Multiple Attribute Decision-Making Method under Hesitant Single Valued Neutrosophic Uncertain Linguistic Environment

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Abstract: Motivated by the idea of single valued neutrosophic uncertain linguistic sets (SVNULSs) and hesitant fuzzy sets (HFSs), in this article we combine SVNULSs with HFSs to present the idea of hesitant single valued neutrosophic uncertain linguistic sets (HSVNULSs), hesitant single valued neutrosophic uncertain linguistic elements (HSVNULEs) and defined some basic operational laws of HSVNULEs. We also presented the score, accuracy and certainty functions for HSVUNLEs. Then, based on the operational laws for HSVNULEs we presented hesitant single valued neutrosophic uncertain linguistic weighted average (HSVNULWA) operator and hesitant single valued neutrosophic uncertain linguistic weighted geometric average (HSVNULWGA) operator to aggregate the hesitant single valued neutrosophic uncertain linguistic information. Furthermore, some necessary assets of the two operators are scrutinized. Based on HSVNULWA and HSVNULWGA operators we defined a multiple criteria decision-making method to switch multiple criteria decision making problems, in which the criterion values with respect to alternatives take the form of HSVNULEs, under a HSVNULs environment. Finally, a numerical example about investment alternatives is specified to show the efficiency of the developed method.

Keywords: "Hesitant single valued neutrosophic uncertain linguistic set, Hesitant single valued neutrosophic uncertain linguistic element, score function, Hesitant single valued neutrosophic uncertain linguistic weighted average operator, Hesitant single valued neutrosophic uncertain linguistic weighted geometric average operator, Multiple attribute decision-making."

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1. Introduction

Smarandache was the first who introduced the idea of neutrosophic sets (NSs) [2], which is further extension of the concepts of intuitionistic fuzzy sets (IFSs) [34], interval valued intuitionistic fuzzy sets (IVIFSs) [35] from philosophical perspective. The NS is a useful mathematical tool to process the neutrosophic information. NS contains three functions, such as satisfaction degree $\tilde{\mathcal{T}}_{\mathcal{A}}(v)$, dissatisfaction degree $\tilde{\mathcal{F}}_{\mathcal{A}}(v)$, and indeterminacy degree $\tilde{\mathfrak{I}}_{\mathcal{A}}(v)$. In NS all the three functions are subsets of the real standard or non-standard unit interval $]0^{-}, 1^{+}[$, $i.e$; $\tilde{\mathcal{T}}_{\mathcal{A}}(v)\colon \mathbb{U} \to]0^{-}, 1^{+}[$, $\tilde{\mathcal{F}}_{\mathcal{A}}(v)\colon \mathbb{U} \to]0^{-}, 1^{+}[$, $\tilde{\mathfrak{I}}_{\mathcal{A}}(v)\colon \mathbb{U} \to]0^{-}, 1^{+}[$. The NS is a better tool to represent inadequate, incompatible and inexact information. However, the NS is difficult to be used in real life and engineering problem due to containment of non-standard subsets. To blown-away this difficulty some researchers have introduced some subclasses of NS to be easily used in real life and engineering problems by changing the range set from $]0^{-}, 1^{+}[$ into $[0,1]$. The single valued neutrosophic set was introduced by Smarandache in 1998 [2] Wang et al. [5, 6] firstly presented the concepts of interval neutrosophic set (INS) and SVNS, and defined some set theoretic properties of INS and SVNS. INS and SVNS are subclasses of NS. Then, Ye [7-10] presented cross-entropy, correlation coefficient, and improved correlation coefficient for SVNSs and INSs and applied them to multiple attribute decision making (MADM) problem. Biswas et al. [15] presented extended TOPSIS and entropy based-grey relational analysis and applied them to MADM under SVN information. Many researchers presented several distances and similarity measures for NSs, SVNSs and INSs [16, 22-25]. S. . Broumi, et al. [27-31] presented the concepts of Single Valued Neutrosophic Graphs, Isolated Single Valued Neutrosophic Graphs, Single Valued Neutrosophic Graphs: Degree, Order and Size and Bipolar Single Valued Neutrosophic Graphs. Liu et al. [17, 21] presented extended TOPSIS and single valued normalized weighted Bonfferroni mean for INSs and SVNSs respectively and applied them to MADM under IN information and SVN information. Zhang et al. [38] presented the score, accuracy and certainty functions of INSs and defined some aggregation operators for INSs and applied it to MADM.

In decision theory MADM is one of the important research topic and it has been broadly applied in numerous fields, such as economic management and engineering. In complex DM, anyhow, there is a galore of problems, such as lack of knowledge, time pressure, information capabilities and limited attention of the decision makers, because there is a galore of qualitative information, so due to these difficulties we can easily articulate the assessment of the decision maker's by linguistic variable (LV) or uncertain linguistic variables (ULV). Zadeh [1] was the first who presented the idea of the LV and applied it to fuzzy reasoning. After the introduction of LV Herrera et al. [3, 4] presented a model of consensus for group decision making (GDM) under linguistic judgment and also introduced some steps for solving decision making with linguistic information. Then Xu [39, 40] presented the some aggregation operators for LV and uncertain linguistic variables (ULV), such as linguistic hybrid aggregation operators and applied them to GDM under linguistic and uncertain linguistic information.

Wang et al. [32] was the first who presented the idea of intuitionistic linguistic set (ILS), which inhere of an intuitionistic part and a linguistic part and also presented an intuitionistic two-semantic , a Hamming distance between two-semantics, and a ranking method for alternatives in consonance with the exhaustive membership degree to the optimal solution for each alternative. For intuitionistic linguistic variables (ILVs) Wang et al [34] presented some operational rules, the expected value, score function and certainty function of ILVs and proposed a few aggregation operators for ILVs and then presented a GDMM based on these aggregation operators under IL information. Then Liu et al. [18] further presented the concept of intuitionistic uncertain linguistic variables (IULVs) and proposed some aggregation operators for IULVs and applied them to MAGDM under IUL information.

Ye [11, 12] further enlarged the idea of ILSs by introducing single valued neutrosophic linguistic set (SVNLS) and interval neutrosophic linguistic set (INLS) and defined a few operation and aggregation operators for SVNLSs and INLSs and applied them to MADM under SVNL and INL information respectively. Liu et al. [19] further proposed the concept of single valued neutrosophic uncertain linguistic set (SVNULS) and presented some basic operations for SVNULSs and then based on these operation presented some neutrosophic uncertain linguistic Heronian mean operators and their applications in MAGDM. Ye [13] further enlarged the idea IULSs to interval neutrosophic uncertain linguitistic sets (INULSs) and defined a few operational rules for INULSs and aggregation operators based on these operational rules and then applied them to MAGDM. Broumi et al. [26] presented an extended TOPSIS for INULSs and applied it to MADM.

Recently, Ye [14] and Liu et al. [20] presented the ideas of single valued neutrosophic hesitant fuzzy set (SVNHFS) and interval neutrosophic hesitant fuzzy set (INHFS) Based on the combination of hesitant fuzzy set (HFS) [37, 38] and SVNS and INS respectively and then presented a few aggregation operators for SVNHFS and INHFS and then applied them to MADM and MAGDM respectively.

A SNULE defined by Liu et al. [19] is inheres of the ULV expressed by the decision maker's assessment to an calculated object and the subjective calculation value expressed by a SVNE as the accuracy of the given ULV. However, in complex DM problems, when the decision maker's give their evaluations on attributes by the form of SVNULEs, they may also hesitant among several possible SVNULEs. For example, for a pre-ordained linguistic term set $\mathcal{S} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\} = \{extremely\ bad, very\ bad, bad, medium, good, very\ good, extremely\ good\}$, evaluatin the "progress" of a company, we utilize a hesitant single valued neutrosophic uncertain linguistic element (HSVNULE) $\{\langle[s_2, s_3], (0.6,0.2,0.3)\rangle, \langle[s_3, s_4], (0.7,0.1,0.2)\rangle, \langle[s_4, s_5], (0.8,0.1,0.1)\rangle\}$ as its assessment, where as $[s_2, s_3], [s_3, s_4]$ and $[s_4, s_5]$ indicate that the "progress" of the company may be "low, medium", "medium, high" and "high, very high" and the SVNEs (0.6,0.2,0.3), (0.7,0.1,0.2), and (0.8,0.1,0.1) indicate that the "progress" of a company may contain truth degrees, indeterminacy degrees and falsity degrees belonging to $[s_2, s_3]$, $[s_3, s_4]$ and $[s_4, s_5]$ respectively. In this case the existing methods are not suitable for dealing with the DM problems with HSVNUL information. Motivated by the concepts of SVNULSs and HFSs the aim of this article is:

(1) To present the idea of hesitant single valued neutrosophic uncertain linguistic sets (HSVNULSs) and hesitant single valued neutrosophic uncertain linguistic element (HSVNULE) which is composed of a set of SVNULEs.
(2) To define the operational laws for HSVNULEs and the score, accuracy and certainty function respectively.
(3) To develop a few aggregation operators such as hesitant single valued neutrosophic uncertain linguistic weighted averaging (HSVNULWA) operator and hesitant single valued neutrosophic uncertain linguistic weighted geometric (HSVNULWG) operator and to inspect their properties.
(4) To propose a DMM based on the HSVNULWA and HSVNULWG operators to handle MADM problems with hesitant single valued neutrosophic uncertain linguistic information.

To do so, the rest of the article is arranged as follows. Section 2 briefly described some concepts of SVNULSs and HFSs. In section 3 the concept of HSVNULSs and HSVNULEs and a few basic operational laws and the score, accuracy and certainty functions are defined for HSVNULEs. In section 4 we develop the HSVNULWA and HSVNULWG operators and inspect their desired properties. In section 5 we proposed a MADM based on HSVNULWA and HSVNULWG operators the score, accuracy and certainty functions. In section 6 a numerical example is given to illustrate the effectiveness of the proposed method. At the end discussion, conclusion and references are given.

2. Preliminaries

In this section a brief overview of SVNULSs [19] and HFSs [36, 337] and some basic operations and related properties are familiarized to exploit the following analysis:

Let $\check{S} = \{\widehat{s_0}, \widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \widehat{s_4}, \widehat{s_5}, \widehat{s_6}\}$ be a finite ordered discrete linguistic term set with odd cardinality, $\widehat{s_\iota}$ express an available value for a LV and $l+1$ is the cardinality of $\check{S}$. For example, when $l = 6$, we can give a linguistic term set $\check{S} = \{\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \hat{s}_5, \hat{s}_6\} =$

$\{extremely\ bad, very\ bad, bad, \ medium, good, very\ good, extremely\ good\}$.

In a linguistic term set $\check{S}$ any two LVs, $\hat{s}_i$ and , $\hat{s}_j$ must satisfy the following properties [3, 4]:

(a) The set is ordered: , $\hat{s}_i \geq \hat{s}_j$ if $i \geq j$,

(b) Negation operator is $neg(\hat{s}_i) = \hat{s}_{l-i}$,

(c) Maximum operator is $\max(\hat{s}_i, \hat{s}_j) = \hat{s}_i$ if $i > j$,

(d) Minimum operator is $\min(\hat{s}_i, \hat{s}_j) = \hat{s}_j$ if $i > j$.

The continuous linguistic set $\check{\check{S}} = \{s_{\mathfrak{D}} | \mathfrak{D} \in R^+\}$, which can swamped the weakness of the loss of information in the process of calculations, is the extension of the original discrete linguistic set $\check{S} = \{\widehat{s_0}, \widehat{s_1}, \dots, \widehat{s_{l-1}}\}$ and $\check{\check{S}} = \{s_{\mathfrak{D}} | \mathfrak{D} \in R^+\}$ meets the strictly monotonically increasing condition [34, 35]. Some basic operational laws are given below:

(1) $\rho \hat{s}_i = \hat{s}_{\rho \times i}, \rho \geq 0$
(2) $\hat{s}_i + \hat{s}_j = \hat{s}_{i+j}$
(3) $\hat{s}_i \times \hat{s}_j = \hat{s}_{i \times j}$
(4) $(\hat{s}_i)^n = \hat{s}_{i^n}$

Definition 2.1.1 [39, 40] Let us assume that $\hat{s} = [\hat{s}_x, \hat{s}_y]$, $\hat{s}_x, \hat{s}_y \in \check{\check{S}}$ with $x \leq y$ are the lower and upper bound of $\check{\check{S}}$, respectively, then $\hat{s}$ is called an uncertain linguistic variable.

Let $\hat{S}$ be the set of all linguistic variables, $\hat{s}_1 = [\hat{s}_{x_1}, \hat{s}_{y_1}]$ and $\hat{s}_2 = [\hat{s}_{x_2}, \hat{s}_{y_2}]$ be any two uncertain linguistic variables, then the operational laws are defined as given below:

(1) $\hat{s}_1 + \hat{s}_2 = [\hat{s}_{x_1}, \hat{s}_{y_1}] + [\hat{s}_{x_2}, \hat{s}_{y_2}] = [\hat{s}_{x_1+x_2}, \hat{s}_{y_1+y_2}]$
(2) $\hat{s}_1 \times \hat{s}_2 = [\hat{s}_{x_1}, \hat{s}_{y_1}] \times [\hat{s}_{x_2}, \hat{s}_{y_2}] = [\hat{s}_{x_1 \times x_2}, \hat{s}_{y_1 \times y_2}]$
(3) $\rho \hat{s}_1 = \rho [\hat{s}_{x_1}, \hat{s}_{y_1}] = [\hat{s}_{\rho x_1}, \hat{s}_{\rho y_1}]$
(4) $(\hat{s}_1)^\rho = [\hat{s}_{x_1}, \hat{s}_{y_1}]^\rho = [\hat{s}_{x_1^\rho}, \hat{s}_{y_1^\rho}]$

2.2 Single valued neutrosophic uncertain linguistic set

Here we give some basic definitions about SVNULSs introduced by Liu [19].

Definition 2.2.1 [19]. Let $\mathbb{X}$ be a non-empty domain set. A SVNULS in $\mathbb{X}$ is demarcated by,

$$\mathcal{A} = \{r, \langle [s_{\vartheta(r)}, s_{\tau(r)}], (\mathcal{T}_{\mathcal{A}}(r), \mathcal{I}_{\mathcal{A}}(r), \mathcal{F}_{\mathcal{A}}(r))\rangle | r \in \mathbb{X}\}$$

Where $[s_{\vartheta(r)} s_{\tau(r)}] \in \check{S}$,$\mathcal{T}_{\mathcal{A}}(r), \mathcal{I}_{\mathcal{A}}(r), \mathcal{F}_{\mathcal{A}}(r) \subseteq [0,1]$, respectively representing truth-membership, indeterminacy-membership, falsity-membership functions of the $r \in \mathbb{X}$ to the uncertain linguistic variable $[s_{\vartheta(r)} s_{\tau(r)}] \in \check{S}$, with the condition $0 \leq \mathcal{T}_{\mathcal{A}}(r) + \mathcal{I}_{\mathcal{A}}(r) + \mathcal{F}_{\mathcal{A}}(r) \leq 3$ for $r \in \mathbb{X}$.

Then the four tuple $\langle [s_{\vartheta(r),} s_{\tau(r)}], (\mathcal{T}_{\mathcal{A}}(r), \mathcal{I}_{\mathcal{A}}(r), \mathcal{F}_{\mathcal{A}}(r))\rangle$ in $\mathcal{A}$ is called SVNULE. For simplicity, SVNULE can be represented as $r = \langle [s_{\vartheta(r)}, s_{\tau(r)}], (\mathcal{T}(r), \mathcal{I}(r), \mathcal{F}(r))\rangle$.

Definition 2.2.2 [19]. Let $r_1 = \langle [s_{\vartheta(r_1),} s_{\tau(r_1)}], ((\mathcal{T}(r_1)), \mathcal{I}(r_1), \mathcal{F}(r_1))\rangle$ and $r_2 = \langle [s_{\vartheta(r_1),} s_{\tau(r_1)}], (\mathcal{T}(r_2), \mathcal{I}(r_2), \mathcal{F}(r_2))\rangle$ be two SVNULEs and $\zeta \geq 0$, then the operation laws for SVNULNs are defined as follows:

1) $r_1 + r_2 = \{[s_{\vartheta(r_1)+\vartheta(r_2),} s_{\tau(r_1)+\tau(r_2)}], (\mathcal{T}(r_1) + \mathcal{T}(r_2) - \mathcal{T}(r_1)\mathcal{T}(r_2)), \mathcal{I}(r_1)\mathcal{I}(r_2), \mathcal{F}(r_1)\mathcal{F}(r_2)\}$
2) $r_1 \times r_2 = \left\{[s_{\vartheta(r_1)\times\vartheta(r_2),} s_{\tau(r_1)\times\tau(r_2)}], \begin{matrix} (\mathcal{T}(r_1)\mathcal{T}(r_2)), \mathcal{I}(r_1) + \mathcal{I}(r_2) - \mathcal{I}(r_1)\mathcal{I}(r_2), \\ \mathcal{F}(r_1) + \mathcal{F}(r_2) - \mathcal{F}(r_1)\mathcal{F}(r_2) \end{matrix}\right\}$
3) $\zeta r_1 = \left\{\langle [s_{\zeta\vartheta(r_1),} s_{\zeta\tau(r_1)}], \left(1 - (1 - \mathcal{T}(r_1))^{\zeta}, (\mathcal{I}(r_1)^{\zeta}, (\mathcal{F}(r_1)^{\zeta}\right), \rangle\right\}$
4) $r_1{}^{\zeta} = \left\{\langle [s_{\vartheta^{\zeta}(r_1),} s_{\tau^{\zeta}(r_1)}], \left((\mathcal{T}(r_1))^{\zeta}, 1 - (1 - \mathcal{I}(r_1)^{\zeta}, 1 - (1 - \mathcal{F}(r_1)^{\zeta}\right)\rangle\right\}$

2.3 Hesitant fuzzy set

HFS was first presented by Torra et al. [36, 37], they defined HFS as follows:

Definition 2.3.1 [36, 37]. Let $\mathbb{X}$ be a fixed domain set, a HFS $\mathcal{B}$ in $\mathbb{X}$ is defined in terms of a function $h_{\mathcal{B}}(b)$ that when applied to $\mathbb{X}$ returns a discrete subset of [0, 1], mathematically HFS can be expressed as:

$$\mathcal{B} = \{b, h_{\mathcal{B}}(b) | b \in \mathbb{X}\}$$

Where $h_{\mathcal{B}}(b) = \cup_{\rho_{\mathcal{B}}(b) \in h_{\mathcal{B}}(b)} \{\rho_{\mathcal{B}}(b)\}$ is a set of some different numbers in [0,1], representing the possible membership degree of the element $b \in \mathbb{X}$ to $\mathcal{B}$. For simplicity, we call $h_{\mathcal{B}}(b)$ a hesitant fuzzy number denoted by h, which reads $h = \cup_{\rho \in h} \{\rho\}$ for $\rho \in [0,1]$.

Definition 2.3.2 [41] . Let h,h_1 and h_2 be three hesitant fuzzy numbers and $\sigma \geq 0$, then the operational rules defined for hesitant fuzzy elements as follows:

(1) $h_1 + h_2 = \cup_{\rho_1 \in h_1, \rho_2 \in h_2} \{\rho_1 + \rho_2 - \rho_1\rho_2\}$

(2) $h_1 \times h_2 = \cup_{\rho_1 \in h_1, \rho_2 \in h_2} \{\rho_1\rho_2\}$

(3) $h^{\sigma} = \cup_{\rho \in h} \{\rho^{\sigma}\}$

(4) $\sigma\hbar = \cup_{\rho\in\hbar}\{1-(1-\rho)^{\sigma}\}$

3. Hesitant Single valued neutrosophic uncertain linguistic set

Definition 3.1. Let $\mathbb{U}$ be a non-empty domain set and $\mathcal{S}=\{s_1,s_2,\dots,s_{l-1}\}$ be a finite ordered discrete linguistic set. Then, a HSVNULS in $\mathbb{U}$ is a structure of the form which is mathematically represented as:

$$\mathcal{N}=\{\langle a,\mathbb{m}_{\mathcal{N}}(a)\rangle | a\in\mathbb{U}\},$$

In which $\mathbb{m}_{\mathcal{N}}(a)=\cup_{v_{\mathcal{N}}(a)\in\mathbb{m}_{\mathcal{N}}(a)}\{v_{\mathcal{N}}(a)\}$ is a set of SVNULEs, representing the possible SVNULEs of the element $a\in\mathbb{U}$ to the set $\mathcal{N}$, and $v_{\mathcal{N}}(a)=\{\langle[s_{\vartheta(a)},s_{\tau(a)}],(\mathcal{T}_{\mathcal{N}}(a),\mathcal{I}_{\mathcal{N}}(a),\mathcal{F}_{\mathcal{N}}(a))\rangle\}$ is a SVNULE. For simplicity, we shall denote $\mathbb{m}_{\mathcal{N}}(a)=\cup_{v_{\mathcal{N}}(a)\in\mathbb{m}_{\mathcal{N}}(a)}\{v_{\mathcal{N}}(a)\}$ by $\mathbb{m}=\cup_{v\in\mathbb{m}}\{v\}$, where $\mathbb{m}$ is called HSVNULE and $v=\{\langle[s_{\vartheta(a)},s_{\tau(a)}],(\mathcal{T}(a),\mathcal{I}(a),\mathcal{F}(a))\rangle\}$ is called SVNULE. Then, $\mathcal{N}$ is the set of all HSVNULEs.

Definition 3.2. Let $\mathbb{m},\mathbb{m}_1$ and $\mathbb{m}_2$ be any three HSVNULEs and $\zeta\geq 0$, then we defined the following operational rules for HSVNULEs which are stated below:

$$\text{(a)}\ \mathbb{m}_1\otimes\mathbb{m}_2=\cup_{v_1\in\mathbb{m}_1,v_2\in\mathbb{m}_2}\left\{\left\langle \begin{array}{c}[s_{\vartheta(v_1)\times\vartheta(v_2)},s_{\tau(v_1)\times\tau(v_2)}],\\ (\mathcal{T}(v_1)\mathcal{T}(v_2),\mathcal{I}(v_1)+\mathcal{I}(v_1)-\mathcal{I}(v_1)\mathcal{I}(v_1),\mathcal{F}(v_1)+\mathcal{F}(v_1)-\mathcal{F}(v_1)\mathcal{F}(v_1))\end{array}\right\rangle\right\} \quad (1)$$

$$\text{(b)}\ \mathbb{m}_1\oplus\mathbb{m}_2=\cup_{v_1\in\mathbb{m}_1,v_2\in\mathbb{m}_2}\left\{\left\langle \begin{array}{c}[s_{\vartheta(v_1)+\vartheta(v_2)},s_{\tau(v_1)+\tau(v_2)}],\\ (\mathcal{T}(v_1)+\mathcal{T}(v_2)-\mathcal{T}(v_1)\mathcal{T}(v_2),\mathcal{I}(v_1)\mathcal{I}(v_1),\mathcal{F}(v_1)\mathcal{F}(v_1))\end{array}\right\rangle\right\} \quad (2)$$

$$\text{(c)}\ \zeta\mathbb{m}=\cup_{v\in\mathbb{m}}\{\langle[s_{\zeta\vartheta(v)},s_{\zeta\tau(v)}],(1-(1-\mathcal{T}(v))^{\zeta},(\mathcal{I}(v))^{\zeta},(\mathcal{F}(v))^{\zeta})\rangle\} \quad (3)$$

$$\text{(d)}\ \mathbb{m}^{\zeta}=\cup_{v\in\mathbb{m}}\{\langle[s_{\vartheta^{\zeta}(v)},s_{\tau^{\zeta}(v)}],((\mathcal{T}(v))^{\zeta},1-(1-\mathcal{I}(v))^{\zeta},1-(1-\mathcal{F}(v))^{\zeta})\rangle\} \quad (4)$$

Definition 3.3. Let $\mathbb{m}$ be HSVNULE. Then, we defined respectively, the score, accuracy and certainty functions for HSVNULE $\mathbb{m}$, as follows:

$$\mathbb{S}(\mathbb{m})=\frac{1}{\#_{\mathfrak{z}}}\sum_{v\in\mathbb{m}}\frac{(2+\mathcal{T}(v)-\mathcal{I}(v)-\mathcal{F}(v))(\vartheta+\tau)}{6(l-1)} \quad (5)$$

$$\mathbb{E}(\mathbb{m})=\frac{1}{\#_{\mathfrak{z}}}\sum_{v\in\mathbb{m}}\frac{(\mathcal{T}(v)-\mathcal{F}(v))(\vartheta+\tau)}{2(l-1)} \quad (6)$$

$$\mathbb{C}(\mathbb{m})=\frac{1}{\#_{\mathfrak{z}}}\sum_{v\in\mathbb{m}}\frac{(\mathcal{T}(v))(\vartheta+\tau)}{2(l-1)} \quad (7)$$

In which, $\#_{\mathfrak{z}}$ represents the number of SVNULEs in $\mathbb{m}$ and l is the cardinality of the linguistic set $\mathbb{S}$.

Definition 3.4. Let $\mathbb{m}_1$ and $\mathbb{m}_2$ be any two HSVNULEs. Then the comparision between HSVNLEs can be defined as follows:

(a) If $\mathbb{S}(\mathbb{m}_1)>S(\mathbb{m}_2)$ then $\mathbb{m}_1>\mathbb{m}_2$,
(b) If $\mathbb{S}(\mathbb{m}_1)=\mathbb{S}(\mathbb{m}_2)$ and $\mathbb{E}(\mathbb{m}_1)>\mathbb{E}(\mathbb{m}_1)$ then $\mathbb{m}_1>\mathbb{m}_2$,
(c) If $\mathbb{S}(\mathbb{m}_1)=\mathbb{S}(\mathbb{m}_2)$, $\mathbb{E}(\mathbb{m}_1)=\mathbb{E}(\mathbb{m}_1)$ and $\mathbb{C}(\mathbb{m}_1)>C(\mathbb{m}_2)$ then $\mathbb{m}_1>\mathbb{m}_2$,

Definition 3.5. Let $\mathbb{m}_1$ and $\mathbb{m}_2$ be any two HSVNULEs and $\zeta,\zeta_1,\zeta_2\geq 0$. Then, the operational rules defined for HSVNULEs, have the following properties:

(1) $\mathbb{m}_1 \oplus \mathbb{m}_2 = \mathbb{m}_2 \oplus \mathbb{m}_1$ (8)

(2) $\mathbb{m}_1 \otimes \mathbb{m}_2 = \mathbb{m}_2 \otimes \mathbb{m}_1$ (9)

(3) $\zeta(\mathbb{m}_1 \oplus \mathbb{m}_2) = \zeta \mathbb{m}_1 \oplus \zeta \mathbb{m}_2$ (10)

(4) $\zeta_1 \mathbb{m}_1 \oplus \zeta_2 \mathbb{m}_1 = (\zeta_1 + \zeta_2)\mathbb{m}_1, \zeta_1, \zeta_2 \geq 0$ (11)

(5) $\mathbb{m}_1{}^{\zeta_1} \otimes \mathbb{m}_1{}^{\zeta_2} = \mathbb{m}_1{}^{(\zeta_1+\zeta_2)}, \zeta_1, \zeta_2 \geq 0$ (12)

(6) $\mathbb{m}_1{}^{\zeta} \otimes \mathbb{m}_2{}^{\zeta} = (\mathbb{m}_1 \otimes \mathbb{m}_2)^{\zeta}, \zeta \geq 0$ (13)

Proof.(1) and (2) are Obviously true from the definition (3.2) according to the operational rules expressed by Eqs. (1) and (2).

(3) From the L.H.S of (10), we have,

$$\mathbb{m}_1 \oplus \mathbb{m}_2 = \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2} \{[\mathscr{s}_{\vartheta(v_1)+\vartheta(v_2)}, \mathscr{s}_{\tau(v_1)+\tau(v_2)}], (\mathcal{T}(v_1) + \mathcal{T}(v_2) - \mathcal{T}(v_1)\mathcal{T}(v_2), \mathcal{I}(v_1)\mathcal{I}(v_2), \mathcal{F}(v_1)\mathcal{F}(v_2))\}$$

Then

$$\zeta(\mathbb{m}_1 \oplus \mathbb{m}_2) = \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2} \langle [\mathscr{s}_{\zeta(\vartheta(v_1)+\vartheta(v_2))}, \mathscr{s}_{\zeta(\tau(v_1)+\tau(v_2))}], \begin{pmatrix} 1 - \left(1 - \left(\mathcal{T}(v_1) + \mathcal{T}(v_2) - \mathcal{T}(v_1)\mathcal{T}(v_2)\right)\right)^{\zeta} \\ \left(\mathcal{I}(v_1)\mathcal{I}(v_2)\right)^{\zeta}, \left(\mathcal{F}(v_1)\mathcal{F}(v_2)\right)^{\zeta} \end{pmatrix} \rangle \quad (1^*)$$

And from R.H.S of (10), we have

$$\zeta \mathbb{m}_1 = \cup_{v_1 \in \mathbb{m}_1} \langle [\mathscr{s}_{\zeta\vartheta(v_1)}, \mathscr{s}_{\zeta\tau(v_1)}], \left(1 - \left(1 - \mathcal{T}(v_1)\right)^{\zeta}, \left(\mathcal{I}(v_1)\right)^{\zeta}, \left(\mathcal{F}(v_1)\right)^{\zeta}\right) \rangle$$

$$\zeta \mathbb{m}_2 = \cup_{v_2 \in \mathbb{m}_2} \langle [\mathscr{s}_{\zeta\vartheta(v_2)}, \mathscr{s}_{\zeta\tau(v_2)}], \left(1 - \left(1 - \mathcal{T}(v_2)\right)^{\zeta}, \left(\mathcal{I}(v_2)\right)^{\zeta}, \left(\mathcal{F}(v_2)\right)^{\zeta}\right) \rangle$$

Then

$$\zeta \mathbb{m}_1 \oplus \zeta \mathbb{m}_2 = \cup_{v_1 \in \mathbb{m}_1, v \in \mathbb{m}_2} \langle [\mathscr{s}_{\zeta(\vartheta(v_1)+\vartheta(v_2))}, \mathscr{s}_{\zeta(\tau(v_1)+\tau(v_2))}], \Big(1 - \left(1 - \mathcal{T}(v_1)\right)^{\zeta} + 1 - \left(1 - \mathcal{T}(v_2)\right)^{\zeta} \\ - \left(1 - \left(1 - \mathcal{T}(v_1)\right)^{\zeta}\right)\left(1 - \left(1 - \mathcal{T}(v_2)\right)^{\zeta}\right), \left(\mathcal{I}(v_1)\mathcal{I}(v_2)\right)^{\zeta}, \left(\mathcal{F}(v_1)\mathcal{F}(v_2)\right)^{\zeta}\Big) \rangle$$

$$\zeta \mathbb{m}_1 \oplus \zeta \mathbb{m}_2 = \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2} \langle [\mathscr{s}_{\zeta(\vartheta(v_1)+\vartheta(v_2))}, \mathscr{s}_{\zeta(\tau(v_1)+\tau(v_2))}], \begin{pmatrix} 1 - \left(1 - \left(\mathcal{T}(v_1) + \mathcal{T}(v_2) - \mathcal{T}(v_1)\mathcal{T}(v_2)\right)\right)^{\zeta} \\ \left(\mathcal{I}(v_1)\mathcal{I}(v_2)\right)^{\zeta}, \left(\mathcal{F}(v_1)\mathcal{F}(v_2)\right)^{\zeta} \end{pmatrix} \rangle \quad (2^*)$$

From (1*) and (2*) we get

$$\zeta(\mathbb{m}_1 \oplus \mathbb{m}_2) = \zeta \mathbb{m}_1 \oplus \zeta \mathbb{m}_2$$

(4) The proof of (11) is same as the proof of (10), therefore we omitted here.

(5) From the L.H.S of (12), we have

$$\mathbb{m}_1{}^{\zeta_1} = \cup_{v_1 \in \mathbb{m}_1} \langle \left[\mathscr{s}_{(\vartheta(v_1))^{\zeta_1}}, \mathscr{s}_{(\tau(v_1))^{\zeta_1}}\right], \left(\left(\mathcal{T}(v_1)\right)^{\zeta_1}, 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_1}, 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_1}\right) \rangle$$

$$\mathbb{m}_1{}^{\zeta_2} = \cup_{v_1 \in \mathbb{m}_1} \langle \left[\mathscr{s}_{(\vartheta(v_1))^{\zeta_2}}, \mathscr{s}_{(\tau(v_1))^{\zeta_2}}\right], \left(\left(\mathcal{T}(v_1)\right)^{\zeta_2}, 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_2}, 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_2}\right) \rangle$$

Then

$$\mathbb{m}_1{}^{\zeta_1} \otimes \mathbb{m}_1{}^{\zeta_2} = \cup_{v_1 \in \mathbb{m}_1} \langle \left[s_{(\vartheta(v_1))^{\zeta_1+\zeta_2}}, s_{(\tau(v_1))^{\zeta_1+\zeta_2}} \right], \left(\left(\mathcal{T}(v_1)\right)^{\zeta_1+\zeta_2}, 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_1} + 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_2} - \left(1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_1}\right)\left(1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_2}\right), 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_1} + 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_2} - \left(1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_1}\right)\left(1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_2}\right) \right) \rangle$$

$$\mathbb{m}_1{}^{\zeta_1} \otimes \mathbb{m}_1{}^{\zeta_2} = \cup_{v_1 \in \mathbb{m}_1} \langle \left[s_{(\vartheta(v_1))^{\zeta_1+\zeta_2}}, s_{(\tau(v_1))^{\zeta_1+\zeta_2}} \right], \left(\left(\mathcal{T}(v_1)\right)^{\zeta_1+\zeta_2}, 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_1+\zeta_2}, 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_1+\zeta_2} \right) \rangle$$
(3*)

And from the R.H.S, we have

$$\mathbb{m}_1{}^{\zeta_1+\zeta_2} = \cup_{v_1 \in v_1} \langle \left[s_{(\vartheta(v_1))^{\zeta_1+\zeta_2}}, s_{(\tau(v_1))^{\zeta_1+\zeta_2}} \right], \left(\left(\mathcal{T}(v_1)\right)^{\zeta_1+\zeta_2}, 1 - \left(1 - \mathcal{I}(v_1)\right)^{\zeta_1+\zeta_2}, 1 - \left(1 - \mathcal{F}(v_1)\right)^{\zeta_1+\zeta_2} \right) \rangle$$
(4*)

From (3*) and (4*), we have

$$\mathbb{m}_1{}^{\zeta_1} \otimes \mathbb{m}_1{}^{\zeta_2} = \mathbb{m}_1{}^{(\zeta_1+\zeta_2)}$$

(1) The proof of (13) is same as the proof of (12), therefore we omitted here.

4. Hesitant single valued neutrosophic uncertain linguistic weighted aggregation operators

In this section we defined some weighted aggregation operators based on the operational rules of HSVNULEs, such as hesitant single valued neutrosophic uncertain linguistic weighted average operators and hesitant single valued neutrosophic uncertain linguistic weighted geometric operator to aggregate hesitant single valued neutrosophic uncertain linguistics information.

Definition 4.1. Let $\mathbb{m}_z (z = 1,2, \ldots, m)$ be a collection of HSVNULEs. Then, the HSVNULWA operator is defined as,

$$HSVNULWA(\mathbb{m}_1, \mathbb{m}_2, \ldots, \mathbb{m}_m) = \sum_{z=1}^{m} w_z \mathbb{m}_z \quad (14)$$

In which, $w = (w_1, w_2, \ldots, w_z)^T$ is the weight vector of $\mathbb{m}_z (z = 1,2, \ldots, m)$, such that $w_z \in [0,1]$, $\sum_{z=1}^{m} w_z = 1$.

Theorem 4.2 . Let $\mathbb{m}_z (z = 1,2, \ldots, m)$ be a collection of HSVNULEs. Then by Eq.(14) and the operations defined for HSVNULNs, we can get the following result:

$$HSVNULWA(\mathbb{m}_1, \mathbb{m}_2, \ldots, \mathbb{m}_m) = \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}, \ldots, v_m \in \mathbb{m}_m} \left\{ \langle \left[s_{\sum_{z=1}^{m} w_z \vartheta(v_z)}, s_{\sum_{z=1}^{m} w_z \vartheta(v_z)} \right], \begin{pmatrix} \left(1 - \prod_{z=1}^{m} \left(1 - \mathcal{T}(v_z)\right)^{w_k}\right), \\ \prod_{z=1}^{m} \left(\mathcal{I}(v_k)\right)^{w_k}, \prod_{z=1}^{m} \left(\mathcal{F}(v_k)\right)^{w_k} \end{pmatrix} \rangle \right\}$$

(15)

Where $w = (w_1, w_2, \ldots, w_m)^T$ is the importance degree of $\mathbb{m}_z (z = 1,2, \ldots, m)$ satisfying the normalized condition that $w_z \in [0,1]\ for\ (z = 1,2, \ldots, m)$, $\Sigma_{z=1}^{m} w_z = 1$.

Proof. We prove Eq. (15) by using mathematical induction.

(a) When $z = 2$, then

$$\mathcal{w}_1 \mathbb{m}_1 = \cup_{v_1 \in \mathbb{m}_1} \left\{ \langle \left[s_{\mathcal{w}_1 \vartheta(v_1)}, s_{\mathcal{w}_1 \tau(v_1)} \right], \begin{pmatrix} \left(1 - \left(1 - \mathcal{T}(v_1)\right)^{\mathcal{w}_1}\right), \\ \left(\mathcal{I}(v_1)\right)^{\mathcal{w}_1}, \left(\mathcal{F}(v_1)\right)^{\mathcal{w}_1} \end{pmatrix} \rangle \right\}$$

$$\mathcal{w}_2 \mathbb{m}_2 = \cup_{v_2 \in \mathbb{m}_2} \left\{ \langle \left[s_{\mathcal{w}_2 \vartheta(v_2)}, s_{\mathcal{w}_2 \tau(v_2)} \right], \begin{pmatrix} \left(1 - \left(1 - \mathcal{T}(v_2)\right)^{\mathcal{w}_2}\right), \\ \left(\mathcal{I}(v_2)\right)^{\mathcal{w}_2}, \left(\mathcal{F}(v_2)\right)^{\mathcal{w}_2} \end{pmatrix} \rangle \right\}$$

Therefore,

$$HSVNULWA(\mathbb{m}_1, \mathbb{m}_2) = \mathcal{w}_1 \mathbb{m}_1 + \mathcal{w}_2 \mathbb{m}_2$$

$$= \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2} \left\{ \langle \left[s_{\sum_{z=1}^{2} \mathcal{w}_z \vartheta(v_z)}, s_{\sum_{z=1}^{2} \mathcal{w}_z \tau(v_z)} \right], \begin{pmatrix} 1 - \left(1 - \mathcal{T}(v_1)\right)^{\mathcal{w}_1} + 1 - \left(1 - \mathcal{T}(v_2)\right)^{\mathcal{w}_2} - \\ \left(1 - \left(1 - \mathcal{T}(v_1)\right)^{\mathcal{w}_1}\right)\left(1 - \left(1 - \mathcal{T}(v_2)\right)^{\mathcal{w}_2}\right) \\ \left(\left(\mathcal{I}(x_1)\right)^{\mathcal{w}_1}\right)\left(\left(\mathcal{I}(v_2)\right)^{\mathcal{w}_2}\right), \left(\left(\mathcal{F}(v_1)\right)^{\mathcal{w}_1}\right)\left(\left(\mathcal{F}(v_2)\right)^{\mathcal{w}_2}\right) \end{pmatrix} \rangle \right\}$$

$$= \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2} \left\{ \langle \left[s_{\sum_{z=1}^{2} \mathcal{w}_z \vartheta(v_z)}, s_{\sum_{z=1}^{2} \mathcal{w}_z \tau(v_z)} \right], \begin{pmatrix} 1 - \left(\left(1 - \mathcal{T}(v_1)\right)^{\mathcal{w}_1}\right)\left(\left(1 - \mathcal{T}(v_2)\right)^{\mathcal{w}_2}\right) \\ \prod_{z=1}^{2}\left(\left(\mathcal{I}(v_z)\right)^{\mathcal{w}_z}\right), \prod_{z=1}^{2}\left(\left(\mathcal{F}(v_z)\right)^{\mathcal{w}_z}\right) \end{pmatrix} \rangle \right\} \quad (16)$$

(b) Let us assume that the Eq. (15) is true for $z = u$, that is,

$$HSVNLWA(\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_u) = \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2, \dots, v_u \in \mathbb{m}_u} \left\{ \langle \left[s_{\sum_{z=1}^{u} \mathcal{w}_z \vartheta(v_z)}, s_{\sum_{z=1}^{u} \mathcal{w}_z \tau(v_z)} \right], \begin{pmatrix} \left(1 - \prod_{z=1}^{u}\left(1 - \mathcal{T}(v_z)\right)^{\mathcal{w}_z}\right), \\ \prod_{z=1}^{u}\left(\mathcal{I}(v_z)\right)^{\mathcal{w}_z}, \prod_{z=1}^{u}\left(\mathcal{F}(v_z)\right)^{\mathcal{w}_z} \end{pmatrix} \rangle \right\} \quad (17)$$

(c) Now we would like to prove that Eq. () is true for $z = u + 1$. So by applying Eqs. (16) and (17), we have

$$HSVNULWA(\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_{u+1}) =$$

$$\cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2, \dots, v_{u+1} \in \mathbb{m}_{u+1}} \left\{ \langle \left[s_{\sum_{z=1}^{u+1} \mathcal{w}_z \vartheta(v_z)}, s_{\sum_{z=1}^{u+1} \mathcal{w}_z \tau(v_z)} \right], \begin{pmatrix} \left(1 - \prod_{z=1}^{u}\left(1 - \mathcal{T}(v_z)\right)^{\mathcal{w}_z}\right) + 1 - \left(1 - \mathcal{T}(v_{u+1})\right)^{\mathcal{w}_{u+1}} \\ \left(1 - \prod_{z=1}^{u}\left(1 - \mathcal{T}(v_z)\right)^{\mathcal{w}_z}\right)\left(1 - \left(1 - \mathcal{T}(v_{u+1})\right)^{\mathcal{w}_{u+1}}\right) \\ \prod_{z=1}^{u+1}\left(\mathcal{I}(v_z)\right)^{\mathcal{w}_z}, \prod_{z=1}^{u+1}\left(\mathcal{F}(v_z)\right)^{\mathcal{w}_z} \end{pmatrix} \rangle \right\}$$

$$= \cup_{v_1 \in \mathbb{m}_1, v_2 \in \mathbb{m}_2, \dots, v_{u+1} \in \mathbb{m}_{u+1}} \left\{ \langle \left[s_{\sum_{z=1}^{u+1} \mathcal{w}_z \vartheta(v_z)}, s_{\sum_{z=1}^{u+1} \mathcal{w}_z \tau(v_z)} \right], \begin{pmatrix} \left(1 - \prod_{z=1}^{u+1}\left(1 - \mathcal{T}(v_z)\right)^{\mathcal{w}_k}\right) \\ \prod_{z=1}^{u+1}\left(\mathcal{I}(v_z)\right)^{\mathcal{w}_z}, \prod_{z=1}^{u+1}\left(\mathcal{F}(v_z)\right)^{\mathcal{w}_z} \end{pmatrix} \rangle \right\}$$

This shows that Eq. (15) is true for any k.This completes the proof.

From the above Definition (4.1) if the weight $\mathcal{w} = \left(\frac{1}{z}, \frac{1}{z}, \dots, \frac{1}{z}\right)^T$, then HSVNULWA operator reduces to a hesitant single valued neutrosophic uncertain linguistic averaging operator for HSVNULEs.

Now we should discuss some desired properties of the HSVNULWA operator, which are stated below:

(a) **Idempotency:** Let $\mathbb{m}_z (z = 1,2, \dots, m)$ be a collection of HSVNULEs. If $\mathbb{m}_z (z = 1,2, \dots, m)$ is equal and $\mathbb{m}_z = \mathbb{m} = \cup_{v\in\mathbb{m}}\{\langle[\mathcal{s}_{\vartheta(v)}, \mathcal{s}_{\tau(v)}], (\mathcal{T}(v), \mathcal{I}(v), \mathcal{F}(v))\rangle\}\ for\ z = 1,2, \dots, m$and $\mathcal{T}(v), \mathcal{I}(v), \mathcal{F}(v) \subseteq [0,1]$, then there is $HSVNULWA\ (\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_m) = \mathbb{m}$.

(b) **Boundedness:** Let $\mathbb{m}_z (z = 1,2, \dots, m)$ be a collection of HSVNULEs. If

$[\mathcal{s}_{\vartheta^-}, \mathcal{s}_{\tau^-}] = \min_{1\le z\le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in\ \mathbb{m}_z\}$,$[\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}] = \max_{1\le z\le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in\ \mathbb{m}_z\}$,

$\mathcal{T}^- = \min_{1\le z\le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\}$,

$\mathcal{T}^+ = \max_{1\le z\le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^- = \min_{1\le z\le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^+ = \max_{1\le z\le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\}$,

$\mathcal{F}^- = \min_{1\le z\le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{F}^+ = \max_{1\le z\le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, for\ z = 1,2, \dots, m$, then there is

$$\langle[\mathcal{s}_{\vartheta^-}, \mathcal{s}_{\tau^-}], (\mathcal{T}^-, \mathcal{I}^+, \mathcal{F}^+)\rangle \le HSVNULWA(\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_q) \le \langle[\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}], (\mathcal{T}^+, \mathcal{I}^-, \mathcal{F}^-)\rangle.$$

Proof.

(a) If $\mathbb{m}_z (z = 1,2, \dots, m)$ is equal and $\mathbb{m}_z = \mathbb{m} = \cup_{v\in\mathbb{m}}\{\langle[\mathcal{s}_{\vartheta(v)}, \mathcal{s}_{\tau(v)}], (\mathcal{T}(v), \mathcal{I}(v), \mathcal{F}(v))\rangle\}\ for\ z = 1,2, \dots, m$ and $\mathcal{T}(v), \mathcal{I}(v), \mathcal{F}(v) \subseteq [0,1]$, then we have

$$HSVNULWA\ (\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_m) = \cup_{v_1\in\mathbb{m}_1, v_2\in\mathbb{m}_2, \dots, v_m\in\mathbb{m}_m}\left\{\left\langle\left[\mathcal{s}_{\sum_{z=1}^m w_z\vartheta(v_z)}, \mathcal{s}_{\sum_{z=1}^m w_z\tau(v_z)}\right], \left(\begin{matrix}\left(1-\prod_{z=1}^m(1-\mathcal{T}(v_z))^{w_z}\right), \\ \prod_{z=1}^m(\mathcal{I}(v_z))^{w_z}, \prod_{z=1}^m(\mathcal{F}(v_z))^{w_z}\end{matrix}\right)\right\rangle\right\}$$

$$= \cup_{v\in\mathbb{m}}\left\{\left\langle\left[\mathcal{s}_{\vartheta(v_z)\sum_{z=1}^m w_z}, \mathcal{s}_{\tau(v_z)\sum_{z=1}^m w_z}\right], \left(\begin{matrix}\left(1-(1-\mathcal{T}(v))^{\sum_{z=1}^m w_z}\right), \\ (\mathcal{I}(v))^{\sum_{z=1}^m w_z}, (\mathcal{F}(v))^{\sum_{z=1}^m w_z}\end{matrix}\right)\right\rangle\right\}$$

$$= \cup_{v\in\mathbb{m}}\{\langle[\mathcal{s}_{\vartheta(v)}, \mathcal{s}_{\tau(v)}], (\mathcal{T}(v), \mathcal{I}(v), \mathcal{F}(v))\rangle\} = \mathbb{m}$$

(b)As $[\mathcal{s}_{\vartheta^-}, \mathcal{s}_{\tau^-}] = \min_{1\le z\le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in\ \mathbb{m}_z\}$, $[\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}] = \max_{1\le z\le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in\ \mathbb{m}_z\}$,

$\mathcal{T}^- = \min_{1\le z\le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\}$,

$\mathcal{T}^+ = \max_{1\le z\le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^- = \min_{1\le z\le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^+ = \max_{1\le z\le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\}$,

$\mathcal{F}^- = \min_{1\le z\le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{F}^+ = \max_{1\le z\le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, for\ z = 1,2, \dots, m$, then there is

$[\vartheta^-, \tau^-] \le [\vartheta(v_z), \tau(v_z)] \le [\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}], \mathcal{T}^- \le \mathcal{T}(v_z) \le \mathcal{T}^+, \mathcal{I}^- \le \mathcal{I}(v_z) \le \mathcal{I}^+, \mathcal{F}^- \le \mathcal{F}(v_z) \le \mathcal{F}^+\ for\ all\ z =$ $1,2, \dots, m$. We have

$[\vartheta(v), \tau(v)] = \sum_{z=1}^m w_z \ge [\sum_{z=1}^m w_z\vartheta^-, \sum_{z=1}^m w_z\tau^-] = [\vartheta^-, \tau^-]$,

$$\mathcal{T}(v) = \left\{\left(1 - \prod_{z=1}^m\left(1 - \mathcal{T}(v_z)\right)^{w_z}\right)\right\} \ge \left(1 - \prod_{z=1}^m(1 - \mathcal{T}^-)^{w_z}\right) = \mathcal{T}^-$$

$$\mathcal{I}(v) = \left\{\left(\prod_{z=1}^m\left(\mathcal{I}(v_z)\right)^{w_k}\right)\right\} \ge \left(\prod_{z=1}^m(\mathcal{I}^+)^{w_z}\right) = \mathcal{I}^+$$

$$\mathcal{F}(x) = \left\{\left(\prod_{z=1}^m\left(\mathcal{F}(v_z)\right)^{w_z}\right)\right\} \ge \left(\prod_{z=1}^m(\mathcal{F}^+)^{w_z}\right) = \mathcal{F}^+$$

Then there are the following scores:

$$\frac{1}{\#_{\mathfrak{z}}}\sum_{v\in \mathbb{m}}\frac{(2+\mathcal{T}(v)-\mathcal{I}(v)-\mathcal{F}(x))(\vartheta(v)+\tau(v))}{6(l-1)}=$$

$$\frac{1}{\#_{\mathfrak{z}}}\sum_{v\in \mathbb{m}}\left\{\frac{1}{6(l-1)}\left(\left[\sum_{z=1}^{m}w_z\vartheta(v_z),\sum_{z=1}^{m}w_z\tau(v_z)\right]\right)\begin{pmatrix}2+1-\prod_{z=1}^{m}\left(1-\mathcal{T}(v_z)\right)^{w_z}-\prod_{z=1}^{m}\left(\mathcal{I}(v_z)\right)^{w_z}\\ \prod_{z=1}^{m}\left(\mathcal{F}(v_z)\right)^{w_z}\end{pmatrix}\right\}$$

$$\geq \frac{(\vartheta^{-}+\tau^{-})(2+T^{-}-I^{+}-F^{+})}{6(l-1)},$$

Where $\#_{\mathfrak{z}}$ is the number of SVNULEs in $HSVNULWA(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_z)$ and l is the cardinality of the linguistic term set $\mathbb{S}$. Therefore according to the definition (3.3) there is

$$\langle[s_{\vartheta^{-}},s_{\tau^{-}}],(\mathcal{T}^{-},\mathcal{I}^{+},\mathcal{F}^{+})\rangle \leq HSVNULWA\left(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_q\right)$$

In a similar way we can show that there is $HSVNULWA\left(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_q\right) \leq \langle[s_{\vartheta^{+}},s_{\tau^{+}}],(\mathcal{T}^{+},\mathcal{I}^{-},\mathcal{F}^{-})\rangle$.

So we completed the proofs of the desired properties of HSVNULWA.

Definition 4.3. Let $\mathbb{m}_z(z=1,2,\dots,m)$ be a collection of HSVNULEs. Then the HSVNULWG operator is defined as,

$$HSVNULWG\left(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_q\right)=\prod_{z=1}^{m}\mathbb{m}_z{}^{w_z} \qquad (18)$$

Where $w=(w_1,w_2,\dots,w_m)^T$ is the importance degree of $\mathbb{m}_z(z=1,2,\dots,m)$ satisfying the condition that $w_z\in[0,1]\ for\ (z=1,2,\dots,m),\ \Sigma_{z=1}^{m}w_z=1$.

Theorem 4.4. Let $\mathbb{m}_z(z=1,2,\dots,m)$ be a collection of HSVNULEs. Then by Eq. (18) and the operations defined for HSVNULNs, we can get the following result:

$$HSVNULWG(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_z) = \bigcup_{v_1\in\mathbb{m}_1,v_2\in\mathbb{m}_2,\dots,v_z\in m_z}\left\{\left\langle\left[s_{\sum_{z=1}^{m}\vartheta^{w_z}(v_z)},s_{\sum_{z=1}^{m}\vartheta^{w_z}(v_z)}\right],\left(,\begin{pmatrix}\left(\prod_{z=1}^{m}\left(\mathcal{T}(v_z)\right)^{w_k}\right),\\ 1-\prod_{z=1}^{m}\left(1-\mathcal{I}(v_k)\right)^{w_k},1-\prod_{z=1}^{m}\left(1-\mathcal{F}(v_k)\right)^{w_k}\end{pmatrix}\right)\right\rangle\right\} \qquad (19)$$

Where $w=(w_1,w_2,\dots,w_m)^T$ is the importance degree of $\mathbb{m}_z(z=1,2,\dots,m)$ satisfying the condition that $w_z\in[0,1]\ for\ (z=1,2,\dots,m),\ \Sigma_{z=1}^{m}w_z=1$.

Proof. Same as theorem (4.2).

From the above Definition(4.3) if the weight $w=\left(\frac{1}{z},\frac{1}{z},\dots,\frac{1}{z}\right)^T$, then HSVNULWG operator reduces to a hesitant single valued neutrosophic uncertain linguistic geometric operator for HSVNLNs.

Now we should discuss some desired properties of the HSVNULWG operator, which are stated below:

(a) **Idempotency:**Let $\mathbb{m}_z(z=1,2,\dots,m)$ be a collection of HSVNULEs. If $\mathbb{m}_z(z=1,2,\dots,m)$ is equal and $\mathbb{m}_z=\mathbb{m}=\bigcup_{v\in\mathbb{m}}\left\{\left\langle\left[s_{\vartheta(v)},s_{\tau(v)}\right],(\mathcal{T}(v),\mathcal{I}(v),\mathcal{F}(v))\right\rangle\right\}\ for\ z=1,2,\dots,m$ and $\mathcal{T}(v),\mathcal{I}(v),\mathcal{F}(v)\subseteq[0,1]$, then there is $HSVNULWG(\mathbb{m}_1,\mathbb{m}_2,\dots,\mathbb{m}_m)=\mathbb{m}$.

(b) **Boundedness:** Let $\mathbb{m}_z (z = 1,2, \dots, m)$ be a collection of HSVNULEs. If

$[\mathcal{s}_{\vartheta^-}, \mathcal{s}_{\tau^-}] = \min_{1 \le z \le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in \mathbb{m}_z\}, [\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}] = \max_{1 \le z \le m}\{[\mathcal{s}_{\vartheta(v_z)}, \mathcal{s}_{\tau(v_z)}] | v_z \in \mathbb{m}_z\},$

$\mathcal{T}^- = \min\limits_{1 \le z \le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\},$

$\mathcal{T}^+ = \max\limits_{1 \le z \le m} \{\mathcal{T}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^- = \min\limits_{1 \le z \le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{I}^+ = \max\limits_{1 \le z \le m} \{\mathcal{I}(v_z) | v_z \in \mathbb{m}_z\},$

$\mathcal{F}^- = \min\limits_{1 \le z \le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, \mathcal{F}^+ = \max\limits_{1 \le z \le m} \{\mathcal{F}(v_z) | v_z \in \mathbb{m}_z\}, for\ z = 1,2, \dots, m$, then there is

$\langle [\mathcal{s}_{\vartheta^-}, \mathcal{s}_{\tau^-}], (\mathcal{T}^-, \mathcal{I}^+, \mathcal{F}^+) \rangle \le HSVNULWG(\mathbb{m}_1, \mathbb{m}_2, \dots, \mathbb{m}_q) \le \langle [\mathcal{s}_{\vartheta^+}, \mathcal{s}_{\tau^+}], (\mathcal{T}^+, \mathcal{I}^-, \mathcal{F}^-) \rangle.$

Proof. Same as the properties of HSVNLWA, so we omitted here.

5. Multiple Attribute Decision Making Method Based on HSVNULWA and HSVNULWG operators

In the coming section we presented a multiple attribute decision making method based on HSVNULWA and HSVNULWG operators and the score, accuracy and certainty functions of HSVNULEs under hesitant single valued neutrosophic uncertain linguistic environment.

For a multiple attribute decision making problem, let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_g\}$ be a set of alternatives and let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_h\}$ be a set of attributes. Let us assume that, the decision makers entered the importance degree of the criteria $\mathcal{C}_m$ $(m = 1,2, \dots, h)$ is represented by $w_m, w_m \in [0,1]$ and such that $\sum_{m=1}^{h} w_m = 1$.

In the decision process, the assessment information of the alternatives $\mathcal{A}_q$ $(q = 1,2, \dots, g)$ on the criteria $\mathcal{C}_m$ $(m = 1,2, \dots, h)$ is expressed in the form of hesitant single valued neutrosophic uncertain linguistic decision matrix denoted by $\mathbb{D} = (\mathbb{m}_{qm})_{g \times m}$, where $\mathbb{m}_{qm} (q = 1,2, \dots, g, m = 1,2, \dots, h)$ is a HSVNULE $\mathbb{m}_{qm} = \cup_{\mathbb{v}_{qm} \in \mathbb{m}_{qm}} \{\mathbb{v}_{qm}\}$ and $\mathbb{v}_{qm} = \langle [\mathcal{s}_{\vartheta(\mathbb{v}_{qm})}, \mathcal{s}_{\tau(\mathbb{v}_{qm})}], (\mathcal{T}(\mathbb{v}_{qm}), \mathcal{I}(\mathbb{v}_{qm}), \mathcal{F}(\mathbb{v}_{qm})) \rangle$ is a SVNULE.

Then the HSVNULWA operator or HSVNULWG operator is operated to ascertained a multiple attribute decision making method under hesitant single valued neutrosophic uncertain linguistic environment, which contained the following steps:

Step 1. By applying Eq. (15) or Eq. (18), to calculate the individual overall HSVNULE $\mathbb{m}_q$ for alternatives $\mathcal{A}_q (q = 1,2, \dots, g)$, that is;

$$\mathbb{m}_q = \cup_{\mathbb{v}_q \in \mathbb{m}_q} \{\mathbb{v}_q\} = HSVNULWA(\mathbb{m}_{q1}, \mathbb{m}_{q2}, \dots, \mathbb{m}_{qm})$$

$$= \cup_{v_{q1} \in \mathbb{m}_{q1}, v_{q2} \in \mathbb{m}_{q2}, \dots, v_{qm} \in \mathbb{m}_{qm}} \left\{ \left\langle \left[\mathcal{s}_{\sum_{m=1}^{h} w_m \vartheta(v_{qm})}, \mathcal{s}_{\sum_{m=1}^{h} w_{qm} \vartheta(v_{qm})} \right], \left(\begin{array}{c} \left(1 - \prod_{m=1}^{h} \left(1 - \mathcal{T}(v_{qm})\right)^{w_m}\right), \\ \prod_{m=1}^{h} \left(\mathcal{I}(v_{qm})\right)^{w_m}, \prod_{m=1}^{h} \left(\mathcal{F}(v_{qm})\right)^{w_m} \end{array} \right) \right\rangle \right\}$$

Or

$$\mathbb{m}_q = \cup_{\mathbb{v}_q \in \mathbb{m}_q} \{\mathbb{v}_q\} = HSVNULWG(\mathbb{m}_{q1}, \mathbb{m}_{q2}, \dots, \mathbb{m}_{qm})$$

$$= \cup_{v_{q1} \in \mathbb{m}_{q1}, v_{q2} \in \mathbb{m}_{q2}, \dots, v_{qm} \in \mathbb{m}_{qm}} \left\{ \langle \left[\mathcal{s}_{\sum_{m=1}^{h} \vartheta^{w m}(v_{qm})}, \mathcal{s}_{\sum_{m=1}^{h} \tau^{w m}(v_{qm})} \right], \begin{pmatrix} \left(\prod_{m=1}^{h} \left(\mathcal{T}(v_{qm}) \right)^{w m} \right), \\ 1 - \prod_{m=1}^{h} \left(1 - \mathcal{I}(v_{qm}) \right)^{w m}, 1 - \prod_{m=1}^{h} \left(1 - \mathcal{F}(v_{qm}) \right)^{w m} \end{pmatrix} \rangle \right\}$$

Step 2. By using Eqs. (5), (6), (7) to calculate the score function (accuracy function (), certainty function.

Step 3. Rank all the alternatives according to their score values of

Step 4. End.

6 Numerical Example

In this section a numerical example is adapted from Ye [14] to show the effectiveness of the proposed method.

An investment company wants to invest a sum of money in the best option. To invest the money there is a panel with four possible alternatives; $\mathcal{A}_1$(*a car company*), $\mathcal{A}_2$ (*a food company*), $\mathcal{A}_3$ (*a computer company*) and $\mathcal{A}_4$(*an arm company*). They must take a decision according to the following three criteria, $\mathcal{C}_1$ (*the risk*), $\mathcal{C}_2$ (*the growth*) and $\mathcal{C}_3$(*the environmental impact*). The importance degree of the criteria is given as $w = (0.35, 0.25, 0.4)^T$. Let us assume that there are three decision makers, which evaluate the alternatives $\mathcal{A}_i$ $(i = 1,2,3,4)$, with respect to criteria $\mathcal{C}_j$ *for* $j = 1,2,3$ by the form of HINULEs under the linguistic term set $\mathcal{S} = \{\mathfrak{s}_0 = \textit{extremely poor}, \mathfrak{s}_1 = \textit{very poor}, \mathfrak{s}_2 = \textit{poor}, \mathfrak{s}_3 = \textit{medium}, \mathfrak{s}_4 =$ *good, ꞩ5=very good, ꞩ6=extremely good*. For example, the HSVNULE of an alternative *𝒜1* with respect to the criteria $\mathcal{C}_1$ is given as $\{\langle[\mathcal{s}_3, \mathcal{s}_4], (0.5, 0.1, 0.2)\rangle, \langle[\mathcal{s}_4, \mathcal{s}_5], (0.3, 0.2, 0.1)\rangle\}$ by the three decision makers, Which indicate the assessment of the alternative $\mathcal{A}_1$ with respect to the criteria $\mathcal{C}_1$ is about the uncertain linguistic variable $[\mathcal{s}_3, \mathcal{s}_4]$ with truth, falsity and indeterminacy membership degrees 0.5,0.1,0.2 is given by the two experts of them and about the uncertain linguistic variable $[\mathcal{s}_4, \mathcal{s}_5]$ given by the one expert of them. Thus, when the four possible alternatives with respect to the above three criterion are evaluated by the three decision makers, the hesitant single valued neutrosophic uncertain linguistic decision matrix is constructed as shown in Table 1.

Then, the developed approach is utilized to obtain the ranking order of the alternatives and the most desirable one(s), which can be described as following steps:

Table 1 hesitant single valued neutrosophic uncertain linguistic decision matrix

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{A}_1$	$\left\{\begin{matrix}\langle[s_3,s_4],(0.5,0.1,0.2)\rangle,\\ \langle[s_4,s_5],(0.3,0.2,0.1)\rangle\end{matrix}\right\}$	$\{\langle[s_4,s_5],(0.5,0.2,0.3)\rangle\}$	$\left\{\begin{matrix}\langle[s_2,s_3],(0.4,0.1,0.3)\rangle,\\ \langle[s_3,s_4],(0.2,0.1,0.5)\rangle\end{matrix}\right\}$
$\mathcal{A}_2$	$\left\{\begin{matrix}\langle[s_4,s_5],(0.7,0.1,0.2)\rangle,\\ \langle[s_5,s_6],(0.6,0.1,0.1)\rangle\end{matrix}\right\}$	$\{\langle[s_4,s_5],(0.6,0.0,0.2)\rangle\}$	$\left\{\begin{matrix}\langle[s_4,s_5],(0.7,0.0,0.1)\rangle,\\ \langle[s_5,s_5],(0.6,0.1,0.1)\rangle,\\ \langle[s_5,s_6],(0.5,0.1,0.2)\rangle\end{matrix}\right\}$
$\mathcal{A}_3$	$\left\{\begin{matrix}\langle[s_3,s_4],(0.7,0.2,0.1)\rangle,\\ \langle[s_4,s_5],(0.5,0.3,0.2)\rangle\end{matrix}\right\}$	$\{\langle[s_4,s_5],(0.5,0.2,0.3)\rangle\}$	$\left\{\begin{matrix}\langle[s_3,s_4],(0.4,0.1,0.3)\rangle,\\ \langle[s_4,s_5],(0.2,0.1,0.5)\rangle\end{matrix}\right\}$
$\mathcal{A}_2$	$\{\langle[s_4,s_5],(0.7,0.0,0.1)\rangle\}$	$\left\{\begin{matrix}\langle[s_3,s_4],(0.6,0.1,0.2)\rangle,\\ \langle[s_4,s_4],(0.5,0.1,0.3)\rangle,\\ \langle[s_4,s_5],(0.4,0.2,0.1)\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle[s_3,s_4],(0.5,0.1,0.1)\rangle,\\ \langle[s_4,s_5],(0.3,0.0,0.1)\rangle\end{matrix}\right\}$

Step 1: Aggregate all HSVNULEs of $\mathbb{m}_{qm}$ $(q = 1,2,3,4; m = 1,2,3)$ by using the HSVNULWA operator to derive the collective HSVNULE $\mathbb{m}_q$ $(q = 1,2,3,4)$ for an alternative $\mathcal{A}_q(q = 1,2,3,4)$. Taking an alternative $\mathcal{A}_1$ for an example, we have

$$\mathbb{m}_1 = HSVNULWA(\mathbb{m}_{11}, \mathbb{m}_{12}, \mathbb{m}_{13})$$

$$= \bigcup_{v_{11}\in\mathbb{m}_{11}, v_{12}\in\mathbb{m}_{12}, v_{13}\in\mathbb{m}_{13}} \left\{ \left\langle \left[s_{\sum_{m=1}^{3} w_m \vartheta(v_{1m})}, s_{\sum_{m=1}^{3} w_m \tau(v_{1m})} \right], \left(\begin{matrix} \left(1-\prod_{m=1}^{3}\left(1-\mathcal{T}(v_{1m})\right)^{w_m}\right), \\ \prod_{m=1}^{3}\left(\mathcal{I}(v_{1m})\right)^{w_m}, \prod_{m=1}^{3}\left(\mathcal{F}(v_{1m})\right)^{w_m} \end{matrix} \right) \right\rangle \right\}$$

$$=\left\{\begin{matrix}\langle[s_{2.85},s_{3.85}],(0.4622,0.1189,0.2603)\rangle,\langle[s_{3.25},s_{4.25}],(0.5427,0.1189,0.3193)\rangle,\\ \langle[s_{3.2},s_{4.2}],(0.3950,0.1516,0.2042)\rangle,\langle[s_{3.6},s_{4.6}],(0.4855,0.1516,0.2505)\rangle\end{matrix}\right\}$$

Similarly we can derive the following collective HSVNULEs of $\mathbb{m}_q(q = 2,3,4)$

$$\mathbb{m}_2=\left\{\begin{matrix}\langle[s_4,s_5],(0.6776,0.0,0.1516)\rangle,\langle[s_{4.4},s_5],(0.6383,0.0,0.1516)\rangle,\\ \langle[s_{4.4},s_{5.4}],(0.6045,0.0,0.2)\rangle,\langle[s_{4.35},s_{5.35}],(0.6435,0.0,0.1189)\rangle,\\ \langle[s_{4.75},s_{5.35}],(0.6,0.0,0.1189)\rangle,\langle[s_{4.75},s_{5.75}],(0.5627,0.0,0.1569)\rangle\end{matrix}\right\}$$

$$\mathbb{m}_3=\left\{\begin{matrix}\langle[s_{3.25},s_{4.25}],(0.5502,0.1516,0.2042)\rangle,\langle[s_{3.65},s_{4.65}],(0.4954,0.1516,0.2505)\rangle,\\ \langle[s_{3.6},s_{4.6}],(0.4622,0.1747,0.2603)\rangle,\langle[s_4,s_5],(0.3966,0.1747,0.3193)\rangle\end{matrix}\right\}$$

$$\mathbb{m}_4=\left\{\begin{matrix}\langle[s_{3.35},s_{4.35}],(0.6045,0.0,0.1189)\rangle,\langle[s_{3.75},s_{4.75}],(0.5476,0.0,0.1189)\rangle,\\ \langle[s_{3.6},s_{4.35}],(0.5819,0.0,0.1316)\rangle,\langle[s_4,s_{4.75}],(0.5216,0.0,0.1316)\rangle,\\ \langle[s_{3.6},s_{4.6}],(0.5624,0.0,0.1)\rangle,\langle[s_4,s_5],(0.4993,0.0,0.1)\rangle\end{matrix}\right\}$$

Step 2: Calculate the score values of the collective HSVULE $\mathbb{m}_q(1,2,3,4)$ by using Eq. ():

$\mathbb{m}_1$=0.42998, $\mathbb{m}_2$=0.66890, $\mathbb{m}_3$=0.46933, $\mathbb{m}_4$=0.5646

Step 3: Rank all the alternatives according to their score values $\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$. Therefore, the alternative $\mathcal{A}_2$ is the best choice according to the largest score value.

On the other hand if we use HSNULWG operator to utilized in the MCDM problem, the decision making steps are described as follows:

Step 1: Aggregate all HSVNULEs of $\mathbb{m}_{qm}$ $(q = 1,2,3,4; m = 1,2,3)$ by using the HSVNULWA operator to derive the collective HSVNULE $\mathbb{m}_q$ $(q = 1,2,3,4)$ for an alternative $\mathcal{A}_q(q = 1,2,3,4)$.

$$\mathbb{m}_1=\left\{\begin{matrix}\langle[s_{2.7411}, s_{3.7679}], (0.4573,0.1261, 0.2665)\rangle, \langle[s_{3.2237}, s_{4.2295}], (0.5378,0.1261, 0.3589)\rangle, \\ \langle[s_{3.0314}, s_{4.0760}], (0.3824,0.1614, 0.2356)\rangle, \langle[s_{3.5652}, s_{4.5731}], (0.4498,0.1614, 0.3319)\rangle\end{matrix}\right\}$$

$$\mathbb{m}_2=\left\{\begin{matrix}\langle[s_4, s_5], (0.6735,0.0362, 0.1614)\rangle, \langle[s_{4.3734}, s_5], (0.6333,0.0760, 0.1614)\rangle, \\ \langle[s_{4.3734}, s_{5.3783}], (0.5887,0.0760, 0.2)\rangle, \langle[s_{4.3249}, s_{5.3295}], (0.6283,0.0362, 0.1261)\rangle, \\ \langle[s_{4.7287}, s_{5.3295}], (0.6,0.0760, 0.1261)\rangle, \langle[s_{4.7287}, s_{5.7327}], (0.5578,0.0760, 0.1663)\rangle\end{matrix}\right\}$$

$$\mathbb{m}_3=\left\{\begin{matrix}\langle[s_{3.2237}, s_{4.2295}], (0.6735,0.1614, 0.2356)\rangle, \langle[s_{3.6169}, s_{4.6244}], (0.6333,0.1614, 0.3319)\rangle, \\ \langle[s_{3.5652}, s_{4.5731}], (0.5887,0.1997, 0.2665)\rangle, \langle[s_4, s_5], (0.6382,0.1997, 0.3589)\rangle\end{matrix}\right\}$$

$$\mathbb{m}_4=\left\{\begin{matrix}\langle[s_{3.178}, s_{4.3249}], (0.5887,0.0662, 0.1261)\rangle, \langle[s_{3.3178}, s_{4.3249}], (0.4799,0.0260, 0.1261)\rangle, \\ \langle[s_{3.5652}, s_{4.3249}], (0.5625,0.0662, 0.1548)\rangle, \langle[s_4, s_{4.7278}], (0.4585,0.0260, 0.1548)\rangle, \\ \langle[s_{3.5652}, s_{4.5731}], (0.5320,0.0933, 0.1)\rangle, \langle[s_4, s_5], (0.4337,0.0543, 0.1)\rangle\end{matrix}\right\}$$

Scores of the

$\mathbb{m}_1$=0.40825, $\mathbb{m}_2$=0.64145, $\mathbb{m}_3$=0.490225, $\mathbb{m}_4$=0.52626

Step 3: Rank all the alternatives according to their score values $\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$. Therefore, the alternative $\mathcal{A}_2$ is the best choice according to the largest score value.

Obviously, above two kinds of ranking orders are the identical and the same as the one obtained by Ye [14]. Although, the two kinds of ranking orders based on the HSVNULWA and HSVNULG operators are identical, there are different focal point [14] between the HSVNULWA and HSVNULWG operators. The HSVNULWA operator emphasizes the group's major points, while the HSVNULWG operator emphasizes the individual major points. Then, decision makers select one of them according to their preference or real requirements. Compare with the relative decision making methods based on SVNULSs and SVNHFSs. The decision making method in this paper use HSVNUL information, While the decision making methods in Ye [12, 14] use SVNHF information.

Since HSVNULSs is further generalization of the SVNULSs and SVNHFSs. HSVNUL information includes SVNUL information and SVNHF information, and also the SVNULWA, SVNHFWA and SVNHFWA, SVNHFWG operators are special cases of the HSVULWA and HSVNULWG operators. Therefore the proposed method in this article can deal not only HSVNUL information but also can handle SVNHF information. In some extent the proposed decision making method in this article is good and suitable to handle HSVNUL information.

Conclusion

The paper presented the concept of HSVNULSs based on the combination of both HFSs and SVNULSs and is a further generalization of these fuzzy concepts and defined some operational rules and properties of HSVNULEs, and the score, accuracy and certainty functions of HSVNULEs. Then, we presented the HSVNULWA operator and the HSVNULWG operator to aggregate HSVNUL information and investigate some desired properties of these aggregation operators. Furthermore, the HSVNULWA and HSVNULWG operators were applied to a multiple attribute decision making method in which the criteria values take the form of HSVNULEs with respect to the alternatives and the criteria weights are known information. We utilize the score function (accuracy and certainty functions) to rank the alternatives and determine the best one(s). Finally, an example is illustrated about investment alternatives to show the effectiveness of the proposed decision making method. The main advantage of the proposed method is that it described the incomplete, indeterminate and inconsistent information by several SVNULNs in which uncertain linguistic variable indicate that whether criteria is good or bad in qualitative decision making and SVNNs are adopted to demonstrate the satisfaction degree, dissatisfaction degree, indeterminacy degree to a uncertain linguistic variable in quantitative. Therefore, the proposed decision making method under HSVNUL environment is more suitable for real scientific and engineering problems.

In future, we shall develop some more aggregation operators and similarity measure and applied them to MCGDM, medical diagnosis, fault diagnosis and information fusion.

7. Compliance with Ethical Structure

We declare that we have no commercial or associative interest that represents a conflict of interest in connection with work submitted.

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