MULTI-CRITERIA DECISION MAKING METHOD BASED ON SIMILARITY MEASURES UNDER SINGLE VALUED NEUTROSOPHIC REFINED AND INTERVAL NEUTROSOPHIC REFINED ENVIRONMENTS

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ABSTRACT. In this paper, we propose three similarity measure methods for single valued neutrosophic refined sets and interval neutrosophic refined sets based on Jaccard, Dice and Cosine similarity measures of single valued neutrosophic sets and interval neutrosophic sets. Furthermore, we suggest two multi-criteria decision making method under single valued neutrosophic refined environment and interval neutrosophic refined environment, and give applications of proposed multi-criteria decision making methods. Finally we suggested a consistency analysis method for similarity measures between interval neutrosophic refined sets and give an application to demonstrate process of the method.

1. INTRODUCTION

To overcome situations containing uncertainty and inconsistency data has been very important matter for researchers that study on mathematical modeling and decision making which is very important in some areas such as operations research, social economics, and management science, etc. From past to present many studies on mathematical modeling have been performed. Some of well-known approximations are fuzzy set (FS) theory proposed by Zadeh [24], intuitionistic fuzzy set (IFS) theory introduced by Atanassov [1] and interval valued intuitionistic fuzzy set theory suggested by Atanassov and Gargov [2]. A FS is identified by its membership function, IFS which is a generalization of the FSs is characterized by membership and nonmembership functions. Even though these set theories are very successful to model some decision making problems containing uncertainty and incomplete information, but they may not suffice to model indeterminate and inconsistent information encountered in real world. Therefore, Smarandache [19] introduced the concept of neutrosophic set which is very useful to model problems containing indeterminate and inconsistent information based on neutrosophy which is a branch of philosophy. A neutrosophic set is characterized by three functions called truth-membership function $(T(x))$, indeterminacy-membership function $(I(x))$ and falsity membership function $(F(x))$. These functions are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$, i.e., $T(x) : X \to]^{-}0, 1^{+}[, I(x) : X \to]^{-}0, 1^{+}[$, and $F(x) : X \to]^{-}0, 1^{+}[$. Basic of the neutrosophic set stands up to the non-standard analysis given by Abraham Robinson in 1960s [17]. Smarandache [20] discussed comparisons between neutrosophic set, paraconsistent set and intuitionistic fuzzy set and he shown that the neutrosophic set is a generalization of paraconsistent set and intuitionistic fuzzy set. In some areas such as engineering and real scientific fields, modeling of problems by using real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ may not be easy sometimes, to overcome this issue concepts of single valued neutrosophic set (SVN-set) and interval neutrosophic set (IN-set) were defined by Wang et al. in [22] and [23], respectively. Zhang et al. [25] presented an application of IN-set in multi criteria decision making problems. Some novel operations on interval neutrosophic sets were defined by Broumi and Smarandache [8]. Bhowmik and Pal [6] defined concept of intuitionistic neutrosophic set by combining intuitionistic fuzzy set and neutrosophic set, and gave some set theoretical operations of the intuitionistic neutrosophic set such as complement, union and intersection. Ansari et al. [3] gave an application of neutrosophic set theory to medical AI. Ye [34] proposed concept of trapezoidal neutrosophic set by combining trapezoidal fuzzy set with single valued neutrosophic set. He also presented some operational rules related to this novel sets and proposed score and accuracy function for trapezoidal neutrosophic numbers.

Set theories mentioned above are based on idea which each element of a set appear only one time in the set. However, in some situations, a structure containing repeated elements may be need. For instance, while search in a dad name-number of children-occupation relational basis. To model such cases, a structure called bags

Key words and phrases. Neutrosophic set, single valued neutrosophic set, interval neutrosophic set, neutrosophic refined set, single valued neutrosophic refined set interval neutrosophic refined set, similarity measure, decision making.

was defined by Yager [26]. In 1998, Baowen [5] defined concepts of fuzzy bags and their operations based on Peizhuang's theory of set-valued statistics [16] and Yager's bags theory [26]. Concept of intuitionistic fuzzy bags (multi set) and its operations were defined by Shinoj and Sunil [18], and they gave an application in medical diagnosis under intuitionistic fuzzy multi environment.

In 2013, Smarandache [21] put forward n-symbol or numerical valued neutrosophic logic which is as generalization of n-symbol or numerical valued logic that is most general case of 2-valued Boolean logic, Kleene's and Lukasiewicz' 3-symbol valued logics and Belnap's 4-symbol valued logic. Although existing set theoretical approximations are generally successful in order to model some problems encountered in real world, in some cases they may not allow modeling of problems. For example, when elements in a set are evaluated by SVN-values in different times as $t_1, t_2, ..., t_p$, SVN-set may not be sufficient in order to express such a case. Therefore, Ye and Ye [29] defined concept of single valued neutrosophic multiset (refined) (SVNR-set) as a generalization of single valued neutrosophic sets, and gave operational rules for proposed novel set. In a SVNR-set, each of truth membership values, indeterminacy membership values and falsity membership values are expressed sequences called truth membership sequence, indeterminacy membership sequence and falsity membership sequence, respectively. SVNR-set allows modeling of problems containing changing values with respect to times under SVN-environment. In this regard, SVNR-set is an important tool to model some problems. Bromi et al. [11] proposed concept of n-valued interval neutrosophic set and set theoretical operations on n-valued interval neutrosophic set (or interval neutrosophic set) such as union, intersection, addition, multiplication, scalar multiplication, scalar division, truth-favorite. Also they developed a multi-criteria group decision making method and gave its an application in medical diagnosis.

Similarity measure has an important role many areas such as medical diagnosis, pattern recognition, clustering analysis, decision making and so on. There are many studies on similarity measures of neutrosophic sets and IN-sets. For example, Broumi and Smarandache [7] developed some similarity measure methods between two neutrosophic sets based on Hausdorff distances and used these methods to calculate similarity degree between two neutrosophic sets. Ye [28] proposed three similarity measure methods used simplified neutrosophic sets (SN-sets) which is a subclass of neutrosophic set that is more useful than neutrosophic set some applications in engineering and real sciences. He also applied the these methods to decision making problem under SN-environment. Ye and Zhang [27] suggested similarity measure between SVN-sets based on minimum and maximum operators. They also developed a multi-attribute decision making method based on weighted similarity measure of SVN-sets, and gave applications to demonstrate effectiveness of the proposed methods. Ye [31] proposed two similarity measures between SVN-sets by defining a generalized distance measure, and presented a clustering algorithm based on proposed similarity measure. In 2015, Ye [34] pointed out some drawbacks of similarity measures given in [28] and proposed improved cosine similarity measures of simplified neutrosophic sets (SN-sets) based on cosine function. Moreover, he defined weighted cosine similarity measures of SN-sets and gave an application in medical diagnosis problem containing SN-information. Ye and Fub [32] proposed a similarity measure of SVN-sets based on tangent function and put forward a medical diagnosis method called multi-period medical diagnosis method based on suggested similarity measure and weighted aggregation of multi-period information. They also gave a comparison tangent similarity measures of SVN-sets with existing similarity measures of SVN-sets. Furthermore, Ye [33] introduced a similarity measure of SVN-sets based on cotangent function and gave an application in the fault diagnosis of steam turbine, and he gave comparative analysis between cosine similarity measure and cotangent similarity measure in the fault diagnosis of steam turbine. Majumdar and Samanta [13] defined notion of distance between two SVN-sets and investigated its some properties. They also put forward the a measure of entropy for a SVN-set. Aydoğdu [4] introduced a similarity measure between two SVN-sets and developed an entropy of SVN-sets. Bromi and Smarandache [10] extended similarity measures proposed in [31] to IN-sets. Ye [30] proposed a similarity measure between two IN-sets based on Hamming and Euclidian distances and gave a multi-criteria decision making method.

Similarity measure on the NR-sets was studied Bromi and Smarandache [9]. Bromi and Smarandache extended improved cosine similarity measure of SVN-sets to NR-sets and gave its an application in medical diagnosis. Mondal and Pramanik [14] introduced cotangent similarity measure of NR-sets and studied on its properties, and applied cotangent similarity measure to educational stream selection. Also they proposed a similarity measure method [15] for NR-sets based on tangent function and gave an application in multi-attribute decision making. In 2015. Bromi and Smarandache [10] presented a new similarity measure method by extending the Hausdorff distance to NR-sets, and gave an application of proposed method in medical diagnosis.

In this paper, we propose three similarity measure methods for single valued refined sets (SVNR-sets) and interval neutrosophic refined sets (INR-sets) by extending Jaccard, Dice and Cosine similarity measures under SVN-value and IN-value given by Ye in [28]. Also we give two multi-criteria decision making methods by defining ideal solutions for best and cost criteria under SVNR-environment and INR-environment. Furthermore, to determine which similarity measure under INR-environment is more appropriate for considered problems, we give a consistency analysis method based on developed similarity measure methods. To demonstrate processes of the similarity measure methods and consistency analysis method, we present real examples based on criteria and attributes given in [28]. The rest of the article is organized as follows. In section 2 some concepts related to the SVN-sets and IN-sets and formulas Jaccard, Dice and Cosine similarity measures under SVN-environment are given. In section 3 for SVNR-set and INR-sets similarity measures methods are developed as an extension of vector similarity measures between SVN-sets and between IN-sets given in [28]. In section 4 multicritera decision making methods are developed under SVNR-environment and INR-environment, and given examples related to the developed methods. In section 5 for similarity measures between two INR-sets, a consistency analysis method is suggested and an application of this method is given. In section 6 conclusions of the paper and studies that can be made in future are presented.

2. Preliminary

In this section, concepts of SVN-set, IN-set, SVNR-set and INR-set and some set theoretical operations of them are presented required in subsequent sections.

Throughout the paper, X denotes initial universe, E is a set of parameters and $I_p = \{1, 2, ...p\}$ is an index set.

Definition 1. [23] *Let X be a nonempty set (initial universe), with a generic element in X denoted by x. A single-valued neutrosophic set (SVN-set) A is characterized by a truth membership function $t_A(x)$, an indeterminacy membership function $i_A(x)$, and a falsity membership function $f_A(x)$ such that $t_A(x)$, $i_A(x)$, $f_A(x) \in [0, 1]$ for all $x \in X$, as follows:*

When X is continuous, a SVN-sets A can be written as follows:

$$A = \int_X \langle t_A(x), i_A(x), f_A(x)\rangle /x, \ \ \text{for all } x \ \in X.$$

If X is crisp set, a SVN-set A can be written as follows:

$$A = \sum_x \langle t_A(x), i_A(x), f_A(x)\rangle /x, \ \ \text{for all } x \ \in X.$$

Also, finite SVN-set A can be presented as follows:
$A = \{\langle x_1, t_A(x_1), i_A(x_1), f_A(x_1)\rangle, \dots, \langle x_M, t_A(x_M), i_A(x_M), f_A(x_M)\rangle\}$ for all $x \in X$.

Here $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$ for all $x \in X$.

Throughout this paper, initial universe will be considered as a finite and crisp set.

From now on set of all SVN-sets over X will be denoted by SVN_X.

Definition 2. [23] *Let $A, B \in SVN_X$. Then,*

1. *$A \subseteq B$ if and only if $t_A(x) \leq t_B(x),\ i_A(x) \geq i_B(x),\ f_A(x) \geq f_B(x)$ for any $x \in X$.*
2. *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$ for all $x \in X$.*
3. *$A^c = \{\langle x, f_A(x), 1 - i_A(x), t_A(x)\rangle : x \in X\}$.*
4. *$A \cup B = \{\langle x, (t_A(x) \vee t_B(x)), (i_A(x) \wedge i_B(x)), (f_A(x) \wedge f_B(x))\rangle : x \in X\}$*
5. *$A \cap B = \{\langle x, (t_A(x) \wedge t_B(x)), (i_A(x) \vee i_B(x)), (f_A(x) \vee f_B(x))\rangle : x \in X\}$.*

Definition 3. [29] *Let X be a nonempty set with generic elements in X denoted by x. A single valued neutrosophic refined set (SVNR-set) $\tilde{A}$ is defined as follows:*

$$\tilde{A} = \Big\{\Big\langle x, (t^1_A(x), t^2_A(x), ..., t^p_A(x)), (i^1_A(x), i^2_A(x), ..., i^p_A(x)), (f^1_A(x), f^2_A(x), ..., f^p_A(x))\Big\rangle : x \in X\Big\}.$$

Here, $t^1_A, t^2_A, ..., t^p_A : X \to [0, 1]$, $i^1_A, i^2_A, ..., i^p_A : X \to [0, 1]$ and $f^1_A, f^2_A, ..., f^p_A : X \to [0, 1]$ such that $0 \leq t^i_A(x) + i^i_A(x) + f^i_A(x) \leq 3$ for all $x \in X$ and $i \in I_p$. $(t^1_A(x), t^2_A(x), ..., t^p_A(x))$, $(i^1_A(x), i^2_A(x), ..., i^p_A(x))$ and

$(f_A^1(x), f_A^2(x), ..., f_A^p(x))$ are the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x. These sequences may be in decreasing or increasing order. Also p is called the dimension of single valued neutrosophic refined set $\tilde{A}$.

A $SVNR$-set A can be represented as follows:

$$\tilde{A} = \Big\{ \big\langle x, t_A^i(x), i_A^i(x), f_A^i(x) \big\rangle : x \in X, i \in I_p \Big\}.$$

From now on, set of all single valued neutrosophic refined sets over X will be denoted by $SVNR_X$ and considered SVNR-sets will be accepted as p dimension SVNR-set.

Definition 4. [29] *Let $\tilde{A}, \tilde{B} \in SVNR_X$. Then,*

(1) If $t_A^i(x) \leq t_{\tilde{B}}^i(x)$, $i_A^i(x) \geq i_{\tilde{B}}^i(x)$, $f_A^i(x) \geq f_{\tilde{B}}^i(x)$ for all $i \in I_p$ and $x \in X$, then $\tilde{A}$ is said to be SVNR-subset of $\tilde{B}$ and denoted by $\tilde{A} \subseteq \tilde{B}$.

(2) $\tilde{A} \tilde{\subseteq} \tilde{B}$ and $\tilde{B} \tilde{\subseteq} \tilde{A}$ if and only if $\tilde{A} = \tilde{B}$;

(3) The complement of $\tilde{A}$, denoted by $\tilde{A}^c$, is define as follows:

$$\tilde{A} = \Big\{ \big\langle x, f_A^i(x), 1 - i_A^i(x), t_A^i(x) \big\rangle : x \in X, i \in I_p \Big\}.$$

Definition 5. [12] *Let $\tilde{A} \in SVNR_X$. Then,*

(1) if $t_A^i(x) = 0$, $i_A^i(x) = 1$ and $f_A^i(x) = 1$ for all $i \in I_p$ and $x \in X$, $\tilde{A}$ is called a null SVNR-set, and denoted by $\tilde{\Phi}$,

(2) if $t_A^i(x) = 1$, $i_A^i(x) = 0$ and $f_A^i(x) = 0$ for all $i \in I_p$ and $x \in X$, $\tilde{A}$ is called universal SVNR-set, and denoted by $\tilde{X}$.

Definition 6. [29] *Let $\tilde{A}, \tilde{B} \in SVNR_X$. Then,*

(1) union:

$$\tilde{A} \tilde{\cup} \tilde{B} = \Big\{ \big\langle x, t_A^i(x) \vee t_B^i(x), i_A^i(x) \wedge i_B^i(x), f_A^i(x) \wedge f_B^i(x) \big\rangle : x \in X, i \in I_p \Big\},$$

(2) intersection:

$$\tilde{A} \tilde{\cap} \tilde{B} = \Big\{ \big\langle x, t_A^i(x) \wedge t_B^i(x), i_A^i(x) \vee i_B^i(x), f_A^i(x) \vee f_B^i(x) \big\rangle : x \in X, i \in I_p \Big\}.$$

Example 1. *Consider SVNR-sets $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are given as follows:*

$$\tilde{A} = \left\{ \begin{array}{c} \Big\langle x_1(.1,.2,.4),(.1,.4,.6),(.0,.3,.3) \Big\rangle, \\ \Big\langle x_2,(.3,.3,.5),(.2,.3,.7),(.1,.5,.6) \Big\rangle, \\ \Big\langle x_3,(.2,.4,.8),(.1,.3,.3),(.5,.6,.9) \Big\rangle \end{array} \right\}, \quad \tilde{B} = \left\{ \begin{array}{c} \Big\langle x_1(.5,.6,.7),(.4,.6,.7),(.3,.3,.4) \Big\rangle, \\ \Big\langle x_2,(.2,.4,.4),(.2,.5,.8),(.2,.6,.7) \Big\rangle, \\ \Big\langle x_3,(.1,.6,.6),(.1,.5,.5),(.3,.4,.7) \Big\rangle \end{array} \right\}$$

and

$$\tilde{C} = \left\{ \begin{array}{c} \Big\langle x_1(.3,.3,.5),(.4,.5,.6),(.1,.3,.4) \Big\rangle, \\ \Big\langle x_2,(.0,.1,.3),(.2,.3,.6),(.1,.4,.6) \Big\rangle, \\ \Big\langle x_3,(.1,.4,.7),(.1,.3,.4),(.3,.3,.5) \Big\rangle \end{array} \right\}.$$

Then, $$\tilde{A} \tilde{\cup} \tilde{B} = \left\{ \begin{array}{c} \Big\langle x_1(.5,.6,.7),(.4,.6,.7),(.3,.3,.4) \Big\rangle, \\ \Big\langle x_2,(.2,.3,.4),(.2,.3,.7),(.1,.5,.6) \Big\rangle, \\ \Big\langle x_3,(.1,.4,.6),(.1,.3,.3),(.3,.4,.7) \Big\rangle \end{array} \right\}, \quad \tilde{A} \tilde{\cap} \tilde{B} = \left\{ \begin{array}{c} \Big\langle x_1(.1,.2,.4),(.1,.4,.6),(.0,.3,.3) \Big\rangle, \\ \Big\langle x_2,(.3,.4,.5),(.2,.5,.8),(.2,.6,.7) \Big\rangle, \\ \Big\langle x_3,(.2,.6,.8),(.1,.5,.7),(.5,.6,.9) \Big\rangle \end{array} \right\},$$

and $\tilde{C} \tilde{\subseteq} \tilde{B}$.

Definition 7. [22] *Let $D[0,1]$ be the set of all closed sub-intervals of the interval $[0,1]$ and X be an ordinary finite non-empty set. An IN-set $\hat{A}$ over X is set of quadruple given as follows:*

$$\hat{A} = \{ \langle x, t_A(x), i_A(x), f_A(x) \rangle \,|x \in X \},$$

where, $t_A(x) \in D[0,1]$, $i_A(x) \in D[0,1]$, and $f_A(x) \in D[0,1]$ with the relation

$$0 \leq sup\, t_A(x) + sup\, i_A(x) + sup\, f_A(x) \leq 3, \ for\ all\, x \in X.$$

Here intervals $t_A(x)=\left[t_A^L(x), t_A^U(x)\right] \subset [0,1]$, $i_A(x)=\left[i_A^L(x), i_A^U(x)\right] \subset [0,1]$, $f_A(x)=\left[f_A^L(x), f_A^U(x)\right] \subset [0,1]$ *denote, respectively the degree of truth, indeterminacy, and falsity membership of* $x \in X$ *in* $\tilde{A}$*; moreover* $t_A^L(x)= inf t_A(x)$, $t_A^U(x)= sup t_A(x)$, $i_A^L(x)= inf i_A(x)$, $i_A^U(x)= sup i_A(x)$, $f_A^L(x)= inf f_A(x)$, $f_A^U(x)= sup f_A(x)$ *for every* $x \in X$*. Thus, the interval neutrosophic set* $\hat{A}$ *can be expressed in the following interval format:*

$$\hat{A} = \left\{\left\langle x, \left[t_A^L(x), t_A^U(x)\right] \left[i_A^L(x), i_A^U(x)\right] \left[f_A^L(x), f_A^U(x)\right]\right\rangle | x \in X\right\}$$

where, $0 \leq sup\, t_A^U(x) + sup\, i_A^U(x) + sup\, f_A^U(x) \leq 3$, $T_A^L(x) \geq 0$, $I_A^L(x) \geq 0$ *and* $F_A^L(x) \geq 0$ *for all* $x \in X$*. Henceforth set of all* IN*-sets over* X *will be denoted by* IN_X*.*

Definition 8. [11] *Let* X *be a nonempty initial universe whose elements are discrete. An n-valued interval neutrosophic refined set (or interval neutrosophic refined set)* $\ddot{A}$ *is defined as follows:*

$$\ddot{A} \;=\; \left\{\left\langle x, ([{t^i}_A^L(x), {t^i}_A^U(x)]), ([{i^i}_A^L(x), {i^i}_A^U(x)]), ([{f^i}_A^L(x), {f^i}_A^U(x)])\right\rangle : x \in X \text{ and } i \in I_p\right\},$$

where ${t^i}_A^L(x) \leq {t^i}_A^U(x), {i^i}_A^L(x) \leq {i^i}_A^U(x)$ *and* ${f^i}_A^L(x) \leq {f^i}_A^U(x)$ *for all* $x \in X$*.*

Here, $0 \leq {t^i}_A^L(x) + {i^i}_A^L(x) + {f^i}_A^L(x) \leq 3$ *and* $0 \leq {t^i}_A^U(x) + {i^i}_A^U(x) + {f^i}_A^U(x) \leq 3$ *for all* $x \in X$ *and* $i \in I_p$*.*

$$([{t^1}_A^L(x), {t^1}_A^U(x)], [{t^2}_A^L(x), {t^2}_A^U(x)], ..., [{t^p}_A^L(x), {t^p}_A^U(x)]),$$

$$([{i^1}_A^L(x), {i^1}_A^U(x)], [{i^2}_A^L(x), {i^2}_A^U(x)], ..., [{i^p}_A^L(x), {i^p}_A^U(x)])$$

and

$$([{f^1}_A^L(x), {f^1}_A^U(x)], [{f^2}_A^L(x), {f^2}_A^U(x)], ..., [{f^p}_A^L(x), {f^p}_A^U(x)])$$

are the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x*, respectively.* p *is called the dimension of n-valued neutrosophic sets* $\ddot{A}$*.*

Henceforth, considered INR-sets will be accepted p dimension n-valued interval neutrosophic set, and set of all interval neutrosophic refined sets over X will be denoted by INR_X. Also notion of interval neutrosophic refined set (INR-set) will be used instead of notion of n-valued interval neutrosophic set.

2.1. **Similarity measures of SVN-sets and IN-sets.** Jaccard, Dice, and Cosine similarity measures are given between two $SVN-$sets and between two $IN-$sets defined in [28].

Definition 9. *Let* A *and* B *be two SVN-sets in a universe of discourse* X $=\{x_1, x_2, \ldots, x_n\}$*. Then the Jaccard similarity measure between SVN-sets A and B in the vector space is defined as follows:*

$$(A,B)_J = \frac{1}{n}\sum_{i=1}^{n} \frac{t_A(x_i)t_B(x_i) + i_A(x_i)i_B(x_i) + f_A(x_i)f_B(x_i)}{\left(\begin{array}{c} (t_A^2(x_i) + i_A^2(x_i) + f_A^2(x_i)) + (t_B^2(x_i) + i_B^2(x_i) + f_B^2(x_i)) \\ -(t_A(x_i)t_B(x_i) + i_A(x_i)i_B(x_i) + f_A(x_i)f_B(x_i) \end{array}\right)} \tag{2.1}$$

Definition 10. *Let* A *and* B *be two SVN-sets on a universe* X $=\{x_1, x_2, \ldots, x_n\}$*. Then the Dice similarity measure between SVN-sets A and B in the vector space is defined as follows:*

$$(A,B)_D = \frac{1}{n}\sum_{i=1}^{n} \frac{2\,(t_A(x_i)t_B(x_i) + i_A(x_i)i_B(x_i) + f_A(x_i)f_B(x_i))}{[(t_A^2(x_i) + i_A^2(x_i) + f_A^2(x_i)) + (t_B^2(x_i) + i_B^2(x_i) + f_B^2(x_i))]}. \tag{2.2}$$

Definition 11. *Let* A *and* B *be two SVN-sets in a universe of discourse* X $=\{x_1, x_2, \ldots, x_n\}$*. Then the cosine similarity measure between SVN-sets A and B in the vector space is defined as follows:*

$$(A,B)_C = \frac{1}{n}\sum_{i=1}^{n} \frac{(t_A(x_i)t_B(x_i) + i_A(x_i)i_B(x_i) + f_A(x_i)f_B(x_i))}{\left[\sqrt{(t_A^2(x_i) + i_A^2(x_i) + f_A^2(x_i))}.\sqrt{(t_B^2(x_i) + i_B^2(x_i) + f_B^2(x_i))}\right]}. \tag{2.3}$$

In some applications, each element $x_i \in X$ may have different weights. Let $w_1, w_2, ..., w_n$ be the weights of elements $x_1, x_2, ..., x_n \in X$ such that $w_j \geq 0 (\forall j \in I_n)$ and $\sum_{j=1}^{n} w_j = 1$, respectively. Then, formulas of Jaccard, Dice and Cosine similarity measures between A and B can be extended to weighted Jaccard, Dice and Cosine similarity measures are defined as follows: defined as follows:

$$(2.4)\qquad W(A,B)_J=\sum_{i=1}^{n} w_i \frac{t_A(x_i)t_B(x_i)+i_A(x_i)i_B(x_i)+F_A(x_i)F_B(x_i)}{\begin{pmatrix} (t_A^2(x_i)+i_A^2(x_i)+f_A^2(x_i))+(t_B^2(x_i)+i_B^2(x_i)+f_B^2(x_i)) \\ -(t_A(x_i)t_B(x_i)+i_A(x_i)i_B(x_i)+f_A(x_i)f_B(x_i)) \end{pmatrix}},$$

$$(2.5)\qquad W(A,B)_D=\sum_{i=1}^{n} w_i \frac{2\left(t_A(x_i)t_B(x_i)+t_A(x_i)t_B(x_i)+f_A(x_i)f_B(x_i)\right)}{((t_A^2(x_i)+i_A^2(x_i)+f_A^2(x_i))+(t_B^2(x_i)+i_B^2(x_i)+f_B^2(x_i))]}$$

and

$$(2.6)\qquad W(A,B)_C=\sum_{i=1}^{n} w_i \frac{(t_A(x_i)t_B(x_i)+i_A(x_i)i_B(x_i)+f_A(x_i)f_B(x_i))}{\left(\sqrt{(t_A^2(x_i)+i_A^2(x_i)+f_A^2(x_i))}.\sqrt{(t_B^2(x_i)+i_B^2(x_i)+f_B^2(x_i))}\right]},$$

respectively.

3. Similarity measures under SVNR and INR-environments

In this section, similarity measures between two SVNR-sets and between two INR-sets are defined based on similarity measures between two SVN-sets and similarity measures between two IN-sets given in [28].

Definition 12. *Let $\tilde{A},\tilde{B}\in SVNR_X$. Then, the Jaccard similarity measure between SVNR-sets $\tilde{A}$ and $\tilde{B}$ is defined as follows:*

$$(3.1)\qquad (\tilde{A},\tilde{B})_J=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{\begin{pmatrix} ([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)+([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2) \\ -[t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j)] \end{pmatrix}},$$

Definition 13. *Let $\tilde{A},\tilde{B}\in SVNR_X$. Then the Dice similarity measure between SVNR-sets $\tilde{A}$ and $\tilde{B}$ is defined as follows:*

$$(3.2)\qquad (\tilde{A},\tilde{B})_D=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{2(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{\left([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)+([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2\right)}.$$

Definition 14. *Let $\tilde{A},\tilde{B}\in SVNR_X$. Then, the cosine similarity measure between SVNR-sets $\tilde{A}$ and $\tilde{B}$ is defined as follows:*

$$(3.3)\qquad (\tilde{A},\tilde{B})_C=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{\left(\sqrt{([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)}.\sqrt{([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2)}\right)}.$$

If $w_j\in[0,1]$ be the weight of each element x_j for $j=1,2,\ldots,n$ such that $\sum_{j=1}^{n}w_j=1$, then the weighted Jaccard, Dice and Cosine similarity measures between SVNR-sets $\tilde{A}$ and $\tilde{B}$ are defined as follows:

$$(3.4)\qquad W(\tilde{A},\tilde{B})_J=\sum_{j=1}^{n}\sum_{i=1}^{p}w_j\frac{(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{\begin{pmatrix} ([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)+([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2) \\ -[t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j)] \end{pmatrix}},$$

$$(3.5)\qquad W(\tilde{A},\tilde{B})_D=\sum_{j=1}^{n}\sum_{i=1}^{p}w_j\frac{2(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{(([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)+([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2))}$$

and

$$(3.6)\qquad W(\tilde{A},\tilde{B})_C=\sum_{j=1}^{n}\sum_{i=1}^{p}w_j\frac{(t_A^i(x_j)t_B^i(x_j)+i_A^i(x_j)i_B^i(x_j)+f_A^i(x_j)f_B^i(x_j))}{\left(\sqrt{([t_A^i(x_j)]^2+[i_A^i(x_j)]^2+[f_A^i(x_j)]^2)}.\sqrt{([t_B^i(x_j)]^2+[i_B^i(x_j)]^2+[f_B^i(x_j)]^2)}\right)},$$

respectively.

TABLE 1. Similarity measure under SVNR-environment

Similarity measures	*Values*
$(\tilde{A},\tilde{B})_J$	0.834
$(\tilde{A},\tilde{B})_D$	0.908
$(\tilde{A},\tilde{B})_C$	0.928

TABLE 2. Weighted similarity measures under SVNR-environment

Similarity measure	*Values*
$W(\tilde{A},\tilde{B})_J$	0.786
$W(\tilde{A},\tilde{B})_D$	0.879
$W(\tilde{A},\tilde{B})_C$	0.429

Example 2. *Consider SVNR-sets $\tilde{A}$ and $\tilde{B}$ given in Example 1. Then, by using Eqs.(3.1),(3.2) and (3.3), similarity measures between SVNR-sets $\tilde{A}$ and $\tilde{B}$ are obtained as in Table 1.*

If weights of the x_1, x_2 and x_3 are taken as $w_1 = .7$ $w_2 = .2$ and $w_3 = .1$, respectively. Then, by using Eqs.(3.4),(3.5) and (3.6), weighted similarity measures are obtained as in Table 3:

Proposition 1. *Let $\tilde{A}$ and $\tilde{B}$ be two $SVNR$-sets. Then, each similarity measure $(\tilde{A},\tilde{B})_\Lambda(\Lambda = J, D, C)$ satisfies the following properties:*

(1) $0 \le (\tilde{A},\tilde{B})_\Lambda \le 1$
(2) $(\tilde{A},\tilde{B})_\Lambda = (\tilde{B},\tilde{A})_\Lambda$;
(3) $(\tilde{A},\tilde{B})_\Lambda = 1$ *if* $\tilde{B} = \tilde{A}$ *i.e.* ${t^i}_A(x_i) = {t^i}_B(x_i)$, ${i^i}_A(x_i) = {i^i}_B(x_i)$, *and* ${f^i}_A(x_i) = {f^i}_B(x_i)$ *for every* $x_j \in X$ *and* $i \in I_p$.

Proof. (1) For $p = 1$ Eq. (3.1), (3.2) and (3.3) are reduce to Eq. (2.1), (2.2) and (2.3), respectively. For all $i \in I_p(p > 1)$ according to inequality $x^2 + y^2 \ge 2xy$, for any $x_j \in X$ we know that

$$\sum_{i=1}^{p}([t^i_A(x_j)]^2 + [t^i_B(x_j)]^2) \ge 2\sum_{i=1}^{p}([t^i_A(x_j)].[t^i_B(x_j)]),$$

$$\sum_{i=1}^{p}([i^i_A(x_j)]^2 + [i^i_B(x_j)^2]) \ge 2\sum_{i=1}^{p}([i^i_A(x_j)].[i^i_B(x_j)])$$

$$\sum_{i=1}^{p}([f^i_A(x_j)]^2 + [f^i_B(x_j)]^2) \ge 2\sum_{i=1}^{p}([f^i_A(x_j)].[f^i_B(x_j)]),$$

and
$\sum_{i=1}^{p}([t^i_A(x_j)]^2+[t^i_B(x_j)]^2+[i^i_A(x_j)]^2+[i^i_B(x_j)]^2+[f^i_A(x_j)]^2+[f^i_B(x_j)]^2) \ge 2\sum_{i=1}^{p}([t^i_A(x_j)].[t^i_B(x_j)]+[i^i_A(x_j)].[i^i_B(x_j)]+[f^i_A(x_j)].[f^i_B(x_j)])$. Thus,

$$\sum_{i=1}^{p}\frac{(t^i_A(x_j)t^i_B(x_j) + i^i_A(x_j)i^i_B(x_j) + f^i_A(x_j)f^i_B(x_j))}{\begin{pmatrix}([t^i_A(x_j)]^2 + [i^i_A(x_j)]^2 + [f^i_A(x_j)]^2) + ([t^i_B(x_j)]^2 + [i^i_B(x_j)]^2 + [f^i_B(x_j)]^2) \\ -[t^i_A(x_j)t^i_B(x_j) + i^i_A(x_j)i^i_B(x_j) + f^i_A(x_j)f^i_B(x_j)]\end{pmatrix}} \le 1. \tag{3.7}$$

and for all $x_j \in X$

$$\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{(t^i_A(x_j)t^i_B(x_j) + i^i_A(x_j)i^i_B(x_j) + f^i_A(x_j)f^i_B(x_j))}{\begin{pmatrix}([t^i_A(x_j)]^2 + [i^i_A(x_j)]^2 + [f^i_A(x_j)]^2) + ([t^i_B(x_j)]^2 + [i^i_B(x_j)]^2 + [f^i_B(x_j)]^2) \\ -[t^i_A(x_j)t^i_B(x_j) + i^i_A(x_j)i^i_B(x_j) + f^i_A(x_j)f^i_B(x_j)]\end{pmatrix}} \le n. \tag{3.8}$$

Similarly, Eq.(3.2) and Eq. (3.3) are true.

(2) The proof is clear.

(3) Let $A = B$. Then, $t^i{}_A(x_j) = t^i{}_B(x_j)$, $i^i{}_A(x_j) = i^i{}_B(x_j)$, and $f^i{}_A(x_j) = f^i{}_B(x_j)$ for all $x_j \in X$ and $i \in I_p$ and

$$\begin{aligned}(\tilde{A},\tilde{B})_J &= \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{[t^i_A(x_j]^2+[i^i_A(x_j)]^2+[f^i_A(x_j)]^2}{(2[t^i_A(x_j)]^2+2[i^i_A(x_j)]^2+2[f^i_A(x_j)]^2-([t^i_A(x_j)]^2+[i^i_A(x_j)]^2+[f^i_A(x_j)]^2))}\\ &= \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n}1=1.\end{aligned}$$

For Dice and Cosine similarity measures, the proofs of can be made with similar way. ∎

Each similarity measure between two SVNR-sets $\tilde{A} = \left\{\langle x, t^i_A(x), i^i_A(x), f^i_A(x)\rangle : x \in X, i \in I_p\right\}$ and $\tilde{B} = \left\{\langle x, t^i_B(x), i^i_B(x), f^i_B(x)\rangle : x \in X, i \in I_p\right\}$ are undefined when $t^i{}_A(x) = i^i{}_A(x) = f^i{}_A(x) = 0$ and $t^i{}_B(x) = i^i{}_B(x) = f^i{}_B(x) = 0$ for all $x \in X$ and $i \in I_p$.

Proposition 2. *Let $\tilde{A}, \tilde{B} \in SVNR_X$. Then, each weighted similarity measure $W(\tilde{A},\tilde{B})_\Lambda (\Lambda = J, D, C)$ satisfies the following properties:*

(1) $0 \le W(\tilde{A},\tilde{B})_\Lambda \le 1$,
(2) $W(\tilde{A},\tilde{B})_\Lambda = W(\tilde{B},\tilde{A})_\Lambda$,
(3) $W(\tilde{A},\tilde{B})_\Lambda = 1$ *if* $\tilde{B} = \tilde{A}$ *i.e.* $t^i{}_A(x_j) = t^i{}_B(x_j)$, $i^i{}_A(x_j) = i^i{}_B(x_j)$, *and* $f^i{}_A(x_j) = f^i{}_B(x_j)$ *for every* $x_j \in X$ *and* $i \in I_p$.

Proof. The proofs can be made similar way to proof of Proposition 1. ∎

Note that, if $w_j (j = 1, 2, ..., n)$ values take as $\frac{1}{n}$, Eqs. (3.4), (3.5) and (3.6) are reduced Eqs. (2.4), (2.5) and (2.6), respectively.

Now similarity measures between two INR-sets will be defined as a extension of similarity measures between two IN-sets given in [28].

For convenience, $\ddot{A} = \left\{\left\langle x, ([t^{iL}_A(x), t^{iU}_A(x)]), ([i^{iL}_A(x), i^{iU}_A(x)]), ([f^{iL}_A(x), f^{iU}_A(x)])\right\rangle : x \in X \text{ and } i \in I_p\right\}$ will be meant by $\ddot{A} \in INR_X$

Definition 15. *Let $\ddot{A}, \ddot{B} \in INR_X$. Then, the Jaccard similarity measure between INR-sets $\ddot{A}$ and $\ddot{B}$ is defined as follows:*

$$(\ddot{A},\ddot{B})_J = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{\begin{pmatrix}(t^{iL}_A(x_j)t^{iL}_B(x_j)+t^{iU}_A(x_j)t^{iU}_B(x_j))\\ +(i^{iL}_A(x_j)i^{iL}_B(x_j)+i^{iU}_A(i_j)i^{iU}_B(i_j))\\ +(f^{iL}_A(x_j)f^{iL}_B(x_j)+f^{iU}_A(x_j)f^{iU}_B(x_j))\end{pmatrix}}{\begin{pmatrix}([t^{iL}_A(x_j)]^2+[i^{iL}_A(x_j)]^2+[f^{iL}_A(x_j)]^2)+([t^{iU}_A(x_j)]^2+[i^{iU}_A(x_j)]^2+[f^{iU}_A(x_j)]^2)\\ +([t^{iL}_B(x_j)]^2+[i^{iL}_B(x_j)]^2+[f^{iL}_B(x_j)]^2)+([t^{iU}_B(x_j)]^2+[i^{iU}_B(x_j)]^2+[f^{iU}_B(x_j)]^2)\\ -\left[t^{iL}_A(x_j)t^{iL}_B(x_j)+i^{iL}_A(x_j)i^{iL}_B(x_j)+f^{iL}_A(x_j)f^{iL}_B(x_j)\right]\\ -\left[t^{iU}_A(x_j)t^{iU}_B(x_j)+i^{iU}_A(x_j)i^{iU}_B(x_j)+f^{iU}_A(x_j)f^{iU}_B(x_j)\right]\end{pmatrix}}. \tag{3.9}$$

Definition 16. *Let $\ddot{A}, \ddot{B} \in INR_X$. Then, the Dice similarity measure between INR-sets $\ddot{A}$ and $\ddot{B}$ is defined as follows:*

$$(\ddot{A},\ddot{B})_D = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{2\begin{pmatrix}(t^{iL}_A(x_j)t^{iL}_B(x_j)+t^{iU}_A(x_j)t^{iU}_B(x_j))\\ +(i^{iL}_A(x_j)i^{iL}_B(x_j)+i^{iU}_A(i_j)i^{iU}_B(i_j))\\ +(f^{iL}_A(x_j)f^{iL}_B(x_j)+f^{iU}_A(x_j)f^{iU}_B(x_j))\end{pmatrix}}{\begin{pmatrix}([t^{iL}_A(x_j)]^2+[i^{iL}_A(x_j)]^2+[f^{iL}_A(x_j)]^2)+([t^{iU}_A(x_j)]^2+[i^{iU}_A(x_j)]^2+[f^{iU}_A(x_j)]^2)\\ +([t^{iL}_B(x_j)]^2+[i^{iL}_B(x_j)]^2+[f^{iL}_B(x_j)]^2)+([t^{iU}_B(x_j)]^2+[i^{iU}_B(x_j)]^2+[f^{iU}_B(x_j)]^2)\end{pmatrix}}. \tag{3.10}$$

Definition 17. *Let $\ddot{A}, \ddot{B} \in INR_X$. Then the cosine similarity measure between $\ddot{A}$ and $\ddot{B}$ is defined as follows:*

$$(\ddot{A},\ddot{B})_C = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{p}\frac{\begin{pmatrix}(t^{iL}_A(x_j)t^{iL}_B(x_j)+t^{iU}_A(x_j)t^{iU}_B(x_j))\\ +(i^{iL}_A(x_j)i^{iL}_B(x_j)+i^{iU}_A(i_j)i^{iU}_B(i_j))\\ +(f^{iL}_A(x_j)f^{iL}_B(x_j)+f^{iU}_A(x_j)f^{iU}_B(x_j))\end{pmatrix}}{\begin{pmatrix}\sqrt{([t^{iL}_A(x_j)]^2+[i^{iL}_A(x_j)]^2+[f^{iL}_A(x_j)]^2)+([t^{iU}_A(x_j)]^2+[i^{iU}_A(x_j)]^2+[f^{iU}_A(x_j)]^2)}\\ \sqrt{([t^{iL}_B(x_j)]^2+[i^{iL}_B(x_j)]^2+[f^{iL}_B(x_j)]^2)+([t^{iU}_B(x_j)]^2+[i^{iU}_B(x_j)]^2+[f^{iU}_B(x_j)]^2)}\end{pmatrix}}. \tag{3.11}$$

If $w_j \in [0,1]$ be the weight of each element x_j for $j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} w_j = 1$, then the weighted Jaccard, Dice and Cosine similarity measures between INR-sets $\ddot{A}$ and $\ddot{B}$ is defined as follows:

$$W(\ddot{A},\ddot{B})_J = \sum_{j=1}^{n}\sum_{i=1}^{p} w_j \frac{\begin{pmatrix} (t{i}_A^L(x_j)t{i}_B^L(x_j) + t{i}_A^U(x_j)t{i}_B^U(x_j)) \\ +(i{i}_A^L(x_j)i{i}_B^L(x_j) + i{i}_A^U(i_j)i{i}_B^U(i_j)) \\ +(f{i}_A^L(x_j)f{i}_B^L(x_j) + f{i}_A^U(x_j)f{i}_B^U(x_j)) \end{pmatrix}}{\begin{pmatrix} ([t{i}_A^L(x_j)]^2 + [i{i}_A^L(x_j)]^2 + [f{i}_A^L(x_j)]^2) + ([t{i}_A^U(x_j)]^2 + [i{i}_A^U(x_j)]^2 + [f{i}_A^U(x_j)]^2) \\ +([t{i}_B^L(x_j)]^2 + [i{i}_B^L(x_j)]^2 + [f{i}_B^L(x_j)]^2) + ([t{i}_B^U(x_j)]^2 + [i{i}_B^U(x_j)]^2 + [f{i}_B^U(x_j)]^2) \\ -[t{i}_A^L(x_j)t{i}_B^L(x_j) + i{i}_A^L(x_j)i{i}_B^L(x_j) + f{i}_A^L(x_j)f{i}_B^L(x_j)] \\ -[t{i}_A^U(x_j)t{i}_B^U(x_j) + i{i}_A^U(x_j)i{i}_B^U(x_j) + f{i}_A^U(x_j)f{i}_B^U(x_j)] \end{pmatrix}}, \tag{3.12}$$

$$W(\ddot{A},\ddot{B})_D = \sum_{j=1}^{n}\sum_{i=1}^{p} w_j \frac{2\begin{pmatrix} (t{i}_A^L(x_j)t{i}_B^L(x_j) + t{i}_A^U(x_j)t{i}_B^U(x_j)) \\ +(i{i}_A^L(x_j)i{i}_B^L(x_j) + i{i}_A^U(i_j)i{i}_B^U(i_j)) \\ +(f{i}_A^L(x_j)f{i}_B^L(x_j) + f{i}_A^U(x_j)f{i}_B^U(x_j)) \end{pmatrix}}{\begin{pmatrix} ([t{i}_A^L(x_j)]^2 + [i{i}_A^L(x_j)]^2 + [f{i}_A^L(x_j)]^2) + ([t{i}_A^U(x_j)]^2 + [i{i}_A^U(x_j)]^2 + [f{i}_A^U(x_j)]^2) \\ +([t{i}_B^L(x_j)]^2 + [i{i}_B^L(x_j)]^2 + [f{i}_B^L(x_j)]^2) + ([t{i}_B^U(x_j)]^2 + [i{i}_B^U(x_j)]^2 + [f{i}_B^U(x_j)]^2) \end{pmatrix}}. \tag{3.13}$$

and

$$W(\ddot{A},\ddot{B})_C = \sum_{j=1}^{n}\sum_{i=1}^{p} w_j \frac{\begin{pmatrix} (t{i}_A^L(x_j)t{i}_B^L(x_j) + t{i}_A^U(x_j)t{i}_B^U(x_j)) \\ +(i{i}_A^L(x_j)i{i}_B^L(x_j) + i{i}_A^U(i_j)i{i}_B^U(i_j)) \\ +(f{i}_A^L(x_j)f{i}_B^L(x_j) + f{i}_A^U(x_j)f{i}_B^U(x_j)) \end{pmatrix}}{\begin{pmatrix} \sqrt{([t{i}_A^L(x_j)]^2 + [i{i}_A^L(x_j)]^2 + [f{i}_A^L(x_j)]^2) + ([t{i}_A^U(x_j)]^2 + [i{i}_A^U(x_j)]^2 + [f{i}_A^U(x_j)]^2)} \\ \sqrt{([t{i}_B^L(x_j)]^2 + [i{i}_B^L(x_j)]^2 + [f{i}_B^L(x_j)]^2) + ([t{i}_B^U(x_j)]^2 + [i{i}_B^U(x_j)]^2 + [f{i}_B^U(x_j)]^2)} \end{pmatrix}}, \tag{3.14}$$

respectively.

Proposition 3. $\ddot{A}, \ddot{B} \in INR_X$. *Then, each similarity measure* $(\ddot{A},\ddot{B})_\Lambda(\Lambda = J, D, C)$ *satisfies the following properties:*

(1) $0 \leq (\ddot{A},\ddot{B})_\Lambda \leq 1$
(2) $(\ddot{A},\ddot{B})_\Lambda = (\ddot{B},\ddot{A})_\Lambda$;
(3) $(\ddot{A},\ddot{B})_\Lambda = 1$ *if* $\ddot{B} = \ddot{A}$ *i.e.* $[t{i}_A^L(x_j), t{i}_A^U(x_j)] = [t{i}_B^L(x_j), t{i}_B^U(x_j)]$, $[i{i}_A^L(x_j), i{i}_A^U(x_j)] = [i{i}_B^L(x_j), i{i}_B^U(x_j)]$, *and* $[f{i}_A^L(x_j), f{i}_A^U(x_j)] = [f{i}_B^L(x_j), f{i}_B^U(x_j)]$ *for all* $x_j \in X$ *and* $i \in I_p$.

Proof. The proofs can be made similar way to proof of Proposition 1. ▪

Proposition 4. $\ddot{A}, \ddot{B} \in INR_X$. *Then, each weighted similarity measure* $W(\ddot{A},\ddot{B})_\Lambda(\Lambda = J, D, C)$ *satisfies the following properties:*

(1) $0 \leq W(\ddot{A},\ddot{B})_\Lambda \leq 1$
(2) $W(\ddot{A},\ddot{B})_\Lambda = W(\ddot{B},\ddot{A})_\Lambda$;
(3) $W(\ddot{A},\ddot{B})_\Lambda = 1$ *if* $\tilde{B} = \tilde{A}$ *i.e.* $[t{i}_A^L(x_j), t{i}_A^U(x_j)] = [t{i}_B^L(x_j), t{i}_B^U(x_j)]$, $[i{i}_A^L(x_j), i{i}_A^U(x_j)] = [i{i}_B^L(x_j), i{i}_B^U(x_j)]$, *and* $[f{i}_A^L(x_j), f{i}_A^U(x_j)] = [f{i}_B^L(x_j), f{i}_B^U(x_j)]$ *for every* $x_j \in X$ *and* $i \in I_p$.

Proof. The proofs can be made similar way to proof of Proposition 1. ▪

Note that if $[t{i}_A^L(x), t{i}_A^U(x)] = [0,0]$, $[i{i}_A^L(x), i{i}_A^U(x)] = [0,0]$, $[f{i}_A^L(x), f{i}_A^U(x)] = [0,0]$ and $[t{i}_B^L(x), t{i}_B^U(x)] = [0,0]$, $[i{i}_B^L(x), i{i}_B^U(x)] = [0,0]$, $[f{i}_B^L(x), f{i}_B^U(x)] = [0,0]$, each similarity measure between two INR-sets $\ddot{A}$ and $\ddot{B}$ are undefined.

4. Similarity measure based multicriteia decision making under SVNR-environment and INR-environment

In this section, applications of weighted similarity measures in multicriteia decision making problems under SVNR-environment and INR-environment are given.

Let us consider a MCDM problem with k alternatives and r criteria. Let $A = \{A_1, A_2, ..., A_k\}$ be a set of alternatives and $C = \{C_1, C_2, ..., C_r\}$ be the set of criteria and $w = \{w_1, w_2, ...w_r\}$ be weights of the criteria $C_j (j = 1, 2, ..., r)$ such that $w_j \geq 0 (j = 1, 2, ..., r)$ and $\sum_{i=}^{r} w_i = 1$.

4.1. **Multi-criteria decision making under SVNR-environment.** Let $\{A_1, A_2, ..., A_k\}$ be a set of alternatives and $\{C_1, C_2, ..., C_r\}$ be a set of criterion. Alternatives $A_i (i = 1, 2, ..., k)$ are characterized by SVNR-values for each $C_j (i = 1, 2, ..., r)$ as follows:

$$A_i = \Big\{ \langle C_j, (t^1_{A_i}(C_j), ..., t^p_{A_i}(C_j)), (i^1_{A_i}(C_j), ..., i^p_{A_i}(C_j)), (f^1_{A_i}(C_j), ..., f^p_{A_i}(C_j)) \rangle : C_j \in C \Big\},$$

for the sake of shortness, $(t^1_{A_i}(C_j), ..., t^p_{A_i}(C_j))$, $(i^1_{A_i}(C_j), ..., i^p_{A_i}(C_j))$ and $(f^1_{A_i}(C_j), ..., f^p_{A_i}(C_j))$ are denoted by $(t^1_{ij}, ..., t^p_{ij})$, $(i^1_{ij}, ..., i^p_{ij})$ and $(f^1_{ij}, ..., f^p_{ij})$, respectively. Thus, the evaluation of the alternative A_i with respect to the criteria C_j made by expert or decision maker can be briefly written as $\gamma_{ij} = \langle (t^1_{ij}, ..., t^p_{ij}), (i^1_{ij}, ..., i^p_{ij}), (f^1_{ij}, ..., f^p_{ij}) \rangle (i = 1, 2, ..., k; j = 1, 2, ..., r)$. Hence, SVNR-decision matrix $D = [\gamma_{ij}]_{k\times r}$ can be constructed.

In MCDM environment, to characterize the best alternative properly in the decision set the notion of the ideal point is used. To evaluate the criteria, two type modifiers called benefit criteria (BC) and cost criteria (CC) are generally used.

In this study, for benefit criteria (BC) and cost criteria (CC) ideal SVNR-values denoted by A^* are defined as follows:

- $\gamma^*_j = \langle ({t^1}^*_j, ..., {t^p}^*_j), ({i^1}^*_j, ..., {i^p}^*_j), ({f^1}^*_j, ..., {f^p}^*_j) \rangle = \left\langle \begin{array}{c} (max_i(t^1_{ij}), ..., max_i(t^p_{ij})), (min_i(i^1_{ij}), ..., min_i(i^p_{ij})), \\ (min_i(f^1_{ij}), ..., min_i(f^p_{ij})) \end{array} \right\rangle$
- $\gamma^*_j = \langle ({t^1}^*_j, ..., {t^p}^*_j), ({i^1}^*_j, ..., {i^p}^*_j), ({f^1}^*_j, ..., {f^p}^*_j) \rangle = \left\langle \begin{array}{c} (min_i(t^1_{ij}), ..., min_i(t^p_{ij})), (max_i(i^1_{ij}), ..., max_i(i^p_{ij})), \\ (max_i(f^1_{ij}), ..., max_i(f^p_{ij})) \end{array} \right\rangle,$

respectively. Here equations are called positive ideal solution and negative ideal solution, respectively.

Algorithm

- **Step 1: Determination of BC and CC criteria.**
- **Step 2: Determination of ideal SVNR-values** A^*
- **Step 3: Calculation of weighted similarity measures**
  In this step, using one of the Eq. (3.4), Eq. (3.5) or Eq.(3.6) weighted similarity measures between the ideal alternative A^* and $A_i (i = 1, 2, ..., k)$ are calculated.
- **Step 4: Ranking of the alternative**
  Considering the values obtained using one of the Eq. (3.4), Eq. (3.5) or Eq.(3.6), the ranking order of all the alternatives can be easily determined.

Illustrative example 1

Let us consider the decision making problem given in [30]. We adapt this decision making problem to SVNR-set. There is an investment company, which wants to invest a sum money in the best option. There is a panel with four possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the three criteria (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is an environmental impact, The weights of criteria C_1, C_2 and C_3 are given by $w_1 = 0.35, w_2 = 0.25$ and $w_3 = 0.40$, respectively. The four alternatives are to evaluated under the criteria by SVNR-values provided by decision maker. These values are shown in SVNR-decision matrix as follows:

$$D = [\gamma_{ij}]_{k\times r} = \begin{pmatrix} \langle (.1, .2, .4), (.3, .3, .5), (.2, .4, .8) \rangle & \langle (.1, .4, .6), (.2, .3, .7), (.1, .3, .3) \rangle & \langle (.0, .3, .3), (.1, .5, .6), (.5, .6, .9) \rangle \\ \langle (.5, .6, .7), (.2, .4, .4), (.1, .6, .6) \rangle & \langle (.4, .6, .7), (.2, .5, .8), (.1, .5, .5) \rangle & \langle (.3, .3, .4), (.2, .6, .7), (.3, .4, .7) \rangle \\ \langle (.3, .3, .5), (.0, .1, .3), (.1, .4, .7) \rangle & \langle (.4, .5, .6), (.2, .3, .6), (.1, .3, .4) \rangle & \langle (.1, .3, .4), (.1, .4, .6), (.3, .3, .5) \rangle \\ \langle (.2, .4, .9), (.1, .5, .6), (.3, .5, 1) \rangle & \langle (.0, .2, .4), (.1, .5, .7), (.6, .7, .9) \rangle & \langle (.8, .8, .9), (.3, .4, .4), (.6, .6, .8) \rangle \end{pmatrix}$$

- **Step 1:** Let us consider C_1 and C_2 as benefit criteria and C_3 as cost criterion.
- **Step 2:** From SVNR-decision matrix, ideal alternative A^* can be obtained as follows:

$$A^* = \Big\{ \big\langle (.5, .6, .9), (.0, .1, .3), (.1, .4, .6) \big\rangle, \big\langle (.4, .6, .7), (.1, .3, .6), (.1, .3, .3) \big\rangle, \big\langle (.0, .3, .3), (.3, .6, .7), (.6, .6, .9) \big\rangle \Big\}.$$

- **Step 3:** By using the Eqs. (3.1), (3.4), (3.2), (3.5), (3.3) and (3.6), for $\Lambda \in \{J, D, C\}$, similarity measures and weighted similarity measures are obtained as in Table 3
- **Step 4:** Rankings of the alternatives are shown in last columnn of Table 3.

TABLE 3. Similarity measure values under INR-environment

Similarity measure	Values	Ranking order
$(A^*, A_i)_J$	$(A^*, A_1)_J = 0.83489$ $(A^*, A_2)_J = 0.90254$ $(A^*, A_3)_J = 0.86578$ $(A^*, A_4)_J = 0.65791$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$(A^*, A_i)_D$	$(A^*, A_1)_D = 0.90283$ $(A^*, A_2)_D = 0.94872$ $(A^*, A_3)_D = 0.92618$ $(A^*, A_4)_D = 0.78961$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$(A^*, A_i)_C$	$(A^*, A_1)_C = 0.90937$ $(A^*, A_2)_C = 0.95841$ $(A^*, A_3)_C = 0.96019$ $(A^*, A_4)_C = 0.80492$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$W(A^*, A_i)_J$	$W(A^*, A_1)_J = 0.83534$ $W(A^*, A_2)_J = 0.75035$ $W(A^*, A_3)_J = 0.85113$ $W(A^*, A_4)_J = 0.66726$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$W(A^*, A_i)_D$	$W(A^*, A_1)_D = 0.90259$ $W(A^*, A_2)_D = 0.94726$ $W(A^*, A_3)_D = 0.91794$ $W(A^*, A_4)_D = 0.79671$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$W(A^*, A_i)_C$	$W(A^*, A_1)_C = 0.90911$ $W(A^*, A_2)_C = 0.95613$ $W(A^*, A_3)_C = 0.95695$ $W(A^*, A_4)_C = 0.81158$	$A_3 \succ A_2 \succ A_1 \succ A_4$

4.2. **Multi-attribute decision making under INR-environment.** Let alternatives $A_i(i = 1, 2, ..., k)$ are characterized by INR-values for each criterion $C_j(i = 1, 2, ..., r)$ as follows:

$$A_i = \left\{ \begin{array}{c} \langle C_j, ([t1^L_{A_i}(C_j), t1^U_{A_i}(C_j)], ..., [tp^L_{A_i}(C_j), tp^U_{A_i}(C_j)]), ([i1^L_{A_i}(C_j), i1^U_{A_i}(C_j)], ..., [ip^L_{A_i}(C_j), ip^U_{A_i}(C_j)]), \\ ([f1^L_{A_i}(C_j), f1^U_{A_i}(C_j)], ..., [fp^L_{A_i}(C_j), fp^U_{A_i}(C_j)])\rangle : C_j \in C \end{array} \right\}.$$

For convenience, $([t1^L_{A_i}(C_j), t1^U_{A_i}(C_j)], ..., [tp^L_{A_i}(C_j), tp^U_{A_i}(C_j)]), ([i1^L_{A_i}(C_j), i1^U_{A_i}(C_j)], ..., [ip^L_{A_i}(C_j), ip^U_{A_i}(C_j)])$ and $([f1^L_{A_i}(C_j), f1^U_{A_i}(C_j)], ..., [fp^L_{A_i}(C_j), fp^U_{A_i}(C_j)])$ are denoted by $([t1^L_{ij}], ..., [tp^U_{ij}])$, $([i1^L_{ij}], ..., [ip^U_{ij}])$ and $([f1^L_{ij}], ..., [fp^U_{ij}])$, respectively. So INR-value $\theta_{ij} = \langle([t1^L_{ij}, t1^U_{ij}], ..., [tp^L_{ij}, tp^U_{ij}]), ([i1^L_{ij}, i1^U_{ij}], ..., [ip^L_{ij}, ip^U_{ij}]), ([f1^L_{ij}, f1^U_{ij}], ..., [fp^L_{ij}, fp^U_{ij}])\rangle(i = 1, 2, ..., k; j = 1, 2, ..., r)$ which is generally obtained from the evaluation of the alternative A_i with related to the criteria C_j by opinion of expert or decision maker. Thus, INR-decision matrix $D = [\theta_{ij}]_{k\times r}$ can be constructed.

In this study, for benefit criteria (BC) and cost criteria (CC) ideal INR-values denoted by A^* are defined as follows:

- $\theta^*_j = \langle(t1^*_j, ..., tp^*_j), (i1^*_j, ..., ip^*_j), (f1^*_j, ..., fp^*_j)\rangle = \left\langle \begin{array}{c} ([max_i(t1^L_{ij}), max_i(t1^U_{ij})], ..., [max_i(tp^L_{ij}), max_i(tp^U_{ij})]), \\ ([min_i(i1^L_{ij}), min_i(i1^U_{ij})], ..., [min_i(ip^L_{ij}), min_i(ip^U_{ij})]), \\ ([min_i(f1^L_{ij}), min_i(f1^U_{ij})], ..., [min_i(fp^L_{ij}), min_i(fp^U_{ij})]) \end{array} \right\rangle$

- $\theta^*_j = \langle(t1^*_j, ..., tp^*_j), (i1^*_j, ..., ip^*_j), (f1^*_j, ..., fp^*_j)\rangle = \left\langle \begin{array}{c} ([min_i(t1^L_{ij}), min_i(t1^U_{ij})], ..., [min_i(tp^L_{ij}), min_i(tp^U_{ij})]), \\ ([max_i(i1^L_{ij}), max_i(i1^U_{ij})], ..., [max_i(ip^L_{ij}), max_i(ip^U_{ij})]), \\ ([max_i(f1^L_{ij}), max_i(f1^U_{ij})], ..., [max_i(fp^L_{ij}), max_i(fp^U_{ij})]) \end{array} \right\rangle$,

respectively.

ALGORITHM

- **Step 1: Determination of BC and CC criteria.**

- **Step 2: Determination of ideal INR-values A^* solution**
- **Step 3: Calculation of weighted similarity measures**
  In this step, using one of the Eq. (3.12), Eq. (3.13) or Eq.(3.14) weighted similarity measures between the ideal alternative A^* and INR-sets $A_i (i = 1, 2, ..., k)$ are calculated.
- **Step 4: Ranking of the alternative** According to the values obtained using one of the Eq. (3.12), Eq. (3.13) or Eq.(3.14), the ranking order of all the alternatives can be easily determined.

Illustrative example 2

In this example, alternatives and criteria given in previous illustrative example will be considered under INR-environment. The four alternatives are to evaluated under the criteria by INR-values provided by decision maker. These values are shown in INR-decision matrix as follows:

$$D = [\gamma_{ij}]_{k\times r} = \begin{pmatrix} \left\langle \begin{matrix} ([.2,.3],[.2,.5],[.4,.7]),\\ ([.3,.4],[.3,.6],[.5,.9]),\\ ([.2,.5],[.4,.7],[.8,.8]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.1,.5],[.4,.5],[.6,1]),\\ ([.2,.4],[.3,.7],[.7,.8]),\\ ([.1,.2],[.3,.8],[.3,.8]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.0,.3],[.3,.5],[.3,9]),\\ ([.1,.2],[.5,.6],[.6,.6]),\\ ([.5,.5],[.6,.7],[.9,.9]) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} ([.1,.2],[.2,.8],[.4,.8]),\\ ([.4,.5],[.3,.6],[.5,.7]),\\ ([.1,.3],[.4,.5],[.8,.8]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.1,.4],[.4,.5],[.6,.6]),\\ ([.2,.3],[.3,.4],[.7,.8]),\\ ([.1,.5],[.3,.6],[.3,.7]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.0,.3],[.3,.4],[.3,5]),\\ ([.1,.6],[.5,.6],[.6,.7]),\\ ([.5,.8],[.6,.8],[.9,1]) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} ([.1,.4],[.2,.5],[.4,.6]),\\ ([.3,.4],[.3,.4],[.6,.7]),\\ ([.2,.3],[.4,.5],[.8,1]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.2,.3],[.4,.5],[.6,.7]),\\ ([.2,.5],[.3,.6],[.7,.8]),\\ ([.1,.2],[.3,.4],[.4,.5]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.0,.1],[.3,.3],[.3,.4]),\\ ([.1,.2],[.5,.6],[.6,.7]),\\ ([.5,.6],[.6,.7],[.9,.9]) \end{matrix} \right\rangle \\ \left\langle \begin{matrix} ([.1,.4],[.2,.4],[.4,.4]),\\ ([.3,.5],[.3,.6],[.5,.6]),\\ ([.2,.5],[.4,.6],[.8,.9]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.1,.5],[.4,.5],[.4,.6]),\\ ([.2,.4],[.3,.5],[.7,.9]),\\ ([.2,.2],[.3,.4],[.3,.4]) \end{matrix} \right\rangle & \left\langle \begin{matrix} ([.0,.2],[.3,.4],[.3,.5]),\\ ([.1,.4],[.5,.6],[.6,.8]),\\ ([.5,.6],[.6,.7],[.9,1]) \end{matrix} \right\rangle \end{pmatrix}$$

- **Step 1:** Let us consider C_1 and C_2 as benefit criteria and C_3 as cost criterion.
- **Step 2:** From INR-decision matrix, ideal alternative A^* can be obtained as follows:

$$\begin{aligned} A^* \quad = \quad & \Big\{\langle([.2,.4],[.2,.8],[.4,.8]),([.3,.4],[.3,.4],[.5,.6]),([.1,.3],[.4,.5],[.8,.8])\rangle, \\ & \langle([.2,.5],[.4,.5],[.6,1]),([.2,.3],[.3,.4],[.7,.8]),([.1,.2],[.3,.4],[.3,.4])\rangle \\ & \langle([.0,.1],[.3,.3],[.3,.4]),([.1,.6],[.5,.6],[.6,.8]),([.5,.8],[.6,.8],[.9,1])\rangle\Big\}. \end{aligned}$$

- **Step 3:** By using the Eq. (3.12), Eq. (3.13) and Eq. (3.14) similarity measures and weighted similarity measures are obtained as shown in Table 4.
- **Step 4:** Rankings of the alternatives are shown in last column of Table 4.

5. Consistency analysis of similarity measures based INR-sets

In this section, to determine which similarity measure gives more consistent results, a method is given.

Let $A = \{A_1, A_2, ..., A_n\}$ be a set of alternatives, $C = \{C_1, C_2, ..., C_k\}$ be a set of criteria and A^* be set of ideal alternative values obtained from decision matrix defined in illustrative example of similarity measures based on INR-set. Then, consistency of the similarity measures based INR-values is define by as follows:

$$C(A^*, A_i)_\Lambda = \frac{1}{n}\sum_{i=1}^{n} |(A^{*L}, A_i{}^L)_\Lambda - (A^{*U}, A_i{}^U)_\Lambda|.$$

Here, A^{*L} and A^{*U} are determined with help of INR-decision matrix using formula of benefit criteria (BC) and cost criteria (CC) given as follows: For $\Delta \in \{L = lower, U = upper\}$

- $\delta_j^{*\Delta} = \langle(t^{1*\Delta}_{j}, ..., t^{p*\Delta}_{j}), (i^{1*\Delta}_{j}, ..., i^{p*\Delta}_{j}), (f^{1*\Delta}_{j}, ..., f^{p*\Delta}_{j})\rangle = \left\langle \begin{matrix} (max_i(t^{1\Delta}_{ij}), ..., max_i(t^{p\Delta}_{ij})), \\ (min_i(i^{1\Delta}_{ij}), ..., min_i(i^{p\Delta}_{ij})), \\ (min_i(f^{1\Delta}_{ij}), ..., min_i(f^{p\Delta}_{ij})]) \end{matrix} \right\rangle$
- $\delta_j^{*\Delta} = \langle(t^{1*\Delta}_{j}, ..., t^{p*\Delta}_{j}), (i^{1*\Delta}_{j}, ..., i^{p*\Delta}_{j}), (f^{1*\Delta}_{j}, ..., f^{p*\Delta}_{j})\rangle = \left\langle \begin{matrix} (min_i(t^{1\Delta}_{ij}), ..., min_i(t^{p\Delta}_{ij})), \\ (max_i(i^{1\Delta}_{ij}), ..., max_i(i^{p\Delta}_{ij})), \\ (max_i(f^{1\Delta}_{ij}), ..., max_i(f^{p\Delta}_{ij})) \end{matrix} \right\rangle,$

TABLE 4. Similarity measure values and ranking of alternatives under INR-environment

Similarity measure	Values	Ranking order
$(A^*, A_i)_J$	$(A^*, A_1)_J = 0.83489$ $(A^*, A_2)_J = 0.95699$ $(A^*, A_3)_J = 0.95304$ $(A^*, A_4)_J = 0.94042$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(A^*, A_i)_D$	$(A^*, A_1)_D = 0.90283$ $(A^*, A_2)_D = 0.97768$ $(A^*, A_3)_D = 0.97595$ $(A^*, A_4)_D = 0.96903$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$(A^*, A_i)_C$	$(A^*, A_1)_C = 0.90937$ $(A^*, A_2)_C = 0.97790$ $(A^*, A_3)_C = 0.97758$ $(A^*, A_4)_C = 0.97007$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$W(A^*, A_i)_J$	$W(A^*, A_1)_J = 0.83534$ $W(A^*, A_2)_J = 0.96355$ $W(A^*, A_3)_J = 0.95420$ $W(A^*, A_4)_J = 0.94270$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$W(A^*, A_i)_D$	$W(A^*, A_1)_D = 0.90259$ $W(A^*, A_2)_D = 0.98114$ $W(A^*, A_3)_D = 0.97656$ $W(A^*, A_4)_D = 0,97021$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$W(A^*, A_i)_C$	$W(A^*, A_1)_C = 0.90911$ $W(A^*, A_2)_C = 0.98138$ $W(A^*, A_3)_C = 0.97849$ $W(A^*, A_4)_C = 0.97112$	$A_2 \succ A_3 \succ A_4 \succ A_1$

respectively.
Also $A_i{}^L = \left\{ \left\langle (t1^L_{A_i}(C_j), ..., tp^L_{A_i}(C_j)), (i1^L_{A_i}(C_j), ..., ip^L_{A_i}(C_j)), (f1^L_{A_i}(C_j), ..., fp^L_{A_i}(C_j)) \right\rangle : C_j \in C \text{ and } i \in I_p \right\}$
and $A_i{}^U = \left\{ \left\langle (t1^U_{A_i}(C_j), ..., tp^U_{A_i}(C_j)), (i1^U_{A_i}(C_j), ..., ip^U_{A_i}(C_j)), (f1^U_{A_i}(C_j), ..., fp^U_{A_i}(C_j)) \right\rangle : C_j \in C \text{ and } i \in I_p \right\}$.

Example 3. *Let us consider the Example 4.2. Then for all $\Lambda \in \{J, D, C\}$, results and orderings are obtained as in Table 5,*

Note that, $C(A^*, A_i)_J \geq C(A^*, A_i)_D \geq C(A^*, A_i)_C$. Since consistency degree of Jaccard similarity measure under INR environment is higher than consistency degrees of Dice and Cosine similarity measures, it is more convenient using the Jaccard similarity measure for discussed problem.

6. CONCLUSION

In this paper, for SVNR-set and INR-sets three similarity measures method developed based on Jaccard, Dice and Cosine similarity measures. Furthermore, applications of proposed similarity measure methods are given in multi-criteria decision making and a method is developed to compare similarity measures of INR-sets, and an application of this method is given. However, I hope that the main thrust of proposed formulas will be in the field of equipment evaluation, data mining and investment decision making. Also in future, similarity measure methods for INR-sets can be proposed based on the methods other than Jaccard, Dice and Cosine similarity measures.

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TABLE 5. Consistency degrees of similarity measures under INR environment

Similarity measure	(A^{*L}, A_i^L)	(A^{*U}, A_i^U)	$C(A^*, A_i)_\Lambda$
Jaccard	0,99220	0,87306	0,07269
	0,99313	0,93997	
	0,99118	0,93188	
	0,98014	0,92097	
Dice	0,99608	0,93187	0,03870
	0,99655	0,96832	
	0,99556	0,96472	
	0,98987	0,95835	
Cosine	0,99649	0,94012	0,03545
	0,99660	0,96869	
	0,99587	0,96947	
	0,99089	0,95975	
Weighted Jaccard	0,99207	0,86958	0,07157
	0,99353	0,94931	
	0,99145	0,93316	
	0,98376	0,92248	
Weighted Dice	0,99602	0,92984	0,03811
	0,99674	0,97337	
	0,99569	0,96541	
	0,99173	0,95912	
Weighted Cosine	0,99649	0,93678	0,03494
	0,99679	0,97377	
	0,99599	0,97097	
	0,99249	0,96049	

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