Prolate Spheroidal Wave Function as Exact solution of the Schrödinger Equation

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Abstract

In quantum mechanics, the wave function and energy are required for the complete characterization of fundamental properties of a physical system subject to a potential energy. It is proved in this work, the existence of a Schrödinger equation with position-dependent mass having the prolate spheroidal wave function as exact solution, resulting from a classical quadratic Liénard-type oscillator equation. This fact may allow the extension of the current one-dimensional model to three dimensions and increase the understanding of analytical features of quantum systems.

Keywords: Schrödinger equation, position-dependent mass, quadratic Liénard-type oscillator equation, prolate spheroidal wave function.

Introduction

The prolate spheroidal wave functions (PSWFs) originally appeared during the resolution of the Helmholtz equation in the spheroidal coordinate system by variables separation [1-6]. Later it is discovered that they are essential for the description of electromagnetic wave propagation, for the signal processing and for many other physical phenomena [1-6]. It is therefore reasonable to be interested in expressing the solution of the Schrödinger equation in terms of prolate spheroidal wave functions. The solution of the Schrödinger equation is fundamental for capturing the behavior of a physical system in quantum mechanics. Many useful modern physical and engineering applications in quantum mechanics are carried out on the basis of the harmonic oscillator with position-dependent mass for more performance [7]. The problem of quantum characterization of such oscillators is not completely resolved since the traditional quantum mechanics approach no longer works. This results in the fact that the generalized momentum operator does not commute with the mass. So, the topic of fundamental features of quantum harmonic oscillator with position-dependent mass is still an attractive mathematical physics research field. In this regard, a lot of work has been focused on the quantization of quadratic Liénard type oscillator equations since they admit a position-dependent mass description

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[1-4]. Recent works in quantum theory have shown that the Schrödinger equation with position-dependent mass may be expressed in terms of the prolate spheroidal wave functions (PSWFs) [1-3]. Under these conditions one may ask whether there is a Schrödinger equation whose exact eigensolutions are the prolate spheroidal wave functions. It appears then logic to investigate the existence of a Schrödinger equation which admits the PSWFs as exact analytical solutions. Recently, Monsia et al. [8-10] have shown the existence of a class of quadratic Liénard type oscillator equations exhibiting an exact harmonic periodic solution but also with amplitude-dependent frequency. The class of quadratic nonlinear oscillator equations under question may be written as

$$\ddot{x} - a\,\varphi'(x)\,\dot{x}^2 + bx\exp[2a\,\varphi(x)] = 0 \tag{1}$$

where a and $b \geq 0$ are arbitrary parameters, and $\varphi(x)$ an arbitrary function, prime stands for differentiation with respect to x, and dot denotes differentiation with respect to time. The general solution to (1) takes the form

$$x(t) = A\sin\left(\sqrt{b}\,\phi(t) + \theta_0\right) \tag{2}$$

with

$$\phi(t) = \tau\,,\ \frac{d\tau}{dt} = \exp\left[a\,\varphi\left(y(\tau)\right)\right] \ \text{ and } \ y(\tau) = A\sin(\sqrt{b}\tau + \theta_0) \tag{3}$$

By choosing $\varphi(x) = -\frac{1}{2}\ln\left(1+\mu x^2\right)$, the equation (1) becomes

$$\ddot{x} + \frac{a\mu x}{1+\mu x^2}\dot{x}^2 + \frac{bx}{\left(1+\mu x^2\right)^a} = 0 \tag{4}$$

Substituting $a = 0$, into (4) yields the linear harmonic oscillator equation. For $a = 1$, and $\mu > 0$, the equation (4) gives the equation of motion of a particle on a rotating parabola. The question to answer in this context is then to know whether the quantization of (4) may lead to the PSWFs as exact solution of the Schrödinger equation under a specified parametric choice. To be more precise the question to answer becomes: Can the quantization of (4) yield the prolate spheroidal wave function as exact solution of the Schrödinger equation? The present work purposes to demonstrate the existence of such a Schrödinger equation exhibiting the prolate spheroidal wave function as exact solution. This prediction is effectively relevant since the PSWFs are deeply studied in mathematical physics and are classically defined as a series of normalized Legendre polynomials which are well known as special functions of mathematics. This prediction according to [4] may also be extended to three dimensions and discrete formulation. Therefore such a solution may contribute to analytically deepen the understanding of fundamental features of Schrödinger wave functions. It is also interesting to mention that the current prediction may allow the assessment of the quadratic nonlinear dissipation effects on the quantum properties of the harmonic oscillator equation. That being so, to reach the fixed objective, it is needed to first express the Schrödinger equation with position-dependent mass associated to (4) (section 2), and secondly determine the exact solution and discuss its properties (section 3). Finally, a conclusion for the work is addressed.

2. **Schrödinger equation with position-dependent mass**

The quantum mechanics of harmonic oscillator with position-dependent mass is usually carried out by using the Hamiltonian operator developed by von Roos [11]. However, the question of quantization is not wholly resolved given that this Hamiltonian leads to a problem of ambiguity parameters. So, an adequate choice of ambiguity parameters consists of a prerequisite for the satisfactory Hermitian Hamiltonian to be applied.

2.1 Hamiltonian operator

To establish the Schrödinger equation in question, the following quantum Hamiltonian [12]

$$H = -\frac{\hbar^2}{2M(x)}\frac{\partial^2}{\partial x^2} + \frac{\hbar^2 M'(x)}{2M(x)^2}\frac{\partial}{\partial x} + V(x) \tag{5}$$

is considered. In (5), $M(x)$ denotes the position-dependent mass, $M'(x)$ is its first derivative with respect to x and $V(x)$ the potential energy. So with that, the Schrödinger equation under consideration may be specified.

2.2 Time-independent Schrödinger equation

By application of (5) to the Schrödinger wave function $\psi(x)$, and designating by E the energy, that is

$$H\psi = E\psi$$

one may obtain the following Schrödinger equation

$$\psi''(x) - \frac{2\mu a x}{1+\mu x^2}\psi'(x) + \left(1+\mu x^2\right)^a \left[2E - bx^2\right]\psi(x) = 0 \tag{6}$$

where prime designates differentiation with respect to x, $\hbar = 1, M(x) = \left(1+\mu x^2\right)^a$, and the potential $V(x) = \frac{1}{2}bx^2$. It seems that the general solution to (6) under arbitrary parameters a and μ is not feasible. In this regard, to attain the predicted result, there is a requirement to make a satisfactory parametric choice to identify the solution $\psi(x)$ for the given eigenvalue E to the prolate spheroidal wave functions.

3. **Prolate spheroidal wave function as exact solution**

The above shows clearly that the parametric choice $a=-1$, and $\mu=-1$, suffices to reduce (6) to the well known prolate spheroidal wave equation [4-6]

$$\psi''(x)-\frac{2x}{1-x^2}\psi'(x)+\left(\frac{2E-bx^2}{1-x^2}\right)\psi=0 \tag{7}$$

under the requirement that $|x|<1$ and $\psi(-1)=\psi(1)=0$. The desired exact solution to (7) corresponding to the eigenvalue E_n may immediately be written in the classical series form [4-6]

$$\psi_n(x)=\sum_{k=0}^{\infty}\beta_k^{(n,\omega)}\overline{P_k}(x) \tag{8}$$

where $n\geq 0$ is an integer, $\omega=\sqrt{b}$ is an arbitrary parameter, and the coefficients $\beta_k^{(n,\omega)}$ are such that $\beta_k^{(n,\omega)}=\int_{-1}^{1}\psi_n(x)\overline{P_k}(x)dx$, with $k=0,1,2,....$ The functions $\overline{P_k}(x)$ mean the normalized Legendre polynomials [4, 6]. The real $\omega>0$ ensures the existence of eigenvalues E_n associated to the wave functions $\psi_n(x)$. These eigenvalues form a set of positive numbers strictly increasing [4-6], $E_0<E_1<...$, such that for each non-negative integer $n\geq 0$, the eigensolutions $\psi_n(x)$ are continuous and bounded on [-1, 1]. According to [4] such a definition of (8) is suited for large values of the function order n compared to ω, termed here as the band-limit. In this way, the authors in [4] have developed asymptotic expressions for the wavefunctions $\psi_n(x)$ and their associated eigenvalues, which are successfully tested by numerical experiments. The same authors in [5], investigated analytically as well as numerically the behavior of PSWFs in the case of large values of the band-limit, here, ω, compared to the order of the function, n, to finally obtain an Hermite series for their formula. In this perspective, the authors in [5] developed asymptotic expressions for PSWFs and their associated eigenvalues. According to [5], as $\omega\rightarrow\infty$, the eigenvalues associated to the solutions $\psi_n(x)$ of the Schrödinger equation (7) take the form $E_n=\omega(n+\frac{1}{2})+O(1)$, so that one recovers, knowing $\hbar=1$, the energy eigenvalues of the harmonic oscillator. This demonstrates the effect of quadratic nonlinear dissipation on the quantum mechanics of the harmonic oscillator so that the latter may be viewed in a certain sense as the limiting case of the quantum model developed in this work. An important implication of this fact is that the ground state energy obtained for $n=0$, is not zero, as a consequence of the Heisenberg relation which is well known to be connected to the PSWFs [4].

4. **Conclusion**

The quantum mechanics of a quadratic Liénard type oscillator equation is carried out. The resulting Schrödinger equation with position-dependent mass is found to exhibit the prolate spheroidal wave function as exact solution. The energy spectrum is given by the prolate eigenvalues. As a result, the energy spectrum of the harmonic oscillator is recovered from the current quantum model as the angular frequency that is to say, the band-limit tends to infinity. It is finally to worth noting that the three dimensions and discrete versions of the developed one-dimensional Schrödinger equation may be performed due to the fact that the latter exhibits PSWFs as eigensolutions.

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