Basic Properties Of Second Smarandache Bol Loops *†

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Abstract

The pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$. A special loop $(G_H, \cdot)$ is called a second Smarandache Bol loop($S_{2^{nd}}BL$) if and only if it obeys the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H. The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{nd}}BLs$ generalize Bol loops. The basic properties of $S_{2^{nd}}BLs$ are studied. These properties are all Smarandache in nature. The results in this work generalize the basic properties of Bol loops, found in the Ph.D. thesis of D. A. Robinson. Some questions for further studies are raised.

1 Introduction

The study of the Smarandache concept in groupoids was initiated by W. B. Vasantha Kandasamy in [23]. In her book [21] and first paper [22] on Smarandache concept in loops, she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. The present author has contributed to the study of S-quasigroups and S-loops in [5, 6, 7, 8, 9, 10, 11, 12] by introducing some new concepts immediately after the works of Muktibodh [14, 15]. His recent monograph [13] gives inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed in the aforementioned papers.

But in the quest of developing the concept of Smarandache quasigroups and loops into a theory of its own just as in quasigroups and loop theory(see [1, 2, 3, 4, 16, 21]), there is the need to introduce identities for types and varieties of Smarandache quasigroups and loops. For now, a Smarandache loop or Smarandache quasigroup will be called a first Smarandache loop($S_{1^{st}}$-loop) or first Smarandache quasigroup($S_{1^{st}}$-quasigroup).

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Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Furthermore, if there exists a unique element $e = e_\rho = e_\lambda$ in L called the identity element such that for all x in L, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. A loop is called a right Bol loop(Bol loop in short) if and only if it obeys the identity

$$(xy \cdot z)y = x(yz \cdot y).$$

This class of loops was the first to catch the attention of loop theorists and the first comprehensive study of this class of loops was carried out by Robinson [18].

The aim of this work is to introduce and study the basic properties of a new class of loops called second Smarandache Bol loops($S_{2^{nd}}$BLs). The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{nd}}$BLs generalize Bol loops. The basic properties of $S_{2^{nd}}$BLs are studied. These properties are all Smarandache in nature. The results in this work generalize the basic properties of Bol loops, found in the Ph.D. thesis [18] and the paper [19] of D. A. Robinson. Some questions for further studies are raised.

2 Preliminaries

Definition 2.1 *Let $(G, \cdot)$ be a quasigroup with an arbitrary non-trivial subquasigroup $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special quasigroup with special subquasigroup $(H, \cdot)$. If $(G, \cdot)$ is a loop with an arbitrary non-trivial subloop $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special loop with special subloop $(H, \cdot)$. If $(H, \cdot)$ is of exponent 2, then $(G_H, \cdot)$ is called a special loop of Smarandache exponent 2.*

A special quasigroup $(G_H, \cdot)$ is called a second Smarandache right Bol quasigroup($S_{2^{nd}}$-right Bol quasigroup) or simply a second Smarandache Bol quasigroup($S_{2^{nd}}$-Bol quasigroup) and abbreviated $S_{2^{nd}}RBQ$ or $S_{2^{nd}}BQ$ if and only if it obeys the second Smarandache Bol identity($S_{2^{nd}}$-Bol identity) i.e $S_{2^{nd}}BI$

$$(xs \cdot z)s = x(sz \cdot s) \text{ for all } x, z \in G \text{ and } s \in H. \tag{1}$$

Hence, if $(G_H, \cdot)$ is a special loop, and it obeys the $S_{2^{nd}}BI$, it is called a second Smarandache Bol loop($S_{2^{nd}}$-Bol loop) and abbreviated $S_{2^{nd}}BL$.

Remark 2.1 *A Smarandache Bol loop(i.e a loop with at least a non-trivial subloop that is a Bol loop) will now be called a first Smarandache Bol loop($S_{1^{st}}$-Bol loop). It is easy to see that a $S_{2^{nd}}BL$ is a $S_{1^{nd}}BL$. But the reverse is not generally true. So $S_{2^{nd}}BLs$ are particular types of $S_{1^{nd}}BL$. There study can be used to generalise existing results in the theory of Bol loops by simply forcing H to be equal to G.*

Definition 2.2 *Let $(G,\cdot)$ be a quasigroup(loop). It is called a right inverse property quasigroup(loop)[RIPQ(RIPL)] if and only if it obeys the right inverse property(RIP) $yx\cdot x^\rho=y$ for all $x,y\in G$. Similarly, it is called a left inverse property quasigroup(loop)[LIPQ(LIPL)] if and only if it obeys the left inverse property(LIP) $x^\lambda\cdot xy=y$ for all $x,y\in G$. Hence, it is called an inverse property quasigroup(loop)[IPQ(IPL)] if and only if it obeys both the RIP and LIP.*

$(G,\cdot)$ is called a right alternative property quasigroup(loop)[RAPQ(RAPL)] if and only if it obeys the right alternative property(RAP) $y\cdot xx=yx\cdot x$ for all $x,y\in G$. Similarly, it is called a left alternative property quasigroup(loop)[LAPQ(LAPL)] if and only if it obeys the left alternative property(LAP) $xx\cdot y=x\cdot xy$ for all $x,y\in G$. Hence, it is called an alternative property quasigroup(loop)[APQ(APL)] if and only if it obeys both the RAP and LAP.

The bijection $L_x:G\to G$ defined as $yL_x=x\cdot y$ for all $x,y\in G$ is called a left translation(multiplication) of G while the bijection $R_x:G\to G$ defined as $yR_x=y\cdot x$ for all $x,y\in G$ is called a right translation(multiplication) of G.

$(G,\cdot)$ is said to be a right power alternative property loop(RPAPL) if and only if it obeys the right power alternative property(RPAP)

$$xy^n=\underbrace{(((xy)y)y)y\cdots y}_{n\text{-times}}\ \text{i.e. } R_{y^n}=R_y^n \text{ for all } x,y\in G \text{ and } n\in\mathbb{Z}.$$

The right nucleus of G denoted by $N_\rho(G,\cdot)=N_\rho(G)=\{a\in G: y\cdot xa=yx\cdot a\ \forall\ x,y\in G\}$.

Let $(G_H,\cdot)$ be a special quasigroup(loop). It is called a second Smarandache right inverse property quasigroup(loop)[$S_{2^{nd}}$RIPQ($S_{2^{nd}}$RIPL)] if and only if it obeys the second Smarandache right inverse property($S_{2^{nd}}$RIP) $ys\cdot s^\rho=y$ for all $y\in G$ and $s\in H$. Similarly, it is called a second Smarandache left inverse property quasigroup(loop)[$S_{2^{nd}}$LIPQ($S_{2^{nd}}$LIPL)] if and only if it obeys the second Smarandache left inverse property($S_{2^{nd}}$LIP) $s^\lambda\cdot sy=y$ for all $y\in G$ and $s\in H$. Hence, it is called a second Smarandache inverse property quasigroup(loop)[$S_{2^{nd}}$IPQ($S_{2^{nd}}$IPL)] if and only if it obeys both the $S_{2^{nd}}$RIP and $S_{2^{nd}}$LIP.

$(G_H,\cdot)$ is called a third Smarandache right inverse property quasigroup(loop)[$S_{3^{rd}}$RIPQ($S_{3^{rd}}$RIPL)] if and only if it obeys the third Smarandache right inverse property($S_{3^{rd}}$RIP) $sy\cdot y^\rho=s$ for all $y\in G$ and $s\in H$.

$(G_H,\cdot)$ is called a second Smarandache right alternative property quasigroup(loop)[$S_{2^{nd}}$RAPQ($S_{2^{nd}}$RAPL)] if and only if it obeys the second Smarandache right alternative property($S_{2^{nd}}$RAP) $y\cdot ss=ys\cdot s$ for all $y\in G$ and $s\in H$. Similarly, it is called a second Smarandache left alternative property quasigroup(loop)[$S_{2^{nd}}$LAPQ($S_{2^{nd}}$LAPL)] if and only if it obeys the second Smarandache left alternative property($S_{2^{nd}}$LAP) $ss\cdot y=s\cdot sy$ for all $y\in G$ and $s\in H$. Hence, it is called an second Smarandache alternative property quasigroup(loop)[$S_{2^{nd}}$APQ($S_{2^{nd}}$APL)] if and only if it obeys both the $S_{2^{nd}}$RAP and $S_{2^{nd}}$LAP.

$(G_H,\cdot)$ is said to be a Smarandache right power alternative property loop(SRPAPL) if and only if it obeys the Smarandache right power alternative property(SRPAP)

$$xs^n=\underbrace{(((xs)s)s)s\cdots s}_{n\text{-times}}\ \text{i.e. } R_{s^n}=R_s^n \text{ for all } x\in G,\ s\in H \text{ and } n\in\mathbb{Z}.$$

The Smarandache right nucleus of G_H denoted by $SN_\rho(G_H,\cdot) = SN_\rho(G_H) = N_\rho(G)\cap H$. G_H is called a Smarandache right nuclear square special loop if and only if $s^2 \in SN_\rho(G_H)$ for all $s \in H$.

Remark 2.2 *A Smarandache; RIPQ or LIPQ or IPQ(i.e a loop with at least a non-trivial subquasigroup that is a RIPQ or LIPQ or IPQ) will now be called a first Smarandache; RIPQ or LIPQ or IPQ($S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ). It is easy to see that a $S_{2^{st}}$RIPQ or $S_{2^{st}}$LIPQ or $S_{2^{st}}$IPQ is a $S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ respectively. But the reverse is not generally true.*

Definition 2.3 *Let $(G,\cdot)$ be a quasigroup(loop). The set $SYM(G,\cdot) = SYM(G)$ of all bijections in G forms a group called the permutation(symmetric) group of G. The triple (U,V,W) such that $U,V,W \in SYM(G,\cdot)$ is called an autotopism of G if and only if*

$$xU \cdot yV = (x\cdot y)W \ \forall\ x,y \in G.$$

The group of autotopisms of G is denoted by $AUT(G,\cdot) = AUT(G)$.

Let $(G_H,\cdot)$ be a special quasigroup(loop). The set $SSYM(G_H,\cdot) = SSYM(G_H)$ of all Smarandache bijections(S-bijections) in G_H i.e $A \in SYM(G_H)$ such that $A : H \to H$ forms a group called the Smarandache permutation(symmetric) group[S-permutation group] of G_H. The triple (U,V,W) such that $U,V,W \in SSYM(G_H,\cdot)$ is called a first Smarandache autotopism($S_{1^{st}}$ autotopism) of G_H if and only if

$$xU \cdot yV = (x\cdot y)W \ \forall\ x,y \in G_H.$$

If their set forms a group under componentwise multiplication, it is called the first Smarandache autotopism group($S_{1^{st}}$ autotopism group) of G_H and is denoted by $S_{1^{st}}AUT(G_H,\cdot) = S_{1^{st}}AUT(G_H)$.

The triple (U,V,W) such that $U,W \in SYM(G,\cdot)$ and $V \in SSYM(G_H,\cdot)$ is called a second right Smarandache autotopism($S_{2^{nd}}$ right autotopism) of G_H if and only if

$$xU \cdot sV = (x\cdot s)W \ \forall\ x \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second right Smarandache autotopism group($S_{2^{nd}}$ right autotopism group) of G_H and is denoted by $S_{2^{nd}}RAUT(G_H,\cdot) = S_{2^{nd}}RAUT(G_H)$.

The triple (U,V,W) such that $V,W \in SYM(G,\cdot)$ and $U \in SSYM(G_H,\cdot)$ is called a second left Smarandache autotopism($S_{2^{nd}}$ left autotopism) of G_H if and only if

$$sU \cdot yV = (s\cdot y)W \ \forall\ y \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second left Smarandache autotopism group($S_{2^{nd}}$ left autotopism group) of G_H and is denoted by $S_{2^{nd}}LAUT(G_H,\cdot) = S_{2^{nd}}LAUT(G_H)$.

Let $(G_H, \cdot)$ be a special quasigroup(loop) with identity element e. A mapping $T \in SSYM(G_H)$ is called a first Smarandache semi-automorphism($S_{1^{st}}$ semi-automorphism) if and only if $eT = e$ and

$$(xy \cdot x)T = (xT \cdot yT)xT \text{ for all } x, y \in G.$$

A mapping $T \in SSYM(G_H)$ is called a second Smarandache semi-automorphism($S_{2^{nd}}$ semi-automorphism) if and only if $eT = e$ and

$$(sy \cdot s)T = (sT \cdot yT)sT \text{ for all } y \in G \text{ and all } s \in H.$$

A special loop $(G_H, \cdot)$ is called a first Smarandache semi-automorphism inverse property loop($S_{1^{st}}SAIPL$) if and only if J_ρ is a $S_{1^{st}}$ semi-automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache semi-automorphism inverse property loop($S_{2^{nd}}SAIPL$) if and only if J_ρ is a $S_{2^{nd}}$ semi-automorphism.

Let $(G_H, \cdot)$ be a special quasigroup(loop). A mapping $A \in SSYM(G_H)$ is a

1. *first Smarandache pseudo-automorphism($S_{1^{st}}$ pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{1^{st}}AUT(G_H)$. c is reffered to as the first Smarandache companion($S_{1^{st}}$ companion) of A. The set of such As' is denoted by $S_{1^{st}}PAUT(G_H, \cdot) = S_{1^{st}}PAUT(G_H)$.*

2. *second right Smarandache pseudo-automorphism($S_{2^{nd}}$ right pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}RAUT(G_H)$. c is reffered to as the second right Smarandache companion($S_{2^{nd}}$ right companion) of A. The set of such As' is denoted by $S_{2^{nd}}RPAUT(G_H, \cdot) = S_{2^{nd}}RPAUT(G_H)$.*

3. *second left Smarandache pseudo-automorphism($S_{2^{nd}}$ left pseudo-automorphism) of G_H if and only if there exists a $c \in H$ such that $(A, AR_c, AR_c) \in S_{2^{nd}}LAUT(G_H)$. c is reffered to as the second left Smarandache companion($S_{2^{nd}}$ left companion) of A. The set of such As' is denoted by $S_{2^{nd}}LPAUT(G_H, \cdot) = S_{2^{nd}}LPAUT(G_H)$.*

3 Main Results

Theorem 3.1 *Let the special loop $(G_H, \cdot)$ be a $S_{2^{nd}}BL$. Then it is both a $S_{2^{nd}}RIPL$ and a $S_{2^{nd}}RAPL$.*

Proof

1. In the $S_{2^{nd}}$BI, substitute $z = s^\rho$, then $(xs \cdot s^\rho)s = x(ss^\rho \cdot s) = xs$ for all $x \in G$ and $s \in H$. Hence, $xs \cdot s^\rho = x$ which is the $S_{2^{nd}}$RIP.

2. In the $S_{2^{nd}}$BI, substitute $z = e$ and get $xs \cdot s = x \cdot ss$ for all $x \in G$ and $s \in H$. Which is the $S_{2^{nd}}$RAP.

Remark 3.1 *Following Theorem 3.1, we know that if a special loop $(G_H,\cdot)$ is a $S_{2^{nd}}BL$, then its special subloop $(H,\cdot)$ is a Bol loop. Hence, $s^{-1}=s^{\lambda}=s^{\rho}$ for all $s\in H$. So, if $n\in\mathbb{Z}^+$, define xs^n recursively by $s^0=e$ and $s^n=s^{n-1}\cdot s$. For any $n\in\mathbb{Z}^-$, define s^n by $s^n=(s^{-1})^{|n|}$.*

Theorem 3.2 *If $(G_H,\cdot)$ is a $S_{2^{nd}}BL$, then*

$$xs^n=xs^{n-1}\cdot s=xs\cdot s^{n-1} \tag{2}$$

for all $x\in G$, $s\in H$ and $n\in\mathbb{Z}$.

Proof
Trivialy, (2) holds for $n=0$ and $n=1$. Now assume for $k>1$,

$$xs^k=xs^{k-1}\cdot s=xs\cdot s^{k-1} \tag{3}$$

for all $x\in G$, $s\in H$. In particular, $s^k=s^{k-1}\cdot s=s\cdot s^{k-1}$ for all $s\in H$. So, $xs^{k+1}=x\cdot s^ks=x(ss^{k-1}\cdot s)=(xs\cdot s^{k-1})s=xs^k\cdot s$ for all $x\in G$, $s\in H$. Then, replacing x by xs in (3), $xs\cdot s^k=(xs\cdot s^{k-1})s=x(ss^{k-1}\cdot s)=x(s^{k-1}s\cdot s)=x\cdot s^ks=xs^{k+1}$ for all $x\in G$, $s\in H$.(Note that the $\mathrm{S}_{2^{nd}}$BI has been used twice.)

Thus, (2) holds for all integers $n\geq 0$.

Now, for all integers $n>0$ and all $x\in G$, $s\in H$, applying (2) to x and s^{-1} gives $x(s^{-1})^{n+1}=x(s^{-1})^n\cdot s^{-1}=xs^{-n}\cdot s^{-1}$, and (2) applied to xs and s^{-1} gives $xs\cdot(s^{-1})^{n+1}=(xs\cdot s^{-1})(s^{-1})^n=xs^{-n}$. Hence, $xs^{-n}=xs^{-n-1}\cdot s=xs\cdot s^{-n-1}$ and the proof is complete.(Note that the $\mathrm{S}_{2^{nd}}$RIP of Theorem 3.1 has been used.)

Theorem 3.3 *If $(G_H,\cdot)$ is a $S_{2^{nd}}BL$, then*

$$xs^m\cdot s^n=xs^{m+n} \tag{4}$$

for all $x\in G$, $s\in H$ and $m,n\in\mathbb{Z}$.

Proof
The desired result clearly holds for $n=0$ and by Theorem 3.2, it also holds for $n=1$.

For any integer $n>1$, assume that (4) holds for all $m\in\mathbb{Z}$ and all $x\in G$, $s\in H$. Then, using Theorem 3.2, $xs^{m+n+1}=xs^{m+n}\cdot s=(xs^m\cdot s^n)s=xs^m\cdot s^{n+1}$ for all $x\in G$, $s\in H$ and $m\in\mathbb{Z}$. So, (4) holds for all $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$. Recall that $(s^n)^{-1}=s^{-n}$ for all $n\in\mathbb{Z}^+$ and $s\in H$. Replacing m by $m-n$, $xs^{m-n}\cdot s^n=xs^m$ and, hence, $xs^{m-n}=xs^m\cdot(s^n)^{-1}=xs^m\cdot s^{-n}$ for all $m\in\mathbb{Z}$ and $x\in G$, $s\in H$.

Corollary 3.1 *Every $S_{2^{nd}}BL$ is a SRPAPL.*

Proof
When $n=1$, the SRPAP is true. When $n=2$, the SRPAP is the SRAP. Let the SRPAP be true for $k\in\mathbb{Z}^+$; $R_{s^k}=R_s^k$ for all $s\in H$. Then, by Theorem 3.3, $R_s^{k+1}=R_s^kR_s=R_{s^k}R_s=R_{s^{k+1}}$ for all $s\in H$.

Lemma 3.1 *Let $(G_H,\cdot)$ be a special loop. Then, $S_{1^{st}}AUT(G_H,\cdot) \leq AUT(G_H,\cdot)$, $S_{2^{nd}}RAUT(G_H,\cdot) \leq AUT(H,\cdot)$ and $S_{2^{nd}}LAUT(G_H,\cdot) \leq AUT(H,\cdot)$. But, $S_{2^{nd}}RAUT(G_H,\cdot) \nleq AUT(G_H,\cdot)$ and $S_{2^{nd}}LAUT(G_H,\cdot) \nleq AUT(G_H,\cdot)$.*

Proof
These are easily proved by using the definitions of the sets relative to componentwise multiplication.

Lemma 3.2 *Let $(G_H,\cdot)$ be a special loop. Then, $S_{2^{nd}}RAUT(G_H,\cdot)$ and $S_{2^{nd}}LAUT(G_H,\cdot)$ are groups under componentwise multiplication.*

Proof
These are easily proved by using the definitions of the sets relative to componentwise multiplication.

Lemma 3.3 *Let $(G_H,\cdot)$ be a special loop.*

1. *If $(U,V,W) \in S_{2^{nd}}RAUT(G_H,\cdot)$ and G_H has the $S_{2^{nd}}RIP$, then $(W,J_\rho VJ_\rho,U) \in S_{2^{nd}}RAUT(G_H,\cdot)$.*

2. *If $(U,V,W) \in S_{2^{nd}}LAUT(G_H,\cdot)$ and G_H has the $S_{2^{nd}}LIP$, then $(J_\lambda U,W,V) \in S_{2^{nd}}LAUT(G_H,\cdot)$.*

Proof

1. $(U,V,W) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H,\cdot)$ implies that $xU \cdot sV = (x \cdot s)W$ for all $x \in G$ and $s \in H$. So, $(xU \cdot sV)(sV)^\rho = (x \cdot s)W \cdot (sV)^\rho \Rightarrow xU = (xs^\rho)W \cdot (s^\rho V)^\rho \Rightarrow (xs)U = (xs \cdot s^\rho)W \cdot (s^\rho V)^\rho \Rightarrow (xs)U = xW \cdot sJ_\rho VJ_\rho \Rightarrow (W,J_\rho VJ_\rho,U) \in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H,\cdot)$.

2. $(U,V,W) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H,\cdot)$ implies that $sU \cdot xV = (s \cdot x)W$ for all $x \in G$ and $s \in H$. So, $(sU)^\lambda \cdot (sU \cdot xV) = (sU)^\lambda \cdot (s \cdot x)W \Rightarrow xV = (sU)^\lambda \cdot (sx)W \Rightarrow xV = (s^\lambda U)^\lambda \cdot (s^\lambda x)W \Rightarrow (sx)V = (s^\lambda U)^\lambda \cdot (s^\lambda \cdot sx)W \Rightarrow (sx)V = sJ_\lambda UJ_\lambda \cdot xW \Rightarrow (J_\lambda U,W,V) \in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H,\cdot)$.

Theorem 3.4 *Let $(G_H,\cdot)$ be a special loop. $(G_H,\cdot)$ is a $S_{2^{nd}}BL$ if and only if $(R_s^{-1},L_sR_s,R_s) \in S_{1^{st}}AUT(G_H,\cdot)$.*

Proof
G_H is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL iff $(xs \cdot z)s = x(sz \cdot s)$ for all $x,z \in G$ and $s \in H$ iff $(xR_s \cdot z)R_s = x(zL_sR_s)$ iff $(xz)R_s = xR_s^{-1} \cdot zL_sR_s$ iff $(R_s^{-1},L_sR_s,R_s) \in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$.

Theorem 3.5 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. G_H is a $S_{2^{nd}}SAIPL$ if and only if G_H is a $S_{3^{rd}}RIPL$.*

Proof
Keeping the $\mathrm{S}_{2^{\mathrm{nd}}}$BI and the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP in mind, it will be observed that if G_H is a $\mathrm{S}_{3^{\mathrm{rd}}}$RIPL, then $(sy \cdot s)(s^\rho y^\rho \cdot s^\rho) = [((sy \cdot s)s^\rho)y^\rho]s^\rho = (sy \cdot y^\rho)s^\rho = ss^\rho = e$. So, $(sy \cdot s)^\rho = s^\rho y^\rho \cdot s^\rho$. The proof of the necessary part follows by the reverse process.

Theorem 3.6 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. If $(U,T,U)\in S_{1^{st}}AUT(G_H,\cdot)$. Then, T is a $S_{2^{nd}}$ semi-automorphism.*

Proof

If $(U,T,U)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$, then, $(U,T,U)\in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H,\cdot)\cap \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H,\cdot)$.

Let $(U,T,U)\in \mathrm{S}_{2^{\mathrm{nd}}}RAUT(G_H,\cdot)$, then $xU\cdot sT=(xs)U$ for all $x\in G$ and $s\in H$. Set $s=e$, then $eT=e$. Let $u=eU$, then $u\in H$ since $(U,T,U)\in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H,\cdot)$. For $x=e$, $U=TL_u$. So, $xTL_u\cdot sT=(xs)TL_u$ for all $x\in G$ and $s\in H$. Thus,

$$(u\cdot xT)\cdot sT=u\cdot (xs)T. \tag{5}$$

Replace x by sx in (5), to get

$$[u\cdot (sx)T]\cdot sT=u\cdot (sx\cdot s)T. \tag{6}$$

$(U,T,U)\in \mathrm{S}_{2^{\mathrm{nd}}}LAUT(G_H,\cdot)$ implies that $sU\cdot xT=(sx)U$ for all $x\in G$ and $s\in H$ implies $sTL_u\cdot xT=(sx)TL_u$ implies $(u\cdot sT)\cdot xT=u\cdot (sx)T$. Using this in (6) gives $[(u\cdot sT)\cdot xT]\cdot sT=u\cdot (sx\cdot s)T$. By the $\mathrm{S}_{2^{\mathrm{nd}}}$BI, $u[(sT\cdot xT)\cdot sT]=u\cdot (sx\cdot s)T\Rightarrow (sT\cdot xT)\cdot sT=(sx\cdot s)T$.

Corollary 3.2 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$ that is a Smarandache right nuclear square special loop. Then, $L_sR_s^{-1}$ is a $S_{2^{nd}}$ semi-automorphism.*

Proof

$s^2\in SN_\rho(G_H)$ for all $s\in H$ iff $xy\cdot s^2=x\cdot ys^2$ iff $(xy)R_{s^2}=x\cdot yR_{s^2}$ iff $(xy)R_s^2=x\cdot yR_s^2(\because$ of $\mathrm{S}_{2^{\mathrm{nd}}}$RAP) iff $(I,R_s^2,R_s^2)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$ iff $(I,R_s^{-2},R_s^{-2})\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$. Recall from Theorem 3.4 that, $(R_s^{-1},L_sR_s,R_s)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$. So, $(R_s^{-1},L_sR_s,R_s)(I,R_s^{-2},R_s^{-2})=(R_s^{-1},L_sR_s^{-1},R_s^{-1})\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)\Rightarrow L_sR_s^{-1}$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$ semi-automorphism by Theorem 3.6.

Corollary 3.3 *If a $S_{2^{nd}}BL$ is of Smarandache exponent 2, then, $L_sR_s^{-1}$ is a $S_{2^{nd}}$ semi-automorphism.*

Proof

These follows from Theorem 3.2.

Theorem 3.7 *Let $(G_H,\cdot)$ be a $S_{2^{nd}}BL$. Let $(U,V,W)\in S_{1^{st}}AUT(G_H,\cdot)$, $s_1=eU$ and $s_2=eV$. Then, $A=UR_s^{-1}\in S_{1^{st}}PAUT(G_H)$ with $S_{1^{st}}$ companion $c=s_1s_2\cdot s_1$ such that $(U,V,W)=(A,AR_c,AR_c)(R_s^{-1},L_sR_s,R_s)^{-1}$.*

Proof

By Theorem 3.4, $(R_s^{-1},L_sR_s,R_s)\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)$ for all $s\in H$. Hence, $(A,B,C)=(U,V,W)(R_{s_1}^{-1},L_{s_1}R_{s_1},R_{s_1})=(UR_{s_1}^{-1},VL_{s_1}R_{s_1},WR_{s_1})\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)\Rightarrow A=UR_{s_1}^{-1}$, $B=VL_{s_1}R_{s_1}$ and $C=WR_{s_1}$. That is, $aA\cdot bB=(ab)C$ for all $a,b\in G_H$. Since $eA=e$, then setting $a=e$, $B=C$. Then for $b=e$, $B=AR_{eB}$. But $eB=eVL_{s_1}R_{s_1}=s_1s_2\cdot s_1$. Thus, $(A,AR_{eB},AR_{eB})\in \mathrm{S}_{1^{\mathrm{st}}}AUT(G_H,\cdot)\Rightarrow A\in \mathrm{S}_{1^{\mathrm{st}}}PAUT(G_H,\cdot)$ with $\mathrm{S}_{1^{\mathrm{st}}}$ companion $c=s_1s_2\cdot s_1\in H$.

Theorem 3.8 *Let $(G_H, \cdot)$ be a $S_{2^{nd}}BL$. Let $(U, V, W) \in S_{2^{nd}}LAUT(G_H, \cdot) \cap S_{2^{nd}}RAUT(G_H, \cdot)$, $s_1 = eU$ and $s_2 = eV$. Then, $A = UR_s^{-1} \in S_{2^{nd}}LPAUT(G_H) \cap S_{2^{nd}}RPAUT(G_H)$ with $S_{2^{nd}}$ left companion and $S_{2^{nd}}$ right companion $c = s_1 s_2 \cdot s_1$ such that $(U, V, W) = (A, AR_c, AR_c)(R_s^{-1}, L_s R_s, R_s)^{-1}$.*

Proof
The proof of this is very similar to the proof of Theorem 3.7.

Remark 3.2 *Every Bol loop is a $S_{2^{nd}}BL$. Most of the results on basic properties of Bol loops in chapter 2 of [18] can easily be deduced from the results in this paper by simply forcing H to be equal to G.*

Question 3.1 *Let $(G_H, \cdot)$ be a special quasigroup(loop). Are the sets $S_{1^{st}}PAUT(G_H)$, $S_{2^{nd}}RPAUT(G_H)$ and $S_{2^{nd}}LPAUT(G_H)$ groups under mapping composition?*

Question 3.2 *Let $(G_H, \cdot)$ be a special quasigroup(loop). Can we find a general method(i.e not an "acceptable" $S_{2^{nd}}BL$ with carrier set $\mathbb{N}$) of constructing a $S_{2^{nd}}BL$ that is not a Bol loop just like Robinson [18], Solarin and Sharma [20] were able to use general methods to construct Bol loops.*

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