ is a commutative semigroup. If $(S, \bullet)$ is not a commutative semigroup, then S is a non-commutative semiring.*

DEFINITION 1.4.3: *If in the semiring $(S, +, \bullet)$, $(S, \bullet)$ is a monoid, that is, there exists $1 \in S$ such that $a \bullet 1 = 1 \bullet a = a$ for all $a \in S$; the semiring is defined as a semiring with unit.*

DEFINITION 1.4.4: *Let $(S, +, \bullet)$ be a semiring. The semiring is of characteristic m if $ms = s + \ldots + s$ (m times) equal to zero for all $s \in S$. If no such m exists, the characteristic of the semiring S is 0 and denoted as characteristic $S = 0$. In case S has characteristic m then it is denoted by characteristic $S = m$.*

However in this thesis only distributive finite lattices are taken as semirings. All distributive lattices including the chain lattices and Boolean algebras are semirings.

DEFINITION 1.4.5: *Let S be a semiring. P a subset of S. P is said to be a subsemiring of S if P itself is a semiring.*

DEFINITION 1.4.6: *Let S be a semiring. I be a non-empty subset of S. I is a right (left) ideal of S if*

1. *I is a subsemiring.*
2. *For all $i \in I$ and $s \in S$, $is \in I$ ($si \in I$).*

DEFINITION 1.4.7: *Let S be a semiring. A non-empty subset I of S is said to be an ideal of S if I is simultaneously a right and left ideal of S.*

DEFINITION 1.4.8: *Let S be a semiring. S is a strict semiring if $a + b = 0$ implies $a = 0$ and $b = 0$.*

DEFINITION 1.4.9: *Let S be a semiring with unit 1. An element x is invertible or has an inverse if there exists a $y \in S$ such that $xy = yx = 1$.*

For more about concept and properties of semirings, refer [61, 89, 100].

Next the notion of semigroup rings and groups rings are briefly recalled in this section.

1.5 GROUP RINGS AND SEMIGROUP RINGS AND THEIR PROPERTIES

The study of group rings is nearly seventy-five years old and a systematic research has been carried out by several researchers like [74, 75, 92, 98]. Moreover, semigroup rings was defined in [52, 67, 70, 94, 95, 98].

Here in this thesis just the definitions and a few properties are given.

DEFINITION 1.5.1: *Let R be a commutative ring with unit 1 and G be a multiplicative group. The group ring, RG of the group G over the ring R consists of all finite formal sums of the form $\sum_i \alpha_i g_i$ (i-runs over a finite number) where $\alpha_i \in R$ and $g_i \in G$ satisfying the following conditions:*

i) $\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \beta_i g_i \Leftrightarrow \alpha_i = \beta_i$ *for* $i = 1, 2, \ldots, n$, $g_i \in G$.

ii) $\left(\sum_{i=1}^{n} \alpha_i g_i\right) + \left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i$; $g_i \in G$.

iii) $\left(\sum_{i} \alpha_i g_i\right)\left(\sum_{j} \beta_i g_i\right) = \sum_{k} \gamma_k m_k$ *where* $\gamma_k = \sum \alpha_i \beta_j$, $g_i h_j = m_k$.

iv) $r_i m_i = m_i r_i$ *for all* $r_i \in R$ *and* $m_i \in G$.

v) $r \sum_{i=1}^{n} r_i g_i = \sum_{i=1}^{n} (r r_i) g_i$ *for* r_i, $r \in R$ *and* $\sum r_i g_i \in RG$.

RG is a ring with $0 \in R$ as its additive identity. Since $1 \in R$; $G = 1.G \subset G$ and $R.e = R \subseteq RG$ where e is the identity of G. Clearly if the group G is replaced by a semigroup with identity (monoid) S then RS is the semigroup ring of the semigroup S over the ring R.

For more about these refer [52, 67, 74, 75, 92, 94, 98].

Finally the concept of Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotents have been studied in [98, 100].

1.6 MOTIVATION AND OVERVIEW OF THE RESEARCH WORK

The main motivation for this study is that so far in general, researchers have studied semigroups by studying and analyzing almost all the properties a ring satisfies; like ideals, zero divisors, idempotents and so on. However the study of semigroups as a generalization of groups is meager [66]. So, in this thesis, a study of this type is made, that is, semigroups of finite order which satisfy partially or does not satisfy the classical theorems on finite groups. The scope of this study pertains only to finite semigroups which are not abstractly defined. Since semilattices are semigroups; in particular, idempotent semigroups which are not monoids are also studied.

Further, the semirings which can be realized as semigroups with two different binary operations defined on the same set, but related by the distributive law are studied. The large class of such semirings of finite order comes from the distributive lattices and in this thesis only finite distributive lattices are taken as semirings and the algebraic structure semigroup semirings are built. These semigroups are finite and moreover, all semirings taken in this study are only finite distributive lattices.

Likewise, group semirings of finite groups over these semirings are also studied in this thesis, where semirings are only finite distributive lattices.

The main motivation for this study is because it has been long ignored and unexplored. In mathematical literature, one can find a lot of research in the study of group rings and semigroup rings, however, study of group semirings and semigroup semirings is meager. Study of groupsemirings and semigroup semirings taking semirings as distributive lattices is carried out in [100]. Since this sort of research is not done to the best of our knowledge, such a study new analysis carried out in this thesis.

1.7 SUMMARY AND SCOPE OF THE THESIS

This thesis mainly studies and analyses finite semigroups for the properties of finite groups. Such a study is new and innovative. As all groups are semigroups and semigroups happen to be the generalization of groups such a study is a relevant research.

The scope of this study is limited to finite semigroups, that too, non-abstract semigroups. By non-abstract semigroups this thesis includes semigroups built on modulo integers under product (multiplication), symmetric semigroups *S(n)* and matrix semigroups built using Z_n under natural product $\times_n$; $2 \leq n \leq \infty$. The thesis does not study abstract inverse semigroups, regular semigroups or semigroups got as functions.

1.7.1 Outline of the thesis

This thesis studies for the first time semigroups as a generalization of groups. Here all classical theorems for finite groups are analysed, adopted and studied for finite semigroups. This study has led to the definition of many new concepts. Here, all semigroups considered are non-abstract finite semigroups which can be visualized. The semigroups are $(Z_n \times)$; modulo integers under product and *S(n),* symmetric semigroups obtained from mapping (1, 2, 3, …, n) to (1, 2, 3, …, n). Apart from these matrix semigroups under natural product $\times_n$, defined and developed in [90].

For the first time special elements like Smarandache units, Smarandache idempotents, Smarandache nilpotents and Smarandache zero divisors are defined for semigroups in this thesis. Condition for semigroups to contain these special elements are obtained in this thesis.

Further group rings and semigroup rings have been widely studied by researchers but the study of group semirings and semigroup semirings has not been systematically carried out.

Semirings are nothing but two distinct semigroups connected by the distributive laws. All distributive lattices are semirings. So, in this thesis semigroup semirings and group semirings are analysed for the first time using only distributive lattices.

This thesis consists of six chapters. The first chapter introduces the basic concepts essential to make this thesis a self-contained one. The motivation for such a study and the outline of the thesis is given in this chapter.

Chapter two gives a brief literature survey.

Chapter three introduces all properties on finite semigroups associated with finite groups. Conditions for finite semigroups to satisfy classical theorems like Lagrange's theorem, Cayley's theorem, Cauchy theorem and Sylow theorems for finite groups are systematically analysed. This has paved the way for the introduction of new definitions and new properties associated with using semigroups.

Special elements like S-units, S-zero divisors, S-idempotents and S-nilpotents are defined for the first time for finite semigroups. Characterization of all these are obtained. Chapter three is the backbone of the thesis.

In chapter four, the study of semigroup semirings is carried out in a systematic way using only finite distributive lattices as semirings. Several interesting results in this direction are obtained.

In chapter five of the thesis, a systematic study of group semirings using distributive lattices as semirings is carried out. If the lattice is a chain lattice and G is a commutative group, then the group semiring is a semifield. If instead a non-commutative group is used, then the group semiring is a semidivision ring. This structure has no units as the semiring used is distributive. Conditions for the group semiring SG to contain S-idempotents, S-zero divisors and S-antizero divisors are obtained. The final chapter gives conclusions of the work carried out in this thesis.

CHAPTER TWO

A BRIEF LITERATURE SURVEY AND DESCRIPTION OF THE RESEARCH PROBLEM

2.1 Literature Survey

As the name suggests, a semigroup is a generalization of a group; because a semigroup need not in general have an element which has an inverse. The algebraic structure enjoyed by a semigroup is a non-empty set together with an associative closed binary operation. The study of semigroups started in the early 20th century, Rees [82] studied semigroups as early as 1940. However it was the Russian mathematician Anton Suschkewitsch (1928) who carried out research on semigroups [88]. He obtained the structure of finite simple semigroups and proved that the minimal ideal (Green's relation) of a finite semigroup is simple. In 1954, Preston [80] defined and developed the concept of inverse semigroups. In fact he also gave the representation of inverse semigroups. In 1961, Preston [81] described about congruences on completely 0-simple semigroups. Free inverse semigroups have been studied by Preston [79] in 1973. Munn W. D. as early as 1955 [67] introduced the notion of semigroup algebras. He has done research in a different direction [67].

Kimura (1957) carried out studies on idempotent semigroups [51]. He has studied about semigroups very widely and vividly. He has further researched idempotent semigroups which satisfies some identities. Moreover, idempotent semigroups have also been studied by McLean David [65] in (1954).

Yamada in 1958 analysed idempotent semigroups [101]. Levi [60] basically an algebraist worked on semigroups in 1944. Green J.A authored a classical paper on the structure of semigroups in 1951 [36] and in 1952 [35] with Rees; he studied those semigroups in which $x^r = x$.

For over 3 decades, Howie John [46-49] worked on embedding theorems for semigroups in his book [47]. In 1992, he along with Munn and Weirert have edited a proceeding of the conference on semigroups and their applications [46]. His contributions to semigroup theory is very significant. In this period of over 3 decades, semigroup theorists like Petrich [76-77], McAlister [63-64], Alan L.T.P [4], Lawson M.V [59] and Lajos [57] have done lots of research on special class of semigroups like inverse semigroups, free semigroups, etc. and their properties. In 1998, Okninski [71] published a book on semigroups of matrices.

Several researchers has worked on and developed more properties of these types of semigroups mentioned earlier, and many have given applications to finite automation and formal languages [28, 29, 45, 48, 58, 87]. As recently as 2014, researchers have worked on semigroups relative to commutative orders revisited [6], on extensions of completely simple semigroups by groups, [66] is a piece of work which is innovative; ordered semigroups of size at most 7 and linearly ordered semigroups of size at most 10 in [30] have analysed a specific size of ordered semigroups.

Kunz et al have studied geometrical illustration of numerical semigroups and some of their invariants [55]. Every group is a maximal subgroup of a naturally occurring free idempotent generated semigroup; by Gould and Yang [34] is an important piece of research. The structures of generalized inverse semigroups by Kudryavtseva and Lausa [54] is also a recent work on inverse semigroups. Permutations of a semigroup that maps to inverses have been researched by P.M. Higgins in [43]. The variety of unary semigroups with associative inverse subsemigroup by Billhardt et al [12] is yet another view on inverse subsemigroups. Thus the recent research in 2014 is in a way more elaborate research work on the earlier work done in the period 1940 to till date.

In a conference held in 2005, John Meakin had given lectures on groups and semigroups, exploring their connections and contrasts [66]. Such study was important and he clearly acknowledged that in this past decade, group theory and semigroup theory have developed in different directions. Cayley's theorem makes one realize all groups as groups of permutations of some set whereas semigroups are represented as semigroups of functions on a set to itself. However, significant research has been carried out both in group theory and semigroup theory in a varied or in a different direction. But, in reality, several concepts in modern semigroup theory are closely related to group theory. For instance, automata theory and formal language theory turn out to be related; [45, 48] have discussed the connection between group theory and semigroup theory. For more of the relations and contrasts please refer the excellent survey research article by [66]. Meakin discusses about cancellative semigroups embeddable in a group. However he has given both necessary and sufficient conditions for the embeddability of a semigroup in a group. Lots of research in this direction is carried out [1, 19, 33, 72] by Adian, Cho et al, Garside and Paris. Regular and inverse monoid properties discussed by Preston [79-80] have proved that every inverse monoid embeds a symmetric inverse monoid. Finally properties of free inverse monoids are extensively studied by [26] and presentation of inverse monoid is discussed in detail in [26].

Here it is important to record that [99] in the book on Smarandache semigroups have studied the conditions for a semigroup S to contain a proper subset A which under the operations of S is a group. Based on these properties, the classical theorems for finite groups have been extended for these Smarandache semigroups [99]. This sort of research is completely different from the research in [66]. Now the brief description of the problem and the relevance for such study is given in the following section.

2.2 Description of the Problem

From the literature survey on semigroups it is clear that not significant amount of research has been done in which study is based on how far the finite semigroups satisfy the classical theorems for finite groups. To the best of one's knowledge such a study is absent. So here the problem of analyzing how far a finite semigroup relates itself to the properties of classical theorems enjoyed by finite group is carried out in this thesis. For instance, Lagrange's theorem for finite groups is true for all finite groups. But its converse is not true in general for A_4 but A_4 satisfies the Lagrange's theorem. Likewise finite semigroups can satisfy partially any classical theorem for finite groups as well as not satisfy fully a classical theorem.

This problem/study leads to the definition of several new and interesting concepts in semigroups. This study is new and innovative. Further, special elements like S-idempotents, S-units, S-zerodivisors, S-nilpotents and S-anti zero divisors are defined for semigroups for the first time. Conditions on these semigroups to contain these special elements is obtained.

Secondly, study of group rings and semigroups rings dates back to 1940's and 1955's respectively in [44, 52, 67]. However study of group semirings and semigroup semirings is very meager [100]. Further semirings are the algebraic structures built using on the same set two semigroups with two distinct binary operations. The two binary operations are connected by the distributive law.

Here, the study of semigroup semirings and group semirings is carried out for the first time using only distributive finite lattices. As finite distributive lattices are nothing but two distinct idempotent semigroups connected by distributive laws. Several innovative and interesting results are obtained in this direction for the first time.

The marked difference between group rings and group semirings are mentioned in this thesis. This study has led to several new properties and a new approach to the study of semigroups.

CHAPTER THREE

SEMIGROUPS AND THEIR SPECIAL PROPERTIES

3.1 INTRODUCTION

In this chapter semigroups are analysed as generalization of groups. Most of the researchers have studied semigroups as the algebraic structure akin to rings. That is why several properties like ideals, idempotents, inverses, units, zero divisors; enjoyed by rings are studied or analysed for semigroups. Here the study is different and distinct for the study seeks to find out those properties which are common in semigroups and groups and those properties that are distinct.

At the outset the notion of idempotents and zero divisors of semigroups can by no means by related with groups. Further for group must have identity but there are semigroups which do not have identity. Only monoids are semigroups that have identity. As every group is a semigroup the notion of Smarandache semigroups have been systematically studied by W.B. Vasantha in [99]. Just for the sake of easy reference a Smarandace semigroup S is nothing but a group which has a proper non empty subset A such A under the operation of S is a group. For more about these notions refer [99].

This chapter has six sections. Section one is introductory in nature. Here in this thesis finite semigroups which satisfy the basic classical theorems for finite groups and those semigroups which do not satisfy the classical theorems of groups is analysed in section two.

In section three for the first time some special properties of substructures in finite semigroups are analysed. Section four analysis the special elements of these semigroups. Section five studies all the semigroup properties enjoyed by matrix semigroups under the natural product $\times_n$ [89]. The final section gives the conclusions of this chapter.

3.2 CLASSICAL THEOREMS ON FINITE GROUPS IN CASE OF FINITE SEMIGROUPS – A STUDY

In this section the study of classical theorems for finite groups; viz., Lagrange's theorem, converse of Lagrange's theorem, Cauchy theorem, Cayleys theorem and Sylow theorems are analysed or partly adopted in the case of finite semigroups. This study is relevant as semigroups are nothing but a generalization of groups; as every group is a semigroup however a semigroup in general is not a group. Further the study of Smarandache semigroups only finds or characterizes those semigroups which contain subsets which are subgroups under the operation of the semigroups.

Now the classical theorem for finite groups, viz., Lagranges's theorem is first analyzed in case of finite semigroups. First by a few examples then defining new notions and obtaining the resulting theorems.

Lagrange's theorem states; "If G is a finite group and H is a proper subgroup of G then $o(H) / o(G)$. However if $t / o(G)$; G need not in general have a subgroup of order t that is the converse of the theorem is not true [39, 56].

Example 3.2.1: Let $S = \{Z_{15}, \times\}$ be the semigroup of order 15. $x = 14 \in S$ but $14 \times 14 = 1 \pmod{15}$. Thus $B_1 = \{1, 14\}$ is a subsemigroup of S and $o(B_1) = 2$; but $2 \nmid 15$. However, B_1 is also a group of order two as B_1 is isomorphic to a cyclic group of order two.

Take $B_2 = \{1, 10\} \subseteq S$, clearly $10^2 = 100 \pmod{15}$; that is $10^2 = 10 \pmod{15}$; B_2 is a subsemigroup of order two which is not a subgroup of S and $o(B_2) \nmid o(S)$. Thus Lagrange's theorem is not true for finite semigroups in general.

In view of this the following definition is made:

Definition 3.2.1: *Let $\{S, \times\}$ be a semigroup of finite order say n. If S has at least one proper subsemigroup B such that B is not a group and B is only a subsemigroup and $o(B) \nmid o(S)$, then S is said to satisify or possess anti Lagrange's property.*

In the following a class of semigroups which satisfy the anti Lagrange's property is described.

Proposition 3.2.1: *Let $S = \{Z_n, \times\}$ be a semigroup of order n having nilpotent elements of order two and idempotents, then S satisfies anti Lagrange's property.*

Proof: Two cases arise, n even or n-odd.

Case i: When n is even. Let $a \in Z_n \setminus \{0\}$ be such that a is a nilpotent element of order two then, $a^2 = 0$; and the set $P = \{0, 1, a\}$ is a subsemiring of order 3 and $3 \nmid n$. Hence the claim.

If $a \in Z_n \setminus \{1, 0\}$ is an idempotent that is $a^2 = a$ then the set $B = \{0, 1, a\} \subseteq \{Z_n, \times\}$ is a subsemigroup and $| B | = 3$ and $3 \nmid n$.

Thus for in case of even n the claim is true.

Case ii: Let n be odd. Now let $a \in Z_n \setminus \{0\}$ be such that $a^2 = 0$, that is a is a nilpotent element of order two; then $T = \{0, a\} \in S$ is a subsemigroup of order two and $o(T) \not| \; n$. Hence the claim. Let $a_1 \in S$ be an idempotent of S; that is $a_1^2 = a_1$ then $T_1 = \{0, a_1\}$ (or $T_2 = \{1, a_1\}$) is a subsemigroup of S and $o(T_1) = 2$ (and $o(T_2) = 2$) so $o(T_i) \not| \; o(S); i = 1, 2$. Hence the claim.

Thus $S = \{Z_n, \times\}$ satisfies the anti Lagrange's property.

This will be illustrated by an example or two.

Example 3.2.2: Let $S = \{Z_{20}, \times\}$ be the semigroup under $\times$ modulo 20.

Now $5 \in S$ is such that $5^2 \equiv 5 \pmod{20}$ and $10 \in Z_{20}$ is such that $10^2 = 0 \pmod{20}$; hence Z_{20} has a nontrivial idempotent and a nilpotent element of order two and $o(Z_{20}) = 20$, that is $n = 20$ is even.

Now $P_1 = \{0, 10, 1\} \subseteq S$ is a subsemigroup of S and $o(P_1) = 3$. Further $3 \not| \; 20$. Take $P_2 = \{0, 5, 1\} \subseteq S$; P_2 is a subsemigroup of S and $o(P_2) = 3 \not| \; 20$. Hence the proposition is verifield.

Now other than these take $T_1 = \{0, 5, 10\} \subseteq S$, T_1 is again a subsemigroup of S such that $o(T_1) \not| \; 20$.

Example 3.2.3: Let $S = \{Z_{35}, \times\}$ be the semigroup of order 35. $15 \in S$ is such that $15^2 = 15 \pmod{35}$. Thus $M = \{0, 15, 1\} \subseteq S$ is a subsemigroup of order three and $|M| \not| \; o(S)$. N

$= \{0, 15\} \subseteq S$ is also a subsemigroup of S and $|N| \not| \, o(S)$. $T = \{15, 1\} \subseteq S$ is a subsemigroup of S such that $|T| \not| \, o(S)$. Now $7, 5 \in S$ is such that $7 \times 5 = 0 \pmod{35}$, but is not a nilpotent element of order two.

Take $x = 21 \in Z_{35}$, $21^2 \equiv 21 \pmod{35}$.

$B = \{1, 21, 0\} \subseteq S$ is a subsemigroup of Z_{35}; $o(B) \not| \, 35$; S satisfies the anti-Lagrange's property.

Some of these semigroups may not contain nilpotent elements of order two.

Example 3.2.4: Let $S = \{Z_{25}, \times\}$ be the semigroup of order 25. $x = 5 \in S$ is such that $x^2 = 0$. $M = \{0, 5\} \subseteq S$ is a subsemigroup of order two and $o(M) \not| \, o(S)$. Take $N_1 = \{1, 5, 0\} \subseteq S$; N_1 is also a subsemigroup such that $o(N_1) \not| \, o(S)$. Now $P = \{0, 10\} \subseteq S$ is a subsemigroup such that $o(P) \not| \, o(S)$.

Thus $D = \{0, 5, 10\} \subseteq S$ is a subsemigroup of S such that $o(D) \not| \, o(S)$. $E = \{0, 5, 10,1\} \subseteq S$ is a subsemigroup such that $o(E) \not| \, o(S)$. Hence S is a semigroup which satisfies the anti Lagrange's property.

In fact there are semigroups which have only idempotents but has no nilpotent elements of order two, still those semigroups satisfy anti Lagrange's property.

Now for a finite semigroup S to satisfy weak Lagrange's property is made as a definition in the following:

Definition 3.2.2: *Let S be a semigroup of finite order. If S contains atleast a proper subsemigroup P such $o(P) / o(S)$; then S is said to satisfy the weak Lagrange's property.*

This is illustrated by some examples.

Example 3.2.5: Let $S = \{Z_{21}, \times\}$ be the semigroup. Let $B_1 = \{0, 1, 7, 14\} \subseteq S$; be the subsemigroup of S. Clearly $o(B_1) \nmid o(S)$. Consider $L = \{0, 3, 6, 9, 12, 15, 18\} \subseteq S$ be the subsemigroup of S; $o(L) / o(S)$. Now take $M = \{1, 0, 3, 6, 9, 12, 15, 18\} \subseteq S$ is a subsemigroup and $o(M) \nmid o(S)$.

$W = \{0, 8, 1\} \subseteq S$ is a subsemigroup such that $o(W) / o(S)$. However S has subsemigroup $H = \{0, 1, 6, 15\} \subseteq S$ is such that $o(H) \nmid o(S)$. So S is a anti Lagrange's semigroup. S also satisfies weak Lagrange's property.

Example 3.2.6: Let $M = \{Z_{12}, \times\}$ be the semigroup of order 12. $B = \{0, 6, 1, 3, 9\} \subseteq M$ is a subsemigroup of order 5. $o(B) \nmid o(M)$. $C = \{0, 3, 6, 9\} \subseteq M$ is a subsemigroup of order 4. $o(C) / o(M)$. Thus M satisfies both a anti Lagrange's property as well as weak Lagrange's property.

In view of this the following proposition is important:

Proposition 3.2.2: *Let $S = \{Z_n, \times\}$ be the semigroup of order n and n is not a prime. S satisfies both anti Lagrange's property as well as weak Lagrange's property.*

Proof: Consider any ideal I of Z_n say of order m; the largest number that divides n, m / n take $I \cup \{1\}$; then $m + 1 \nmid n$; hence the claim. Second part of the theorem is proved.

S has a subsemigroup of order t; or to be more precise if $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$; p_i's are distinct primes, $\alpha_i \geq 1$ then S has subsemigroups of order $p_1, \ldots, p_s$ and their respective powers also. All these subsemigroups have order which divides n. Hence the claim.

This situation will be illustrated by the following example:

Example 3.2.7: Let $S = \{Z_{180}, \times\}$ be the finite semigroup. $180 = 2^2 \times 3^2 \times 5$; to show S has subsemigroups of order. 2, 4, 3, 9, 5, 10, 20, 45, 15, 18, 6, 30, 12 and 36. Let

$H_1 = \{0, 90\}$, $H_2 = \{0, 45, 90, 135\}$,

$H_3 = \{0, 60, 120\}$,

$H_4 = \{0, 20, 40, 60, 80,100,120, 140. 160\}$,

$H_5 = \{0, 36, 72, 108, 144\}$,

$H_6 = \{0, 18, 36, 54, 72, 90, 108, 126, 144, 162\}$,

$H_7 = \{0, 9\ 18, 27, \ldots, 171\}$,

$H_8 = \{0, 4, 8, 12, 16, \ldots, 176\}$,

$H_9 = \{0, 12, 24, 36, \ldots, 168\}$

$H_{10} = \{0, 10, 20, 30, \ldots, 170\}$,

$H_{11} = \{0, 30, 60, 90,120, 150\}$,

H_{12} = {0, 5, 10, 15, 20, …, 175},

H_{13} = {0, 2, 4, 6, 8, 10, …, 178} and so on.

are all subsemigroups of S.

In view of this one can say all semigroups $S = \{Z_n, \times, n$ not a prime$\}$ satisfy weak Lagrange's property, or they will be known as weak Lagrange semigroups.

Next there is a class of semigroups known as the symmetric semigroups $S(n)$ and $S(n)$ behaves in a very different way.

Some examples in this direction are given.

Example 3.2.8: Let $S(3)$ be the symmetric semigroup of degree three. $S(3)$ has idempotents. $S(3)$ satisfies both anti Lagrange's property as well as weak Lagrange's property.

For take

$$B_1 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\} \subseteq S(3);$$

B_1 is a subgroup of order three.

Let

$$B_2 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\right\} \subseteq S(3);$$

B_2 is a subsemigroup of order two. $o(B_2) \nmid o(S_3))$ as $o(S(3)) = 3^3$.

$$B_3 = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 2 & 2\end{pmatrix}\right\} \subseteq S(3);$$

is a subsemigroup of $S(3)$ and $|B_3| = 3$ and $3/3^3$.

Take

$$M = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}\right\} \subseteq S(3);$$

M is a subgroup and $o(M) \nmid o(S(3))$.

$S(3)$ has subsemigroups of order 9. For take

$$N = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix} \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 3 & 3\end{pmatrix}\right\} \subseteq S(3) \;;$$

N is a subsemigroup of order 9 and $9/3^3$. Hence $S(3)$ satisfies weak Lagrange's property.

Thus

$$\begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 2 & 2\end{pmatrix} \text{ and } \begin{pmatrix}1 & 2 & 3\\3 & 3 & 3\end{pmatrix}$$

are idempotents in $S(3)$.

For

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Likewise for other two elements.

Take

$$P_1 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\right\},\ P_2 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}\right\} \text{ and}$$

$$P_3 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\right\};$$

they are subsemigroups of order two; $o(P_i) \not\mid o(S(3))$ for $i = 1, 2, 3$. Hence $S(3)$ enjoys anti Lagrange's property also.

In view of this the following proposition is proved:

Proposition 3.2.3: *Let S(n), be the symmetric semigroup or degree n (n odd). S(n) satisfies anti Lagrange's property.*

Proof: Given n is odd so $o(S(n)) = n^n$ and $2 \not\mid o(S(n))$.

Let

$$P_1 = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 2 & 3 & 4 & \dots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}\right\} \subseteq S(n);$$

P_1 is a subsemigroup of order 2. $o(P_1) \not\mid o(S(n))$ so $S(n)$ satisfies the anti Lagrange's property.

Proposition 3.2.4: *Let S(n) be the symmetric semigroup of degree n; n a even integer. S(n) satisfies the weak Lagrange's property.*

Proof: $o(S(n)) = (n)^n$ (n an even integer). Thus 2 / $o(S(n))$.

Take

$$D_1 = \left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \ldots & n \end{pmatrix}\right\} \subseteq S(n);$$

D_1 is a subsemigroup of order 2 and $o(D_1)$ / $o(S(n))$. Thus $S(n)$ satisfies the weak Lagrange's property.

Theorem 3.2.1: *Every symmetric semigroup S(n) of degree n satisfies anti Lagrange's property;that is S(n)is a weak Lagrange's semigroup.*

Proof: Let n be a prime; $S(n)$ the symmetric semigroup of degree n. $o(S(n)) = n^n$; n a prime ($n > 2$).

Let

$$P = \left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \ldots & n \end{pmatrix}\right\} \subseteq S(n)$$

be a subsemigroup such that $o(P) \nmid n^n$. Hence $S(n)$ satisfies the anti Lagrange's property.

Suppose n is not a prime consider the subsemigroup;

$$M = S_n \cup \left\{ \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 2 & \ldots & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \ldots & 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n & n & \ldots & n \end{pmatrix} \right\} \subseteq S(n)$$

is a subsemigroup of order $n! + n = n\,((n-1)! + 1)$.

Clearly $n((n - 1)! + 1) \not\mid n^n$. Hence $S(n)$ in this case also satisfies the anti Lagrange's property.

Theorem 3.2.2: *Every symmetric semigroup S(n) of degree n satisfies the weak Lagrange's property.*

Proof: Let $S(n)$ be the symmetric semigroup of order n^n.

$$B = \left\{ \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 2 & \ldots & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \ldots & 3 \end{pmatrix}, \ldots, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n & n & \ldots & n \end{pmatrix} \right\} \subseteq S(n)$$

is a subsemigroup of order n and $o(B) \,/\, o(S(n))$. Thus the symmetric semigroup $S(n)$ satisfies the weak Lagrange's property.

It is important to make a mention that these two properties are not in any way related to the notion of Smarandache Lagrange's semigroup or Smarandache weakly Lagrange's semigroup defined in [99], for both are related to subgroups of a semigroup.

Further here one says it is a property satisfied by a semigroup so a semigroup can satisfy more than one property or both the properties simultaneously.

3.3 CAUCHY PROPERTY IN FINITE SEMIGROUPS

Next the concept of semigroups satisfying Cauchy property and those semigroups that satisfy anti Cauchy property is defined and analysed in the following. For Cauchy property refer [39, 56].

Definition 3.3.1: *Let S be a semigroup with unit or monoid of finite order. If these exist an element $\alpha \in S$ such that $\alpha^m = 1$ and if $m / o(S)$ then S satisfies the Cauchy property. If $m \not| o(S)$ then S is said to satisfy anti Cauchy property.*

These properties enjoyed by semigroups will be described by some examples in the following:

Example 3.3.1: Let $S = \{Z_{29}, \times\}$ be the semigroup. Let $x = 28 \in S$; $x^2 = 1 \in S$ but $2 \not| o(S)$ so S satisfies anti Cauchy property.

In fact as $o(S) = 29$, a prime number only any element $y \in S$ with $y^{29} = 1$ alone will enjoy Cauchy property but that is an impossibility in S.

Example 3.3.2: Let $S = \{Z_7, \times\}$ be the semigroup. $o(S) = 7$; $x = 6 \in S$ is such that $x^2 = 1$ and $2 \not| 7$ so S satisfies anti Cauchy property. For $y = 2 \in S$ is such that $y^3 = 2 \times 2 \times 2 = 8 = 1 \pmod 7$ and $3 \not| 7$ so y satisfies anti Cauchy property. Let $z = 3 \in S$ is such that $3^6 =$

1 and 6 $\not|$ $o(S)$ hence the claim. S has no Cauchy element. Thus S does not enjoy Cauchy property.

In view of this the following result is true:

Theorem 3.3.1: *A semigroup $S = \{Z_n, \times\}$ does not satisfy Cauchy property if n is a prime.*

Proof: Follows from the fact any $a \in Z_n$ is such that $a^2 = 1$ or so on $a^{n-1} = 1$, n a prime and there does not exist, a such that $a^n = 1$ for then alone $n / | Z_n |$; This is impossible as the highest power of any element in Z_n is $n - 1$.

Corollary 3.3.1: *All semigroups, $S = \{Z_n, \times\}$; n even, satisfy Cauchy property. For $x = (n - 1) \in S$ is such that $(n - 1)^2 = 1 \ (\text{mod } n)$ and $2 / n$ as n is even.*

Example 3.3.3: Let $S = \{Z_{15}, \times\}$ be a semigroup. S has no element x such that $x^n = 1$ and $n \not|$ 15.

Example 3.3.4: Let $S = \{Z_{21}, \times\}$ be a semigroup of order 21.

$5 \in Z_{21}$ is such that $5^6 = 1$ (mod 21),

$7^2 = 7$ (mod 21) ; $8^2 = 1$ (mod 21).

but 2 $\not|$ 21 ; $10^6 = 1$ (mod 21).

But 6 $\not|$ 21; $11^6 = 1$ (mod 21).

$13^2 = 1$ (mod 21); 2 $\not|$ 21, $15 \in Z_{21}$;

$15^2 = 5 \pmod{21}$.

$16^3 = 1 \pmod{21}$ and $3 / 21$.

So S satisfies Cauchy property, but $17 \in Z_{21}$ is such that $17^6 \equiv 1 \pmod{21}$. $19 \in Z_{21}$ is such that $19^6 \equiv 1 \pmod{21}$ and $20 \in Z_{21}$ is such that $20^2 \equiv 1 \pmod{21}$ and $2 \nmid 21$.

Example 3.3.5: Let $S = \{Z_{25}, \times\}$ be the semigroup and $6^5 \equiv 1 \pmod{25}$ and $5 / 25$. So S satisfies Cauchy property.

In view of all these the following result is proved.

Proposition 3.3.1: *Let $S = \{Z_n, \times\}$ be a semigroup in which $n = p^2$ where p is a prime; S satisfies Cauchy property.*

Proof: $(p + 1)^p = 1 \pmod{p^2}$ as p / p^2; hence the claim.

Example 3.3.6: Let $S = \{Z_{27}, \times\}$ be the semigroup. $10 \in 27$; $10^3 \equiv 1 \pmod{27}$. So S satisfies Cauchy property.

The following problem is left for future study:

Problem: Let $S = \{Z_n, \times\}$; $n = p^t$, p a prime be a semigroup ($t \geq 2$). Does S satisfy Cauchy property? (p a large prime).

For this problem needs more knowledge about modulo integers in particular and number theory in general.

Next the study of symmetric semigroups *S(n)* is analysed for the Cauchy property and anti Cauchy property.

First a few examples, in this direction are given.

Example 3.3.7: Let *S(8)* be the symmetric semigroup of degree 8. Clearly $o(S(8)) = 8^8$. Now consider

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 8 \end{pmatrix} \in S(8).$$

Clearly

$$x^5 = \begin{pmatrix} 1 & 2 & 3 & \ldots & 8 \\ 1 & 2 & 3 & \ldots & 8 \end{pmatrix} = 1.$$

Thus x is a not a Cauchy element of *S(8)* as $5 \nmid 8^8$. So *S(8)* satisfies the anti Cauchy property.

Consider

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} \in S(8);$$

$y^4 = 1$ and $4 / 8^8$ so y satisfies the Cauchy property also. Thus *S(8)* is the symmetric semigroup which satisfies both anti Cauchy property as well as Cauchy property.

Example 3.3.8: Let *S(15)* be the symmetric semigroup of order 15^{15}. Let

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 15 \\ 2 & 3 & 4 & 1 & 5 & \dots & 15 \end{pmatrix} \in S(15).$$

Clearly $x^4 = 1$ and $4 \not| \; 15^{15}$; so x is a anti Cauchy element of *S(15)*.

Take

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & 15 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & \dots & 15 \end{pmatrix} \in S(15).$$

Clearly $y^5 = 1$ and $5/15^5$. Thus *S(15)* satisfies Cauchy property.

In view of this the following result is proved.

Proposition 3.3.2: *Let S(n) be the symmetric semigroup of degree n ($3 < n < \infty$); S(n) has Cauchy elements as well as S(n) satisfies the anti Cauchy property.*

Proof: Let *S(n)* be the symmetric semigroup of finite order; n^n ($n < \infty$). Let n be odd and choose a $m < n$ and m even; such a choice is always possible.

Let

$$1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & n \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & n \end{pmatrix} \in S(n).$$

Consider

$$x = \begin{pmatrix} 1 & 2 & 3 & \dots & m-1 & m & m+1 & \dots & n \\ 2 & 3 & 4 & \dots & m & 1 & m+1 & \dots & n \end{pmatrix} \in S(n);$$

clearly $x^m = 1$ and $m \nmid n^n$ as n is odd and m is even. Hence x is an anti Cauchy element, so S is satisfies anti Cauchy property. Let p be a number less than n and p/n. Such an element is possible as n is not a prime.

Let

$$y = \begin{pmatrix} 1 & 2 & 3 & \dots & p-1 & p & p+1 & \dots & n \\ 2 & 3 & 4 & \dots & p & 1 & p+1 & \dots & n \end{pmatrix} \in S(n);$$

clearly $y^p = 1$ and p/n^n (as p/n). Thus S satisfies the Cauchy property.

Let n be an even number. Let $s < n$ such that $s \nmid n$ and s is odd.

Let

$$z = \begin{pmatrix} 1 & 2 & 3 & \dots & s-1 & s & s+1 & \dots & n \\ 2 & 3 & 4 & \dots & s & 1 & s+1 & \dots & n \end{pmatrix} \in S(n)$$

and $z^s = 1$ and $s \nmid n^n$ thus z is an anti Cauchy element of $S(n)$.

Finally if $n = p$, p a prime $S(p)$ be the symmetric semigroup.

Take

$$x = \begin{pmatrix} 1 & 2 & 3 & \dots & p-1 & p \\ 2 & 3 & 4 & \dots & p & 1 \end{pmatrix} \in S(p);$$

$x^p = 1$ and p / p^p so x is a Cauchy element in $S(n)$. However for any $m < p$;

$$y = \begin{pmatrix} 1 & 2 & 3 & \dots & m-1 & m & m+1 & \dots & p \\ 2 & 3 & 4 & \dots & m & 1 & m+1 & \dots & p \end{pmatrix} \in S(p),$$

$y^m = 1$ but $m \not| \, p^p$ so y is a anti Cauchy element of $S(p)$. Thus $S(n)$ satisfies both Cauchy property and anti Cauchy property.

Next the class of finite semilattice under $\cup$ or $\cap$ will be studied. Both semilattices are idempotent semigroups and are not monoids unless otherwise it is made into a monoid.

This situation will first be described by some examples.

Example 3.3.9: Let S = {{ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}; ∩} be a semilattice as well as an idempotent semigroup which is not a monoid.

$|S| = 7$ and $H_1 = \{\{a\}, \{b\}, \{\phi\}\}$,

$H_2 = \{\{a\}, \{a,b\}, \{b\}, \{\phi\}\}$,

$H_3 = \{\{a\}, \{b\}, \{c\}, \{\phi\}\}$, and $H_4 = \{\{a\}, \phi\}$

are subsemigroups of S and the order of none of these H_i's ($1 \leq i \leq 4$) can divide 7 so S satisfies anti Lagrange's property.

For no subsemigroup of S is such that its order can divide 7 (of course order one subsemigroups are considered as trivial subsemigroups).

Example 3.3.10: Let $S = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \cap\}$ be the semilattice (semigroup) under '$\cap$'. Clearly $o(S) = 6$.

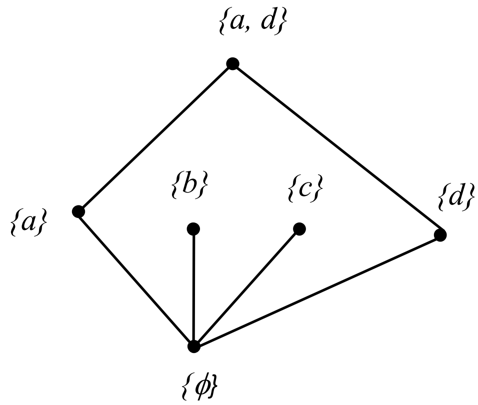


Figure: 3.3.1

S is an idempotent semigroup which is not a monoid. Let $P_1 = \{\phi, \{a\}\}$ and $P_2 = \{\{a\}, \{b\}, \phi\}$ be subsemigroups of order two and three respectively. $o(P_2)/6$ and $o(P_1)/6$; so S satisfies Lagrange's property. Consider $P_3 = \{\{a\}, \{d\}, \{a, d\}, \phi\} \subseteq S$; P_3 is a subsemigroup of order 4 but $o(P_3) \nmid o(S)$. Thus S also satisfies the anti Lagrange's property.

Hence S satisfies both anti Lagrange's property as well as weak Lagrange's property.

In view of these examples the following result is proved:

Proposition 3.3.3: *Let $S = \{\phi, \{a_1\}, \{a_2\}, \ldots, \{a_{p-1}\}, \cap\}$ be a semilattice of order p, p is a prime. S is an idempotent semigroup which is not a monoid. S satisfies only anti Lagrange's property and not weak Lagrange's property.*

Proof: S has several subsemigroups of order 2, 3, 4, 5, …, $p - 1$. However as $o(S) = p$, p a prime none of the orders of the subsemigroups divide order of S.

Thus there are semigroups which satisfy only anti Lagrange's property and not weak Lagrange's property.

Proposition 3.3.4: *Let $\{S, \cap\}$ be a semilattice or the idempotent semigroup which is not a monoid. $o(S) = n$ (n not a prime). S satisfies both anti Lagrange's property as well as weak Lagrange's property.*

Proof: Given $o(S) = n$ if n is even all sets $P_i = \{\phi, a_i\}$ are subsemigroups of order two; $1 \leq i \leq n - 1$ and $o(P_i) / n$.

If n is odd certainly there exists subsemigroups M_i such that $o(M_i) / n$. Hence the result.

Similarly there exists subsemigroups of odd order in S which does not divide order of n (n even) and even order subsemigroups which does not divide n; where n odd.

On similar lines results regarding semilattice under '$\cup$' can be proved.

Here one or two examples are given.

Example 3.3.11: Let $S = \{\{a, b, c\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \cup\}$ be the semilattice (semigroup) under the operation '$\cup$'. Clearly $|S| = 7$ so whatever subsemilattice or subsemigroup is taken from S, the order of it will not divide order of S as $o(S)$ is a prime number 7. Thus this idempotent semigroup is not a monoid.

Further $P_1 = \{\{a\}, \{b\}, \{a, b\}, \cup\}$ is a subsemigroup of S and $o(P_1) \nmid 7$. Likewise $\{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ is a subsemigroup of order 4 and $4 \nmid 7$. Thus there is a class of semigroups which satisfy anti Lagrange's property.

In fact these class of semigroups are not Smarandache semigroups. These class of semigroups has no relevance to Cauchy property or anti Cauchy property as these semigroups are not monoids and they are idempotent semigroups.

Next the study of Cayley's theorem is tested for these semigroups. In case of *S*-semigroups a new type of *S*-semigroup homomorphisms and isomorphisms are defined and described in [99] . However in this work certain class of semigroups which can satisfy Cayley's theorem is described and semigroups which has zero divisors certainly will not satisfy Cayley's theorem.

In view of this some examples are given.

Example 3.3.12: Let $S = \{Z_{12}, \times\}$ be the semigroup. Clearly this S cannot be embedded in the symmetric group $S(n)$ for any n as $S(n)$ has no zero divisors but S has zero divisors.

Example 3.3.13: Let $S = \{\phi, \{1\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \cap\}$ be the semilattice which is an idempotent semigroup and not a monoid.

Now when one tries to embed S in $S(n)$ one wants to see how best ϕ the empty set can be embedded.

So $S(n) \cup \{\phi\}$ is defined as the extended symmetric semigroup; here $\{\phi\}$ is the permutation on the empty set, so is empty and for any $\alpha \in S(n)$; $\alpha \text{ o } \phi = \phi \text{ o } \alpha = \phi$. This symmetric semigroup $S(n) \cup \{\phi\}$ is defined the extended symmetric semigroup.

Now using this extended symmetric semigroup can embedding of semigroups be possible?

This will be first illustrated by some examples.

Example 3.3.14: Let $S = \{1, \phi, a_1, a_2, a_3, a_4, \cap\}$ be the semilattice under $\cap$. This S is an idempotent semigroup. Now this S can be embedded in the extended symmetric semigroup $S(4) \cup \{\phi\}$ in the following way:

Let $\eta : S \to S(4)$ be an embedding defined by;

$$\eta(\phi) = \phi, \ \eta(1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1 \text{ of } S(4)$$

$$\eta(a_1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \ \eta(a_2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix},$$

$$\eta(a_3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix} \text{ and } \eta(a_4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

η is only a map for under the composition of mappings. η is not properly defined.

$$\phi = a_1 \cap a_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \neq \phi.$$

So the embedding η fails to give the empty permutation.

$$a_2 \cap a_1 = \phi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \neq \phi$$

$$\neq a_1 \cap a_2.$$

So the semilattice/ semigroup structure on $\eta(S)$ can by no means be achieved.

Thus Cayley theorem fails to be true even using the extended symmetric semigroup.

So as far as semilattices are concerned it is impossible to get even something near to embedding in $S(n)$ or $S(n) \cup \{\phi\}$.

So semigroups constructed using Z_n or semilattices can never be embedded if n of the Z_n is a composite number.

Can $S = \{Z_p, \times\}; p$ a prime be embedded in a suitable $S(n)$?

First this will be tried using some examples.

Example 3.3.15: Let $S(2)$ be the symmetric semigroup of degree 2.

$S = Z_3 = \{0, 1, 2\}$ be the semigroup. Can $Z_3 = S$ be embedded in $S(2) \cup \{\phi\}$?

Define a map η: $S \to S(2) \cup \{\phi\}$ as follows:

$$\eta(0) = \phi, \ \ \eta(1) = 1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } \eta(2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then η is an embedding of Z_3 in the extended symmetric semigroup.

Example 3.3.16: Let $S(4)$ be the symmetric semigroup. $S(4) \cup \{\phi\}$ the extended semigroup. $S = \{Z_5, \times\}$ be the semigroup. Let η be a map from S to $S(4) \cup \{\phi\}$ defined by $\eta\ (0) = \phi$,

$$\eta(1) = 1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},\ \eta(2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

$$\eta(3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \text{ and } \eta(4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Clearly η embeds S into $S(4) \cup \{\phi\}$. Thus for this semigroup extended Cayley's theorem is true.

In view of this the following theorem is proved:

Theorem 3.3.2: *Let $S = \{Z_p, \times\}$ be the semigroup under product, p a prime $S(p-1) \cup \{\phi\}$ be the extended symmetric semigroup of degree $(p-1)$. Extended Cayley's theorem is true for this S.*

(That is $S \to S(p-1) \cup \{\phi\}$, in other words S is embedded in the extended symmetric semigroup $S(p-1) \cup \{\phi\}$. This sort embedding of semigroups is known as extended embedding Cayley's theorem or extended Cayley's theorem).

Proof: Let $\eta\colon S \to S(p-1) \cup \{\phi\}$ be defined as

$$\eta(0) = \phi,\ \eta(1) = 1 = \begin{pmatrix} 1 & 2 & 3 & \ldots & p-1 \\ 1 & 2 & 3 & \ldots & p-1 \end{pmatrix}$$

$$\eta(t) = x \in \left\{ \begin{pmatrix} 1 & 2 & 3 & \ldots & p-1 \\ 2 & 3 & 4 & \ldots & p-1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & p-1 \\ 3 & 4 & 5 & 6 & \ldots & 2 \end{pmatrix}, \ldots,\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & \ldots & p-1 \\ p-1 & 1 & 2 & \ldots & p-2 \end{pmatrix}\right\}$$

for every $t \in Z_p \setminus \{0, 1\}$. It is verified η is an embedding; hence extended Cayley's theorem is true.

This is explained by another example.

Example 3.3.17: Let $S = \{Z_7, \times\}$ be the semigroup under $\times$. $S(6) \cup \{\phi\}$ be the extended symmetric semigroup.

Define $\eta : S \to S(6) \cup \{\phi\}$ by $\eta(0) = \phi$, $\eta(1) = 1 = \begin{pmatrix} 1 & 2 & \ldots & 6 \\ 1 & 2 & \ldots & 6 \end{pmatrix}$

$$\eta(x) \in \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}\right\}$$

This map η is an embedding. Thus extended Cayley's theorem is true. Thus only for this class of finite semigroups described and used in this thesis; Cayley's extended theorem is true.

Now the definition of restricted weak Cayley's extended theorem is given.

Definition 3.3.2: *Let $S = \{Z_n, \times\}$ be a semigroup of order n; (n is not a prime) $S(n-1) \cup \{\phi\}$ be the extended symmetric semigroup. Let H be a subsemigroup of S. If there is an* *embedding of H in a subsemigroup of S(n − 1) ∪ {φ} then S is said to satisfy restricted weak extended Cayley's theorem.*

This is illustrated by the following example:

Example 3.3.18: Let $S = \{Z_{15}, \times\}$ be the semigroup. Take $M = \{0, 1, 3, 6, 9, 12, 14\} \subseteq S$; M is a subsemigroup of order 7. 7 a prime and S does not have subsemigroup M_1 of order 8 such that $M \subseteq M_1$.

Consider $N = \{0, 5, 10\} \subseteq S$ is a subsemigroup of S and $o(N) / o(S)$ and S has subsemigroup of order 4. So N is doubly not a subsemigroup sought.

In view of this example a few definitions are made.

Definition 3.3.3: *Let S be a semigroup of finite order say n. Let p be a prime* $0 < p < n$ *such that* $p \not| \; n$. *If S has a subsemigroup H of order p and H is not a subgroup of S and there does not exist a proper subsemigroup* H_1 *of order* $p + 1$ *such that* $H \subseteq H_1$ *then H is defined as the pseudo p-Sylow subsemigroup of S.*

Examples of this is given in the following:

Example 3.3.19: Let $S = \{Z_6, \times\}$ be a semigroup. 3/6 so the only prime is 5. $P = \{0, 1, 2, 4, 5\} \subseteq S$ is a subsemigroup of S and S has no proper subsemigroup of order 6 as $o(S) = 6$. Further P is only a semigroup as 2 and 4 are zero divisors in S. Thus P is a pseudo 5-Sylow subsemigroup of S.

Example 3.3.20: Let $S = \{Z_8, \times\}$ be the semigroup. The prime, less than 8 are 3, 5 and 7. Clearly S has subsemigroups of order 4 so $p = 3$ is ruled out. Consider $P_1 = \{0, 2, 4, 6, 1\}$

is a subsemigroup of order five. Take $P_2 = \{0, 2, 4, 6, 1, 7\} \subseteq S$ is again a subsemigroup of order six. So P_1 is not a pseudo 5-Sylow subsemigroup of S.

But $M_1 = \{1, 5, 3, 0, 7\}$ is a subsemigroup of order 5. For this M_1 one can get a subsemigroup M_2 of order 6 such that $M_1 \subseteq M_2$. Hence M_1 is a not a pseudo 5-Sylow subsemigroup of S. For $M_2 = \{1, 5, 3, 4, 0, 7\}$ contains M_1. S has no pseudo 7-Sylow subsemigroup.

Study in this direction is new and innovative.

Example 3.3.21: Let $S = \{Z_9, \times\}$ be a semigroup. The primes less than 9 and not divisible by 9 are 5 and 7. $P_1 = \{0, 1, 3, 6, 8\} \subseteq S$ is a subsemigroup of S. P_1 is a pseudo 5-Sylow subsemigroup of S.

Example 3.3.22: $S = \{Z_{10}, \times\}$ be the semigroup. S has no pseudo 7-Sylow subsemigroup given by $P = \{1, 0, 2, 4, 6, 8, 5\}$ as $P_1 = \{0, 1, 2, 4, 6, 8, 5, 9\}$ is a subsemigroup of S such that $P \subseteq P_1$.

Definition 3.3.4: *Let S be a semigroup of order n. Let p be a prime $0 < p < n$ and $p \not| \; n$. If P_1 is a subsemigroup of order p and if $P_1 \subseteq M$ and M is a proper subsemigroup of S of order p+1 then P_1 is defined as the quasi pseudo p-Sylow subsemigroup of S. Clearly if M is a pseudo p-Sylow subsemigroup of S then M is not a quasi pseudo p-Sylow subsemigroup of S.*

Some examples are given to describe this situation.

Example 3.3.23: Let $S = \{Z_{24}, \times\}$ be the semigroup. $0 < 5, 7, 11, 13, 17, 19, 23 < 24$ are the primes less than 24 and not divisible by 24. Let $P_1 = \{0, 1, 6, 12, 18\} \subseteq S$ be a subsemigroup of S of order 5. $M_1 = \{0, 1, 6, 12, 18, 5\} \subseteq S$ is a subsemigroup of S and $P_1 \subset M_1$. Clearly P_1 is a quasi pseudo 5-Sylow subsemigroup of S.

Let $P_2 = \{1, 23, 0, 5, 19\} \subseteq S$ be the subsemigroup of S of order 5. $M_2 = \{1, 23, 0, 5, 19, 12\} \subseteq S$ is again a subsemigroup of S of order 6. $P_2 \subset M_2$, so P_2 is a quasi pseudo 5-Sylow subsemigroup of S.

This example shows S can have more than one quasi pseudo 5-Sylow subsemigroup of order 5.

$P_3 = \{0, 1, 8, 16, 23\} \subseteq S$ is a subsemigroup of order 5. $M_3 = \{0, 1, 8, 12, 16, 23\} \subseteq S$ is a subsemigroup such that $P_3 \subset M_3$; thus P_3 is a quasi pseudo 5-Sylow subsemigroup of S.

Let $R_1 = \{0, 4, 8, 12, 16, 20, 1\}$ be a subsemigroup of order 7. $B_1 = \{0, 1, 4, 8, 12, 16, 20, 23\} \subseteq S$ is a subsemigroup of order 8 such that $R_1 \subset B_1$, so R_1 is a quasi pseudo 7-Sylow subsemigroup of S. $B_2 = \{0, 1, 4, 8, 12, 16, 20, 5\} \subseteq S$ is a subsemigroup of S such that $R_1 \subset B_2$.

Thus a quasi pseudo p-Sylow subsemigroups H of order p, may have more than one subsemigroup H of order $p + 1$ so that H is a proper subset of those subsemigroups.

This is illustrated by R_1 in example 3.3.23 for R_1 is contained in both B_1 and B_2.

Next the concept of conjugate subsemigroups of a semigroup are defined in the following:

Definition 3.3.5: *Let {S, ×} be a semigroup. P_1 and P_2 be any two subsemigroups of S of same order. P_1 is said to be the conjugate of P_2 and vice versa if there exist x, y ∈ S; such that $xP_1 = yP_2$ ($=P_2\,y$). It is interesting to note that two of the pseudo p-Sylow subsemigroups may or may not be conjugate; not as in case of p-Sylow subgroups of a group.*

In the same way two quasi pseudo p-Sylow subsemigroups may or may not be conjugate to each other in general in a semigroup S.

This is the marked deviation from the usual *p*-Sylow subgroups of a group and pseudo *p*-Sylow subsemigroups or quasi pseudo *p*-Sylow subsemigroups of a semigroup.

Example 3.3.24: Let $S = \{Z_{15}, \times\}$ be the semigroup of order 15. Consider $P_1 = \{0, 3, 6, 9, 1, 12, 14\} \subseteq S$, is a quasi 7-Sylow subsemigroup of *S*.

Now consider $P_2 = \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12\} \subseteq S$ is a pseudo 11-Sylow subsemigroup of *S*.

Next define the notion pseudo conjugate pseudo *p*-Sylow subsemigroup and a quasi pseudo *p*-Sylow subsemigroup of a semigroup *S*.

Definition 3.3.6: *Let S be a semigroup of finite order. Let P be a pseudo p-Sylow subsemigroup of S and Q a quasi pseudo p-Sylow subsemigroup of S. P and Q are said to be pseudo conjugate if P = aQb or Q = cPd for some a, b, c, d ∈ S.*

Example 3.3.25: Let $S = \{Z_6, \times\}$ be a semigroup. $P_1 = \{0, 1, 2, 4, 3\}$ and $P_2 = \{0, 1, 2, 4, 5\}$ are quasi pseudo 5-Sylow subsemigroups of S but they can never be conjugate subsemigroups of S.

In view of this the following result is proved:

Theorem 3.3.3: *Let $\{S, \times\}$ be a semigroup. If P_1 and P_2 are two quasi pseudo p-Sylow subsemigroups of S then P_1 and P_2 need not in general be conjugate to each other.*

Proof: Follows from the example 3.3.25.

In fact the first part of Sylow theorem is true for any semigroup also. The only change will be there may be a subsemigroup of order $p^{\alpha+1}$ even though $p^{\alpha} / o(S)$ and $p^{\alpha+1} \not| \; o(S)$ where S is the finite semigroup.

However the second part is not true for here only definition of pseudo conjugate subsemigroups are made. For the third part of the p-Sylow theorem one has to define the notion of cosets and double cosets this is carried out in the following:

However all these study has been modified, studied in case of Smarandache semigroups in [99].

Definition 3.3.7: *Let $\{S, \times\}$ be any semigroup. Let P be a semigroup if for any $x \in S \setminus P$; $xP \neq P$ then the semigroup S has coset xP associated with it.*

This is illustrated by some examples.

Example 3.3.26: Let $S = \{Z_{15}, \times\}$ be the semigroup. Let $P = \{0, 5, 10, 1, 11\} \subseteq S$ be a subsemigroup. For $2 \in 5$

$$2P = \{0, 10, 5, 2, 7\},$$

$$3P = \{0, 0, 0, 3\} \quad \text{and} \quad o(3P) = 2.$$

P is not an ideal only a subsemigroup of order 5 in S.

$$4P = \{0, 5, 10, 4, 14\},$$

$$6P = \{0, 0, 6\},$$

$$7P = \{0, 7, 5, 10, 2\},$$

$$8P = \{0, 8, 5, 10, 13\},$$

$$9P = \{0, 9, 0\},$$

$$12P = \{0, 12\},$$

$$13P = \{13, 0, 5, 10, 8\} \text{ and}$$

$$14P = \{14, 0, 10, 5, 4\}.$$

From this example the following are observed;

$$xP \cap yP \neq \phi.$$

For some values of x and y

$$xP \cap yP = \{0\};$$

for some $x, y \in S$.

$$2P \cap 4P = \{0, 5, 10\};$$

$$7P \cap 2P = \{0, 7, 5, 10, 2\},$$

$$14P \cap 4P = \{0, 4, 5, 10, 14\} \text{ and}$$

$$8P \cap 13P = \{0, 8, 5, 10, 13\}.$$

Let $I = \{0, 3, 6, 9, 12\} \subseteq S$ be the subsemigroup of S.

$$1I = I,\ 0I = 0,\ 2I = \{0, 6, 12, 3, 9\},$$

$$3I = \{0, 9, 3, 12, 6\},\ 4I = \{0, 12, 9, 6, 3\},\ 5I = \{0\},$$

$$7I = \{0, 6, 12, 3, 9\},\ 8I = \{0, 9, 3, 12, 6\},\ 10I = \{0\},$$

$$11I = \{0, 3, 6, 12, 9\},\ 13I = \{0, 3, 6, 12, 9\} \text{ and}$$

$$14I = \{0, 3, 6, 12, 9\}.$$

Clearly $aI = 0$ or $aI = I$ for the subsemigroup I of S if I is an ideal thus $aI = I$ *or* $\{0\}$ for every $a \in \{Z_{15}, \times\}$.

Let $M = \{0, 1, 3, 6, 9, 12, 5, 10\} \subseteq S$ be a subsemigroup of S.

$$0.M = 0,\ \ 2M = \{0, 2, 6, 12, 3, 9, 10, 5\},$$

$$4M = \{0, 4, 12, 9, 6, 3, 5, 10\},\ 7M = \{0, 7, 6, 12, 3, 9, 5, 10\},$$

$$8M = \{0, 8, 9, 3, 12, 6, 10, 5\},\ 11M = \{0, 11, 3, 6, 9, 1, 10, 5\},$$

$$13M = \{0, 13, 9, 3, 12, 5, 10, 6\},\ 14M = \{0, 14, 12, 9, 3, 6, 10, 5\} \text{ and}$$

$$xM \cap yM = \{0, 3, 12, 10, 9, 6, 5\}.$$

Thus the property of cosets related to semigroups is different from that of a group.

For in case of a group the cosets of subgroup, are either disjoint or identical; they partition the group.

In view of these observations the following result is proved:

Theorem 3.3.4: *Let $\{S, \times\}$ be the semigroup of finite order with unit and not a semilattice.*

i. If H is a subsemigroup of S and H is an ideal of S then cosets of H in S is either H or 0.

ii. If H is a subsemigroup and contains the zero of S then $P = x_1H \cap x_2H \neq \phi$ and $P \not\subset H$ in general if $1 \in H$.

Proof: If H is an ideal of S clearly the cosets of H in S is $\{0\}$ or H. Hence (i) is true. Clearly if $\{0\} \subseteq H$; then $x_1H \cap x_2H \neq \phi$ for $x_1H \cap x_2H = \{0\}$ is the least possibility. If $1 \in H$; $x_1H \cap x_2H = P \not\subseteq H$.

As said earlier the scope of this study is to analyse only finite semigroups which have non abstract representation.

$\{Z_n, \times\}$ and $S(n)$ are the two semigroups of finite order; which are mainly used in this thesis.

Now all matrices with entries from Z_n under natural product $\times_n$ of matrices are also semigroups. [89]

Next some results about the cosets of $\{Z_n, \times\}$ and $S(n)$ are carried out in the following:

Example 3.3.27: Let $S(5)$ be the symmetric semigroup.

Let

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix} \right\}$$

be the subsemigroup of $S(5)$.

Clearly $xP = Px = P$ for all $x \in S(n)$.

If

$$P_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix} \right\}$$

$\subseteq S(n)$ is a subsemigroup with identity.

$xP_1 = P_1x$ but $xP_1 \neq P_1$ infact $xP_1 \cap yP_1 \neq P_1$ for all $x, y \in S(n)$.

This is the special feature enjoyed by this particular subsemigroup P_1 of $S(n)$.

Consider

$$M = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \right\};$$

a subsemigroup of $S(5)$.

Let

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix} \in S(5).$$

$$xM = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \right\}.$$

Consider

$$Mx = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}\right\}.$$

Clearly $Mx \neq xM$. Further $xM \cap Mx \neq \phi$.

Let

$$B = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 4 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 5 & 5 \end{pmatrix}\right\} \subseteq S(5)$$

be a subsemigroup of $S(5)$.

Let

$$Y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \in S(n).$$

$$YB = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 5 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 4 & 4 & 3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 5 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 5 & 5 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 5 & 5 & 2 \end{pmatrix}\right\}.$$

Now

$$BY = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 5 & 5 \end{pmatrix},\right.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 1 & 1 \end{pmatrix}, \left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 1 & 1 \end{pmatrix}\right\}.$$

Clearly $BY \neq YB$.

In view of all these observations the following results related to the symmetric semigroup *S(n)* are proved:

Theorem 3.3.5: *Let S(n) be the symmetric semigroup of finite order.*

i. *S(n) has subsemigroups P such that* $Px = xP = P$.

ii. *S(n) has subsemigroups M such that* $Mx \neq xM$.

Proof: Let

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 2 & \dots & 2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & 3 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & n & n & \dots & n \end{pmatrix} \right\} \subseteq S(n)$$

be a subsemigroup of *S(n)*. Order of *P* is *n*.

Clearly $xP = Px = P$ for every $x \in S(n)$. Hence the claim.

Let

$$M = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 4 & 4 & 4 & 5 & \dots & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & n \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & \dots & 3 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ 5 & 5 & 5 & 5 & \ldots & 5 \end{pmatrix}$$

$$\ldots \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & n \\ 3 & 3 & 3 & 7 & 7 & 7 & 8 & \ldots & 8 \end{pmatrix}, \ldots, \left.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n \\ n & n & n & n & n & 7 & \ldots & 7 \end{pmatrix}\right\}$$

be a subsemigroup. That is none of the map is a permutation but M under composition of maps is closed. It is easily verified $xM \neq Mx$ for $x \in S(n)$.

This situation is already illustrated by the example.

Next some examples of the subsemigroups of $S(n)$ for which double coset of the subsemigroup is obtained is given in the following:

Example 3.3.28: Let $S(9)$ be the symmetric semigroup of degree nine.

Let

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & 9 \\ 1 & 1 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 2 & 2 & 2 & 2 & \ldots & 2 \end{pmatrix} \right\}$$

and

$$Q = \left\{ \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 3 & 3 & 3 & \ldots & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 5 & 5 & 5 & \ldots & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 8 & 8 & 8 & \ldots & 8 \end{pmatrix} \right\}$$

be two subsemigroups of $S(9)$. $P\ x\ Q = \{p\ x\ q\ /\ p \in P,\ q \in Q\}$ for some $x \in S(9)$ is defined as the double coset of P and Q.

If

$$x = \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 4 & 4 & 4 & \ldots & 4 \end{pmatrix}$$

is in S(9).

$$P\,x\,Q = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 3 & 3 & 3 & 3 & \ldots & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 5 & 5 & 5 & \ldots & 5 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 8 & 8 & 8 & 8 & \ldots & 8 \end{pmatrix} \right\} = Q.$$

$$Q\,x\,P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 2 & 2 & 2 & 2 & \ldots & 2 \end{pmatrix} \right\} = P.$$

Thus $Q\,x\,P \neq P\,x\,Q$. The cardinality of both the double cosets are not the same.

Example 3.3.29: Let *S(4)* be the symmetric semigroup of degree four.

Let

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

and

$$Q = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix} \right\}$$

be subsemigroups of *S(4)*.

Let

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 3 \end{pmatrix} \in S(4).$$

The double coset;

$$P\,x\,Q = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\3 & 3 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\right\} \times \left\{\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 4 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\2 & 2 & 4 & 4\end{pmatrix}\right.$$

$$\left.\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\2 & 2 & 3 & 3\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1 & 2 & 3 & 4\\4 & 4 & 4 & 4\end{pmatrix} \begin{pmatrix}1 & 2 & 3 & 4\\3 & 3 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 2 & 3 & 4\\2 & 2 & 2 & 2\end{pmatrix}\right\} \qquad \mathrm{I}$$

I denotes the double coset of $P\,x\,Q$. Consider.

$$QxP \quad = (Qx)P = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\3 & 3 & 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 3 & 3\end{pmatrix}\right\} \times$$

$$\left.\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\2 & 2 & 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\3 & 3 & 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\4 & 4 & 4 & 4\end{pmatrix}\right\}$$

$$= \quad \left\{\begin{pmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\2 & 2 & 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\3 & 3 & 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\4 & 4 & 4 & 4\end{pmatrix}\right\} \qquad \mathrm{II}$$

Now I and II are the same for this *x; P x Q = Q x P = P.*

Suppose

$$y = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 1\end{pmatrix} \in S(4).$$

Consider

$$Py\,Q = (Py)\,Q$$

$$= \left\{\begin{pmatrix}1&2&3&4\\2&2&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\right\}\times$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&4&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&4&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix}\right\} \qquad \mathrm{I}$$

Now

$$QyP = (Qy)\,P$$

$$= \left\{\begin{pmatrix}1&2&3&4\\2&2&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&4&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&4&4\end{pmatrix}\right\}\times$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&2&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}\right\} \qquad \mathrm{II}$$

I and II are identical hence the double coset of P and Q for this y is equal to P.

In some cases only $P \times Q = Q \times P$, the double cosets of the subsemigroups are equal.

The result in general is not true for all subsemigroups of $S(4)$. It is important to record if

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

then for every subsemigroup *Q* of *S(4)* and for any *x* in *S(4);*

$$P\, x\, Q = Q\, x\, P = P.$$

In view of this the following result is true:

Theorem 3.3.6: *Let S(n) be the symmetric semigroup of degree n. The double coset of the subsemigroup.*

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 2 & 2 & \dots & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 3 & \dots & 3 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ n & n & n & n & \dots & n \end{pmatrix} \right\} \in S(n);$$

of S(n) is such that PxQ = QxP = P for all x $\in$ S(n) and for every subsemigroup Q of S(n) and P is an ideal of S(n).

Proof:

$$P = \left\{ \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 2 & \dots & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & 3 \end{pmatrix}, \dots, \right.$$

$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ n & n & n & n & \dots & n \end{pmatrix} \right\}$$

is a subsemigroup of $S(n)$ which is also an ideal of $S(n)$ so if P is present in any double coset of two subsemigroups then $Px = xP$, so $P\,x\,Q = QxP = P$ for all $x \in S(n)$ and for all subsemigroups $Q \subseteq S(n)$. Hence the result.

Example 3.3.30: Let $\{Z_{12}, \times\} = S$ be the semigroup of order 12.

$$P = \{1, 11, 5, 7, 0\} \text{ and } Q = \{0, 3, 6, 9\}$$

be the subsemigroups of S. Let $2 \in S$.

$$P\,2\,Q = \{1, 11, 5, 7, 0\} \times 2 \times \{0, 6, 9, 3\}$$

$$= \{2, 10, 0\} \times \{0, 6, 9, 3\} = \{0, 6\} \qquad \text{I}$$

Here it is important to note that $|\,P\,| = 4$ and $|\,Q\,| = 4$ but order of the double coset of the subsemigroup is two and it is also a subsemigroup of S. Take $8 \in S$;

$$P\,8\,Q = \{8, 4, 0\} \times \{0, 3, 6, 9\} = \{0\}. \qquad \text{II}$$

Thus the double coset associated with 8 is different from that of 2.

Next to find

$$P\,10\,Q = \{10, 2, 0\} \times \{0, 3, 6, 9\} = \{6, 0\}.$$

Now consider

$$P_1 = \{0, 4, 8\} \text{ and } Q = \{0, 3, 6, 9\}$$

be two subsemigroups of S.

The double coset

$$P_1 5Q = \{0, 8, 4\} \times \{0, 3, 6, 9\} = \{0\}.$$

Thus if both P and Q are ideals such that $P \cap Q = \{0\}$ then $PxQ = \{0\}$ for all $x \in S$.

This is proved by the following theorem:

Theorem 3.3.7: *Let $S = \{Z_n, \times\}$ be a semigroup (n not a prime). If P and Q are ideals such that $P \cap Q = \{0\}$ and $PQ = \{0\}$ where $\langle p \rangle = P$ and $Q = \langle q \rangle$ then the double coset $PxQ = \{0\}$ for all $x \in S$.*

Proof: Follows from the fact

$Px = P$ and $P \times Q = \{0\} = P \cap Q$ as P and Q are finite sets and ideals of S.

As $P \cap Q = \{0\}$ and $P = \langle p \rangle$, $Q = \langle q \rangle$; $P \times Q = \{0\}$.

This is illustrated by an example or two.

Example 3.3.31: Let $S = \{Z_{36}, \times\}$ be the semigroup. Let $P = \{\langle 4 \rangle\}$ and $Q = \{\langle 3 \rangle\}$ be subsemigroups of S; which are ideals of S.

$$P = \{0, 4, 8, 12, 16, 20, 24, 28, 32\} \text{ and}$$

$$Q = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\}$$

be two subsemigroups of S.

Clearly

$$P \cap Q = \{0, 12, 24\}.$$

$$P\ 5\ Q = \{0, 20, 4, 24, 8, 12, 28, 32, 16\} \times \{0, 3, 6, \ldots, 33\} = \{0, 24, 12\}.$$

In view of this the following result is made.

Corollary 3.3.2: *Let $S = \{Z_n, \times\}$ be a semigroup, n not a prime. P and Q be ideals of S such that $P \cap Q \neq \{0\} = H$; H an ideal of S. $P\ x\ Q = H$ for all $x \in S$. Thus when subsemigroups of a semigroup are taken as ideals; the double coset results in a subsemigroup (ideal).*

This is the marked difference between groups and semigroups.

Example 3.3.32: Let $S = \{Z_{24}, \times\}$ be a semigroup.

Let $P = \{0, 1, 5, 12\}$ and $Q = \{1, 7, 5, 11\}$ be two subsemigroups of S. For $2 \in S$;

$$P\ 2\ Q = \{0, 2, 10\} \times \{1, 7, 5, 11\} = \{0, 2, 10, 14, 22\} \subseteq S.$$

Clearly $P\ 2\ Q$ the double coset semigroup is only a subset further $|P| = 4$ and $|Q| = 4$ but $|P\ 2\ Q| = 5$; only a subset of S.

Thus the question of whether the double coset will divide order of S; $S = \{Z_n, \times\}$ into equivalence classes is not true.

For P and $P2Q$ have 0 alone to be the common element. Thus it is to be noted that double cosets of subsemigroups behave in an entirely different way from that of the double cosets of subgroups of a group. This is one of the main contributions of this thesis.

So that part of Sylows theorem which is based on double coset property is not true in case of finite semigroups.

For the first property of decomposing the semigroup into double cosets P where P is a anti-Sylow subsemigroup will not be possible. This situation is also described by the following example:

Example 3.3.33: Let $S = \{Z_{12}, \times\}$ be the semigroup of order 12.

Let $P = \{0, 1, 3, 5, 7, 9, 11\} \subseteq S$ be a subsemigroup of order 7. P is a anti 7-Sylow subsemigroup of S. Consider $x = 2 \in S$;

$P\,2\,P = \{0, 10, 6, 2\} \times \{0, 5, 9, 11, 1, 7, 3\} = \{0, 10, 6, 2\}$.

Take $x = 6$, $P\,6\,P = \{0, 6\} \times \{0, 1, 3, 5, 7, 9, 11\} = \{0, 6\}$.

Take $x = 4$, $P\,x\,P = \{0, 8, 4\} \times \{0, 5, 9, 1, 7, 3\} = \{0, 4, 8\}$.

Take $x = 8$, $P\,x\,P = \{0, 4, 8\}$.

Take $x = 10$, $P\,x\,P = \{0, 2, 6, 10\} \times \{0, 5, 9, 11, 1, 7, 3\} = \{0, 10, 6, 2\}$.

Clearly

$$\bigcup_{x \in \{2,4,8,3,10\}} PxP = \mathrm{S}.$$

Thus the Sylow theorems cannot be easily extended to semigroups. $M = \{0, 5, 1, 9, 6\}$ is a quasi anti 5-Sylow subsemigroup of S.

For $M \subseteq \{0, 5, 1, 9, 6, 3\} = N$; N is a subsemigroup of order six.

Let $x = 2$;

$$M\,x\,M = \{0, 10, 2, 6\} \times \{0, 5, 1, 9, 6\} = \{0, 10, 2, 6\}.$$

Take $x = 3$;

$$M\,3\,M = \{0, 3, 6\} \times \{0, 5, 1, 9, 6\} = \{0, 3, 6\}.$$

Take $x = 4$;

$$M\,4\,M = \{0, 8, 4\} \times \{0, 5, 1, 9, 6\} = \{0, 4, 8\}.$$

Let $x = 7$;

$$M\,7\,M = \{0, 11, 7, 3, 6\} \times \{0, 5, 1, 9, 6\} = \{0, 7, 3, 6, 11\}.$$

$$M\,8\,M = \{0, 4, 8\} \times \{0, 5, 1, 9\ 5\} = \{0, 8, 4\}.$$

Take $x = 10$;

$$M10M = \{0, 2, 10, 6\} \times \{5, 0, 1, 9, 6\} = \{0, 2, 10, 6\}.$$

For $x = 11$;

$$M11M = \{0, 7, 11, 3, 6\} \times \{0, 1, 5, 9, 6\} = \{0, 7, 11, 3, 6\}.$$

For $x = 7$ and $x = 11$;

$$MxM = \{0, 7, 11, 3, 6\}$$

which is only a subset and not a subsemigroup of S.

For $x = 2$; $M\,x\,M = \{0, 10, 2, 6\}$ which is only a subset and not a subsemigroup of S.

For $x = 3$; $M\,x\,M$ is only a subset of S. Only for $x = 4$ and 8

$M\,x\,M = \{0, 4, 8\}$ is a subsemigroup as well as an ideal of S.

Clearly

$$S \neq \bigcup_{x \in S \setminus M} MxM .$$

Thus if S has either quasi pseudo anti p-Sylow subsemigroup or a anti-p-Sylow subsemigroup still the double coset property is not satisfied.

3.4 SPECIAL ELEMENTS IN SEMIGROUPS

For the first time this thesis defines the notion of special elements like Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents for semigroups whenever applicable. These concepts are introduced and studied in case of rings and semirings [98, 100]. These concepts are illustrated by examples. Conditions for these elements to exist in a semigroup is determined.

Definition 3.4.1: *Let S be a semigroup with unit and zero divisors. $x, y \in S$ is said to be a Smarandache zero divisor (S-zero divisor) if $x \cdot y = 0$ and there exists $a, b \in S \setminus \{x, y, 0\}$ with*

1) *$xa = 0$ or $ax = 0$,*
2) *$yb = 0$ or $by = 0$ and*
3) *$ab \neq 0$ or $ba \neq 0$.*

Examples of S-zero divisors are given only in case of $S = \{Z_n, \times\}$ for $S(n)$ the symmetric semigroup has no zero divisors.

Example 3.4.1: Let $S = \{Z_{20}, \times\}$ be the semigroup.

10, 16 $\in S$ are zero divisors as $10 \times 16 = 0 \pmod{20}$ and is also a S-zero divisor for 5, 6 $\in Z_{20} \setminus \{0, 10, 16\}$ is such that

$$5 \times 16 = 0 \pmod{20},\ 6 \times 10 = 0 \pmod{20} \text{ and } 6 \times 5 \neq 0 \pmod{20}.$$

It is important to note all semigroups built using $\{Z_n, \times\}$, n a composite number has zero divisors but it need not in general be S-zero divisors.

Example 3.4.2: Let $S = Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the semigroup under $\times$.

2, 5 $\in Z_{10}$ is such that $2 \times 5 = 0 \pmod{10}$ is a zero divisor and is not a S-zero divisor.

In view of this the following result is true:

Proposition 3.4.1: *Let S be a semigroup. Every S-zero divisor is a zero divisor but a zero divisor in general is not a S-zero divisor.*

Proof: One way is evident from the definition of a S-zero divisor. Example 3.4.2 proves the other part of the result.

Consider $S(n)$; this is a semigroup which has no zero divisors; so S-zero divisor has no relevance to this semigroup $S(n)$.

Next the notion of S-units is defined for semigroups.

Definition 3.4.2: *Let S be a semigroup with unit (monoid). $x \in S \setminus \{1\}$ is defined as the Smarandache unit (S-unit) if there exists $y \in S$ with*

1) $xy = 1$ there exist $a, b \in S \setminus \{x, y, 1\}$.

2) i) $xa = y$ or $ax = y$ or

ii) $yb = x$ or $by = x$ and

iii) $ab = 1$.

(2(i) or 2(ii) is satisfied it is enough to make a S-unit).

This is represented by the following examples:

Example 3.4.3: Let $S = \{Z_{15}, \times\}$ be the semigroup.

Now $2 \in Z_{15}$

$$2.8 = 1 \text{ (mod 15)},$$

Consider $4 \in Z_{15}$

$$4^2 \equiv 1 \text{ and } 2.4 = 8.$$

Thus (2, 8) is a S-unit of the semigroup S.

Proposition 3.4.2: *Every S-unit in a semigroup S is a unit. However all units in general are not S-units in S.*

Proof: Consider $4 \in Z_{15}$ in the above example 3.4.3 which is a unit in Z_{15}; but 4 is not a S-unit for in this case $x = y = 4$.

$$4a \equiv 4 \text{ or } 4b \equiv 4 \text{ with } a \cdot b = 1.$$

In view of this, as in case of S-units in a ring [98] the following result is proved for semigroups.

Theorem 3.4.1: *Let S be a monoid. If $x \in S \setminus \{1\}$ is a S-unit; $xy = 1$ then $x \neq y$.*

Proof: The proof is similar to rings. Let $x \in S \setminus \{0\}$ be a S-unit, this implies $xy = 1$ with $xa = y$ or $ax = y$ $(by = x$ or $yb = x)$ and $ab = 1$ if $x = y$ then $x^2 = 1$; $xa = x$; $x^2 a = x^2$ forcing $a = 1$; as $x^2 = 1$ a contradiction.

Now for the first time the notion of S-idempotents in rings is adopted to semigroups in this thesis.

Definition 3.4.3: *Let S be a semigroup. $x \in S \setminus \{0, 1\}$ is defined as a Smarandache idempotent of S if $x^2 = x$ and there exist $y \in S \setminus \{0, 1, x\}$ such that $y^2 = x$ and $yx = x$ or $xy = y$. y is defined as the Smarandache coidempotent (S coidempotent) and the pair is denoted by (x, y).*

Example 3.4.4: Let $S = \{Z_{12}, \times\}$ be the semigroup. $4 \in S$ is such that $4^2 = 4 \pmod 8$. $8^2 = 4$ and $8 \times 4 = 8$ so 4 is a S-idempotent. Clearly if x is an idempotent.

But every idempotent in a semigroup need not be a S-idempotent.

Example 3.4.5: Let $S(4)$ be the symmetric semigroup. $S(4)$ has no zero divisors but has units and idempotents.

Here the study pertains to finding S-idempotents and S-units if any in $S(4)$.

Take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in S(4);$$

clearly $x^2 = x;$ let

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} \in S(4).$$

$$y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x$$

$$\text{and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x.$$

Thus x is an S-idempotent of $S(4)$. Thus the symmetric semigroup $S(4)$ has S-idempotents.

Take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \in S(4).$$

$$x \times y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1.$$

Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \in S(4);$$

$$x \times a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = y.$$

Now

$$a \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1.$$

Thus x is a S-unit of *S(4)*. Hence *S(4)* has both S-idempotents and S-units. However as *S(4)* has no zero divisors. *S(4)* cannot have S-zero divisors as every S-zero divisor is a zero divisor.

In view of these the following result is proved.

Theorem 3.4.2: *Let S(n) be the symmetric semigroup of degree n.*

i) *S(n) has no S-zero divisors,*

ii) *S(n) has S-units and*

iii) *S(n) has S-idempotents.*

Proof. Since *S(n)* is the symmetric semigroup of degree *n* and has no zero divisors. Since every S-zero divisor is a zero divisor hence *S(n)* cannot have S-zero divisors.

S(n) has S-units.

For take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix} \in S(n).$$

$$x \circ y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 2 & 3 & 4 & 5 & \dots & n \end{pmatrix} = 1;$$

the identity element of *S(n)*.

Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix} \in S(n)$$

$$x \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix} = y$$

and

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 2 & 3 & 4 & 5 & \dots & n \end{pmatrix} = 1 \in S(n).$$

Thus x is a S-unit of *S(n)*. Hence (ii) is true.

Now to prove *S* has S-idempotents.

Let

$$x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n).$$

$$x_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1.$$

Hence x_1 is an idempotent of $S(n)$.

Take

$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 3 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n).$$

$$y_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1$$

$$y_1 \cdot x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 3 & 1 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1.$$

Thus x_1 is an S-idempotent of $S(n)$ hence (iii) is proved.

The next natural question would be; will the co-idempotents in $S(n)$ be unique. The answer is no.

This is proved by the following result:

Proposition 3.4.3: *Let S(n) be the symmetric semigroup of degree n; the S-coidempotents of an S-idempotent in S(n) in general are not unique.*

Proof: The result is proved by a counter example.

$$x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 1 & 1 & 1 & 5 & \ldots & n \end{pmatrix} \in S(n)$$

is an S-idempotent of *S(n)*.

The S-coidempotent of x_1 is

$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 3 & 1 & 1 & 5 & \ldots & n \end{pmatrix} \text{ in } S(n).$$

Consider

$$y_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 4 & 1 & 1 & 5 & \ldots & n \end{pmatrix} \in S(n).$$

$$y_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 1 & 1 & 1 & 5 & \ldots & n \end{pmatrix} \in S(n)$$

and

$$y_2 \cdot x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 4 & 1 & 1 & 5 & \ldots & n \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 1 & 1 & 1 & 5 & \ldots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 1 & 1 & 1 & 5 & \ldots & n \end{pmatrix} = x_1 \in S(n).$$

Thus y_2 is also a S-coidempotent of x_1 in *S(n)*. The coidempotents in general in *S(n)* for a given S-idempotent is not unique.

However the notion of semi idempotents and S-semi idempotents in case of rings has no relevance to semigroups of finite order under the product operation.

Next the notion of nilpotent elements and S-nilpotent elements are defined in case of semigroups. At the outset it is clear that only semigroups which has zero divisors can have nilpotent elements. Hence the symmetric semigroup $S(n)$ has no zero divisors so has no nilpotents.

Thus the only class of finite non abstract semigroups which has zero divisors is the class of semigroups $S = \{Z_n, \times\}$; n not a prime number.

Definition 3.4.4: *Let S be a semigroup under product with zero divisors. $x \in S \setminus \{0\}$ is said to be a Smarandache nilpotent element if $x^n = 0$ and there exists a $y \in S \setminus \{0, x\}$ such that $x^r y = 0$ or $yx^s = 0$, $r, s, > 0$ and $y^m \neq 0$ for any integer $m > 1$.*

First this situation will be described by some examples.

Example 3.4.6: Let $S = \{Z_{12}, \times\}$ be the semigroup. Clearly $6^2 = 0 \pmod{12}$; $8 \in S$ is such that $6 \times 8 \equiv 0 \pmod{12}$ but $8^m \neq 0 \pmod{12}$ for $m > 1$ as $8^3 \equiv 8 \pmod{12}$. Thus 6 is a S-nilpotent element of S.

Example 3.4.7: Let $S = \{Z_8, \times\}$ be the semigroup. S has nilpotents but none of them are S-nilpotents of S.

For $2^3 \equiv 0 \pmod 8$; $4^2 \equiv 0 \pmod 8$. There are no S-nilpotents in S.

In view of this one has the following result:

Proposition 3.4.4: *Let $\{S, \times\}$ be a semigroup with nilpotents.*

i) *Every S-nilpotent element of S is a nilpotent element of S.*

ii) If x is a nilpotent element of S, x need not in general be S-nilpotent.

Proof: Proof of (i) follows from the very definition of the S-nilpotent element of *S*.

Proof of (ii) follows from the above example 3.4.7 for $2 \in S = \{Z_8, \times\}$ is a nilpotent element of *S* but 2 is not a S-nilpotent of *S*.

Example 3.4.8: Let $S = \{Z_{27}, \times\}$ be the semigrouop. 3 is a nilpotent element of *S*. 6 is a nilpotent element of *S*. 12 is a nilpotent element of *S*. But *S* has no S-nilpotent elements.

In view of this the following interesting result is proved:

Theorem 3.4.3: *Let* $S = \{Z_{p^n}, \times\}$ *where p is a prime* $n \geq 2$*; S has no S-nilpotent elements.*

Proof: $x \in S$ is a nilpotent element if and only if p / x and $x^n = (0)$. Further $x^t y = 0$ if and only if p^{n-t} / y and hence $y^m = 0$ for some m. Hence it is not possible to find a y such that $y^m \neq 0$ and $x^t y = 0$. Hence the claim.

Corollary 3.4.1: *Let* $S = \{Z_{p^n}, \times\}$*, p a prime; be a semigroup. Then the nilpotent elements of S are p, 2p, 3p, …,* $(p^{n-1} - 1)p$*). That is there are* $(p^{n-1} - 1)$ *number of nilpotents.*

Proof: Follows from simple number theoretic argument.

This is illustrated by an example.

Example 3.4.9: *Let* $S = \{Z_{3^5} = Z_{243}, \times\}$ be a semigroup. The nilpotent elements of S are 3, 6, 9, 12, 15, 18, 21, 24, …, 240 = $(3^4 - 1)3$. Thus there are $3^4 - 1$ number of nilpotents in S none of them are S-nilpotents of S.

Example 3.4.10: Let $S = \{Z_{5^{10}}, \times\}$ be the semigroup. S has $(5^9 - 1)$ number of nilpotents; none of them are S-nilpotents of S.

Thus there exists a class of semigroups which has only nilpotent elements and none of them are S-nilpotents.

In fact this class has infinite number of finite semigroups of the form $S = \{Z_{p^n}, \times\}$ where $2 \leq n < \infty$ and p any prime. So for a fixed prime; one has infinite number of such semigroups.

Further for the number of primes is also infinite so this class of semigroups has undoubtedly infinite cardinality.

Next using these semigroups $S = \{Z_n, \times\}$ matrix semigroups of all orders is constructed in the following section.

3.5 MATRIX SEMIGROUPS USING Z_n

In this section matrix semigroups are studied using row matrix, column matrix, square matrix and a $s \times t$ matrix $s \neq t$; $t \neq 1$, $s \neq 1$; using the natural product $\times_n$ defined in [89]. For more about the natural product refer [89]. The natural product $\times_n$ on row matrices coincides with the usual product $\times$. However in case of square matrices both the operations can be performed and usual product $\times$ is non commutative and the other the natural product $\times_n$ is commutative.

First some examples of these matrix semigroups are given.

Example 3.5.1: Let $S = \{(x_1, x_2, x_3) \,/\, x_i \in Z_{12}, 1 \leq i \leq 3, \times\}$ be the row matrix semigroup of finite order. S has zero divisors and idempotents.

For $x = (6, 0, 6) \in S$ is such that $x^2 = (0\ 0\ 0)$. Take $x = (11, 1, 2)$ and $y = (0, 0, 6) \in S$; $x \times y = (0, 0, 0)$. Clearly $(1, 1, 1)$ acts as the multiplicative identity.

For

$$(x_1, x_2, x_3) \times (1, 1, 1) = (1, 1, 1) \times (x_1, x_2, x_3) = (x_1, x_2, x_3).$$

This S is a finite commutative monoid.

Let $A = \{(0, x, 0) \,/\, x \in Z_{12}\} \subseteq S$; A is an ideal of S.

Take $p = (4, 9, 1) \in S$; $p^2 = p = (4, 9, 1)$ is an idempotent of S.

Example 3.5.2: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \, a_i \in Z_{15}\ 1 \leq i \leq 4, \times_n \right\}$$

be the column matrix semigroup under the natural product $\times_n$,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ is the zero of } S \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in S$$

is the identity of S with respect to $\times_n$.

Let

$$x = \begin{bmatrix} 3 \\ 7 \\ 2 \\ 5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 7 \\ 10 \\ 12 \\ 1 \end{bmatrix} \in S;$$

$$x \times_n y = \begin{bmatrix} 3 \\ 7 \\ 2 \\ 5 \end{bmatrix} \times \begin{bmatrix} 7 \\ 10 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 9 \\ 5 \end{bmatrix} \in S.$$

Clearly S is a commutative monoid of finite order.

Example 3.5.3: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in Z_{40},\ 1 \le i \le 10,\ \times_n \right\}$$

be the 5×2 matrix semigroup under the natural product $\times_n$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is the zero of } M \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is the identity of M with respect to $\times_n$. M has zero divisors. M is a finite commutative monoid.

$$p = \begin{bmatrix} 0 & 0 \\ 5 & 7 \\ 4 & 14 \\ 25 & 7 \\ 0 & 39 \end{bmatrix} \text{ and } q = \begin{bmatrix} 27 & 33 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 17 & 0 \end{bmatrix} \in M$$

is such that

$$p \times_n q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

is a zero divisor of *M. M* has ideals as well as subsemigroups which are not ideals.

Example 3.5.4: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} a_i \in Z_{24}, 1 \le i \le 15 , \times_n \right\}$$

be the commutative monoid of finite order. S has units, zero divisors and idempotents. S has subsemigroups as well as ideals.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ is the identity element of } S.$$

Example 3.5.5: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \right| a_i \in Z_{20},\ 1 \leq i \leq 20,\ \times_n \right\}$$

be the 5×4 matrix semigroup under the natural product $\times_n$. M is a commutative monoid of finite order.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

acts as the unit (or identity) element of M under the natural product $\times_n$.

This semigroup also has zero divisors and units.

$$x = \begin{bmatrix} 19 & 1 & 1 & 9 \\ 1 & 1 & 9 & 19 \\ 9 & 9 & 9 & 9 \\ 19 & 9 & 11 & 11 \\ 11 & 11 & 19 & 9 \end{bmatrix} \in M \text{ is such that } x \times_n x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Example 3.5.6: Let

$$\text{S} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \middle| a_i \in Z_{43},\ 1 \leq i \leq 9,\ \times_n \right\}$$

be the 3 × 3 square matrix semigroup under the natural product $\times_n$. S is a finite commutative monoid having units, zero divisors and idempotents. If '$\times_n$'; the natural product is replaced by × on S, S is a non commutative semigroup with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ as the multiplicative identity.}$$

It is clear $A \times_n B \neq A \times B$ in general for some $A, B \in S$.

Consider

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 0 & 1 \\ 1 & 6 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{pmatrix} \in S.$$

$$A \times_n B = \begin{pmatrix} 3 & 2 & 15 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{pmatrix} \qquad \text{I}$$

$$A \times B = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 0 & 1 \\ 1 & 6 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 28 & 6 & 13 \\ 7 & 4 & 6 \\ 1 & 2 & 27 \end{pmatrix} \qquad \text{II}$$

Clearly I and II are distinct; further S is commutative monoid with respect to $\times_n$ and a non commutative monoid with respect to $\times$.

Now

$$B \times A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ 2 & 0 & 1 \\ 1 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 19 & 7 \\ 4 & 24 & 0 \\ 15 & 5 & 25 \end{pmatrix} \qquad \text{III}$$

Clearly II and III are distinct. Only in case of square matrices there can be two semigroups one under natural product $\times_n$ which is commutative and under the usual product $\times$; S is non commutative.

Having seen examples of them now this concept is formally defined.

Definition 3.5.1: *Let $S = \{m \times n$ matrix with entries from Z_s; $m = n$ or $m = 1$ and $n \neq 1$ or $m \neq 1$ and $n = 1$, $\times_n\}$ be the matrix semigroup under natural product $\times_n$. S is defined as a commutative finite matrix monoid under the natural product $\times_n$.*

This situation has been illustrated by many examples.

The following theorems are proved:

Theorem 3.5.1: *Let* $S = \{(x_1, \ldots, x_n) \mid 2 < n < \infty;\ x_i \in Z_m;\ 1 \leq i \leq n, \times\}$ *be a finite semigroup of* $1 \times n$ *row matrices.*

i. *S has subsemigroups which are not ideals.*

ii. *S has subsemigroups which are ideals.*

Proof: Consider $P = \{(x_1, \ldots, x_n) \mid x_i \in \{1, 0, m-1\}, 1 \leq i \leq n, \times\} \subseteq S$. Clearly P is a subsemigroup and is not an ideal. Hence the claim.

Consider $Q = \{(x_1, 0, \ldots, 0) \mid x_1 \in Z_m;\ \times\} \subseteq S$, clearly Q is an ideal of S. Hence the claim.

Corollary 3.5.1: *Let*

$$M = \left\{ \left. \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \right| y_i \in Z_n;\ 1 \leq i \leq m, \times_n \right\}$$

be the semigroup. M has subsemigroups which are not ideals as well as subsemigroups which are ideals.

Proof: Let

$$P_1 = \left\{ \left. \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \right| y_i \in \{0, 1, n-1\}, 1 \leq i \leq m, \times_n \right\} \subseteq M$$

be a subsemigroup which is not an ideal. Likewise

$$R_1 = \left\{ \left[\begin{array}{c} y_1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \middle| \; y_i \in Z_n, \times_n \right\} \subseteq M$$

is an ideal.

It is interesting to note M has atleast ${}_mC_1 + {}_mC_2 + \ldots + {}_mC_{m-1}$ number subsemigroups which are ideals.

Corollary 3.5.2: *Let $N = \{m \times n$ matrix $m \neq n$, $(m \neq 1$ and $n \neq 1)$ with entries from Z_s; $\times_n\}$ be the $m \times n$ matrix semigroup. N has atleast ${}_{m\times n}C_1 + {}_{m\times n}C_2 + \ldots + {}_{m\times n}C_{(m \times n - 1)}$ number of ideals and has atleast ${}_{m\times n}C_1 + {}_{m\times n}C_2 + \ldots + {}_{m\times n}C_{m \times n}$ number of subsemigroups which are not ideals.*

Proof: Let $N = \{(m \times n)$ matrix $M = (a_{ij})$ in which only the first entry $a_{11} \neq 0$ and $a_{11} \in \{0, 1, s-1\}$ all other $a_{ij} = 0$; $2 \leq i \leq m$ and $2 \leq j \leq n\}$. T is a subsemigroup of order three and T is not an ideal of N. $V = \{m = (m_{ij}) \mid m_{ij} \in Z_s$; $m_{11} \neq 0$ all other m_{ij}'s are zero$\} \subseteq S$ is an ideal of N and $|V| = s$.

Corollary 3.5.3: *Let $W = \{m \times n$ matrices with entries from Z_s, $\times_n\}$ be the semigroup. W has atleast ${}_{m\times n}C_1 + \ldots + {}_{m\times n}C_{m \times n}$ number subsemigroups and atleast ${}_{m\times n}C_1 + \ldots + {}_{m\times n}C_{(m \times n - 1)}$ number of ideals.*

Proof: This is similar to earlier corollary.

Note of the natural product of $\times_n$ in Corollary 3.5.3 is replaced by $\times$ the usual product then the above claim is not true.

All these situations are described by examples.

Example 3.5.7: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in Z_7, 1 \leq i \leq 7, \times_n \right\}$$

be the semigroup of finite order.

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1, 6\}, \times_n \right\}$$

is a subsemigroup of order 9, which is not an ideal of M.

In fact M has atleast ${}_5C_1 + {}_5C_2 + \ldots + {}_5C_5$ number of such subsemigroups which are not ideals of M.

Consider

$$B = \left\{ \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in Z_7, \times_n \right\}$$

a subsemigroup of M.

Clearly B is an ideal of M. In fact M has atleast ${}_5C_1 + {}_5C_2 + \ldots + {}_5C_4$ number of such ideals.

Example 3.5.8: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ where } a_i \in Z_{12},\ 1 \leq i \leq 9,\ \times_n \right\}$$

be the square matrix semigroup of finite order.

$$S_1 = \left\{ \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & a_4 \\ a_5 & 0 & a_6 \end{bmatrix} \middle|\ a_i \in \{0, 1, 13\},\ 1 \leq i \leq 6,\ \times_n \right\} \subseteq T$$

is a subsemigroup of finite order which is not an ideal of T. Clearly T has atleast ${}_9C_1 + {}_9C_2 + \ldots + {}_9C_9$ number of subsemigroups.

Consider

$$C_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_4 & 0 & 0 \end{bmatrix} \middle|\ a_i \in Z_{12};\ 1 \leq i \leq 4,\ \times_n \right\} \subseteq T;$$

C_1 is a subsemigroup of T which is also an ideal of T.

If

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_4 & 0 & 0 \end{bmatrix} \middle|\ a_i \in \{0, 2, 4, 6, 8, 10\},\ 1 \leq i \leq 4,\ \times_n \right\} \subseteq T;$$

D is also an ideal of T but D_1 and C_1 are distinct or different ideals of S.

It is important to note

$$E_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_4 & 0 & 0 \end{bmatrix} \middle| \; a_i \in \{0, 3, 6, 9\}; 1 \le i \le 4, \times_n \right\} \subseteq T$$

is a subsemigroup which is also an ideal of S. E_1 is different from D_1 and C_1.

Let

$$F_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_4 & 0 & 0 \end{bmatrix} \middle| \; a_i \in \{0, 4, 8\}; 1 \le i \le 4, \times_n \right\} \subseteq T$$

is a again an ideal of T different from C_1, D_1 and E_1. In fact

$$E_1 \cap F_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Thus if n of Z_n is a composite number one has more number of subsemigroups which are ideals. In fact for each ideal L in Z_n there exist atleast ${}_9C_1 + {}_9C_2 + \ldots + {}_9C_9$ number of ideals in Z_n for that specific ideal L. The ideals of Z_{12} are $L_1 = \{0, 6\}$, $L_2 = \{0, 2, 4, 6, 8, 10\}$, $L_3 = \{0, 4, 8\}$ and $L_4 = \{0, 3, 6, 9\}$ so apart from ideals of the form given by C_1. T has ideals got from L_1, L_2, L_3 and L_4 and all these contribute to $4\ ({}_9C_1 + {}_9C_2 + \ldots + {}_9C_9)$ number of distinct ideals.

However it is important to note that if Z_p is taken where p; a prime number, then the number of ideals will be less. Thus if Z_n is a composite number then Z_n contributes to more number of ideals.

Further it is interesting to note ideals of this form are also possible in T.

Let

$$R = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \middle| \; a_1, a_2, a_3 \in L_1 = \{0, 6\}, a_4, a_5, a_6 \in L_3 = \{0, 4, 8\} \right.$$

and $a_7, a_8, a_9 \in L_4 = \{0, 3, 6, 9\}; \times_n\} \subseteq T$ be an ideal of T.

Thus the operation $\times_n$, the natural product alone can yield such types of ideals also. If in T, the natural product $\times_n$ is replaced by $\times$ certainly R will not be an ideal. It may not in general be a subsemigroup.

Now having seen examples of ideals and subsemigroups; the following result is proved.

Theorem 3.5.2: *Let $S = \{m \times n$ matrix with entries from Z_s, $\times_n\}$ be the semigroup under natural product $\times_n$.*

i. If s is a non prime, S has more number of subsemigroups as well as ideals.

ii. If s is a prime, the number of ideals and subsemirings of S are less in number.

Proof: Can be proved for any given prime p and a non prime q.

The following corollary is an observation:

Corollary 3.5.4: *The number of ideals in S will depend on the number of ideals in Z_p.*

Proof is simple using number theoretic arguments.

Example 3.5.9: Let

$$U = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i \in Z_{60};\ 1 \le i \le 18,\ \times_n \right\}$$

be the matrix semigroup of finite order.

$B_1 = \{0, 30\}$, $B_2 = \{0, 20, 40\}$, $B_3 = \{0, 15, 30, 45\}$, $B_4 = \{0, 10, 20, 30, 40, 50\}$, $B_5 = \{0, 5, 10, \ldots, 55\}$, $B_6 = \{0, 4, 8, \ldots, 56\}$, $B_7 = \{0, 3, 6, 9, \ldots, 57\}$, $B_8 = \{0, 6, 12, \ldots, 54\}$, $B_9 = \{0, 12, 24, 36, 48\}$ and $B_{10} = \{0, 2, 4, \ldots, 56, 58\}$ are ideals of Z_{60}.

Thus each ideal can atleast lead to ${}_{18}C_1 + {}_{18}C_2 + \ldots + {}_{18}C_{17}$ number of ideals that is $10({}_{18}C_1 + {}_{18}C_2 + \ldots + {}_{18}C_{18})$ number of ideals apart from the ideals of the form:

$$Q = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_1, a_2, a_3 \in B_1 = \{0, 30\},\ a_4, a_5, a_6 \in B_2 = \{0, 20, 40\}, \right.$$

$a_7, a_8, a_9 \in B_3 = \{0, 15, 30, 45\}$, $a_{10}, a_{11}, a_{12} \in B_4 = \{0, 10, 20, 30, 40\}$, $a_{13}, a_{14}, a_{15} \in B_5 = \{0, 5, 10, \ldots, 55\}$ *and* $a_{16}, a_{17}, a_{18} \in B_6 = \{0, 4, 8, \ldots, 56\}, \times_n\}$; which is again an ideal of U.

In fact these types of ideals are described and are not taken into account of in this sum.

$$Y = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_1, a_2 \in B_1, a_3, a_4 \in B_2, a_5, a_6 \in B_3, a_7, a_8 \in B_4, \right.$$

$$a_{10}, a_{11} \in B_5, a_{12}, a_{13} \in B_6, a_{14}, a_{15} \in B_7, a_{16}, a_{17} \in B_8, a_{18} \in B_9; \times_n \} \subseteq U$$

is an ideal of U. Such types of ideals can also be constructed.

By improvising the operation the natural product $\times_n$ on matrices one is in a position to get several types of ideals all of them are of finite order.

Thus the more ideals in Z_n the more ideals contributed by the matrix semigroup under the natural product $\times_n$. Likewise the natural product has made the existence of S-zero divisors, S-units, S-idempotents and S-nilpotents in case of matrix semigroups.

Thus the following characterization theorem is given:

Theorem 3.5.3: *Let $M = \{(a_{ij}) \mid a_{ij} \in Z_q;\ 1 \leq i \leq n$ and $1 \leq j \leq m;\ \times_n\}$ be the $m \times n$ matrix semigroup under the natural product $\times_n$. M has more ideals than the number of ideals in Z_q (q a composite number).*

Proof: Follows from simple number theoretic techniques.

Next those matrix semigroups under natural product which has S-units are illustrated first by examples.

Example 3.5.10: Let $P = \{(x_1, x_2, x_3, x_4) / x_i \in Z_q, 1 \leq i \leq 4, \times_n\}$ be the row matrix semigroup. The S-units of P are as follows:

First (1, 1, 1, 1) is the identity element of P. Let $x = (2, 2, 2, 2) \in P$ is a unit as $y = (5, 5, 5, 5) \in P$ gives $x \times_n y = (1, 1, 1, 1)$.

Consider $a = (7, 7, 7, 7)$ and $b = (4, 4, 4, 4) \in P$ such that $x \times_n a = (5, 5, 5, 5) = y$ and $y \times_n b = (2, 2, 2, 2) = x$ and $a \times_n b = (1, 1, 1, 1)$. Thus $x = (2, 2, 2, 2)$ is a S-unit of P.

Example 3.5.11: Let

$$L = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in Z_5; 1 \leq i \leq 10; \times_n \right\}$$

be the 5×2 matrix semigroup under natural product $\times_n$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is the unit of } L.$$

Consider

$$x = \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix} \in L; x \text{ is a S-unit of } Z_5.$$

For $y = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \in L$ is such that $x \times_n y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Clearly there exists

$$a = \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \in L$$

such that

$$y \times_n a = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \times_n \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix} = x$$

and

$$a \times_n x = \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} = y;$$

further

$$a \times_n a = \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$x = \begin{bmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix}$$

is a S-unit of L.

Example 3.5.12: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \text{where } a_i \in Z_{15},\ 1 \leq i \leq 16,\ \times_n \right\}$$

be the square matrix semigroup under natural product $\times_n$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is the identity element of } V.$$

Consider

$$x = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \in V$$

is a S-unit of V as

$$y = \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} \in V$$

is such that

$$x \times_n y = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \times_n \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and take

$$a = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \in V; \quad a \times_n a = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$x \times_n a = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \times_n \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} = y.$$

Thus x is a S-unit of V.

Example 3.5.13: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \{Z_7, \times\};\ 1 \leq i \leq 12, \times_n \right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the multiplication identity of M.

Let $x = 3$ and $y = 5 \in Z_7$ is such that $x.y = 3.5 = 1 \pmod 7$.

Now $a = 2 \in Z_7$ such that $5.2 = 3$ and $4.3 = 5$ and $2.4 = 1$. Thus x is a S-unit of Z_7.

Consider

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

be the unit in M for

$$B = \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} \in M$$

is such that

$$A \times_n B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Take

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \in M.$$

$$B \times_n C = \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} \times_n \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix} = A \in M.$$

Consider

$$D = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \in M;$$

$$D \times_n A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{bmatrix} \in M.$$

Thus A is a S-unit of M. Hence if Z_7 has a S-unit; certainly M has a S-unit.

However finding the converse question is a difficult task; that is M has a S-unit does it imply Z_7 has a S-unit?

Here certainly $A = (a_{ij})$ is such that all a_{ij}*'s* are not the same for if they have same entry, surely M has a S-unit imply Z_7 has a S-unit. These S-units of M are defined as the inherited S-units from Z_7.

Let $M = \{n \times n$ matrix with entries in $Z_s, \times_n\}$ be the matrix semigroup under the natural product $\times_n$. Suppose M has A to be a S-unit with all elements the same; say $a \in Z_s$ is a S-unit of Z_s. Clearly A is not a S-unit with respect to $\times$, the usual product. So S-units under natural product $\times_n$ are not in general S-units in $\times$.

Will S-units under the usual product be S-units of $\times_n$. The answer for this is no.

This is illustrated by the following:

However it is easily argued the very units of a matrix under the natural product $\times_n$ are different from that of the matrix under the usual product $\times$; so the S-units are different.

Example 3.5.14: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_5;\ 1 \le i \le 4,\ \times_n \right\}$$

and

$$N = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_5;\ 1 \le i \le 4,\ \times \right\}$$

be the matrix semigroups under the natural product and the usual product respectively.

Let

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \in M,$$

A is a S-unit of M; for

$$B = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \in M$$

is such that

$$A \times_n B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Further

$$C = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \in M$$

is such that

$$A \times_n C = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = B$$

and

$$B \times_n C = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = A$$

with

$$C \times_n C = C^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now for the same $A, B \in N$;

$$A \times B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the identity of N. For

$$C = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \in N,\ C \times C = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Further

$$A \times C = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq B.$$

Finally

$$B \times C = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \times \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Hence a S-unit of M is not a S-unit of N.

Now consider

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in N,$$

X is a S-unit of N for

$$Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in N$$

is such that

$$X \times Y = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the unit element of N.

Let

$$P = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \in N;$$

$$A \times P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = Y$$

and

$$B \times P = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = X.$$

Finally

$$P \times P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a unit in N. Hence the claim.

Now for these $X, Y \in M;$

$$X \times_n Y = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times_n \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is not a unit in M.

Let

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in M$$

$$A \times_n P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times_n \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = B.$$

Now

$$B \times_n P = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \times_n \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = B$$

but

$$P \times_n P = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times_n \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

the unit of M. Hence the claim.

Thus in case of usual product a S-unit is not a S-unit in case of natural product as the very unit elements are different.

In view of all this the following theorem is true:

Theorem 3.5.4: *Let $M = \{ (a_{ij})_{m \times n} / a_{ij} \in Z_s;\ 1 \leq i \leq m \text{ and } 1 \leq j \leq n,\ \times_n \}$ be a matrix semigroup under the natural product $\times_n$. $A = (a)$ is a S-unit if and only if $a \in Z_s$ is a S-unit.*

Proof: Follows from the fact $A = (a)$ is a S-unit there exist $B = (b) \in M$ with $A \times_n B = (1)$ and there exists $X, Y \in M$ with $A \times_n X = B$ and $B \times_n Y = A$ and $X \times_n Y = (1)$ where $X = (x)$ and $Y = (y)$. Hence $a \in Z_s$ must be a S-unit by the very operation $\times_n$ on M.

Conversely if $a \in Z_s$ is a S-unit then $A = (a) \in M$ is a S-unit.

Next the concept of S-idempotents in these matrix semigroups under natural product are analysed.

First this situation is illustrated by some examples.

Example 3.5.15: Let $B = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_6;\ 1 \leq i \leq 5,\ \times\}$ be the row matrix semigroup.

Let $X = (4, 4, 4, 4, 4) \in B$; $X \times X = (4, 4, 4, 4, 4)$ and $Y = (2, 2, 2, 2, 2) \in B$ is such that $Y \times Y = (4, 4, 4, 4, 4) = X$ and $X \times Y = (2, 2, 2, 2, 2) = Y$. Clearly $X = (4, 4, 4, 4, 4) \in B$ is an S-idempotent.

Consider $A = (3, 3, 3, 3, 3) \in B$ clearly $A^2 = (3, 3, 3, 3, 3)$ but A is not a S-idempotent of B.

Example 3.5.16: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in Z_{30};\ 1 \le i \le 6,\ \times_n \right\}$$

be the column matrix semigroup under natural product $\times_n$.

Take

$$X = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$

in M,

$$X \times_n X = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \text{ is an idempotent of } M.$$

Consider

$$Y = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix} \in M; \qquad Y \times_n Y = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} = X$$

and

$$X \times_n Y = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix} \times_n \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix} = Y.$$

Thus X is a S-idempotent of M.

Let

$$A = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \in M$$

is such that $A \times_n A = A$ is an idempotent.

Consider

$$B = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} \in M; \qquad B \times_n B = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} \times_n \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = A;$$

$$A \times_n B = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \times_n \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix};$$

hence A is an S-idempotent of M.

Consider

$$P = \begin{bmatrix} 10 \\ 6 \\ 6 \\ 10 \\ 10 \\ 6 \end{bmatrix} \in M,$$

clearly

$$P \times_n P = \begin{bmatrix} 10 \\ 6 \\ 6 \\ 10 \\ 10 \\ 6 \end{bmatrix} \in M.$$

$$\text{Take } R = \begin{bmatrix} 20 \\ 24 \\ 24 \\ 20 \\ 20 \\ 24 \end{bmatrix} \in M\, ;\ R \times_n R = \begin{bmatrix} 10 \\ 6 \\ 6 \\ 10 \\ 10 \\ 6 \end{bmatrix} = P \in M;$$

$$R \times_n P = \begin{bmatrix} 200 \\ 144 \\ 144 \\ 200 \\ 200 \\ 144 \end{bmatrix} (mod\ 30) = \begin{bmatrix} 20 \\ 24 \\ 24 \\ 20 \\ 20 \\ 24 \end{bmatrix} = R.$$

Thus P is an S-idempotent of M. By this method we can have more idempotents.

Let

$$C = \begin{bmatrix} 25 \\ 25 \\ 0 \\ 1 \\ 25 \\ 0 \end{bmatrix} \in M;$$

$$C \times_n C = \begin{bmatrix} 25 \\ 25 \\ 0 \\ 1 \\ 25 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 25 \\ 25 \\ 0 \\ 1 \\ 25 \\ 0 \end{bmatrix} = \begin{bmatrix} 625 \pmod{30} \\ 625 \pmod{30} \\ 0 \\ 1 \\ 625 \pmod{30} \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 25 \\ 0 \\ 1 \\ 25 \\ 0 \end{bmatrix} = C.$$

Let

$$D = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix} \in M; \; D \times_n C = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 25 \\ 25 \\ 0 \\ 1 \\ 25 \\ 0 \end{bmatrix} = \begin{bmatrix} 125 \pmod{30} \\ 125 \pmod{30} \\ 0 \\ 1 \\ 125 \pmod{30} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix} = C.$$

Thus C is a S-idempotent of M.

Hence using S-idempotents of Z_n one can build many S-idempotents in M. Likewise for S-units.

Example 3.5.17: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \text{where } a_i \in S = \{Z_{15}, \times\};\ 1 \leq i \leq 15, \times_n \right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$A = \begin{pmatrix} 6 & 0 & 1 & 0 & 10 \\ 10 & 10 & 0 & 6 & 0 \\ 0 & 1 & 6 & 1 & 6 \end{pmatrix} \in M;$$

clearly $A \times_n A = A$.

$$B = \begin{pmatrix} 9 & 0 & 1 & 0 & 5 \\ 5 & 5 & 0 & 9 & 0 \\ 0 & 1 & 9 & 1 & 9 \end{pmatrix} \in M$$

is such that

$$A \times_n B = \begin{pmatrix} 6 & 0 & 1 & 0 & 10 \\ 10 & 10 & 0 & 6 & 0 \\ 0 & 1 & 6 & 1 & 6 \end{pmatrix} \times_n \begin{pmatrix} 9 & 0 & 1 & 0 & 5 \\ 5 & 5 & 0 & 9 & 0 \\ 0 & 1 & 9 & 1 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 54 & 0 & 1 & 0 & 50 \\ 50 & 50 & 0 & 54 & 0 \\ 0 & 1 & 54 & 1 & 54 \end{pmatrix} (mod\ 15) = \begin{pmatrix} 9 & 0 & 1 & 0 & 5 \\ 5 & 5 & 0 & 9 & 0 \\ 0 & 1 & 9 & 1 & 9 \end{pmatrix} = B$$

and

$$B \times_n B = \begin{pmatrix} 9 & 0 & 1 & 0 & 5 \\ 5 & 5 & 0 & 9 & 0 \\ 0 & 1 & 9 & 1 & 9 \end{pmatrix} \times_n \begin{pmatrix} 9 & 0 & 1 & 0 & 5 \\ 5 & 5 & 0 & 9 & 0 \\ 0 & 1 & 9 & 1 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 1 & 0 & 10 \\ 10 & 10 & 0 & 6 & 0 \\ 0 & 1 & 6 & 1 & 6 \end{pmatrix} = A.$$

Thus A is a S-idempotent of M. One can construct many such S-idempotents using 6, 10, 1 and 0.

Let

$$P = \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix} \in M;$$

$$P^2 = \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix} \times_{\mathrm{n}} \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix} \in M.$$

Choose

$$Q = \begin{pmatrix} 9 & 9 & 0 & 5 & 5 \\ 5 & 5 & 1 & 9 & 9 \\ 5 & 9 & 5 & 9 & 5 \end{pmatrix} \in M$$

is such that

$$Q \times_n Q = \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix} = P.$$

$$P \times_n Q = \begin{pmatrix} 6 & 6 & 0 & 10 & 10 \\ 10 & 10 & 1 & 6 & 6 \\ 10 & 6 & 10 & 6 & 10 \end{pmatrix} \times_n \begin{pmatrix} 9 & 9 & 0 & 5 & 5 \\ 5 & 5 & 1 & 9 & 9 \\ 5 & 9 & 5 & 9 & 5 \end{pmatrix} = Q.$$

Thus P is a S-idempotent of M.

In fact M has several such S-idempotents but they are finite in number.

Now having seen S-idempotents in matrix semigroups, here the necessary and sufficient condition for S-idempotents to exists in matrix semigroup is obtained.

Theorem 3.5.5: *Let $S = \{m \times n$ matrix with entries from Z_s, s a composite number; $\times_n\}$ be a matrix semigroup under the natural product $\times_n$.*

S has S-idempotents if and only if Z_s has S-idempotents.

Proof: The results is similar to S-units.

Next the notion of S-zero divisors is analysed and examples of them are given.

Example 3.5.18: Let $B = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_{20};\ 1 \leq i \leq 5,\ \times\}$ be the row matrix semigroup.

Let $X = (10, 16, 0, 0, 10) \in B$; $Y = (10, 10, 0, 0, 10)$ is such that

$$X \times Y = (0\ 0\ 0\ 0\ 0).$$

Take

$A = (6, 5, 0, 0, 6)$ and $D_1 = (6, 6, 0, 0, 6) \in B$;

clearly

$$X \times A = (10, 16, 0, 0, 10) \times (6, 5, 0, 0, 6) = (0, 0, 0, 0, 0).$$

$$Y \times D_1 = (10, 10, 0, 0, 10) \times (6, 6, 0, 0, 6) = (0, 0, 0, 0, 0)$$

and

$$A \times D_1 = (6, 5, 0, 0, 6) \times (6, 6, 0, 0, 6) = (16, 10, 0, 0, 16) \neq (0, 0, 0, 0, 0).$$

Thus X is a S-idempotent of D_1. Consider $Z = (10, 8, 10, 8, 10)$ and $Y = (8, 10, 8, 10, 8) \in B$. Clearly $Z \times Y = (0\ 0\ 0\ 0\ 0)$.

Take $D = (5, 4, 5, 4, 5)$ and $C = (2, 5, 2, 5, 2) \in B$.

Clearly

$$C \times Z = (0\ 0\ 0\ 0\ 0) \text{ and } D \times Y = (0\ 0\ 0\ 0\ 0)$$

but

$$\begin{aligned} C \times D &= (2, 5, 2, 5, 2) \times (5, 4, 5, 4, 5) \\ &= (10, 0, 10, 0, 10) \neq (0\ 0\ 0\ 0\ 0). \end{aligned}$$

Thus Z is a S-zero divisor of B.

Example 3.5.19: Let $W = \{(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_i \in Z_{10}, 1 \leq i \leq 6, \times\}$ be the row matrix semigroup. W has no S-zero divisors. Clearly Z_{10} has no S-zero divisors.

Example 3.5.20: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in Z_{12};\ 1 \leq i \leq 6,\ \times_n \right\}$$

be a column matrix semigroup.

Let

$$X = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 4 \\ 4 \\ 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 6 \\ 6 \\ 6 \end{bmatrix} \in V;\ \ X \times_n Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a zero divisor.

Let

$$A = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 6 \\ 6 \\ 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 4 \\ 4 \\ 4 \end{bmatrix} \in V;\ X \times_n A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } Y \times_n B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now

$$A \times_n B = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So X is a S-zero divisors of V.

Example 3.5.21: Let

$$N = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in Z_{24};\ 1 \le i \le 12,\ \times_n \right\}$$

be the matrix semigroup under natural product $\times_n$.

Let

$$X = \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \\ 6 & 6 & 6 \\ 12 & 12 & 12 \end{bmatrix} \in N;\ \ Y = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \end{bmatrix} \in N$$

is such that

$$X \times_n Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 8 & 8 & 8 \\ 4 & 4 & 4 \\ 8 & 8 & 8 \\ 4 & 4 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \end{bmatrix} \in N.$$

$$A \times_n X = \begin{bmatrix} 8 & 8 & 8 \\ 4 & 4 & 4 \\ 8 & 8 & 8 \\ 4 & 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$B \times_n Y = \begin{bmatrix} 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A \times_n B = \begin{bmatrix} 8 & 8 & 8 \\ 4 & 4 & 4 \\ 8 & 8 & 8 \\ 4 & 4 & 4 \end{bmatrix} \times_n \begin{bmatrix} 6 & 6 & 6 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \\ 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 16 & 16 & 16 \\ 0 & 0 & 0 \\ 16 & 16 & 16 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus X is a S-zero divisor of N.

In fact N has several such S- zero divisors. N has S-zero divisors as Z_{24} has several S-idempotents.

Example 3.5.22: Let

$$P = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \right| a_i \in Z_{28},\ 1 \leq i \leq 10,\ \times_n \right\}$$

be the matrix semigroup under natural product $\times_n$.

Let

$$X = \begin{pmatrix} 7 & 7 & 7 & 7 & 7 \\ 4 & 0 & 4 & 0 & 4 \end{pmatrix} \in P \text{ then } Y = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 14 & 0 & 14 & 0 & 14 \end{pmatrix} \in P$$

is such that

$$X \times_n Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 14 & 0 & 14 & 0 & 14 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 7 & 7 & 7 & 7 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix} \in P.$$

$$A \times_n X = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 14 & 0 & 14 & 0 & 14 \end{pmatrix} \times_n \begin{pmatrix} 7 & 7 & 7 & 7 & 7 \\ 4 & 0 & 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$B \times_n Y = \begin{pmatrix} 7 & 7 & 7 & 7 & 7 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix} \times_n \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 14 & 0 & 14 & 0 & 14 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider

$$A \times_n B = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 14 & 0 & 14 & 0 & 14 \end{pmatrix} \times_n \begin{pmatrix} 7 & 7 & 7 & 7 & 7 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\neq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus X is a S-zero divisor of P.

In view of all these one has the following result.

Proposition 3.5.1: *Let $S = \{n \times m$ matrix with entries from Z_t; t; a composite number, $\times_n\}$ be the matrix semigroup under the natural product $\times_n$. S has S-zero divisors if and only if Z_t has S-zero divisors.*

Proof: Follows from the fact if S has S-zero divisors under the natural product $\times_n$ each entry in that matrix must be a S-zero divisor. Conversely if Z_t has S-zero divisors certainly S has S-zero divisors. Hence the result.

One can speak of the S-antizero divisors also in case of the matrix semigroups under product $\times_n$.

First this will be illustrated by an example or two.

Example 3.5.23: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \, a_i \in Z_{12}, \times_n \right\}$$

be the matrix semigroup under natural product $\times_n$, S has S-antizero divisors.

For take

$$x = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix} \text{ and } y = \begin{pmatrix} 8 & 8 & 8 \\ 8 & 8 & 8 \end{pmatrix} \in S.$$

Clearly

$$x \times_n y = \begin{pmatrix} 8 & 8 & 8 \\ 8 & 8 & 8 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let

$$a = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix} \in S;\ x \times_n a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$y \times_n a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } a^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus S has S-antizero divisors.

Example 3.5.24: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle|\, a_i \in Z_{20};\ 1 \leq i \leq 5, \times_n \right\}$$

be the matrix semigroup under natural product. Take

$$x = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \text{ and } y = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \in B;\ \ x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Take

$$a = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \in B;\ a \times_n x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \in B$$

is such that

$$b \times_n y = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ but } a \times_n b = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus a is a S-antizero divisor of B. It is clearly observed that if Z_n has S-antizero divisors then clearly B has antizero divisors.

In view of this the following result is proved:

Proposition 3.5.2: *Let S = {m × n matrix with entries from Z_t; t a composite number; $\times_n$} be the matrix semigroup under natural product $\times_n$. S has anti zero divisors if and only if Z_t has antizero divisors.*

Proof: As in case of S-zero divisors the result can be proved.

Next for the first time the concept of Smarandache nilpotent elements was defined for semigroups in this thesis. Examples of S-nilpotents in case of matrix semigroups is given in the following:

Example 3.5.25: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in Z_{12},\ 1 \leq i \leq 15,\ \times_n \right\}$$

be the matrix semigroup under natural product $\times_n$.

Take

$$A = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix} \in W;\ A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \end{bmatrix} \in W$$

is such that

$$A \times_n B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ but } B^3 \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ \text{for } B^3 = \begin{bmatrix} 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \end{bmatrix}.$$

Thus A is a S-nilpotent element of W.

Example 3.5.26: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \text{ where } a_i \in Z_{20};\ 1 \leq i \leq 15, \times_n \right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$A = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \end{bmatrix} \in P$$

is such that

$$A \times_n A = A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider

$$B = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix} \in P$$

is such that

$$A \times_n B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

but

$$B^3 = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus A is a S-nilpotent element of P.

In fact as in case of S-idempotents, S-zero divisors and S-antizero divisors one can prove the following result.

Proposition 3.5.3: *Let N = {Collection of all $n \times m$ matrices with entries from Z_t, t a non prime, $\times_n$} be the matrix semigroup under natural product $\times_n$. N has S-nilpotents if and only if Z_t has non trivial S-nilpotents.*

Proof: As in case of S-zero divisors used S-antizero divisors.

3.6 Conclusions

The following conclusions are based on results obtained in this chapter. Thus in this chapter, semigroups in the first two sections; are viewed as algebraic structures which satisfy some of the classical theorems for finite groups.

In view of this the new notion of anti Lagrange's property and weak Lagrange's property are defined and a class of finite semigroups which satisfy these properties are given.

Cauchy theorem for finite semigroups is adopted to finite semigroups; this study leads to the definition of anti Cauchy property and Cauchy property.

It is proved symmetric semigroups satisfies both Cauchy property as well as anti Cauchy property.

Special study of finite semilattices $\cup$ and $\cap$ which are idempotent semigroups are analysed. Semilattices which satisfy both the Lagrange's and anti Lagrange's property are obtained.

Finally the notion of extended symmetric semigroups are made with a fond hope of embedding semilattices in these extended symmetric semigroups. However one faces several hurdles at this stage. It is proved finite semilattices of order n can be embedded in a extended symmetric semigroup $S(n - 1) \cup \phi$.

Further to study Sylow theorems for finite semigroups one is forced to define the two new notions of pseudo p-Sylow subsemigroups and quasi pseudo p-Sylow subsemigroups.

These lead to the concept of pseudo conjugate subsemigroups. Finally the concept of coset and double coset cannot be adopted when the subsemigroups are ideals for they behave in a very different way.

All these are described by examples, theorems related with them are proved. Thus by all means Sylow theorems have many limitations in case of finite semigroups.

Finally this thesis is the first one to introduce the notion of Smarandache units, Smarandache idempotents, Smarandache zero divisors and S-nilpotents for finite semigroups.

Characterization for semigroups to contain these new concepts are obtained. Further a complete study for the class of matrix semigroups under the natural product $\times_n$ are carried out and analysed for these properties.

CHAPTER FOUR

SEMIGROUP SEMIRINGS USING DISTRIBUTIVE LATTICES AS SEMIRINGS

4.1 INTRODUCTION

In this chapter semigroup semirings of finite semigroups over distributive lattices are carried out. This sort of study has been done in [100]. This chapter has five sections. Section one is introductory in nature. Section two studies semigroup semirings using distributive lattices as semirings. Throughout this section C_n denotes the chain lattice of length n. $0 < a_{n-2} < a_{n-1} < \ldots < a_1 < 1$ will denote the chain lattice C_n of length n. S any of the finite semigroups $\{Z_n, \times\}$ or $S(n)$ or semilattices (with operations $\cup$ or $\cap$) or matrix semigroups of finite order under the natural product $\times_n$. Section three obtains properties related to substructures of the semigroup semiring C_nS. Section four uses the distributive lattices (and or) Boolean algebras of finite order which are not chain lattices as semirings in the study of semigroup semirings. The final section gives the conclusions derived from this study.

4.2 SEMIGROUP SEMIRINGS OF SEMIGROUPS OVER CHAIN LATTICES

The definition of semigroup semiring is made only to make this chapter self contained one. Throughout this chapter it is assumed all semigroups are finite and contain identity that is they are finite monoids. Condition for these semigroup semirings to contain special elements like zero divisors etc are obtained. For more about semigroup semirings refer [52, 67, 100].

Definition 4.2.1: *Let S be a finite monoid under product operation* $\times$ *(or operations other than +) and* C_n *be the chain lattice of finite order. The set*

$$C_nS = \left\{ \sum_{i=1}^{n} a_i s_i \;\middle|\; n < \infty;\ s_i \in S,\ a_i \in C_n,\ +,\ \times \right\}$$

with two binary operations '+' and $\times$ *is defined as the semigroup semilattice or semigroup semiring if the following conditions are satisfied:*

i. *If* $\alpha = \sum_{i=1}^{n} a_i s_i$ *and* $\beta = \sum_{i=1}^{n} b_i s_i \in C_nS$ *then* $\alpha = \beta$ *if and only if* $a_i = b_i$ *for* $i = 1, 2, \ldots, n$ *and* $s_i \in S$.

ii. *Let* $\alpha = \sum_{i=1}^{n} a_i s_i$ *and* $\beta = \sum_{i=1}^{n} b_i s_i$ *be in* C_nS *then*

$$\alpha + \beta = \sum_{i=1}^{n} a_i s_i + \sum_{i=1}^{n} b_i s_i = \sum_{i=1}^{n} (a_i + b_i) s_i = \sum_{i=1}^{p} (a_i \cup b_i) s_i$$

is in C_nS.

iii. $$\alpha \times \beta = \sum_{i=1}^{n} a_i s_i \times \sum_{j=1}^{m} b_i s_j = \sum_{k} a_i \times b_j s_k ;\ s_k = s_i \times s_j \in S$$

$$= \sum_{k} (a_i \cap b_j) s_k$$

$$= \sum_{k} y_k s_k ;\ y_k \in C_n$$ *is in* C_nS. *(k runs over finite number).*

iv. $a_i s_i = s_i a_i$ *for all* $a_i \in C_n$ *and* $s_i \in S$.

v. $1.a_i = a_i.1 = a_i$ *for all* $a_i \in C_n$; *1 the identity element of the semigroup of S.*

vi. $1.s_i = s_i.1 = s_i$ *for all* $s_i \in S$ *and* $1 \in C_n$ *is the greatest element of* C_n.

vii. $0.s_i = s_i.0 = 0$ *for all* $s_i \in S$ *and* $0 \in C_n$.

viii. *If* $0 \in S$ *then* $a_i.0 = 0.a_i$ *for all* $a_i \in C_n$.

ix. $\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma$ *for all* $\alpha, \beta, \gamma \in C_nS$.

First this will be illustrated by some examples.

Example 4.2.1: Let $C_9 = 0 < a_7 < a_6 < \ldots < a_1 < 1$ be the chain lattice of order 9. $S = \{Z_{10}, \times\}$ be the semigroup. C_9S be the semigroup semiring of S over the chain lattice C_9. C_9S has zero divisors and $o(C_9S) < \infty$.

For take

$$x = a_8 5 \text{ and } y = (a_2 4 + a_3 8 + a_4 2) \in C_9S.$$

$$\begin{aligned}
x + y &= a_8 5 + a_2 4 + a_3 8 + a_4 2 \in C_9S. \\
x \times y &= (a_8 5) \times (a_2 4 + a_3 8 + a_4 2) \\
&= (a_8 \cap a_2)(5 \times 4) + (a_8 \cap a_3)(5 \times 8) + (a_8 \cap a_4)(5 \times 2) \\
&= a_8 \times 0 + a_8 \times 0 + a_8 \times 0 \\
&= 0.
\end{aligned}$$

Thus C_9S has zero divisors since the semigroup semiring has zero divisors C_9S is not a semifield. Further as S is a commutative so is C_9S.

Example 4.2.2: Let $C_{12} = 0 < a_{10} < a_9 < \ldots < a_1 < 1$ be a chain lattice of order 12. $S = \{Z_{13}, \times\}$ be the semigroup of order 13. $C_{12}S$ be the semigroup semiring of finite order which is commutative.

Clearly if

$$x = a_1 10 + a_3 5 + a_2 3 + a_6 7 + a_{10} \text{ and } y = a_1 5 + a_7 3 + a_5 4 + a_6 \in C_{12}S.$$

$$\begin{aligned}
x + y &= (a_1 10 + a_3 5 + a_2 3 + a_6 7 + a_{10}) + (a_1 5 + a_7 3 + a_5 4 + a_6) \\
&= a_1 10 + (a_3 \cup a_1)5 + (a_2 \cup a_7)3 + a_6 7 + a_5 4 + a_{10} \cup a_6 \\
&= a_1 10 + a_1 5 + a_2 3 + a_6 7 + a_5 4 + a_6 \text{ is in } C_{10}S.
\end{aligned}$$

Let

$$a = a_7 10 + a_5 2 + a_8 \text{ and } b = a_5 9 + a_6 11 + a_2 12 + a_1 \in C_{12}S;$$

$$\begin{aligned}
a \times b &= (a_7 10 + a_5 2 + a_8) \times (a_5 9 + a_6 11 + a_2 12 + a_1) \\
&= (a_7 \cap a_2)\ 10 \times 12 + (a_5 \cap a_2)\ 2 \times 12 + (a_8 \cap a_2)\ 1 \times 12 + (a_7 \cap a_5)\ 10 \times 9 + (a_5 \cap a_5)\ 2 \times 9 + (a_8 \cap a_5)\ 1 \times 9 + (a_7 \cap a_6)\ 10 \times 11 + (a_5 \cap a_6)\ 2 \times 11 + (a_8 \cap a_6)\ 11 + (a_7 \cap a_1)\ 10 \times 1 + (a_5 \cap a_1)\ 2 \times 1 + (a_8 \cap a_1) \\
&= a_7 3 + a_5 11 + a_8 12 + a_7 12 + a_5 5 + a_8 9 + a_7 6 + a_6 9 + a_8 11 + a_7 10 + a_5 2 + a_8 \\
&= a_7 3 + (a_8 \cup a_5)11 + (a_8 \cup a_7)12 + a_5 5 + (a_8 \cup a_6)9 + a_7 6 + a_7 10 + a_5 2 + a_8 \\
&= a_7 3 + a_5 11 + a_7 12 + a_5 5 + a_6 9 + a_7 6 + a_7 10 + a_5 2 + a_8.
\end{aligned}$$

This is the way product is performed. Clearly $a + b = 0$ is impossible in $C_{12}S$. Thus $C_{12}S$ is a semifield as $a \times b \neq 0$ for any $a, b \in C_{12}S$.

Example 4.2.3: Let C_{11} be the chain lattice of order 11. Let $S = \{Z_6, \times\}$ be the semigroup of order 6. $C_{11}S$ be the semigroup semiring of finite order. Clearly $C_{11}S$ is commutative but is not a semifield.

For take $x = a_9 3 \in C_{11}S.$

Clearly

$$x \times x = a_9 3 \times a_9 3 = (a_9 \cap a_9)\ (3 \times 3) = a_9 3.$$

Thus x is an idempotent element of $C_{11}S$.

Consider $y = a_3 4 \in C_{11}S$;

$$y \times y = a_3 4 \times a_3 4 = (a_3 \cap a_3)\ (4 \times 4) = a_3 4 = y.$$

Thus y is also an idempotent of $C_{11}S$.

Let $a = (a_{10} 3 + a_5 4) \in C_{11}S$

$$\begin{aligned} a \times a &= (a_{10} 3 + a_5 4) \times (a_{10} 3 \times a_5 4) \\ &= (a_{10} \cap a_{10})\ 3 \times 3 + (a_5 \cap a_{10})\ 4 \times 3 + (a_{10} \cap a_5)\ 3 \times 4 + (a_5 \cap a_5)\ 4 \times 4 \\ &= a_{10} 3 + a_5 4 = a. \end{aligned}$$

Thus a is also an idempotent of $C_{11}S$. $C_{11}S$ has zero divisors.

For take

$$p = a_4 2 + a_5 4 \text{ and } q = a_5 3 \in C_{11}S.$$

$$\begin{aligned} p \times q &= (a_4 2 + a_5 4) \times a_5 3 \\ &= (a_4 \cap a_5)\ 2 \times 3 + (a_5 \cap a_5)\ 4 \times 3 \\ &= 0, \end{aligned}$$

so $C_{11}S$ has non trivial zero divisors. Hence $C_{11}S$ is not a semifield.

Next consider semigroup semiring which is non commutative.

Example 4.2.4: Let $C_{16} = 0 < a_{14} < a_{13} < \ldots < a_2 < a_1 < 1$ be a chain lattice of order 16 and $S = S(4)$ be the symmetric semigroup of order 4^4. Let $C_{16}S$ be the semigroup semiring of finite order. Clearly $C_{16}S$ is non commutative as $S(4)$ is a non commutative semigroup.

Now how sum and product operations are performed on $C_{16}S$ is described briefly.

Let

$$x = a_4\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} + a_9\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + a_6$$

and

$$y = a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} + a_2\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_7\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_{10} \in C_{16}S.$$

$$\begin{aligned} x + y = \; & (a_4 \cup a_2)\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \\ & + a_9\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \\ & + a_7\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_6 \cup a_{10} \end{aligned}$$

$$\begin{aligned} = \; & a_2\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} + a_9\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + \\ & a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} + a_7\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_6 \in C_{16}S. \end{aligned}$$

$$\begin{aligned} x \times y = \; & [a_4\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} + a_9\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + a_6] \times \\ & [a_{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} + a_2\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_7\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_{10}] \end{aligned}$$

$$
\begin{aligned}
= \quad & (a_4 \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \right] \\
& + (a_4 \cap a_2) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right] \\
& + (a_4 \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right] \\
& + (a_4 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\
& + (a_{10} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \right] \\
& + (a_{10} \cap a_2) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right] \\
& + (a_{10} \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right] \\
& + (a_{10} \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right] \\
& + (a_9 \cap a_{10}) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \right] \\
& + (a_9 \cap a_2) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \right] \\
& + (a_9 \cap a_7) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \right] \\
& + (a_9 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + (a_6 \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \\
& + (a_6 \cap a_2) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + (a_6 \cap a_7) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_6 \cap a_{10}.
\end{aligned}
$$

$$
\begin{aligned}
= \quad & a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 4 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix} \\
& + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}
\end{aligned}
$$

$$+ a_{10}\begin{pmatrix}1&2&3&4\\1&2&4&2\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\1&1&1&4\end{pmatrix}$$
$$+ a_{9}\begin{pmatrix}1&2&3&4\\2&2&2&4\end{pmatrix} + a_{9}\begin{pmatrix}1&2&3&4\\2&2&2&1\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}$$
$$+ a_{10}\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} + a_{6}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_{7}\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix}$$
$$+ a_{10} \in C_{16}S.$$

Thus + and × are performed.

Consider

$$y \times x \quad = \quad [a_{10}\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} + a_{2}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_{7}\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix}$$
$$+ a_{10}] \times [\, a_4 \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} + a_{10}\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} +$$
$$a_{9}\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix} + a_6]$$

$$= \quad (a_{10} \cap a_4)\left[\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$
$$+ (a_{2} \cap a_4)\left[\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$
$$+ (a_{7} \cap a_4)\left[\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right]$$
$$+ (a_{10} \cap a_4)\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$$
$$+ \; (a_{10} \cap a_{10})\left[\begin{pmatrix}1&2&3&4\\1&3&4&1\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\right]$$
$$+ (a_{2} \cap a_{10})\left[\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\right]$$
$$+ (a_{7} \cap a_{10})\left[\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix} \circ \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\right]$$

$$+ (a_{10} \cap a_{10}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$+ (a_{10} \cap a_9) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ (a_2 \cap a_9) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ (a_7 \cap a_9) \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} \right]$$

$$+ (a_{10} \cap a_9) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + (a_{10} \cap a_6) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix}$$

$$+ (a_2 \cap a_6) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + (a_7 \cap a_6) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix}$$

$$+ a_{10} \cap a_6$$

$$= \quad a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 3 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

$$+ a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \end{pmatrix} + a_6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$+ a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix} + a_{10}.$$

Clearly $x \times y \neq y \times x$; thus $C_{16}S(4)$ is a non commutative semigroup semiring.

However $C_{16}S(4)$ has no zero divisors but has units as well as idempotents. For take

$$x = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in C_{16}S(4);$$

$$x \times x = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \times a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = a_7 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x.$$

Thus x is an idempotent of $C_{16}S(4)$. Thus $C_{16}S(4)$ is only a semigroup semiring which is a semidivison ring.

In view of this the following result is important:

Proposition 4.2.1: *Let $C_nS(m)$ be the semigroup semiring of the symmetric semigroup $S(m)$ over the chain lattice C_n.*

i. *$C_nS(m)$ has idempotents.*

ii. *$C_nS(m)$ is only a semidivision ring.*

Proof: Follows from the simple fact

$$x = a_i \begin{pmatrix} 1 & 2 & 3 & \ldots & \mathrm{m} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix} \in C_nS(m)$$

is such that $x \times x = x$. Thus $C_nS(m)$ has idempotents. Since $S(m)$ has no zero divisors and $S(m)$ is non commutative semigroup; $C_nS(m)$ is a semidivision ring.

It is an important and an interesting observation to see that the existence of idempotent in a semiring does not imply the existence of a zero divisor a marked difference between rings and semirings.

Next the properties of semigroups $S = \{Z_n, \times\}$ are characterized.

Proposition 4.2.2: *Let $S = \{Z_n, \times\}$ be the semigroup (n a prime) and C_m be the chain lattice of order m. The semigroup semiring C_mS is a semifield.*

Proof: Follows from the fact S has no zero divisors as n is a prime; hence C_mS is a semifield; for always $a + b = 0$ if and only if $a = b = 0$ for all $a, b \in C_mS$ and $a.b = 0$ is not possible.

Proposition 4.2.3: *Let C_m be the chain lattice $0 < m - 2 < m - 1 < \ldots < m_2 < m_1 < 1$ and $S = \{Z_n, \times\}$ be the semigroup. C_mS be the semigroup semiring of the semigroup S over the chain lattice C_m. C_mS is not a semifield if and only if Z_n has zero divisors.*

Proof: When Z_n has zero divisor clearly C_mZ_n has zero divisors so C_mZ_m is not a semifield. If C_mZ_n has zero divisors since C_m is a chain lattice only zero divisors are contributed by Z_n. This is true from proposition 4.2.2. Hence the result.

Next consider the matrix semigroup using the natural product $\times_n$. For the notion of natural product in matrices refer [90].

Example 4.2.5: Let $C_8 = 0 < a_6 < a_5 < \ldots < a_2 < a_1 < 1$ be the chain lattice of order 8 and $S = \{(a_1, a_2, a_3) \mid a_i \in Z_7, 1 \leq i \leq 3, \times\}$ be the matrix semigroup. C_8S is the semigroup semiring. C_8S has zero divisors and units. (1, 1, 1) is the unit of C_8S. C_8S is not a semifield.

Example 4.2.6: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in Z_{17}, 1 \leq i \leq 5, \times_n \right\}$$

be the column matrix semigroup under the natural product $\times_n$. $C_{10} = 0 < a_8 < a_7 < \ldots < a_2 < a_1 < 1$ be the chain lattice of order 10. $C_{10}S$ be the semigroup semiring. $C_{10}S$ has zero divisors.

All the idempotents are only of the form

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \, a_i \in \{0, 1\};\ 1 \leq i \leq 5,\ \times_n \right\} \subseteq S.$$

However $C_{10}S$ has units and the identity element is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$$x = \begin{bmatrix} 9 \\ 2 \\ 16 \\ 3 \\ 6 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 9 \\ 16 \\ 6 \\ 3 \end{bmatrix} \in C_{10}S \text{ is such that } x \times_n y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let

$$a = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 9 \end{bmatrix} \text{ and } b = \begin{bmatrix} 7 \\ 0 \\ 8 \\ 0 \\ 0 \end{bmatrix} \in C_{10}S. \qquad a \times_n b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a zero divisor.

Apart from this let

$$\alpha = a_3 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + a_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + a_8 \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 12 \\ 0 \end{bmatrix}$$

and

$$\beta = a_7 \begin{bmatrix} 4 \\ 0 \\ 7 \\ 0 \\ 6 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 12 \end{bmatrix} \in C_{10}S.$$

$$\alpha \times_n \beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $C_{10}S$ has zero divisors even though the semigroup $S = \{Z_{17}, \times\}$ has no zero divisors.

Example 4.2.7: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \middle| \, a_i \in Z_{13};\ 1 \leq i \leq 12,\ \times_n \right\}$$

be the matrix semigroup under product $\times_n$. Let $C_2 = \{0, 1\}$ be the chain lattice of order two. C_2M be the semigroup semiring. C_2M has zero divisors however the idempotents are from the subset

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \middle| \, a_i \in \{0, 1\};\ 1 \leq i \leq 12,\ \times_n \right\} \subseteq M.$$

P is the collection of all idempotents from M.

Thus C_2M has idempotents though Z_{13} has no idempotents or zero divisors.

Thus matrix semigroup S under the natural product paves way for several idempotents, units and zero divisors even if S is built using Z_p; p a prime.

Finally this leads to the following result:

Proposition 4.2.4: *Let S = {collection of all $m \times n$ matrices with entries from Z_t; $\times_n$} be the matrix semigroup and $C_s = \{0 < a_{s-2} < a_{s-3} < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of order s. C_sS be the semigroup semiring of S over C_s.*

i. *C_sS has zero divisors, units and idempotents.*

ii. *P = {$m \times n$ matrices with entries from {0, 1}} if t is a prime is the only subset of idempotents of S.*

iii. *Q = {$m \times n$ matrices with entries from {0, 1, collection of all idempotents from Z_t}; if t is a non prime} $\subseteq S$ is the only collection of idempotents of S.*

Proof: Follows from the fact S has zero divisors, units and idempotents. Further (ii) and (iii) can be verified to be true.

This will be illustrated by an example.

Example 4.2.8: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| a_i \in Z_6;\ 1 \leq i \leq 8,\ \times_n \right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ is the unit element of } S.$$

Let $C_3 = \{0 < a_1 < 1\}$ be the chain lattice of order three; C_3S be the semigroup semiring of S over C_3. The units of S are

$$M = \left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{pmatrix} \right\}.$$

That is

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| a_i \in \{1, 5\} \right\} \subseteq S$$

alone are units of S.

Further every $x \in P$ is such that

$$x^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

P is also a subsemigroup of S. Consider

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} \middle| a_i \in \{0, 1, 3, 4\};\ 1 \leq i \leq 8, \times_n \right\} \subseteq S$$

is the collection of all idempotents in S.

Let

$$p = a_1 \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \in C_3S.$$

$$p^2 = \left[a \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \right]^2$$

$$= \quad a \cap a\left[\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix}\right]$$

$$+ \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

$$+ (a_1 \cap 1) \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

$$+ (1 \cap a_1) \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} \times_n \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix}$$

$$= \quad a \begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\neq \quad p.$$

Thus in general p is not an idempotent of C_3S though

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

are idempotents further in this case

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 4 & 0 & 1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is also an idempotent. Thus in this case M is an idempotent subsemigroup of S.

This may not in general be true for all matrix semigroup built using Z_n.

This is proved by the following examples:

Example 4.2.9: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i \in Z_{12},\ 1 \le i \le 8,\ \times_n \right\}$$

be the matrix semigroup under the natural product $\times_n$.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i \in \{0, 1, 4, 9\};\ 1 \le i \le 8,\ \times_n \right\} \subseteq W$$

is a subsemigroup of W. P is an idempotent subsemigroup of W. However if $C_{20}W$ is a semigroup semiring and $C_{20}P$ is a subsemiring. But clearly $C_{20}P$ has elements which are not idempotents of $C_{20}W$.

Next the study of idempotent semigroups under $\cup$ (or $\cap$) is analysed first by examples.

Example 4.2.10: Let $S = \{1,\ a'_1,\ a'_2,\ \ldots,\ a'_8,\ \phi,\ \cap /\ a'_i \cap a'_j = \phi \text{ if } i \neq j\ \ 1 \cap a'_1 = a_i,\ a'_i \cap a'_i = a'_i;\ 1 \le i, j \le 8\}$ be the idempotent semigroup or a semilattice of order 10. $C_{10} = \{0 < a_8 < a_7 < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of order 10. $C_{10}S$ be the semigroup semiring of S over C_{10}.

Let

$$\alpha = a_3 a'_1 + a_6 a'_3 + a_5.$$

$$\begin{aligned} \alpha^2 = {} & a_3 a'_1 + a_6 a'_3 + a_5 + (a_3 \cap a_5)\ a'_1 + (a_6 \cap a_5)\ a'_3 \\ & + (a_3 \cap a_6)\ (a'_1 \cap a'_3) \qquad + (a_5 \cap a_3)\ a'_1 + (a_5 \cap a_6)\ a'_3 \\ & + (a_6 \cap a_3)\ (a'_3 \cap a'_1) \\ = {} & a_3 a'_1 + a_6 a'_3 + a_5 \end{aligned}$$

is an idempotent of $C_{10}S$. This has both zero divisors and idempotents.

It is to be noted that one can in the above definition put $\phi = 0$ without loss of generality. If ϕ is used define $a_i\phi = \phi$ and $\phi.0 = 0$ and ϕ is assumed to be the zero of C_nS.

Example 4.2.11: *Let S be any idempotent semigroup with 1 and 0 such that* $s_i \cap s_i = s_i$ *and* $s_i \cap s_j = 0$, $s_i \cap 1 = s_i$; $i \neq j$. *C_n any chain lattice. The semigroup semiring C_nS is such that every* $x \in C_nS$; $x^2 = x$. *For take*

$$x = \sum_{i=1}^{n} a_i s_i \;;\; s_i \in S \setminus \{0, 1\} \text{ and } a_i \in C_n,$$

$$x \times x = \{\sum (a_i \cap a_i)\, s_i \cap s_i + \sum (a_i \cap a_j)\, s_i \cap s_j\} = \sum_{i=1}^{n} a_i s_i + 0.$$

Hence the claim. Let

$$x = 1 + a_1s_1 + a_3s_2 \in C_nS.$$

$$x^2 = 1 + a_1s_1 + a_3s_2 + a_1s_1 + a_3s_2 + (a_1 \cap a_3)\,(s_1 \cap s_2)$$

$$= x.$$

If 1 is replaced by a_4 that is let

$$y = a_4 + a_1s_1 + a_3s_2$$

$$\begin{aligned} y^2 &= a_4 + a_1s_1 + a_3s_2 + (a_4 \cap a_1)s_1 + (a_4 \cap a_3)\, s_2 + (a_4 \cap a_1)s_1 \\ &\quad + (a_4 \cap a_3)s_2 + (a_1 \cap a_3)\,(s_1 \cap s_2) + (a_3 \cap a_1)\,(s_1 \cap s_2) \\ &= a_4 + a_1s_1 + a_3s_2 \\ &= y. \end{aligned}$$

Thus every element is an idempotent.

In view of this one has the following result in case of idempotent semigroup under $\cap$ of the special form described in Example 4.2.11.

Proposition 4.2.5: *Let $S = \{1, a'_1, a'_2, \ldots, a'_m, 0 \,/\, a_i \cap a_j = 0$ if $i \neq j$ $a_i \cap a_i = a_i$, $1 \cap a'_i = a'_i$ $0 \cap a'_j = 0$; $1 \leq i, j \leq m\}$ be the idempotent semigroup. C_n be any chain lattice. C_nS be the semigroup semiring of the semigroup S over the semiring C_n. Every $\alpha \in C_nS$ is an idempotent of C_nS.*

Proof: Let

$$\alpha = \sum_{i=1}^{t} a_i a'_i \in C_nS.$$

Clearly $\alpha^2 = \alpha$ in C_nS; hence the claim (using $a'_i \cap a'_j = 0$ if $i \neq j$ and $a_i \cap a_j = a_i$ if $i < j$ or a_j if $j < i$ and $a_i \cup a_j = a_i$ if $i > j$ and a_j if $i < j$).

Next the notion of substructures in C_nS is analysed in the following section:

4.3 SUBSTRUCTURES IN SEMIGROUP SEMIRING C_nS

Throughout this section C_nS will denote the semigroup semiring of the semigroup S over the chain lattice C_n. Here the ideals, subsemirings, S-ideals and S-subsemirings of SC_n are analysed.

First a few examples to this effect are given.

Example 4.3.1: Let $S = \{Z_{14}, \times\}$ be the semigroup. $C_{10} = \{0 < a_8 < a_7 < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of order 10. $C_{10}S$ be the semigroup semiring of S over C_{10}. Let $H = \{0, 2, 4, 6, 8, 10, 12\} \subseteq S$ be the subsemigroup of S. $C_{10}H \subseteq C_{10}S$ is a subsemiring of $C_{10}S$. Let $K = \{0, 7\} \subseteq S$; $C_{10}K = \{0, 7 \mid a_j \in C_{10}\}$ is a subsemiring of $C_{10}S$.

Let

$$x = a_8 2 + a_4 6 + a_5 8 \in C_{10}H = \left\{\sum a_i g_i \,|\, a_i \in C_{10}, \; g_i \in \{0, 2, 4, 6, 8, 10, 12\}\right\}.$$

$$x + x = x.$$

$$\begin{aligned} x \times x = & \ a_8 4 + a_4 8 + a_5 8 + (a_8 \cap a_4)\ (2 \times 6) + (a_4 \cap a_8)\ (6 \times 2) \\ & + (a_8 \cap a_5)\ (2 \times 8) + (a_5 \cap a_8)\ (8 \times 2) + (a_4 \cap a_5)\ (6 \times 8) \\ & + (a_5 \cap a_4)\ (8 \times 6) \\ = & \ a_8 4 + (a_4 \cup a_5) 8 + a_8 12 + a_8\, 2 + a_5\, 6 \\ = & \ a_8 4 + a_8 2 + a_4 8 + a_8 12 + a_5 6 \in C_{10}H. \end{aligned}$$

This is the way product operation is performed on $C_{10}H$. Clearly $1 \notin C_{10}H$. $C_{10}H$ is only a subsemiring which has no identity. So $C_{10} \not\subseteq C_{10}H$ but $H \subseteq C_{10}H$ as $1 \in C_{10}$. Clearly $C_{10}H$ has no nontrivial zero divisors; yet $C_{10}H$ is not a semifield as $1 \notin C_{10}H$.

Consider

$$B = \{0, a_i\ (1 + g_7) \mid a_i \in C_{10}\ ; g_7 = 7 \in Z_{14}\}.$$

Consider

$$a_i\ (1 + g_7) + a_j\ (1 + g_7) = a_i\ (1 + g_7) \text{ if } a_i > a_j$$

$$\text{or} = a_j\ (1 + g_7) \text{ if } a_j > a_i.$$

$$a_i\ (1 + g_7) \times a_j\ (1 + g_7) = (a_i \cap a_j)\ (1 + g_7)^2$$

$$= (a_i \cap a_j)\ (1 + g_7 + g_7^2 + g_7)$$

$$= a_k\ (1 + g_7).\ \left\{ a_k = \begin{matrix} a_i \text{ if } i < j \\ a_j \text{ if } i > j \end{matrix} \right\}.$$

Thus $x \times y \in B$ for all $x, y \in B$. Thus B is a subsemiring of $C_{10}S$.

In fact an idempotent subsemiring as $x^2 = x$ for every $x \in B$.

If $x = a_5 (1 + g_7)$; $x + x = x$ and

$$x \times x = a_5 (1 + g_7) \times a_5 (1 + g_7)$$
$$= (a_5 \cap a_5) (1 + g_7)$$
$$= a_5 (1 + g_7) = x.$$

Hence the claim.

Thus these semigroup semirings can contain subsemirings which are idempotent subsemirings.

Let $V = \{0, a_i (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12}) / a_i \in C_{10}$ and $g_2 = 2, g_4 = 4, g_6 = 6, g_8 = 8, g_{10} = 10$ and $g_{12} = 12 \in H \subseteq Z_{14}\} \subseteq C_{10}S.$

V is again an idempotent subsemiring of $C_{10}S$.

Consider

$$x = a_1 (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12}) \in V.$$

$$x^2 = (a_1 \cap a_1) [g_4 + g_2 + g_8 + g_8 + g_2 + g_4 + g_8 + g_{12} + g_4 + g_6 + g_{10} + g_4 + g_{12} + g_6 + g_6 + g_4 + g_2 + g_{10} + g_{12} + g_8]$$

$$= a_1(g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12}) \in V.$$

Thus V is an idempotent subsemiring of S.

However this V is different from the subsemiring $C_{10}H$. But $V \subseteq C_{10}S$.

Can V be an ideal of $C_{10}S$? The answer is yes.

For

$$x = (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12}) \in V;$$

Let

$$y = (a_1g_5 + a_3g_7 + a_5g_{11}) \in C_{10}S.$$

$$\begin{aligned}
x \times y &= (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12}) \times (a_1g_5 + a_3g_7 + a_5g_{11}) \\
&= a_1g_{10} + a_1a_6 + a_1g_2 + a_1g_{12} + a_1g_8 + a_1g_4 + a_3 .0 + a_3 .0 + a_3 . 0 \\
&\quad + a_3 . 0 + a_3 . 0 + a_3 . 0 + a_5g_8 + a_5g_2 + a_5g_{10} + a_5g_4 + a_5g_{12} \\
&\quad + a_5g_6 \\
&= (a_1 \cup a_5)\, g_2 + (a_1 \cup a_5)\, g_4 + (a_1 \cup a_5)\, g_6 + (a_1 \cup a_5)\, g_8 \\
&\quad + (a_1 \cup a_5)\, g_{10} + (a_1 \cup a_5)\, g_{12} \\
&= a_1g_2 + a_1g_4 + a_1g_6 + a_1g_8 + a_1g_{10} + a_1g_{12} \\
&= a_1 (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12})
\end{aligned}$$

is in V. Thus V is an ideal of $C_{10}S$. Likewise B is not an ideal of $C_{10}S$.

Example 4.3.2: Let $S = \{Z_{23}, \times\}$ and $C_{12} = \{0 < a_{10} < a_9 < \ldots < a_2 < a_1 < 1\}$ be the semigroup and chain lattice of orders 23 and 12 respectively.

Let $S = \{g_0 = 0, g_1 = 1, g_2 = 2, \ldots, g_{22} = 22, \times\}$ be used for notational convenience. $C_{12}S$ be the semigroup semiring of S over C_{12}.

Let $P = \{0, a_i (1 + g_2 + \ldots + g_{22}) \mid a_i \in C_{12}\} \subseteq C_{12}S$ be the subsemiring of $C_{12}S$. Clearly P is an ideal of S. $o(P) = 12$.

In view of this the following result is true:

Proposition 4.3.1: *Let* $S = \{Z_p, \times\} = \{g_0 = 0, g_1 = 1, \ldots, g_{p-1} = p - 1, \times\}$ *be the semigroup of order p, p a prime.* C_n *be the chain lattice of order n.*

$$C_n\!: 0 < a_{n-2} < a_{n-3} < \ldots < a_3 < a_2 < a_1 < 1.$$

C_nS be the semigroup semiring of the semigroup S over the semiring C_n. C_nS has an ideal I such that $o(I) = n$.

Proof: Consider $V = \{a_i (1 + g_2 + \ldots + g_{p-1}) \mid a_i \in C_n\} \subseteq C_nS$ be the semiring of C_nS. Clearly V is an ideal as p is a prime and $o(V) = n$, the number of elements in C_n. Hence the claim.

Corollary 4.3.1: If in the semigroup $S = \{Z_p, \times\}$; p is not a prime; V is not an ideal of C_nS.

The semigroup $\{Z_p, \times\}$ is such that p must be a prime for if p is not a prime say $p = 12$ that is $S = \{Z_{12}, \times\} = \{g_0 = 0, g_1 = 1, g_2 = 2, \ldots, g_{11} = 11, \times\}$ be the semigroup.

$$V = \{a_i (1 + g_2 + g_3 + \ldots + g_{11}) / a_i \in C_n\} \subseteq C_nS.$$

Let

$$x = a_7 (1 + g_2 + g_3 + \ldots + g_{11}) \in V.$$

Take

$$y = a_7 g_4 \in C_nS.$$

$$\begin{aligned} x \times y &= a_7 (g_4 + g_8 + 0 + g_4 + g_8 + 0 + g_4 + g_8 + 0 + g_4 + g_8 + 0 \\ &\quad + g_4 + g_8) \\ &= a_7 (g_4 + g_8) \notin V. \end{aligned}$$

Hence V is not an ideal of C_nS if in $S = \{Z_n, \times\}$; n is not a prime.

Example 4.3.3: Let $S = \{Z_{30}, \times\} = \{0, g_1 = 1, g_2 = 2, \ldots, g_{29} = 29, \times\}$ be the semigroup of order 30. Let $C_5 = \{0 < a_3 < a_2 < a_1 < 1\}$ be the chain lattice of order 5. C_5S be the semigroup semiring of S over C_5.

Consider $V_1 = \{0, (1 + g_{15}), a_1(1 + g_{15}), a_2(1 + g_{15}), a_3(1 + g_{15})\} \subseteq C_5S$. Clearly V_1 is only a subsemiring and not an ideal of C_5S.

Let $M = \{d_i g_{15} \mid \ d_i \in C_5\} \subseteq C_5S$ is an ideal of C_5S of order 5. However V_1 is only an idempotent subsemiring of order 5.

Consider $V_2 = \{d_i(g_6 + g_{12} + g_{18} + g_{24}) \,/\, d_i \in C_5\} \subseteq C_5S$ is a subsemiring which is also an ideal of C_5S of order S.

Let $V_3 = \{d_i \{g_2 + g_4 + \ldots + g_{28}) \mid d_i \in C_5\} \subseteq C_5\, S$ is only a subsemigroup and not an ideal of S.

For if

$$y = g_5 \in C_5S,$$
$$x = (g_2 + g_4 + g_6 + g_8 + g_{10} + g_{12} + g_{14} + g_{16} + g_{18} + g_{20} + g_{22} + g_{24} + g_{26} + g_{28}) \in V_3.$$

$$\begin{aligned} xy &= (g_{10} + g_{20} + 0 + g_{10} + g_{20} + 0 + g_{10} + g_{20} + 0 + g_{10} + g_{20} + 0 \\ &\quad + g_{10} + g_{20}) \\ &= (g_{10} + g_{20}) \not\in V_3. \end{aligned}$$

Hence the claim.

Thus if I is an ideal of S; $\left\{\sum g \mid \ g \in I\right\}$ need not in general be an ideal of C_nS. This will be characterized.

Can C_5S have other ideals? The answer is yes.

$V_4 = \{d_i\, (g_{10} + g_{20}) \,/\, d_i \in C_5S\}$ is an ideal of C_5S.

$V_5 = \{d_i\, (g_3 + g_6 + g_9 + \ldots + g_{27}) \,/\, d_i \in C_5\} \subseteq C_5S$ is not an ideal of C_5S.

Let

$$x = (g_3 + g_6 + g_9 + g_{12} + g_{15} + g_{18} + g_{21} + g_{24} + g_{27}) \in V_5 \text{ and } y = 5$$

$$x \times y \ = (g_{15} + 0 + g_{15} + 0 + g_{15} + 0 + g_{15} + 0 + g_{15}) = g_{15} \not\in V_5.$$

Thus V_5 is not an ideal of C_5S.

Example 4.3.4: Let $M = \{Z_{19}, \times\}$ be the semigroup and $C_9 = 0 < a_7 < a_8 < \ldots < a_1 < 1$ be the chain lattice; C_9M be the semigroup semiring of the semigroup M over the semiring C_9. C_9M is of finite order.

C_9M has no zero divisors, C_9M is a semifield.

Let $x = a_i\ (1 + g_1 + \ldots + g_{18}) \in C_9M$ then $x^2 = x\ (a_i \in C_9)$ is the nontrivial idempotent of C_9M. $y = a_i(1 + g_{18}) \in C_9M$ is an idempotent of C_9M.

However $a_i 1 = a_i \in C_9 \subseteq C_9M$ are idempotents of C_9M and these are called as the trivial idempotents of C_9M.

It is interesting to note that this idempotent x generates an ideal of C_9M.

Infact in view of the above example the following theorem is proved.

Theorem 4.3.1: *Let $C_n : \{0 < a_{n-2} < a_{n-3} < \ldots < a_2 < a_1 < 1\}$ be the chain lattice of length n. $M = \{Z_p, \times\}$ be a semigroup; p a prime. C_nM be semigroup semiring of the semigroup M over the semiring C_n. C_nM has an ideal I generated by the idempotent;*

$$x = a_i\ (1 + g_1 + \ldots + g_{p-2}) \text{ where } g_1 = 2,\ g_2 = 3,\ \ldots,\ g_{p-2} = p - 1.$$

Proof: Let $x = a_i\ (1 + g_1 + \ldots + g_{p-2})$ be an idempotent clearly as $x^2 = a_i\ (1 + g_1 + \ldots + g_{p-2}) = x$.

Further $I = \langle x \rangle$ generates an ideal using the property of the chain lattice and the semigroup $M = \{Z_p, \times\}$.

This is represented by the following example.

Example 4.3.5: Let $M = \{Z_7, \times\}$ be the semigroup. $C_3 = 0 < a_1 < 1$ be the chain lattice of length 3. C_3M be the semigroup semiring of the semigroup M over the chain lattice C_3.

Let $x = a_1\ (1 + g_1 + \ldots + g_5)$ that is $x = a_1\ (\overline{1} + \overline{2} + \ldots + \overline{6})$ where $1 = \overline{1}$ $g_1 = \overline{2}, \ldots, g_5 = \overline{6}$.

$$x \times x = a_1 (1 + g_1 + \ldots + g_5)$$

$$= (a_1 \cap a_1) (1 + g_1 + \ldots + g_5)^2$$

$$= a_1 \cap a_1 (1 + g_1 + \ldots + g_5 + g_1 + g_2 + \ldots + 1 + \ldots + g_5 + 1 + g_1 + g_2 + g_3 + g_4)$$

$$= a_1 (1 + g_1 + \ldots + g_5).$$

As $a_1 \cap a_1 = a_1$ and $1 \cup 1 = 1$.

Now consider $x \times y$ where $y = (g_3 + g_4 + 1) \in C_3M;$

$$a_1 (1 + g_1 + g_2 + g_3 + g_4 + g_5) \times (g_3 + g_4 + 1)$$

$$= (a_1 \cap 1) (g_3 + 1 + g_4 + g_1 + g_5 + g_2 + g_4 + g_2 + 1 + g_5 + g_3 + g_1 + 1 + g_1 + g_2 + g_3 + g_4 + g_5)$$

$$= a_1 (1 + g_1 + g_2 + \ldots + g_5)$$
$$= x.$$

$$x . 0 = 0.$$

Thus $\langle x \rangle = \{0, a_1 (1 + g_1 + \ldots + g_5)\}$ is an ideal of order three.

Next some of subsemirings which are not ideals are illustrated by the following example.

Example 4.3.6: Let $S = \{Z_{12}, \times\}$ be the semigroup and $C_5 = 0 < a_3 < a_2 < a_1 < 1$ be the chain lattice of order five.

Let $S = \{0, 1 = g_1, g_2 = 2, \ldots, g_{11} = 11\}$ be used for notational convenience. C_5S be the semigroup semiring of S over C_5.

Let $H = \{0, 1, a_ig_{11}, a_i, a_ig_{11} + a_j;\ 1 \leq i, j \leq 3$ as well as $a_i = 1$ and $a_j = 1$ can also occur$\}$.

H is only a subsemiring and not an ideal of C_5S.

$H = C_5P$ where P is the subsemigroup = $\{0, 1, g_{11}\} = \{0, 1, 11\}$.

Similarly if $K = \{0, 1, 7\} = \{0, g_1 = 1, g_7\}$, then C_5K is a subsemiring which is not an ideal of C_5S. Thus there are subsemirings which are not ideals.

In view of this the following theorem is proved:

Theorem 4.3.2: *Let $S = \{Z_n, \times\}$; n a composite number. C_m be the chain lattice of length m; $C_m = 0 < a_{m-2} < a_{m-1} < \ldots < a_1 < 1$. C_mS be the semigroup semiring of the semigroup S over the semiring C_m.*

(i) If H is a subsemigroup of S and not an ideal of S then C_mH is a subsemiring which is not an ideal of C_mS.

(ii) Every ideal of S contributes to a ideal in the semigroup semiring C_mS.

Proof: Let H be a subsemigroup of S which is not an ideal of S. C_mS be the semigroup semiring. C_mH is only a subsemiring and is not an ideal. For the $H = \{0, 1, n - 1\} \subseteq S$ is only a subsemigroup and not an ideal of S. H is only a subsemigroup of S and not an ideal of S. Hence C_nH is only a subsemiring and not an ideal so is true.

Let K be an ideal of S, then C_mK is an ideal of C_mS; can be verified by using the ideal property of the semigroup S.

Hence the claim.

Next examples of non commutative semigroup semirings are given in the following:

Example 4.3.7: Let $S = S(6)$ be the symmetric semigroup which is non commutative. $C_{12} = 0 < a_{10} < a_9 < \ldots < a_1 < 1$ be the chain lattice. $C_{12}S$ be the semigroup semiring.

Let $C_{12}S$ be non commutative semigroup semiring. Take

$$P_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix} \right\}$$

to be a subsemigroup of $S(6)$. $C_{12}P_1$ is a subsemiring and not an ideal of $C_{12}S$.

Consider

$$P_2 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \right\} \subseteq S(6).$$

P_2 is an ideal of $S(6)$. $C_{12}P_2$ is the subsemiring which is an ideal of $C_{12}S(6)$. Thus if we take subgroups G in $S(6)$ then $C_{12}G$ are not ideals of $C_{12}S$. If ideals I of $S(6)$ are taken then $C_{12}I$ are subsemirings which are also ideals of $C_{12}S$.

In view of this we have the following theorem:

Theorem 4.3.3: *Let $S(n)$ be the symmetric semigroup. C_m be the chain lattice; $0 < a_{m-1} < a_{m-2} < \ldots < a_1 < 1$; $C_mS(n)$ be the semigroup semiring of the semigroup $S(n)$ over the semiring C_m.*

i) $C_mS(n)$ is a noncommutative strict semigroup semiring.

ii) $C_mS(n)$ has subsemirings which are not ideals.

iii) $C_mS(n)$ has ideals.

Proof: i) Since the symmetric semigroup $S(n)$ is non commutative so is the semigroup semiring $C_mS(n)$. Further as C_m is a chain lattice if $a, b \in C_mS(n)$; $a \cdot b \neq 0$ as well as $a + b = 0$ implies $a = 0$ and $b = 0$. Hence $C_mS(n)$ is a strict semiring.

ii) There is a subsemigroup P of $S(n)$ such that C_mP are only subsemirings and not ideals.

iii) $$B = \left\{ \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 2 & 2 & 2 & 2 & 2 & \ldots & 2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ n & n & n & n & \ldots & n \end{pmatrix} \right\} \subseteq S(n)$$

is an ideal of $S(n)$ so C_mB is a subsemiring which is also an ideal of C_mS.

Hence the theorem.

Thus when chain lattices are taken as semirings with $S(n)$ as semigroup the semigroup semiring is semidivision ring.

However if $S = \{Z_n, \times\}$, n a composite number C_mS the semigroup semiring has zero divisors.

Similarly if $M = \{s \times t$ matrix with entries from $Z_n, \times_n\}$ be the semigroup then C_mM the semigroup semiring has zero divisors, ideals and subsemirings.

This is illustrated by an example or two.

Example 4.3.8: Let $S = \{(a_1, a_2, a_3, a_4) \mid a_i \in Z_{12}, i \leq i \leq 4, \times\}$ be the matrix semigroup.

$C_2 = \quad L =$

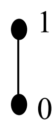


Figure: 4.3.1

be the lattice. LS be the semigroup semiring of the semigroup S over the lattice L. LS has idempotents, ideals, subsemirings, zerodivisors and units.

Example 4.3.9: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in Z_9,\ 1 \leq i \leq 6,\ \times_n \right\}$$

be the column matrix semigroup under natural product $\times_n$.

$C_6 =$

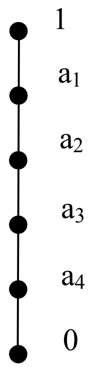


Figure: 4.3.2

C_6S be the semigroup semiring of the semigroup S over the semiring (chain lattice) C_6. C_6S has units, zero divisors and idempotents.

Further if S has S-units, S-zero divisors and S-idempotents then C_nS will have S-units, S-zero divisors and S-idempotents. In fact C_nS has both subsemirings as well as ideals.

The following results can be proved.

Theorem 4.3.4: *Let $S = \{m \times n$ matrices with entries from Z_n, $\times_n\}$ be the semigroup $L = C_n$ be any finite chain lattice. LS be the semigroup semiring.*

i) LS has S-units, S-zero divisors, S-idempotents if and only if Z_n has S-units, S-zero divisors and S-idempotents respectively.

ii) C_nS has ideals if and only if S has ideals.

Proof is similar to that of semigroups.

In fact in next section the study of semigroup semirings over the distributive lattices are studied. Here the distributive lattices can be Boolean algebras or any finite distributive lattice L. Further all distributive lattices are assumed to contain 0 and 1. Here a distributive lattice L has zero divisors if $a \cap b = 0$ for $a, b \in L \setminus \{0\}$. In fact every element is an idempotent in L.

4.4 SEMIGROUP SEMIRINGS USING FINITE DISTRIBUTIVE LATTICES

In this section semirings which are distributive lattices which are not chain lattices are used. Here semigroups as mentioned earlier in the scope of the thesis is $\{Z_n, \times\}$, $S(n)$ and matrix semigroups under the natural products $\times_n$. This is the first time the study of semigroup semirings using disturbutive lattices as semirings is carried out.

Semirings considered in this thesis in general and in particular in this chapter are finite distributive lattices. Such study is done systematically for the first time in this thesis.

In this section semigroup semirings (where semirings are finite distributive lattices) are described. First a few examples to this effect are given. Here distributive lattices are not taken as chain lattices.

Example 4.4.1: Let

$L =$

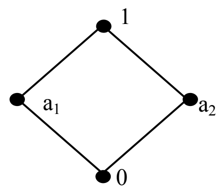


Figure: 4.4.1

be a distributive lattice. $S(10)$ be the symmetric semigroup of degree 10.

$LS(10)$ be the semigroup semiring of $S(10)$ over the lattice L. $LS(10)$ has units, zero divisors and idempotents. $LS(10)$ is a non commutative semiring.

Let

$$p = a_1 \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 10 \\ 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix} \in LS\ (10);$$

$p^2 = p$ is an idempotent of $LS(10)$.

Let

$$p = a_1 \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 10 \\ 2 & 3 & 4 & 5 & \dots & 1 \end{pmatrix} \text{ and } q = a_2 \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 10 \\ 1 & 1 & 1 & 2 & \dots & 2 \end{pmatrix} \in LS(10);$$

$$p \times q = 0.$$

Let

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 10 \\ 2 & 3 & 1 & 4 & 5 & 6 & \dots & 10 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 10 \\ 3 & 1 & 2 & 4 & 5 & 6 & \dots & 10 \end{pmatrix} \in LS(10).$$

$$x \times y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots & 10 \end{pmatrix}$$

is a unit in $LS(10)$.

Example 4.4.2: Let

$L =$

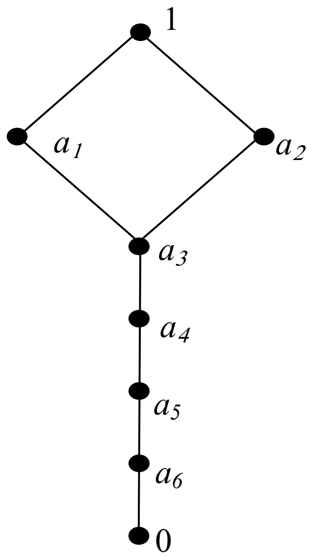


Figure: 4.4.2

be a distributive lattice.

$S = S(4)$ be the symmetric semigroup of degree 4. $LS(4)$ be the semigroup lattice (semigroup semiring), $LS(4)$ is non commutative as $S(4)$ is a non commutative semigroup and $LS(4)$ has no zero divisors but $LS(4)$ has units and idempotents. For

$$m = a_6 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in LS(4)$$

is such that $m^2 = m$. In fact $LS(4)$ is of finite order and $LS(4)$ has idempotents.

Let

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in LS(4)$$

is such that

$$x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1$$

is a unit in *LS(4)*.

Clearly *LS(4)* has sublattices. *LS(4)* is a semidivision ring.

Example 4.4.3: Let

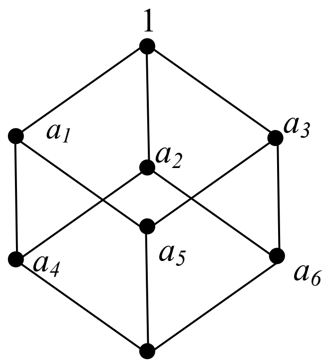
L = 

Figure: 4.4.3

be the Boolean algebra of order *8*. *S(6)* be the symmetric semigroup. *LS(6)* be the semigroup semiring. *LS(6)* is a non commutative finite semiring which is not a semifield.

In fact $o(LS(6)) < \infty$ and has only finite number of zero divisors, units and idempotents.

LS(6) is a Smarandache semigroup semiring as $S_6 \subseteq S(6)$ is the symmetric group of degree 6. *LS(6)* has subsemigroups as well as ideals. Clearly *LS(6)* is not a semidivision ring.

In view of all these the following theorem is proved.

Theorem 4.4.1: *Let L be a distributive lattice other than a chain lattice and S(n) be the symmetric semigroup of degree n. LS(n) be the semigroup lattice (semigroup semiring) of S(n) over the semiring L.*

i) *LS(n) has zero divisors if and only if L has zero divisors.*

ii) *LS(n) has units and idempotents.*

iii) *LS(n) is non commutative and is of finite order.*

iv) *LS(n) is a semidivision ring if and only if L is a lattice without* *zero divisors.*

Proof: Clearly *S(n)* has no zero divisors. Further only if *L* has $a_i \cap a_j = 0$; $a_i, a_j \in L \setminus \{0\}$ then *LS(n)* has zero divisors and vice versa.

Thus if *L* has no zero divisors, *LS(n)* is a semidivision ring as *S(n)* is a non commutative semigroup with no zero divisors. Hence (iii) and (iv) are proved.

S(n) has units and idempotents so *LS(n)* has units and idempotents.

Corollary 4.4.1: *If L in the theorem 4.4.1 is replaced by a Boolean algebra of order greater than two then LS(n) has zero divisors.*

Proof: Follows from the fact all Boolean algebras of order greater than two has zero divisors.

Example 4.4.4: Let

$L =$

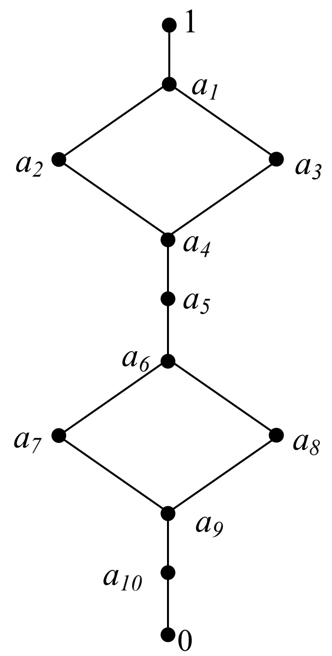


Figure: 4.4.4

be the distributive lattice of order 12. *S(4)* be the semigroup of order 4^4. *LS(4)* be the semigroup semiring. *LS(4)* has no zero divisors, has only idempotents and units.

All

$$x_1 = a_i \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \; x_2 = a_i \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix},$$
$$x_3 = a_i \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \; x_4 = a_i \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \in LS(4)$$

are idempotents $a_i \in L \setminus \{0\}$. *LS(4)* has also units.

For

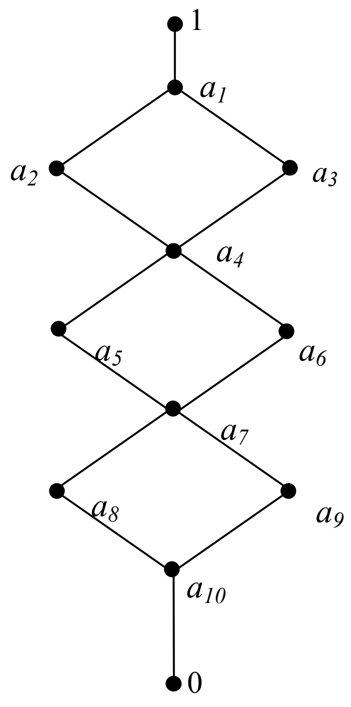
$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \; y_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \; y_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix},$$
$$y_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \; y_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \; y_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

are some of the units of *LS(4)*.

However *LS(4)* has no zero divisors only units and idempotents as *L* has no zero divisors.

Example 4.4.5: Let

$L =$



be a lattice of order 12. $S(3)$ be the symmetric semigroup. $LS(3)$ be the semigroup lattice (semigroup semiring). $LS(3)$ has no zero divisors, but has only idempotents and units.

Since L has no zero divisors so $LS(3)$ has no zero divisors as $S(3)$ has no zero divisors.

Let

$$x = a_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} + a_1$$

and

$$y = a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} + a_6 \in LS(3).$$

$$\begin{aligned} x \times y \quad = \quad & \{a_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} + a_1\} \times \\ & \{a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} + a_6\} \\ = \quad & a_5 \cap a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} \\ & + a_2 \cap a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} \\ & + a_1 \cap a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} + a_5 \cap a_9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \end{aligned}$$

$$+ a_2 \cap a_9 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} + a_1 \cap a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$+ a_5 \cap a_6 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} + a_2 \cap a_6 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} + a_1 \cap a_6$$

$$= \quad a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

$$+ a_9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$+ a_6 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} + a_6 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} + a_6$$

$$= \quad a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} + (a_{10} \cup a_9) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} + a_{10} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} +$$

$$a_9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} + a_9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} + a_6 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} +$$

$$a_6 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} + a_6$$

($\because$ $a_{10} \cup a_9 = a_9$; a_9 is the coefficient of $\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}\right)$.

Clearly $x \times y \neq y \times x$.

Thus $LS(3)$ is a non commutative semiring which is a semidivision ring.

Next semigroups $S = \{Z_n, \times\}$ are used to construct semigroup semirings which is illustrated by the following example.

Example 4.4.6: Let $S = \{Z_{12}, \times\}$ be the semigroup and

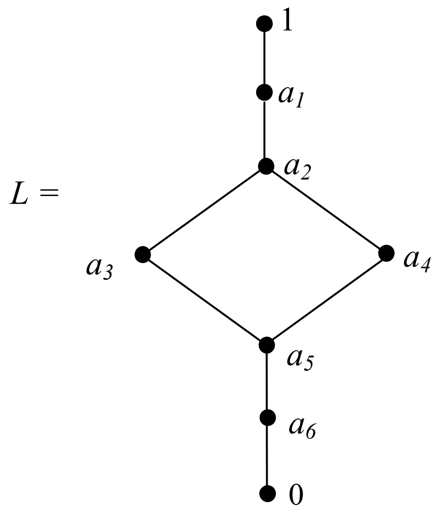


Figure: 4.4.6

LS be semigroup semiring. LS is commutative and is of finite order. LS has idempotents and units.

Let

$x = a_36 + a_53$ and $y = a_68 + a_24 \in LS$. $(6, 3, 4, 8 \in Z_{12}$ and $a_8, a_3, a_5, a_2 \in L)$.

$$\begin{aligned} x \times y &= (a_36 + a_53) \times (a_68 + a_24) \\ &= a_3 \cap a_6\,(6 \times 8) + a_5 \cap a_8\,(3 \times 8) + a_3 \cap a_2\,(6 \times 4) \\ &\quad + a_5 \cap a_2\,(3 \times 4) \\ &= 0. \end{aligned}$$

Thus LS has zero divisors.

Let

$$x = a_54 + a_29 \in LS$$

$$\begin{aligned} x \times x &= (a_54 + a_29) \times (a_54 + a_29) \\ &= a_5 \cap a_5\,(4 \times 4) + a_2 \cap a_5\,(9 \times 4) + (a_5 \cap a_2)\,(4 \times 9) \\ &\quad + a_2 \cap a_2\,(9 \times 9) \\ &= a_54 + a_29 \\ &= x. \end{aligned}$$

Thus x is an idempotent in LS.

Let $x = 15 \in LS; x^2 = 1$ is a unit in LS.

Thus LS has units zero divisors and idempotents. Thus LS is not a semifield.

It is to be noted that L has no zero divisors but S has zero divisors, units and idempotents so that LS has zero divisors, units and idemoptents.

Example 4.4.7: Let $S = \{Z_7, \times\}$ be the semigroup.

$L =$

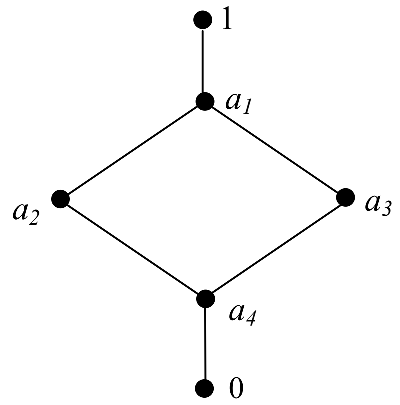


Figure: 4.4.7

be the lattice of order six. LS be the semigroup semiring. LS is of finite order. LS has no zero divisors and the idempotents are contributed by L and not by $LS \setminus L$ and LS has units. $x = 5 \in LS$ is such that

$5^6 = 1(\text{mod } 1)$ is a unit in LS.

$6 \in LS$ is such that $6^2 = 1 \ (\text{mod } 7)$,

$2 \in LS$ is such that $2^3 = 1 \ (\text{mod } 7)$,

$3 \in LS$ is such that $3^6 = 1 \ (\text{mod } 7)$ and

$4 \in LS$ is such that $4^3 = 1 \ (\text{mod } 7)$.

Thus L has units. Elements $a \in L$ are some of the idempotents of LS.

LS is a finite semifield. Further $4.2 = 1 \pmod 7$ and $3.5 = 1 \pmod 7$ also contribute for units.

These semifields have no characteristic associated with them.

Example 4.4.8: Let $S = \{Z_{11}, \times\}$ be the semigroup and

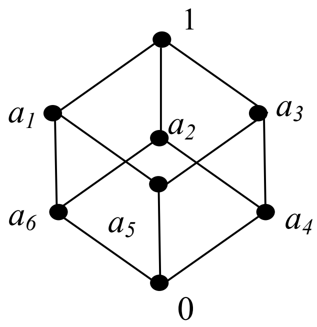
$L =$ 

Figure: 4.4.8

L be the semiring; LS be the semigroup semiring of S over L. LS has units. LS has zero divisors.

Let $10 \in LS$; $10^2 = 1 \pmod{11}$ is a unit in LS.

$3 \times 4 = 1 \pmod{11}$.

In view of all these examples the following theorem is proved:

Theorem 4.4.2: *Let $S = \{Z_n, \times\}$ be the semigroup. L is a distributive lattice (semiring) without zero divisors. LS be semigroup semiring.*

i) *LS has zero divisors if and only if S has zero divisors.*

ii) *LS has nontrivial idempotents.*

iii) *LS has $(n - 2)$ nontrivial units if and only if n is a prime.*

iv) *LS is a semifield if and only if n in Z_n is a prime and L is a lattice without zero divisors.*

Proof: *S* has zero divisors if and only if *LS* has zero divisors as *L* has no zero divisors. Hence (i) is true.

Take $x = (1 + g_2 + \ldots + g_{n-1}) \in LS$ where $g_2 = 2, \ldots, g_{n-1} = n - 1$. $x^2 = x;$ thus *x* is an idempotent of *LS*. Hence (ii) is true.

Thus *LS* has idempotents even if *S* has no idempotents.

Proof of (iii): If *n* is prime then $2 = g_2, \ldots, g_{n-1} = n - 1$ are units in *S* so *LS* has only $n - 2$ units. If *n* is a prime then clearly *LS* is a semifield hence (iv) is true.

Corollary 4.4.2: *If L in the theorem have zero divisors, then LS has zero divisors even if S has no zero divisors.*

Now substructures in *LS(n)* and *LS* where $S = \{Z_n, \times\}$ the semigroup semirings using semigroups *S(n)* and *S* respectively are analysed.

Example 4.4.9: Let *LS(9)* be the semigroup semiring of the semigroup *S(n)* over the semiring *L* (*L* any distributive lattice). *LS(9)* has ideals and subsemirings which are not ideals.

For *LP* where

$$P = \left\{ \begin{pmatrix} 1 & 2 & \ldots & 9 \\ 1 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 2 & 2 & 2 & \ldots & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 3 & 3 & 3 & \ldots & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 4 & 4 & 4 & 4 & \ldots & 4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 9 \\ 5 & 5 & 5 & 5 & \ldots & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 6 & 6 & 6 & \ldots & 6 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 2 & 3 & \ldots & 9 \\ 9 & 9 & 9 & \ldots & 9 \end{pmatrix} \right\} \subseteq S(9)$$

is an ideal of the semiring *LS(9)*.

Let

$$M = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 9 \\ 1 & 2 & 3 & 4 & \dots & 9 \end{pmatrix} \right\} \cup P$$

be the subsemigroup of *S(9)*. *LM* is a subsemiring of *LS(9)* which is not an ideal of *LS(9)*. If *L* has ideals say $T \subseteq L$ then *TP* will be an ideal of *LS*.

However $TP \subseteq LP$.

Example 4.4.10: Let

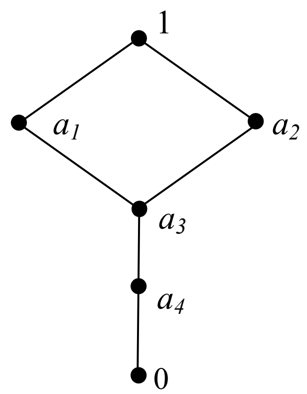
$L =$ 

Figure: 4.4.9

be the semiring of order 6. *S(4)* be the semigroup and *LS(4)* be the semigroup semiring. $T_1 = \{1, a_1\ a_3, a_4, 0\} \subseteq L$ is a sublattice of *L*; $T_1S(4)$ is a subsemiring of *LS(4)* which is also an ideal of *LS(4)*.

Example 4.4.11: Let

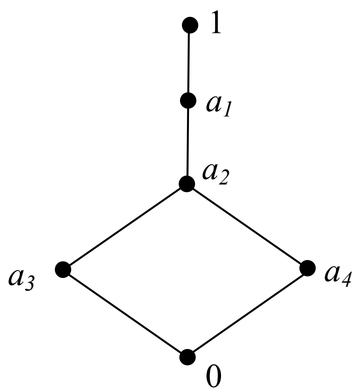
$L =$ 

Figure: 4.4.10

be a semiring and $S = \{Z_{12}, \times\}$ be the semigroup. LS be the semigroup semring.

$P = \{1, a_1, a_2, a_3, 0\} \subseteq L$ be an ideal of L: PS is again an ideal of LS. Every ideal of L leads to an ideal of LS.

In view of all these facts the following result is proved.

Theorem 4.4.3: *Let L be a distributive lattice. $S = S(n)$ (or $(Z_n, \times)$) be the semigroup. $LS(n)$ (or LS) be the semigroup semiring. If L has ideals then $LS(n)$ has ideals.*

Proof: If P is an ideal of L then $P(S(n))$ (PS) is an ideal of $LS(n)$ (or LS). Hence the claim.

Similarly if M is a sublattice of L; then $MS(n)$ (or MS) is a subsemiring of $LS(n)$ (or LS).

Since all semirings used in this thesis are only distributive lattices which includes the chain lattices and the Boolean algebras apart from finite distributive lattices.

This allows one to define filter in lattices which will be extended to filter in semigroup semirings.

This will be illustrated by the following examples.

Example 4.4.12: Let L be the lattice

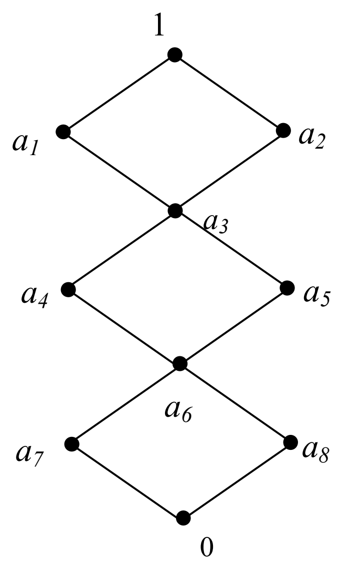


Figure: 4.4.11

$S = \{Z_{12}, \times\}$ be the semigroup. LS be the semigroup semiring. $F = \{1, a_1, a_2, a_3\}$ is a filter of L. FS is a subsemiring.

It is left as future study to find filters in the semigroup semirings.

4.5 CONCLUSIONS

In this chapter for the first time a systematic analysis of semigroup semirings by taking finite semigroups over the finite distributive lattices is carried out. These concepts are defined and condition for the semigroup semirings to contain idempotents, zero divisors and units are obtained.

Also those semidivision rings are characterized and several examples are given which makes the understanding of this abstract theory simple.

Substructures like ideals and subsemirings of these semigroup semirings is studied. Finally conditions for the semigroup semirings to contain ideals are obtained.

CHAPTER FIVE

GROUP SEMIRINGS USING DISTRIBUTIVE LATTICES AS SEMIRINGS

5.1 INTRODUCTION

In this chapter the study of group semirings is carried out using finite or distributive lattices which are semirings. Group rings and semigroup rings have been studied by several authors [24, 74-5]. Both these structures are only studied over rings with unit or a field. In case of group rings several researchers have studied about the zero divisors, idempotents, units etc; [44, 75, 93]. Likewise study of semigroup rings that is semigroups over rings and fields have been studied and also special elements like zero divisors, units have been analysed by several researchers [52, 67, 70].

This study is very new for group semirings have been studied by researchers very sparingly [100]. In this chapter a study of group semirings is carried out in a systematic way.

This chapter has four sections. Section one is introductory in nature. Section two studies group semirings where semirings are taken as chain lattices. Section three introduces the new study of group semirings by taking semirings which are distributive lattices as well as Boolean algebras and the conclusions are given in the final section.

5.2 GROUP SEMIRINGS OF SEMIRINGS WHICH ARE CHAIN LATTICES AND THEIR PROPERTIES

Throughout this section semirings are taken as chain lattices. C_n will denote a chain lattice of length n and G will denote a group under multiplication. First for the sake of completeness the definition of group semiring is recalled.

Definition 5.2.1: *Let S = C_n be the chain lattice that is; a semiring. G be any group. The group semiring C_nG = SG of the group G over the semiring C_n (= S) contains all finite formal sums of the form*

$$\left\{ \sum_{i=1}^{n} s_i g_i \,\middle|\, \begin{array}{l} s_i \in S(=C_n) \\ g_i \in G \end{array} \right\}$$

on which two binary operations '+' (the union in the lattice C_n) and $\times$ (the $\cap$ on the lattice C_n) are defined on SG which is as follows:

Let $\alpha = \sum_{i=1}^{n} s_i \alpha_i$ *and* $\beta = \sum_{i=1}^{n} r_i \alpha_i \in C_nG = SG$*; where* $s_i, r_i \in C_n = S$ *and* $\alpha_i \in G$*;* $\alpha = \beta$ *if and only if each* $s_i = r_i$*;* $1 \leq i \leq n$*.*

i) $\alpha + \beta = \sum_{i=1}^{n} s_i \alpha_i + \sum_{i=1}^{m} r_i \alpha_i$

$$= \sum_{i=1}^{m \text{ or } n} (s_i \cup r_i)\alpha_i \qquad \textit{(which ever m or n is greater)}$$

$$= \sum_{i=1}^{n} m_i \alpha_i \; (m_i = s_i \cup r_i \in C_n = S).$$

ii) $\alpha \times \beta = \left(\sum_{i=1}^{n} s_i \alpha_i\right) \times \left(\sum_{i=1}^{m} r_i \alpha_j\right)$

$$= \sum_{k=1}^{t} (s_i \cap r_j)\alpha_k = \sum_{k=1}^{t} \gamma_k \alpha_k$$

where $\alpha_k = \alpha_i \, \alpha_j$ *and* $\gamma_k = s_i \cap r_j$.

iii) *For* $e = 1 \in G$ *(the identity of* G*)*

$1 \times s_i = s_i \times 1 = s_i$ *for all* $s_i \in C_n$

and $s_i g = g s_i$ *for all* $g \in G$.

iv) *For* $1 \in C_n = S$ *we have*

$1 \cdot g_i = g_i \cdot 1 = g_i$ *for all* $g_i \in G$.

v) *For* $0 \in C_n = S$ *we have*

$0 \cdot g_i = g_i \cdot 0 = 0$ *for all* $g_i \in G$.

vi) *Further* $1 \cdot G \subseteq SG$ *and* $S \cdot 1 \subseteq SG$.

(Here 1 of G and 1 of $C_n = S$ *is defined and denoted by 1).*

The identity element is 1 of $SG = C_nG$. *SG is a semiring.*

Some examples of the group semirings is given in the following:

Example 5.2.1: Let

$C_7 =$

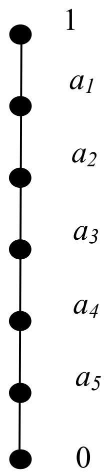


Figure 5.2.1: C_7

be the semiring (chain lattice) and $G = \langle g \mid g^{20} = 1 \rangle$ be the cyclic group of order 20. C_7G be the group semiring of the group G over the semiring C_7.

Clearly number of elements in C_7G is finite so the group semiring is of finite order. Since G is a commutative group so is the group semiring C_7G.

Let

$$\alpha = a_2g^6 + a_3g^2 + a_4g + a_5 \text{ and } \beta = a_3g^7 + a_4g^6 + a_1g + 1 \in C_7G.$$

$$\begin{aligned}
\alpha + \beta &= (a_2g^6 + a_3g^2 + a_4g + a_5) + (a_3g^7 + a_4g^6 + a_1g + 1) \\
&= a_3g^7 + (a_2 \cup a_4)\, g^6 + a_3g^2 + (a_4 \cup a_1)g + a_5 \cup 1
\end{aligned}$$

$$= \quad a_3g^7 + a_2g^6 + a_3g^2 + a_1g + 1 \in SG.$$

$$\alpha \times \beta \quad = \quad (a_2g^6 + a_3g^2 + a_4g + a_5) \times (a_3g^7 + a_4g^6 + a_1g + 1)$$

$$= \quad (a_2 \cap a_3)\, g^6 \times g^7 + (a_3 \cap a_3)\, g^2 \times g^7 + (a_4 \cap a_3)\, g \times g^7$$
$$+ (a_5 \cap a_3)\, g^7 + (a_2 \cap a_4)\, g^6 \times g^6 + (a_3 \cap a_4)\, g^2 \times g^6$$
$$+ (a_4 \cap a_4) g \times g^6 + (a_5 \cap a_4)\, g^6 + (a_2 \cap a_1)\, g^6 \times g$$
$$+ (a_3 \cap a_1) g^2 \times g + (a_4 \cap a_1)\, g \times g + (a_5 \cap a_1)\, g$$
$$+ (a_2 \cap 1)\, g^6 + (a_3 \cap 1)\, g^2 + (a_4 \cap 1)\, g + a_5 \cap 1$$

$$= \quad a_3\, g^{13} + a_3g^9 + a_4\, g^8 + a_5g^7 + a_4g^{12} + a_4g^8 + a_4g^7 + a_5g^6$$
$$+ a_2g^7 + a_3g^3 + a_4g^2 + a_5g + a_2g^6 + a_3g^2 + a_4g + a_5$$

$$= \quad a_3g^{13} + a_4g^{12} + a_3g^9 + a_4g^8 + (a_5g^7 + a_4g^7 + a_2g^7)$$
$$+ (a_5g^6 + a_2g^6) + a_3\, g^3 + (a_3g^2 + a_3g^2) + (a_5g + a_4g) + a_5$$

$$= \quad a_3g^{13} + a_4g^{12} + a_3g^9 + a_4g^8 + a_2g^7 + a_2g^6 + a_3g^3 + a_3g^2 +$$
$$a_4g + a_5 \in C_7G.$$

This is the way sum and product are obtained on the group semiring using the semiring as the chain lattice C_n.

Example 5.2.2: Let $G = S_3$ the permutation group of degree three and $S = C_{10}$ be the chain lattice. SG be the group semiring of the group G over the semiring C_{10}. Clearly order of SG is finite but SG is non commutative.

$$G = S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e = 1, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = p_1, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = p_2, \right.$$

$$\left.\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}=p_3,\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}=p_4,\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}=p_5\right\}$$

be the symmetric group of degree three.

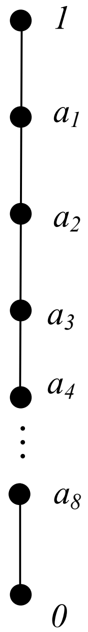


Figure 5.2.2: C_{10}

Let

$$\alpha = a_1p_5 + a_3p_3 + a_2p_1 + a_7$$

and

$$\beta = a_8p_4 + a_2p_3 + a_5p_1 + a_6 \in C_nG = SS_3.$$

$$\begin{aligned}
\alpha + \beta &= (a_1p_5 + a_3p_3 + a_2p_1 + a_7) + (a_8p_4 + a_2p_3 + a_5p_1 + a_6)\\
&= a_1p_5 + a_8p_4 + (a_3p_3 + a_2p_3) + (a_2p_1 + a_5p_1) + (a_7 + a_6)\\
&= a_1p_5 + a_8p_4 + (a_3 \cup a_2)\, p_3 + (a_2 \cup a_5)\, p_1 + a_7 \cup a_6\\
&= a_1p_5 + a_8p_4 + a_2p_3 + a_2p_1 + a_6 \in SS_3.
\end{aligned}$$

Consider

$$\alpha \times \beta = (a_1p_5 + a_3p_3 + a_2p_1 + a_7) \times (a_8p_4 + a_2p_3 + a_5p_1 + a_6)$$

$$= (a_1 \cap a_8)\, p_5 \times p_4 + (a_3 \cap a_8)\, p_3 \times p_4 + (a_2 \cap a_8)\, p_1 \times p_4 + (a_7 \cap a_8)\, p_4 + (a_1 \cap a_2)\, p_5 \times p_3 + (a_3 \cap a_2)\, p_3 \times p_3 + (a_2 \cap a_2)\, p_1 \times p_3 + (a_7 \cap a_2)\, p_3 + (a_1 \cap a_5)\, p_5 \times p_1 + (a_3 \cap a_5)\, p_3 \times p_1 + (a_2 \cap a_5)\, p_1 \times p_1 + (a_7 \cap a_5)\, p_1 + (a_1 \cap a_6)\, p_5 + (a_3 \cap a_6)\, p_3 + (a_2 \cap a_6)\, p_1 + a_7 \cap a_6$$

$$= a_8 + a_8\, p_2 + a_8\, p_3 + a_8\, p_4 + a_2p_2 + a_3 \cdot 1 + a_2p_4 + a_7p_3 + a_5p_3 + a_5p_5 + a_5 + a_7p_1 + a_6\, p_5 + a_6p_3 + a_6p_1 + a_7$$

$$= (a_8 \cup a_7 \cup a_5 \cup a_3) + (a_8 \cup a_2)\, p_2 + (a_8 \cup a_7 \cup a_5 \cup a_6)p_3 + (a_8 \cup a_2)p_4 + (a_5 \cup a_6)\, p_5 + (a_5 \cup a_6 \cup a_7)\, p_1$$

$$= a_3 + a_2p_2 + a_5p_3 + a_2p_4 + a_5p_5 + a_5p_1 \in SS_3. \qquad \text{I}$$

Now find

$$\beta \times \alpha = (a_8p_4 + a_2p_3 + a_5p_1 + a_6) \times (a_1p_5 + a_3p_3 + a_2p_1 + a_7)$$

$$= (a_8 \cap a_1)\, p_4 \times p_5 + (a_2 \cap a_1)\, p_3 \times p_5 + (a_5 \cap a_1)\, p_1 \times p_5 + (a_6 \cap a_1)\, p_5 + (a_8 \cap a_3)\, p_4 \times p_3 + (a_2 \cap a_3)\, p_3 \times p_3 + (a_5 \cap a_3)\, p_1 \times p_3 + (a_6 \cap a_3)\, p_3 + (a_8 \cap a_2)\, p_4 \times p_1 + (a_2 \cap a_2)\, p_3 \times p_1 + (a_5 \cap a_2)\, p_1 \times p_1 + (a_6 \cap a_2)\, p_1 + (a_8 \cap a_7)\, p_4 + (a_2 \cap a_7)\, p_3 + (a_5 \cap a_7)\, p_1 + (a_6 \cap a_7)$$

$= \quad a_8 \cdot 1 + a_2p_1 + a_5p_2 + a_6p_5 + a_8p_1 + a_3 \cdot 1 + a_5p_4 + a_6p_3$
$+ a_8p_2 + a_2p_5 + a_5 \cdot 1 + a_6p_1 + a_8p_4 + a_7p_3 + a_7p_1 + a_7$

$= \quad (a_8 \cup a_3 \cup a_5 \cup a_7) + (a_2 \cup a_8 \cup a_6 \cup a_7)p_1 + (a_5 \cup a_8)p_2 \quad + (a_6 \cup a_2)p_5 + (a_5 \cup a_8)\, p_4 + (a_6 \cup a_7)\, p_3$

$= \quad a_3 + a_2p_1 + a_5p_2 + a_2p_5 + a_5p_4 + a_6p_3.$ II

Clearly I and II are not equal so $\alpha \times \beta \neq \beta \times \alpha$ for this $\alpha, \beta \in SS_3$, hence SS_3 is a non commutative group semiring of finite order.

The following proposition characterizes the group semiring of a group G over the semiring which is a chain lattice.

Proposition 5.2.1: *Let C_n be a finite chain lattice. G be a group and C_nG be the group semiring of the group G over the semiring C_n. C_nG is a commutative group semiring if and only if G is a commutative group.*

Proof: Given C_n is a chain lattice so C_n is a commutative semiring infact a semifield. Let G be a commutative group clearly the group semiring C_nG is a commutative group semiring.

Suppose C_nG be a commutative group semiring of the group G over the lattice C_n. To prove G is a commutative group, it enough to prove for every $g, h \in G; gh = hg$.

Given SG is commutative let $g, h \in SG$ $(g, h \in G)$ then $gh = hg$ as SG is commutative. This is true for every $g, h \in G$ hence G is a commutative group.

Next result proves C_nG is a finite group semiring.

Proposition 5.2.2: *Let C_n be a semiring (chain lattice of order n) and G a group. C_nG be the group semiring. C_nG is of finite order if and only if G is a finite group.*

Proof. Given the group semiring C_nG is of finite order.

Clearly if G is not of finite order, since $G \subseteq C_nG$; C_nG would be of infinite order. Hence G must be a group of finite order.

Suppose G is a group of finite order clearly C_nG the group semiring will be of finite order as C_n is a finite semiring.

Now an example of group semiring of infinite order is given.

Example 5.2.3: Let $G = R \setminus \{0\}$ be the group of reals under product and C_{15} be the semiring (chain lattice of order 15). $C_{15}G$ be the group semiring. Clearly $C_{15}G$ is of infinite order as $R \setminus \{0\}$ is an infinite group; so the group semiring $C_{15}G$ is of infinite order.

$$C_{15} = 0 < a_{13} < a_{12} < \ldots < a_1 < 1.$$

be the semiring.

Let

$$\alpha = a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 \text{ and } \beta = a_5 + a_8\, 0.3 + a_{13}\, 9 \in C_{15}G.$$

$$\begin{aligned} \alpha + \beta &= a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_5 + a_8\, 0.3 + a_{13}\, 9 \\ &= (a_1 \cup a_5) + (a_5 \cup a_8)\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_{13}\, 9 \end{aligned}$$

$$= \quad a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_{13}\, 9 \in C_{15}G.$$

$$\alpha \times \beta \quad = \quad (a_1 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2) \times (a_5 + a_8\, 0.3 + a_{13}\, 9)$$

$$\begin{aligned} = \quad & a_1 \cap a_5 + (a_5 \cap a_5)\, 0.3 + (a_9 \cap a_5)\, 120 + (a_{10} \cap a_5)\, 6.2 \\ & + (a_1 \cap a_8) 0.3 + (a_5 \cap a_8)\, (0.3 \times 0.3) \\ & + (a_9 \cap a_8)\, (120 \times 0.3) + (a_{10} \cap a_8)\, (6.2 \times 0.3) \\ & + (a_1 \cap a_{13})\, 9 + (a_5 \cap a_{13})\, (0.3 \times 9) \\ & + (a_9 \cap a_{13})\, (120 \times 9) + (a_{10} \cap a_{13})\, 6.2 \times 9. \end{aligned}$$

$$\begin{aligned} = \quad & a_5 + a_5\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_8\, 0.3 + a_8\, 0.09 \\ & + a_9\, 36 + a_{10}\, 1.86 + a_{13} 9 + a_{13}\, 2.7 + a_{13}\, 1080 + a_{13}\, 55.8 \end{aligned}$$

$$\begin{aligned} = \quad & a_5 + (a_5 \cup a_8)\, 0.3 + a_9\, 120 + a_{10}\, 6.2 + a_8\, 0.09 + a_9\, 36 \\ & + a_{10}\, 1.86 + a_{13} 9 + a_{13}\, 1080 + a_{13}\, 2.7 + a_{13}\, 55.8 \in C_{15}G. \end{aligned}$$

This is the way product is defined on the infinite group semiring.

There are several group semirings of infinite order.

Just recall all chain lattices are semifields. For more about semifields refer [100].

Theorem 5.2.1: *Let C_n be the semiring which is a semifield and G be a commutative group. C_nG the group semiring is a semifield.*

Proof: Given C_nG is a commutative group semiring. Clearly if $\alpha, \beta \in C_nG;$ $\alpha + \beta = 0$ is possible only when $\alpha = 0$ and $\beta = 0$. For in $C_n;$ $a_i + a_j = 0 = a_i \cup a_j$ if only if $a_i = 0 = a_j$ for

$$C_n = 0 < a_{n-2} < a_{n-3} < \ldots < a_1 < 1.$$

Further $\alpha \times \beta = 0$ is possible in C_nG only if $\alpha = 0$ and $\beta = 0$. For in C_n; $a_ia_j = 0 = a_i \cap a_j$ if and only if $a_i = 0$ or $a_j = 0$.

Thus C_nG is a semifield.

Corollary 5.2.1: *Let C_nG be a group semiring of a group G over the semiring C_n; G is a non commutative group then C_nG is a semidivision ring.*

Proof: Since C_nG has no zero divisors and for every $\alpha, \beta \in C_nG$; $\alpha + \beta = 0$ implies $\alpha = 0$ and $\beta = 0$. C_nG is a semidivision ring as G is a non commutative group.

Example 5.2.4: Let $G = \langle g \mid g^{15} = 1 \rangle$ be the cyclic group of order 15. C_{16} be the chain lattice of order 16. $C_{16}G$ be the group semiring of the group G over the semiring C_{16}. $C_{16}G$ is a semifield of finite order.

Example 5.2.5: Let $G = \langle a, b \mid a^2 = b^7 = 1, bab = a \rangle$ be the dihedral group of order 14. C_{27} be the semiring of order 27. $C_{27}G$ is the group semiring of finite order which is a semidivision ring. This proves that group semiring, C_nG where C_n is a chain lattice has no zero divisors.

This is in contrast with group rings for every group ring of a finite group G over any field; finite or infinite field has zero divisors.

Next the units and idempotents in C_nG are discussed in the following:

C_n, the chain lattice has only idempotents and has no units.

For any $x, y \in C_n$ we have $x \cap y = 1$ is not possible unless $x = y = 1$. Further $g.h = 1$ for $g, h \in G$ are defined as trivial units.

So C_n has no units. Further every element in C_n is an idempotent as in C_n; $a_i \cap a_i = a_i$ for every $a_i \in C_n$ as C_n is a chain lattice.

In view of this the following theorems are proved:

Theorem 5.2.2: *Let C_n be the chain lattice and G any group. The group semiring C_nG has no nontrivial zero divisors.*

Proof: Follows from the fact C_nG is a semifield.

Theorem 5.2.3: *Let C_nG be the group semiring. The only trivial units of C_nG are $1 \cdot g = g$ for every $g \in C_nG$.*

Proof: Follows from the fact $G \subseteq C_nG$ and every $g \in G$ has a unique inverse. However if $\alpha \in C_n \setminus \{1\}$; then it is not a unit only an idempotent as C_n is a chain lattice.

If $\alpha = \sum_{i=1}^{n} \alpha_i g_i$ then $\alpha^2 = 1$ is impossible as C_nG is proved to be a semifield, so no zero divisors to cancel of or add to 1.

Hence the chain.

However C_nG has idempotents if G is a group of finite order.

Example 5.2.6: Let $G = \langle g \mid g^{12} = 1 \rangle$ be the cyclic group of order 12. C_8 be the chain lattice. C_8G be the group semiring.

Consider $\alpha = (1 + g^6) \in G,$

$$
\begin{aligned}
\alpha^2 &= (1 + g^6) \times (1 + g^6) \\
&= (1 + g^6 + g^6 + g^{12}) \\
&= 1 \cup 1 + (1 \cup 1)\, g^6 \\
&= 1 + g^6 \\
&= \alpha.
\end{aligned}
$$

Thus α is an idempotent in C_8G.

Consider

$$\beta = 1 + g^3 + g^6 + g^9 \in C_nG$$

$$\beta^2 = (1 + g^3 + g^6 + g^9) \times (1 + g^3 + g^6 + g^9)$$

$$= 1 + g^3 + g^6 + g^9 + g^3 + g^6 + g^9 + g^{12} + g^6 + g^9 + g^{12} + g^3 + g^9 + g^{12} + g^3 + g^6$$

$$= 1 + g^3 + g^6 + g^9 \text{ (as } 1 \cup 1 = 1)$$

$$= \beta.$$

Thus β is an idempotent. Similarly $\gamma = 1 + g^4 + g^8 \in C_nG$.

Clearly $\gamma^2 = \gamma$ is an idempotent in C_nG,

Finally $\delta = 1 + g + g^2 + \ldots + g^{11} \in C_nG$ is also an idempotent of C_nG.

Thus apart from this, all elements in C_n as $C_n \subseteq C_nG$ are also idempotents of C_nG as $a_i \times a_i = a_i \cap a_i$ for all $a_i \in C_n$; they will be known as trivial idempotents.

In view of this we have the following theorem:

Theorem 5.2.4: *Let C_n be the group semiring of the group G of finite order over the chain lattice C_n.*

i) All $\alpha \in C_n$; $C_n \subseteq C_nG$ are trivial idempotents of C_nG.

ii) If $H_i \subseteq G$ is a subgroup of G and $H_i = \{1, h_1, \ldots, h_t\}$ then $\beta = 1 + h_1 + \ldots + h_t \in C_nG$ is an idempotent of C_nG. This is true for every subgroup of G.

iii) If $G = \{1, g_1, \ldots, g_m\}$ then $\gamma = 1 + g_1 + g_2 + \ldots + g_m \in C_nG$ is an idempotent of C_nG.

Proof: For every $\alpha \in C_n$ it is clear $\alpha \times \alpha = \alpha \cap \alpha = \alpha$ is an idempotent of C_nG as $C_n \subseteq C_nG$.

Further if $| G | < \infty$ and $G = \{1, g_1, \ldots, g_m\}$ then $\beta = 1 + g_1 + \ldots + g_m \in C_nG$ is such that $\beta^2 = \beta$. Finally every H_i, a subgroup in G is of finite order and if $H_i = \{1, h_1, h_2, \ldots, h_t\}$ then $\gamma = 1 + h_1 + \ldots + h_t \in C_nG$ is such that $\gamma^2 = \gamma$.

Hence the theorem.

Next subsemirings and ideals of the group semiring C_nG are discussed in the following:

Example 5.2.7: Let C_9 be a chain lattice and $G = \langle g \mid g^{18} = 1 \rangle$ be the cyclic group of order 18. C_9G be the group semiring.

$A_1 = \{0, 1, (1 + g + \ldots + g^{17})\} \subseteq C_9G$ is a subsemiring of order 3.

$A_2 = \{0, 1, (1 + g^2 + g^4 + \ldots + g^{16})\} \subseteq C_9G$ is again a subsemiring of order 3.

$A_3 = \{0, 1, (1 + g^3 + g^6 + g^9 + \ldots + g^{15})\} \subseteq C_9G$ is also a subsemiring of order 3.

$A_4 = \{0, 1, \{1 + g^6 + g^{12}\}\} \subseteq C_9G$ is also a subsemiring of C_9G.

$A_5 = \{0, 1, \{1 + g^9\}\} \subseteq C_9G$ is a subsemiring of order 3.

Now let $P_1 = \{1, g^2, g^4, \ldots, g^{16}\} \subseteq G$ be a subgroup of G. $C_9P_1 \subseteq C_9G$ is a subsemiring of C_9G.

Let $P_2 = \{1, g^9\} \subseteq G$ be a subgroup of G. $C_9P_2 \subseteq C_9G$ is a subsemiring of C_9G. Let $P_3 = \{1, g^6, g^{12}\} \subseteq G$ be a subgroup of G. $C_9P_3 \subseteq C_9G$ is a subsemiring of C_9G. Let $P_4 = \{1, g^3, g^6, \ldots, g^{15}\} \subseteq G$ be a subgroup; $C_9P_4 \subseteq C_9G$ be a subsemiring of C_9G.

Let $M_1 = \{0, a_5, 1\} \subseteq C_9$ be a sublattice of $C_9 = 0 < a_7 < a_6 < \ldots < a_2 < a_1 < 1$, now $M_1P_1 \subseteq C_9G$ is a subsemiring of C_9G.

Let $M_2 = \{0, a_6, 1\} \subseteq C_9$ be a sublattice of C_9 and M_2P_1, M_2P_2, M_2P_3 and M_2P_4 are all subsemirings of C_9G.

Thus C_9G has several subsemirings but all of them are not ideals of C_9G, only a few of them are ideals.

Further M_2P_1, M_2P_2, M_2P_3 and M_2P_4 are only subsemirings and none of them are ideals of C_9G.

Example 5.2.8: Let C_2 be the chain lattice. $G = \langle g \mid g^3 = 1 \rangle$ be the cyclic group of degree three. C_2G be the group semiring of the group G over the semiring C_2. $P = \{0, 1, 1 + g + g^2\} \subseteq C_2G$ is a subsemiring. This is not an ideal of C_2G.

In view of all these the following proposition is proved:

Proposition 5.2.3: *Let C_nG be the group semiring of the group G over the semiring C_n. If M is a subsemiring of C_nG then M is not an ideal of C_nG.*

Proof: Proved using an example. In the example 5.2.7 of this chapter there are several subsemirings of the group semiring which are not ideals.

Next the concept of right and left ideal exist only when C_nG is a non commutative group semiring.

Consider the following example:

Example 5.2.9: Let C_2S_3 be the group semiring of the symmetric group S_3 over the semiring C_2.

$$P = \{0, 1 + p_1, p_2 + p_5, p_3 + p_4, 1 + p_1 + p_2 + p_5, 1 + p_3 + p_1 + p_4,$$
$$1 + p_1 + p_2 + p_3 + p_4 + p_5, p_2 + p_3 + p_4 + p_5\} \subseteq C_2S_3$$

is a right ideal and is not a left ideal.

Thus as in case of group rings which are non commutative in case of group semirings which are non commutative has right ideals that are not left ideals and vice versa.

Another interesting feature is in case of a field, field has no ideals other than (0) and F but however semifields which are group semirings of the form C_nG has ideals.

In the next section the study of group semirings using distributive lattices which are not chain lattices is carried out.

5.3 STUDY OF GROUP SEMIRINGS USING DISTRIBUTIVE LATTICES WHICH ARE NOT CHAIN LATTICES

In this section a study of group semirings using distributive lattices L which are not chain lattices is carried out. Unlike chain lattices in case of distributive lattices, group semirings in general are not semifields. However in case of certain lattices the group semiring can be a semifield.

Since the replacing of semiring (chain lattice) by a distributive lattice will not alter the definition of a group semiring, so here the definition of group semiring using distributive lattices are not made once again. First a few examples of them are given.

Example 5.3.1: Let L be the lattice whose Hasse diagram is as follows:

be a distributive lattice.

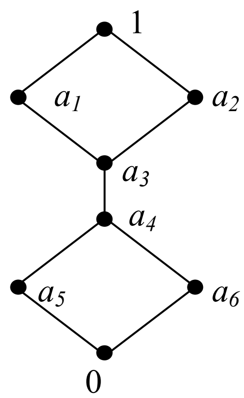


Figure 5.3.1: L

$G = \langle g \mid g^{10} = 1 \rangle$ be the cyclic group of order *10*. *LG* be the group semiring of the group *G* over the semiring *L* which is a distributive lattice. *LG* is not a field. In the first place *L* is a semiring and not a semifield as $a_5 \cap a_6 = 0$. Thus *LG* has zero divisors so *LG* is only a semiring.

Example 5.3.2: Let *L* be the Hasse diagram of the lattice which is as follows:

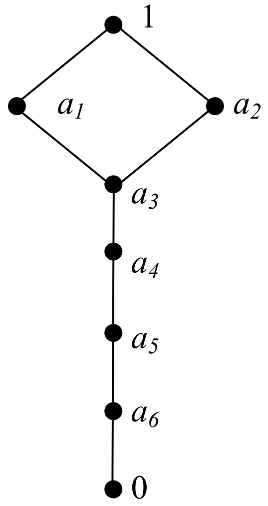


Figure 5.3.2: *L*

be a distributive lattice and $G = S_4$ be the symmetric group of degree 4. *LG* the group semiring is a semidivision ring. Further *L* is not a semifield as *LG* is non-commutative semiring. Thus *LG* has no zero divisors but *LG* is a non-commutative semiring.

Example 5.3.3: Let *B*

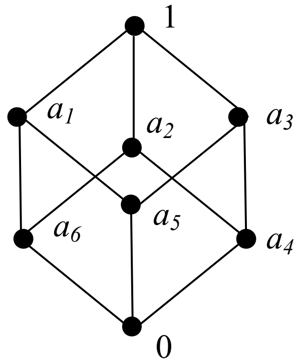


Figure 5.3.3: Boolean Algebra *B*

be the Boolean algebra. $G = \langle g \mid g^{12} = 1 \rangle$ be the cyclic group of order 12. BG the group semiring. BG has zero divisors, units and idempotents. $\alpha = (a_6g^6 + a_5g^2)$ and $\beta = a_4g^2 \in BG$.

$$\alpha\beta \quad = \quad (a_6g^6 + a_5g^2) \times a_4g^2 = \quad 0.$$

Let

$$\alpha = 1 + g^3 + g^6 + g^9 \in BG.$$

Clearly $\alpha^2 = \alpha$ so α is an idempotent in BG. $g^7,\ g^5 \in BG$ is such that $g^7 \times g^5 = 1$. All elements in G are units and $G \subseteq BG$.

Theorem 5.3.1: *Let L be a distributive lattice and G any group. LG be the group semiring of the group G over the semiring L. LG has zero divisors if and only if L is a distributive lattice which is not a semifield.*

Proof: If L is not a semifield. That is there exist $a_i, a_j \in L \setminus \{0\}, a_i \neq a_j$ such that $a_i \cap a_j = 0$.

Take $\alpha = a_ig_1$ and $\beta = a_jg_2 \in LG; g_1, g_2 \in G;$

$$\begin{aligned} \alpha \cap \beta &= a_i g_1 \cap a_j g_2 \\ &= (a_1 \cap a_j)\,(g_1g_2) \\ &= 0\,(g_1g_2) \quad = \quad 0. \end{aligned}$$

Thus LG has zero divisors.

Corollary 5.3.1: *If L is a semifield then the group semiring LG has no zero divisors for all groups G.*

Proof: Follows from the fact L is a semifield and no $\alpha, \beta \in LG \setminus \{0\}$ is such that $\alpha \times \beta = (0)$.

Example 5.3.4: Let L be the lattice given by following diagram:

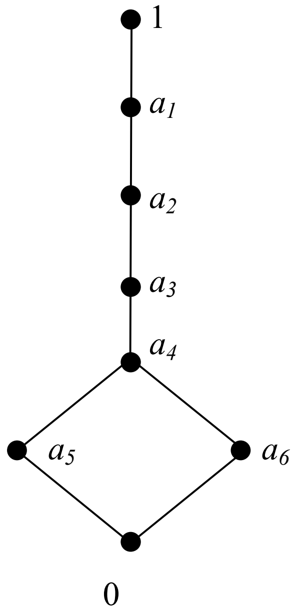


Figure 5.3.4: L

$G = S_3$ be the symmetric group of degree three. LG be the group semiring of the group G over S_3.

Let $R_1 = \{1, p_1\}$ be a subgroup of S_3. LR_1 is a subsemiring which is not an ideal.

Let $R_2 = \{1, p_2\} \subseteq S_3$ be the subgroup of S_3. LR_2 is again a subsemiring.

LR_1 and LR_2 are isomorphic as subsemirings by mapping p_1 to p_2 and rest of the elements to itself.

The following theorem is interesting which describes a semiring which is not a semifield.

Theorem 5.3.2: *Let G be a finite group. L is a Boolean algebra of order greater than or equal to four. LG the group semiring has zero divisors.*

Proof: Follows from the fact all Boolean algebras of order greater than or equal to four has elements $a, b \in L \setminus \{0\}$ with $a \cap b = 0$. This will contribute for zero divisors of the form $\alpha\beta = 0$ when $\alpha = ag_1$ and $\beta = bg_2$ with $\alpha \times \beta = \alpha\beta = ag_1 \times bg_2 = (a \cap b)\, g_1g_2 = 0$.

Theorem 5.3.3: *Let G be a group of finite order and L be a distributive lattice which is not a chain lattice. LG be the group semiring of the group G over the lattice L. LG has non trivial idempotents.*

Proof: Given $|G| = n < \infty$ a finite group. LG be the group semiring.

Take $\alpha = (1 + g_1 + \ldots + g_{n-1}) \in LG$. Clearly $\alpha^2 = \alpha$ so α is an idempotent of LG.

Likewise if $H_1, H_2, \ldots, H_t$ are non- trivial subgroups of order $p_1, p_2, \ldots, p_t$ respectively then $\beta_1 = 1 + h_1 + h_2 + \ldots + h_{p_1 - 1} \in LG$ where $H_1 = \{1, h_1, h_2, \ldots, h_{p_1-1}\} \subseteq G$ is such that $\beta_1^2 = \beta_1$.

Let $\beta_2 = 1 + k_1 + k_2 + ... + k_{p_2 - 1} \in LG$, where $H_2 = \{1, k_1, k_2, ..., k_{p_2 - 1}\} \subseteq G$ is such that $\beta_2^2 = \beta_2$.

Likewise if

$$H_t = \{1, m_1, m_2, ..., m_{p_t - 1}\} \subseteq G$$

be the subgroup, then

$$\beta_t = 1 + m_1 + m_2 + ... + m_{p_t - 1} \in LG$$

is such that $\beta_t^2 = \beta_t$.

Hence the theorem.

The idempotents in L will be called as trivial idempotents. Likewise zero divisors in L will be defined as trivial zero divisors of LG.

Clearly L has no units and units contributed by the group G will be termed as trivial units of LG.

Conditions for Smarandache zero divisors to exist in group semirings; BG where B is a Boolean algebra is obtained in the following:

Example 5.3.5: Let B

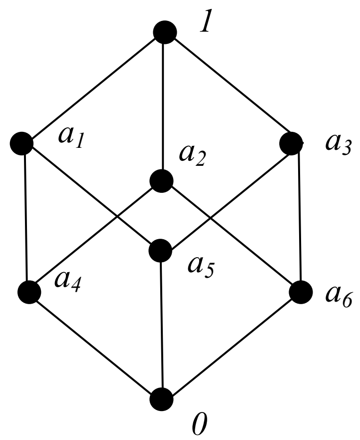


Figure 5.3.5: *B*

be a Boolean algebra. $G = \langle g \mid g^{16} = 1 \rangle$ be the cyclic group of order 16. BG be the group semiring of the group G over the semiring B.

Let $x = a_4 (g^{12} + g^2)$ and $y = a_5 (g^7 + g^5) \in BG.$

$$
\begin{aligned}
x \times y &= a_4 (g^{12} + g^2) \times a_5 (g^7 + g^5) \\
&= (a_4 \cap a_5) (g^{12} + g^2) (g^7 + g^5) \\
&= 0.
\end{aligned}
$$

Let $a = a_6 (g^{10} + g)$ and $b = a_2 (g^3 + g^{11} + g^{13}) \in BG.$ $x \times a = 0$ and $y \times b = 0$ but $a \times b \neq 0.$

Thus $x, y \in BG$ is a Smarandache zero divisor.

Example 5.3.6: Let B be a Boolean algebra

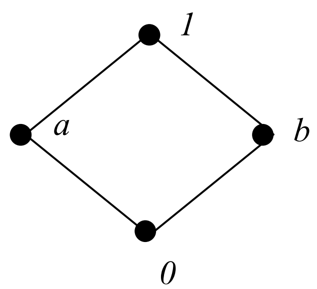


Figure 5.3.6: B

and G be any group. BG be the group semiring of the group G over the semiring B. BG has no S-zero divisors.

In view of this the following theorem is proved:

Proposition 5.3.1: *Let G be any group and B a Boolean algebra; BG the group semiring.*

i. *BG the group semiring has S-zero divisors if* $| B | > 4$.

ii. *BG has only zero divisors and no S-zero divisors if* $| B | = 4$.

iii. *BG has no zero divisors if* $| B | = 2$.

Proof: Proof of i: Follows from the fact if $| B | > 4$ then B has zero divisors as well as S-zero divisors.

Hence BG will have S-zero divisors (refer example 5.3.5).

Proof of ii: If $| B | = 4$ then $B =$

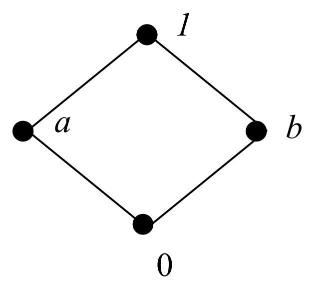


Figure 5.3.7: B

Clearly $a \times b = 0$ is a zero divisor and cannot find another $y \neq 0$ with $a.y = 0$ and $a.\ x \neq 0$ with $bx = 0$ and $xy \neq 0$. Hence the claim.

Proof of iii: If $| B | = 2$ then B is a chain lattice hence BG has no zero divisors.

Next the study about the existence of S-anti zero divisor is discussed.

Example 5.3.7: Let B

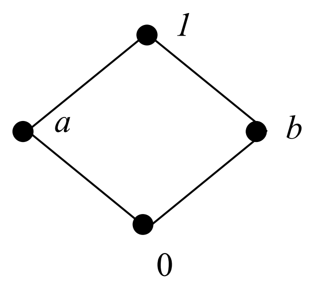


Figure 5.3.8: *B*

be the Boolean algebra of order four; $G = S_3$, the symmetric group of degree 3. BS_3 be the group semiring of the group S_3 over the semiring B.

Let $\alpha = (1 + p_1 + p_2)$, $\beta = (1 + p_4) \in BS_3$. Clearly $\alpha \times \beta \neq (0)$.

Take $x = a\ (p_2 + p_5)$ and $y = b\ (p_5 + p_3 + 1) \in BS_3$.

Consider

$$\begin{aligned} \alpha x &= (1 + p_1 + p_2)\ a\ (p_2 + p_5) \\ &= 1 \cap a\ [(1 + p_1 + p_2) \times (p_2 + p_5)] \\ &= a(p_2 + p_5 + 1 + p_5 + p_2 + p_3) \end{aligned}$$

$$\begin{aligned} &= a\,(p_2 + 1 + p_5 + p_5 + p_2 + p_3) \\ &= a\,(1 + p_5 + p_2 + p_3) \neq 0. \end{aligned}$$

Consider

$$\begin{aligned} \beta y &= (1 + p_4) \times b\,(1 + p_3 + p_5) \\ &= b \cap 1\,[(1 + p_4) \times (1 + p_3 + p_5)] \\ &= b\,(1 + p_4 + p_3 + p_1 + 1 + p_5) \\ &= b\,(1 + p_1 + p_3 + p_4 + p_5) \neq 0. \end{aligned}$$

So $\beta y \neq 0$ and $\alpha x \neq 0$ but

$$\begin{aligned} xy &= a\,(p_2 + p_5) \times b\,(p_5 + p_3 + 1) \\ &= a \cap b\,[(p_2 + p_5) \times (p_5 + p_3 + 1)] \\ &= 0. \end{aligned}$$

Thus x is a Smarandache anti-zero divisor of BG.

In view of this the following proposition is proved:

Proposition 5.3.2: *Let B be a Boolean algebra of order four. G any group and BG be the group semiring of the group G over the semiring B. BG has S-anti zero divisors.*

Proof: Let $\alpha = \Sigma g_i$ and $\beta = \Sigma h_i$, $g_i, h_i \in G$ (all coefficients of g_i in α and h_i in β are 1). Take $x = (\Sigma a k_i)$ and $y = \Sigma b m_j$ $(k_i, m_j \in G)$.

$B =$

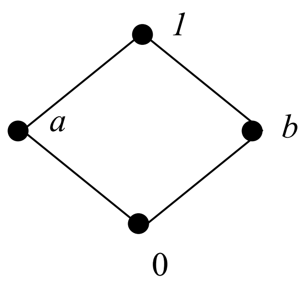


Figure 5.3.9: B

Clearly $\alpha\beta \neq 0$. Further $\alpha x \neq 0$ and $\beta y \neq 0$ but $xy = 0$. Thus x is a S-anti zero divisor in BG.

Theorem 5.3.4: *Let BG be the group semiring of the group G over the Boolean algebra of order four. Let $\alpha \in BG$ be a S-anti zero divisor then α need not be a S-zero divisor.*

Proof: Follows from the examples 5.3.7 and 5.3.6. For x in that example is not a S-zero divisor in BG.

Proposition 5.3.3: *Let BG be the group semiring of the group G over the Boolean algebra of order greater than 4. BG has both S-zero divisors as well as S-anti zero divisors.*

Next the study of Smarandache idempotents in these group semirings is carried out. At first it is important to know that the distributive lattices or for that matter any lattice L will not contain any Smarandache idempotent as every element a in L is such that $a \times a = a \cap a = a^2 = a$ for all $a \in L$.

However it is an interesting feature to analyse whether the group semiring of a group G over a distributive lattice L have Smarandache idempotents. Let BG be the group semiring of the group $G = S_3$ over the semiring B which is a Boolean algebra.

Take $a = (1 + p_4 + p_5)$ and $b = (p_1 + p_2 + p_3)$ we see $a^2 = a$ and $b^2 = a; ab = a$. This group semiring has S-idempotents.

Example 5.3.8: Let $G = \langle g \mid g^8 = 1 \rangle$ be the cyclic group of order 8. B be any Boolean algebra or a distributive lattice. BG be the group semiring of the group G over the semiring B.

Let $\alpha = 1 + g^2 + g^4 + g^6$ and $\beta = g + g^3 + g^5 + g^7 \in BG$. Clearly $\alpha^2 = \alpha$, $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Thus α is a S-idempotent of BG.

In view of all these the following interesting theorem for cyclic groups of even order is given:

Theorem 5.3.5: *Let G =* $\langle g \mid g^{2n} = 1 \rangle$ *be the cyclic group of order 2n. B be a distributive lattice or a Boolean algebra. BG the group semiring has a Smarandache idempotents.*

Proof: Take $\alpha = (1 + g^2 + g^4 + g^6 + g^8 + \ldots + g^{2n-2}) \in BG.$

Let $\beta = (g + g^3 + g^5 + g^7 + \ldots + g^{2n-1}) \in BG$. Clearly $\alpha^2 = \alpha$, $\alpha\beta = \beta$ and $\beta^2 = \alpha$. So α is a Smarandache idempotent.

Example 5.3.9: Let B be a distributive lattice or a Boolean algebra.

$$D = \{a, b \,/\, a^2 = b^{20} = 1,\ bab = a\}$$

be the dihedral group. BD be the group semiring of the group D over the semiring B.

Take $\alpha = (1 + b + b^2 + \ldots + b^{19})$ and $\beta = (a + ab + ab^2 + \ldots + ab^{19}) \in BD$. Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ with $\alpha\beta = \beta$. Then α is a Smarandache idempotent in BD.

In view of this the following theorem:

Theorem 5.3.6: *Let L be a distributive lattice or a Boolean algebra. Let* $G = D_{2n} = \{a,b \,/\, a^2 = b^n = 1;\ bab = a\}$ *be a dihedral group; n an even integer say 2m. LG be the group semiring of the group G over the semiring L. LG has S-idempotents.*

Proof: Consider $\alpha = (1 + b + b^2 + ... + b^{2m-1})$ and $\beta = (a + ab + ab^2 + ... + ab^{2m-1}) \in LG$. Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Thus α is a Smarandache idempotent of LG.

Example 5.3.10: Let A_4 be the alternating subgroup of S_4; L be a distributive lattice or a Boolean algebra. LA_4 be the group semiring.

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in LA_4.$$

$$\alpha^2 = \alpha.$$

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \alpha$$

$$\alpha\beta = \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\right] \times \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$= \beta$.

Thus α is a Smarandache idempotent of LA_4.

In view of all these the following theorems are proved.

Theorem 5.3.7: *Let G be a group of order n and G has a subgroup H of order m (m/n; m an even number). L be any distributive lattice or a Boolean algebra. LG, the group semiring has S-idempotents.*

Proof: Let H be a subgroup of order say $m = 2t$ and let $P = \{1, g_1, \ldots, g_{t-1}\}$ be a subgroup of H. Then take $\alpha = 1 + g_1 + \ldots + g_{t-1}$ and

$$\beta = \sum_{h_i \in H \setminus P} h_i \in LG.$$

Clearly $\alpha^2 = \alpha$ and $\beta^2 = \alpha$ and $\alpha\beta = \alpha$. Thus α is the S-idempotent of LG.

Theorem 5.3.8: *Let S_n be the symmetric group of degree n (n even or odd). L any distributive lattice. LS_n be the group semiring. LS_n has S-idempotents.*

Proof: Case 1: n is even. S_n has a subgroup of order n which is cyclic. Hence using this subgroup; say $G = \{1, g, \ldots, g^{n-1}\}$ contributes to the S-idempotent $\alpha = 1 + g^2 + \ldots + g^{n-2}$ and $\beta = (g + g^3 + \ldots + g^{n-1}) \in LS_n$ is such that $\alpha^2 = \alpha$; $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Let n be odd then $n - 1$ is even. Let H = cyclic group generated by $h \in S_n$ of order $n - 1$.

Now $(1 + h^2 + h^4 + \ldots + h^{n-3}) = \alpha$ and $\beta = (h + h^3 + \ldots + h^{n-1}) \in LS_n$ are such that $\alpha^2 = \alpha$, $\beta^2 = \alpha$ and $\alpha\beta = \beta$. Thus α is a S-idempotent of LS_n. Hence the theorem.

5.4 CONCLUSIONS

In this chapter group semirings of groups over semirings which are distributive lattices is carried out. Further in case of chain lattices C_n the group semiring C_nG is a semifield; in case G is abelian and a semidivision ring in case G is a non-commutative group.

Further if the distributive lattice L has zero divisors then only the group semiring LG will have zero divisors. However grouprings FG have zero divisors if G is a finite group; a marked difference between these two structures.

Finally idempotents which are in $LG \setminus L$ are identified. The concept of Smarandache zero divisors and Smarandache idempotents in group semirings (where semirings are distributive lattices) are carried out and conditions for their existence is also determined in this chapter. However in case of group semirings LG over distributive lattices; it is impossible to find units or S-units in $LG \setminus G$

CHAPTER SIX

CONCULSIONS

Conclusions at the end of each chapter is given. Overall research work carried out in this thesis is briefly described.

Study of properties enjoyed by finite semigroups in par with finite groups is carried out in connection with classical theorems for finite groups in this thesis. This is relevant as semigroups are generalization of groups.

Further study of this type is completely lacking. So this is the first attempt which is made and it is proved that classical Lagrange's theorem for finite groups is not true in general in case of finite semigroups.

For there are some finite semigroups for which the order of the subsemigroup will divide the order of the semigroup and in the same semigroup there are subsemigroups whose order does not in general divide the order of the semigroup. So in this thesis two new properties are defined viz; anti Lagrange's property and weak Lagrange's property. In fact semigroups of prime order have subsemigroups but this class of semigroups satisfy anti Lagrange's property. For instance $S = \{Z_p, \times, p$ a prime$\}$ is a class of semigroups which satisfy anti Lagrange's property.

Further all symmetric semigroups satisfy both the anti Lagrange's property as well as weak Lagrange's property.

Next natural study would be analyzing Cauchy property in case of finite semigroups.

Again the class of semigroups, $S = \{Z_p, \times\}$, p a prime do not satisfy Cauchy property. However $S(n)$ has elements which satisfy Cauchy property as well as elements which do not satisfy Cauchy property. These are characterized in this thesis.

Finally an attempt is made to embed all semigroups in the symmetric semigroup $S(n)$, which happens to be an impossibility. However to overcome all this extended Cayley's theorem was defined and a class of idempotent semigroups (semilattice) happen to satisfy the notion of extended Cayley's theorem. Several interesting results are obtained in the course of this analysis.

Finally the relevance of Sylow theorems was pondered.

Certainly, pseudo p-Sylow subsemigroups in general are not conjugate. Further the concept of partition of a finite semigroup by cosets or double cosets happens to be an impossibility.

However when the subsemigroups are ideals these notions are interesting leading to innovative results. These new results are derived. Thus chapter three happens to be the back bone this thesis, finite semigroups for the first time are analysed for these classical theorems on finite groups. This has led to new definitions and new characterization of finite semigroups. Apart from this only in this chapter special elements like S-units, S-idempotents etc. are introduced for the first time in semigroups. Condition for semigroups to contain these special elements is obtained in this thesis.

Since semirings are nothing but two semigroups on the same set with two distinct binary operations, the two operations connected by the distributive law. The study of semigroup semirings and group semirings has become mandatory. Throughout this thesis finite distributive lattices are taken as semirings of finite order. This includes chain lattices, Boolean algebras and other distributive lattices which are not Boolean algebras or chain lattices. Semigroup semirings using lattices as semirings are studied for the first time elaborately. Conditions for these semigroup semirings to contain ideals, subsemirings are obtained.

Also conditions for these semigroup semirings to be semifields or semidivision rings are obtained. However the presence of idempotents in the semigroup semirings in general do not guarantee the presence of zero divisors which is a marked difference between the semirings and rings.

Finally condition for S-units, S-zero divisors and S-idempotents to be present in a semigroup semiring are characterized.

The chapter on group semirings using finite groups and distributive lattices as semirings is systematically carried out for the first time in this thesis.

Here also presence of idempotents do not guarantee the presence of zero divisors. Further zero divisors are present in group semirings if and only if the distributive lattices (semirings) contain zero divisors.

However every group semiring has idempotents where the group is of finite order. Several results in this direction are obtained. For study of group rings is done systematically but group semirings that too using distributive lattices are meager [100]. Hence the study is important and relevant. Characterization of group semirings to contain S-anti zero divisors, S-idempotents and S-zero divisors are obtained.

A few problems are given for future study for this thesis could not find solutions for them.

The problems for future study is listed below.

1. Can $S = \{Z_n, \times\}$; $n = p^t$, p a prime satisfy Cauchy property?

2. Find filters in the semigroup semirings.

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