Biological n-System with Global Stability

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Abstract: A food web on n living things $x_1, x_2, \cdots, x_n$, i.e., a biological n-system can be mathematically characterized by action flow $\overrightarrow{G}^L$ of order n with surplus flows of growth rates $\dot{x}_i$ of population on vertices v_i, vector flow (x_i, x_j), end-operators $x_i f_{ij}, x_j f'_{ji}$ on edge (v_i, v_j), where f_{ij}, f'_{ji} are 2 variable functions for integers $1 \leq i, j \leq n$ holding with a system of conservation equations

$$\dot{x}_i = x_i \left(\sum_{v_k \in N^-(v_i)} f'_{ki}(x_k, x_i) - \sum_{v_l \in N^+(v_i)} f_{il}(x_i, x_l) \right), \quad 1 \leq i \leq n,$$

which is a system of n differential equations. Certainly, $\mathbf{0} \in \mathbb{R}^n$ is one of its equilibrium points. But the system

$$\sum_{v_k \in N^-(v_i)} f'_{ki}(x_k, x_i) = \sum_{v_l \in N^+(v_i)} f_{il}(x_i, x_l), \quad 1 \leq i \leq n$$

of equations may be solvable or not. However, even if it is non-solvable, it characterizes biological systems also if it can be classified into solvable subsystems. The main purpose of this paper is to characterize the biological behavior of such systems with global stability by a combinatorial approach, i.e., establish the relationship between solvable subsystems of a biological n-system with Eulerian subgraphs of labeling bi-digraph of $\overrightarrow{G}^L$, characterize n-system with linear growth rate and the global stability on subgraphs, and interpret also the biological behavior of G^L-solutions of non-solvable equations, which opened a way for characterizing biological system with species more than 3, i.e., mathematical combinatorics. As we know, nearly all papers discussed biological system with species less or equal to 3 in the past decades.

Key Words: Food web, biological n-system, action flow, Klomogorov model, non-solvable system, Eulerian graph, bi-digraph, Smarandache multispace, mathematical combinatorics.

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§1. Introduction

There is a well-known biological law for living things in the natural world, i.e., the survival of the fittest in the natural selection because of the limited resources of foods. Thus, foods naturally result in connection with living things, i.e., *food chain*, a linear network starting from producer organisms and ending at apex predator species or decomposer species. And biologically, a *food web* is a natural interconnection of food chains, a resultant by a simple ruler ([28]), and generally a graphical representation of what-eats-what in the ecological community such as those shown in Fig.1 for 4 food chains: grass→ladybug→frog→snake→eagle, grass→ladybug→frog →egret, grass→rabbit→ snake→eagle and grass→rabbit→eagle.
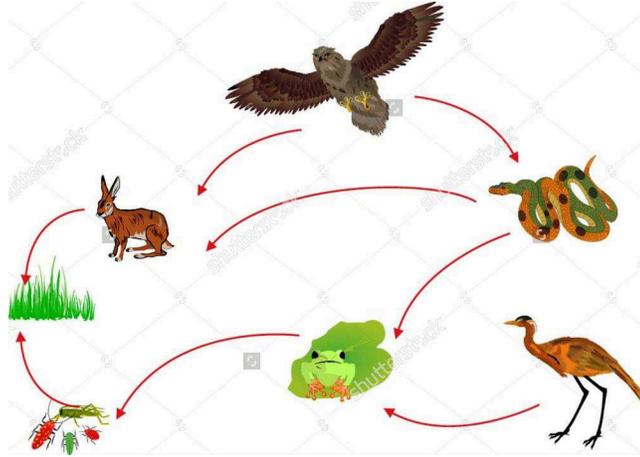

Fig.1

Actually, a food web is an interaction system in physics ([15]-[16], [25]) which can be mathematically characterized by the strength of what action on what. For a biological 2-system, let x, y be the two species with the action strength $F'(x \to y)$, $F(y \to x)$ of x to y and y to x on their growth rate, respectively ([21]). Then, such a system can be quantitatively characterized by differential equations

$$\begin{cases} \dot{x} = F(y \to x) \\ \dot{y} = F'(x \to y) \end{cases}$$

on the populations of species x and y.

Usually, we denote 2 competing things by a directed edge (u, v) labeling with vector flow (x, y) and end-operators F, F' respectively on its center and both ends, where F, F' are action operators with $F(x \to 0) = F'(0 \to y) = 0$ if $y = 0$ or $x = 0$

and the growth rates $\dot{x}, \dot{y}$ of populations on vertices, such as those shown in Fig.2. Particularly, $F = xf, F' = yf'$ in the Kolmogorov model, where f, f' are 2 variable functions, and $f = \lambda - by$, $f' = \mu + cx$ in the Lotka-Volterra model ([2], [20]).
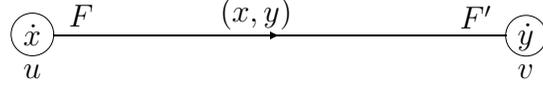

Fig.2

Then, a food web is nothing else but a topological digraph $\overrightarrow{G}$, a 2-tuple $\left(V(\overrightarrow{G}), E(\overrightarrow{G})\right)$ with $E(\overrightarrow{G}) \subset V(\overrightarrow{G}) \times V(\overrightarrow{G})$ and a labeling $L: \overrightarrow{G} \to R \bigcup S$ on $\overrightarrow{G}$ with $L: V(\overrightarrow{G}) \to R$ and $E(\overrightarrow{G}) \to S$, where R and S are predetermined sets ([19]). Particularly, if $R = \{\dot{x}, \dot{y}\}$, the growth rates of populations and $S = \{(F, (x, y), F')\}$, a 3-tuple with action operator F on the initial, F' on the end and vector (x, y) on the middle of edge (u, v), we get the biological 2-system shown in Fig.2.

However, the law of conservation of matter concludes that matter is neither created nor destroyed in chemical reactions. In other words, the mass of any one element at the beginning of a reaction will equal to that of element at the end, i.e., the in and out-action must be conservative with the surplus on each vertex of $\overrightarrow{G}^L$. Thus, a food web is an *action flow* ([18]) further, i.e., a topological digraph $\overrightarrow{G}^L$ labeled with surplus flows of growth rates $\dot{x}_i$ of population on vertices v_i, vector flow (x_i, x_j), initial and end operators F_{ij}, F'_{ij} on edge (v_i, v_j) for integers $1 \leq i, j \leq n$, where $n \geq 2$ holding with a system of conservation equations

$$\dot{x}_i = \sum_{v_k \in N^-(v_i)} F'_{ki}(x_k \to x_i) - \sum_{v_l \in N^+(v_i)} F_{il}(x_i \to x_l), \quad 1 \leq i \leq n$$

and particularly,

$$\dot{x}_i = x_i \left(\sum_{v_k \in N^-(v_i)} f'_{ki}(x_k, x_i) - \sum_{v_l \in N^+(v_i)} f_{il}(x_i, x_l) \right), \quad 1 \leq i \leq n \qquad (1.1)$$

in the *Kolmogorov model*. For example, a biological 4-system shown in Fig.3 is a system of 4 ordinary differential equations

$$\begin{cases} \dot{x}_1 = x_1 \left((b_{52} - a_{11} - a_{12})x_1 - a_{12}x_2 - a_{22}x_3 + b_{51}x_4\right) \\ \dot{x}_2 = x_2 \left(b_{11}x_1 + (b_{12} + b_{62} - a_{31})x_2 - a_{32}x_3 + b_{61}x_4\right) \\ \dot{x}_3 = x_3 \left(b_{21}x_1 + b_{31}x_2 + (b_{22} + b_{32} - a_{41})x_3 - a_{42}x_4\right) \\ \dot{x}_4 = x_4 \left(b_{41}x_3 - a_{52}x_1 - a_{62}x_2 + (b_{42} - a_{51} - a_{61})x_4\right) \end{cases} \qquad (1.2)$$

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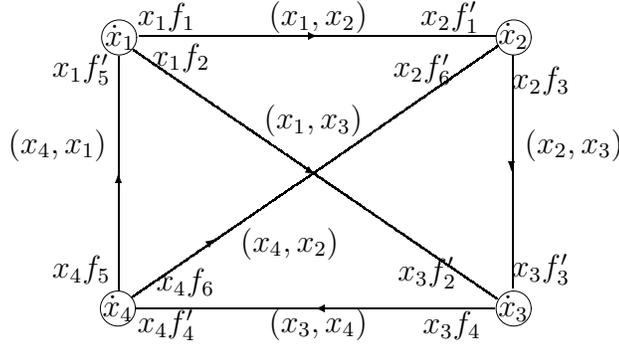
Fig.3

where,

$$f_1(x_1, x_2) = a_{11}x_1 + a_{12}x_2, \quad f_1'(x_1, x_2) = b_{11}x_1 + b_{12}x_2,$$
$$f_2(x_1, x_3) = a_{21}x_1 + a_{22}x_3, \quad f_2'(x_1, x_3) = b_{21}x_1 + b_{22}x_3,$$
$$f_3(x_2, x_3) = a_{31}x_2 + a_{32}x_3, \quad f_3'(x_2, x_3) = b_{31}x_2 + b_{32}x_3,$$
$$f_4(x_3, x_4) = a_{41}x_3 + a_{42}x_4, \quad f_4'(x_3, x_4) = b_{41}x_3 + b_{42}x_4,$$
$$f_5(x_4, x_1) = a_{51}x_4 + a_{52}x_1, \quad f_5'(x_4, x_1) = b_{51}x_4 + b_{52}x_1,$$
$$f_6(x_4, x_2) = a_{61}x_4 + a_{62}x_2, \quad f_6'(x_4, x_2) = b_{61}x_4 + b_{62}x_2.$$

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Definition 1.1 *Let $\overrightarrow{G}^L$ be a labeling topological digraph. A subgraph $\overrightarrow{H}$ of $\overrightarrow{G}$ is said to be a labeling subgraph of $\overrightarrow{G}^L$ if its vertices and edges are labeled by $L|_H$, denoted by $\overrightarrow{H}^L \prec \overrightarrow{G}^L$ and furthermore, if $\overrightarrow{H}^L = \overrightarrow{G}^L\Big|_{V(H)}$, such a labeling subgraph is said to be an induced subgraph of $\overrightarrow{G}^L$, denoted by $\left\langle V(\overrightarrow{H}) \right\rangle_G$.*

For example, the 2 labeling graphs $\overrightarrow{G}_1^L$, $\overrightarrow{G}_2^L$ in Fig.4 are all labeling subgraphs but only $\overrightarrow{G}_1^L$ is an induced subgraph of the graph shown in Fig.3.

Fig.4

Clearly, a labeling subgraph of $\overrightarrow{G}^L$ is also consisting of food chains but it maybe not a food web if it is not an action flow again. Even it is, the sizes of species are

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not the same as they in $\overrightarrow{G}^L$ because the conservative laws are completely changed. For example, the system of conservation equations for the labeling subgraph $\overrightarrow{G}_1^L$ is

$$
\begin{cases}
\dot{x}_1 = x_1 \left((b_{51} - a_{21})x_1 - a_{22}x_3 + b_{51}x_4 \right) \\
\dot{x}_3 = x_3 \left(b_{21}x_1 + (b_{22} - a_{41})x_3 - a_{42}x_4 \right) \\
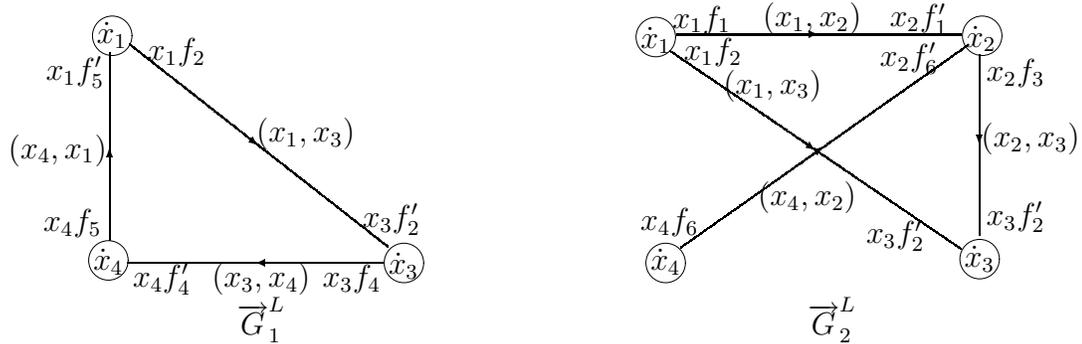\dot{x}_4 = x_4 \left(b_{41}x_3 - a_{52}x_1 + (b_{42} - a_{51})x_4 \right)
\end{cases}
\tag{1.3}
$$

a very different system from that of (1.2).

The following terminologies are useful for characterizing food webs.

Definition 1.2 *Let $\overrightarrow{G}$ be a digraph with $\overleftarrow{G}$ a digraph reversing direction on every edge in $\overrightarrow{G}$. A bi-digraph of $\overrightarrow{G}$ is defined by $\overrightarrow{G} \bigcup \overleftarrow{G}$ and a labeling bi-digraph $\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right)^{\widehat{L}}$ of a labeling digraph $\overrightarrow{G}^L$ is a labeling graph on $\overrightarrow{G} \bigcup \overleftarrow{G}$ with a labeling $\widehat{L} : V(\overrightarrow{G} \bigcup \overleftarrow{G}) \to L\left(V(\overrightarrow{G}) \right)$, $\widehat{L} : E\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right) \to L\left(E\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right) \right)$ by $\widehat{L} : (u,v) \to \{0, (x,y), yf'\}$, $(v,u) \to \{xf, (x,y), 0\}$ if $L : (u,v) \to \{xf, (x,y), yf'\}$ for $\forall (u,v) \in E(\overrightarrow{G})$, such as those shown in Fig.5.*
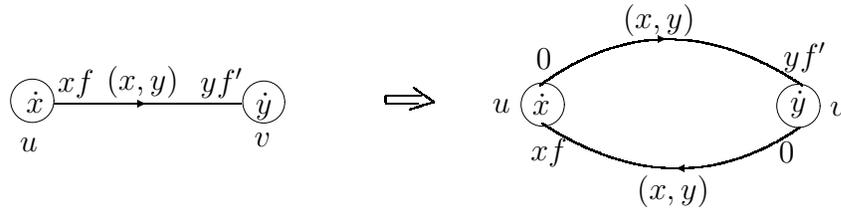
Fig.5

Definition 1.3 *A circuit in a digraph $\overrightarrow{G}$ is a nontrivial closed trail with different edges in $\overrightarrow{G}$ and an Eulerian circuit in digraph $\overrightarrow{G}$ is a circuit of $\overrightarrow{G}$ containing every edge of $\overrightarrow{G}$.*

A digraph $\overrightarrow{G}$ is Eulerian if it contains an Eulerian circuit.

Clearly, a bi-digraph of a digraph is an Eulerian graph. The main purpose of this paper is to characterize the biological behavior of biological n-systems with global stability by a combinatorial approach, i.e., establish the relationship between solvable subsystems of a biological n-system with that of labeling Eulerian subgraphs of labeling bi-digraph $\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right)^{\widehat{L}}$ of $\overrightarrow{G}^L$, characterize conditions of an n-system with linear growth rate become distinct and global stability, and interpret also the biological behavior of G^L-solutions of non-solvable equations, which opened a way

for characterizing biological system with species more than 3, i.e., mathematical combinatorics, or differential equations over graphs.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [25] for interaction particles, [2] and [20] for biological mathematics, [3] for differential equations with stability, [6]-[7] for topological graphs, digraphs and combinatorial geometry, [7] and [26] for Smarandache multispaces.

§2. Geometry Over Equilibrium Points

2.1 Equilibrium Sets

We consider the generalized Kolmogorov model on biological n-system ([2], [20]), i.e., the system (1.1) of differential equations

$$\left\{ \begin{array}{l} \dot{x}_1 = x_1 \left(\sum_{v_k \in N^-(v_1)} f'_{k1}(x_k, x_1) - \sum_{v_l \in N^+(v_1)} f_{1l}(x_1, x_l) \right) \\ \dot{x}_2 = x_2 \left(\sum_{v_k \in N^-(v_2)} f'_{k2}(x_k, x_2) - \sum_{v_l \in N^+(v_2)} f_{2l}(x_2, x_l) \right) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n = x_n \left(\sum_{v_k \in N^-(v_n)} f'_{kn}(x_k, x_n) - \sum_{v_l \in N^+(v_n)} f_{nl}(x_n, x_l) \right) \end{array} \right.$$

satisfying conditions following:

(1) f_{ij}, $f'_{ij} \in \mathbb{C}^1$ for integers $1 \leq i, j \leq n$;

(2) For any integer i, $1 \leq i \leq n$, there is $(x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$ hold with

$$\sum_{v_k \in N^-(v_i)} f'_{ki}(x_k^0, x_i^0) = \sum_{v_l \in N^+(v_i)} f_{il}(x_i^0, x_1^0)$$

but

$$\sum_{v_k \in N^-(v_i)} \left. \frac{\partial f'_{ki}}{\partial x_i} \right|_{(x_k^0, x_i^0)} \neq \sum_{v_l \in N^+(v_i)} \left. \frac{\partial f_{il}}{\partial x_i} \right|_{(x_i^0, x_1^0)}.$$

For any integer i, $1 \leq i \leq n$, define

$$F_i = \sum_{v_k \in N^-(v_i)} f'_{ki}(x_k, x_i) - \sum_{v_l \in N^+(v_i)} f_{il}(x_i, x_l).$$

Then it concludes that

(1) $F_i \in \mathbb{C}^1$ for integers $1 \leq i \leq n$;

(2) $F_i|_{(x_1^0, x_2^0, \cdots, x_n^0)} = 0$ but

$$\frac{\partial F_i}{\partial x_i}\bigg|_{(x_1^0, x_2^0, \cdots, x_n^0)} = \sum_{v_k \in N^-(v_i)} \frac{\partial f_{ki}'}{\partial x_i}\bigg|_{(x_1^0, x_2^0, \cdots, x_n^0)} - \sum_{v_l \in N^+(v_i)} \frac{\partial f_{il}}{\partial x_i}\bigg|_{(x_1^0, x_2^0, \cdots, x_n^0)} \neq 0.$$

Applying the implicity function theorem, each equation

$$F_i(x_1, x_2, \cdots, x_n) = 0$$

is solvable, i.e., *there is a solution manifold S_{F_i} in $\mathbb{R}^n$ for any integer $1 \leq i \leq n$*, and in this case furthermore, *there is a unique solution on the Cauchy problem of system* (1.1) *prescribed with an initial condition* $(x_1(t_0), x_2(t_0), \cdots, x_n(t_0)) = (x_1^0, x_2^0, \cdots, x_n^0)$.

An *equilibrium set* of system (1.1) are all points $(x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$ holding with

$$\begin{cases} x_1^0 \left(\displaystyle\sum_{v_k \in N^-(v_1)} f_{k1}'(x_k^0, x_1^0) - \sum_{v_l \in N^+(v_1)} f_{1l}(x_1^0, x_l^0) \right) = 0 \\[2em] x_2^0 \left(\displaystyle\sum_{v_k \in N^-(v_2)} f_{k2}'(x_k^0, x_2^0) - \sum_{v_l \in N^+(v_2)} f_{2l}(x_2^0, x_l^0) \right) = 0 \\[1em] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[1em] x_n^0 \left(\displaystyle\sum_{v_k \in N^-(v_n)} f_{kn}'(x_k^0, x_n^0) - \sum_{v_l \in N^+(v_n)} f_{nl}(x_n^0, x_l^0) \right) = 0 \end{cases} \tag{2.1}$$

Clearly, only those solutions $x_i^0 \geq 0, 1 \leq i \leq n$ of system (2.1) have the biological meaning, and $(0, 0, \cdots, 0) \in \mathbb{R}^n$ is an obvious equilibrium point. We classify all equilibrium points of system (2.1) into 3 categories following:

(C1) Only $(0, 0, \cdots, 0) \in \mathbb{R}^n$ hold with system (2.1), i.e., the system

$$\begin{cases} \displaystyle\sum_{v_k \in N^-(v_1)} f_{k1}'(x_k^0, x_1^0) = \sum_{v_l \in N^+(v_1)} f_{1l}(x_1^0, x_l^0) \\[1em] \displaystyle\sum_{v_k \in N^-(v_2)} f_{k2}'(x_k^0, x_2^0) = \sum_{v_l \in N^+(v_2)} f_{2l}(x_2^0, x_l^0) \\[1em] \cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \\[1em] \displaystyle\sum_{v_k \in N^-(v_n)} f_{kn}'(x_k^0, x_n^0) = \sum_{v_l \in N^+(v_n)} f_{nl}(x_n^0, x_l^0) \end{cases} \tag{2.2}$$

is non-solvable in $\mathbb{R}^n$.

(C2) Only $(0, \cdots, 0, K_1, 0, \cdots, 0, K_2, 0, \cdots, 0, K_s, 0, \cdots, 0) \in \mathbb{R}^n$ hold system (2.1) with numbers $K_1, K_2, \cdots, K_s > 0$ on columns $i_1, i_2, \cdots, i_s$ respectively, i.e., for any integer $j \notin \{i_1, i_2, \cdots, i_s\}$, the system

$$
\begin{cases}
\displaystyle\sum_{v_k \in N^-(v_{i_1})} f'_{ki_1}(x_k^0, x_{i_1}^0) = \displaystyle\sum_{v_l \in N^+(v_{i_1})} f_{i_1 l}(x_{i_1}^0, x_l^0) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\displaystyle\sum_{v_k \in N^-(v_{i_s})} f'_{ki_s}(x_k^0, x_{i_s}^0) = \displaystyle\sum_{v_l \in N^+(v_{i_s})} f_{i_s l}(x_{i_s}^0, x_l^0) \\
\displaystyle\sum_{v_k \in N^-(v_j)} f'_{kj}(x_k^0, x_j^0) = \displaystyle\sum_{v_l \in N^+(v_j)} f_{jl}(x_j^0, x_l^0)
\end{cases}
\tag{2.3}
$$

is non-solvable in $\mathbb{R}^n$.

(C3) There are $(K_1, K_2, \cdots, K_n) \in \mathbb{R}^n$ hold system (2.1) with $K_i > 0$ for integers $1 \le i \le n$.

2.2 Geometry Over Equations

Usually, one applies differential equations to characterize the reality of things by their solutions. But *can this notion describes the all behavior of things*? Certainly not ([8]-[19]), particularly in biology follows by the discussion following.

For an integer $n \ge 1$, let $u : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable mapping. Its n-dimensional graph $\Gamma[u]$ is defined by the ordered pairs

$$
\Gamma[u] = \{((x_1, \cdots, x_n), u(x_1, \cdots, x_n)) \,|\, (x_1, \cdots, x_n)\}
$$

in $\mathbb{R}^{n+1}$. Clearly, the assumption on $f_{ij}, f'_{ij}, 1 \le i, j \le n$ concludes that the solution manifold S_{F_i} is nothing else but a graph $\Gamma[F_i]$ in $\mathbb{R}^n$.

Geometrically, the system

$$
\begin{cases}
F_1(x_1, x_2, \cdots, x_n) = 0 \\
F_2(x_1, x_2, \cdots, x_n) = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
F_n(x_1, x_2, \cdots, x_n) = 0
\end{cases}
\tag{2.4}
$$

is solvable or not dependent on $\bigcap_{i=1}^{n} S_{F_i} \neq \emptyset$ or not, and conversely, if $\bigcap_{i=1}^{n} S_{F_i} \neq \emptyset$ or not, we can or can not choose point $(x_1, x_2, \cdots, x_n)$ in $\bigcap_{i=1}^{n} S_{F_i}$, a solution of (2.4). We therefore get a simple but meaningful conclusion following.

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Theorem 2.1 *A system of equations*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ F_n(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

under previous assumption is non-solvable or not if and only if $\bigcap\limits_{i=1}^{n} S_{F_i} = \emptyset$ *or* $\neq \emptyset$.

If the intersection $\bigcap\limits_{i=1}^{n} S_{F_i} \neq \emptyset$, *it is said to be a* $\wedge$-*solution of equations* (2.4).

Usually, one characterizes a system S of things $T_1, T_2, \cdots, T_n$ by equations (2.4) with their solutions to hold on the dynamical behavior of these things. *Is it always right?* The answer is negative at least in the non-solvable case of equations (2.4), and even if they are solvable, it can be used only to characterize those of coherent behaviors of things in S, not the individual such as those of discussions on multiverse of particles in [15] and [16]. Then, *what is its basis in philosophy?* It results deeply in an assumption on things, i.e., *the behavior of things discussed is always consistent, i.e., the system* (2.4) *is solvable.* If it holds, the behavior of these things then can be completely characterized by the intersection $\bigcap\limits_{i=1}^{n} S_{F_i}$, i.e., the solution of system (2.4). However, this is a wrong understanding on things because all things are in contradiction in the nature even for human ourselves, and further on different species. This fact also concludes that characterizing things by solvable system (2.4) of equations is only part, not the global, and with no conclusion if it is non-solvable in classical meaning.

Philosophically, things $T_1, T_2, \cdots, T_n$ consist of a group, or a union set $\bigcup\limits_{i=1}^{n} T_i$, and if T_i is characterized by the ith equation in (2.4), they are geometrically equivalent to the union $\bigcup\limits_{i=1}^{n} S_{F_i}$, i.e., a Smarandache multispace, not the intersection $\bigcap\limits_{i=1}^{n} S_{F_i}$.

For example, if things T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' are respectively characterized by systems of equations following

$$(LES_4^N) \begin{cases} x + y & = & 1 \\ x + y & = & -1 \\ x - y & = & -1 \\ x - y & = & 1 \end{cases} \qquad (LES_4^S) \begin{cases} x = y \\ x + y = 2 \\ x = 1 \\ y = 1 \end{cases}$$

then it is clear that (LES_4^N) is non-solvable because $x + y = -1$ is contradictious to $x + y = 1$, and so that for equations $x - y = -1$ and $x - y = 1$, i.e., there are no solutions x_0, y_0 hold with this system. But (LES_4^S) is solvable with $x = 1$ and $y = 1$. *Can we conclude that things T_1', T_2', T_3', T_4' are $x = 1, y = 1$ and T_1, T_2, T_3, T_4 are nothing?* Certainly not because $(x, y) = (1, 1)$ is the intersection of straight line behavior of things T_1', T_2', T_3', T_4' and there are no intersection of T_1, T_2, T_3, T_4 in plane $\mathbb{R}^2$. However, they are indeed exist in $\mathbb{R}^2$ such as those shown in Fig.6.

(LES_4^N) $\qquad\qquad\qquad\qquad$ (LES_4^S)
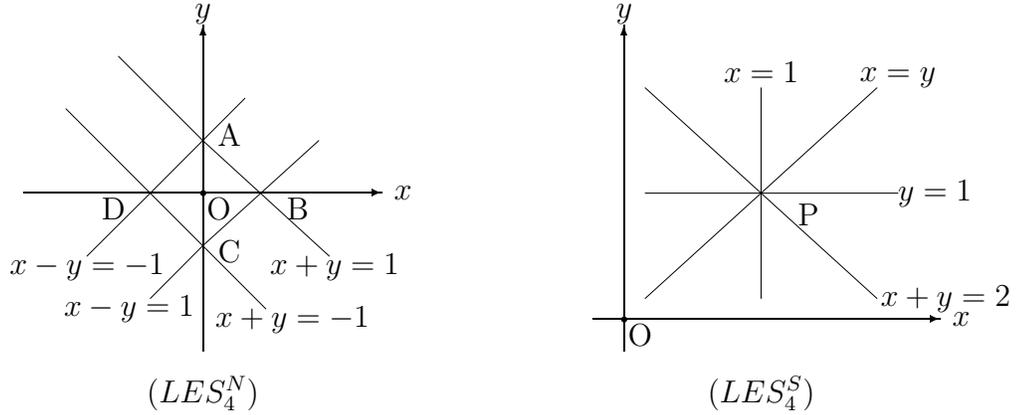
Fig.6

Denoted by the point set

$$L_{a,b,c} = \{(x, y) | ax + by = c, ab \neq 0\}$$

in $\mathbb{R}^2$. Then, we are easily know the straight line behaviors of T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' are nothings else but the unions $L_{1,-1,0} \bigcup L_{1,1,2} \bigcup L_{1,0,1} \bigcup L_{0,1,1}$ and $L_{1,1,1} \bigcup L_{1,1,-1} \bigcup L_{1,-1,-1} \bigcup L_{1,-1,1}$, respectively.

Definition 2.2 *A $\vee$-solution, also called G-solution of system (2.4) is a labeling graph G^L defined by*

$$V(G) = \{S_{F_i},\ 1 \leq i \leq n\};$$

$$E(G) = \left\{(S_{F_i}, S_{F_j})\ if\ S_{F_i} \bigcap S_{F_j} \neq \emptyset\ for\ integers\ 1 \leq i, j \leq n\right\}\ with\ a\ labeling$$

$$L:\ S_{F_i} \to S_{F_i}, \qquad (S_{F_i}, S_{F_j}) \to S_{F_i} \bigcap S_{F_j}.$$

Example 2.3 The $\vee$-solutions of (LES_4^N) and (LES_4^S) are respectively labeling graphs C_4^L and K_4^L shown in Fig.7 following.

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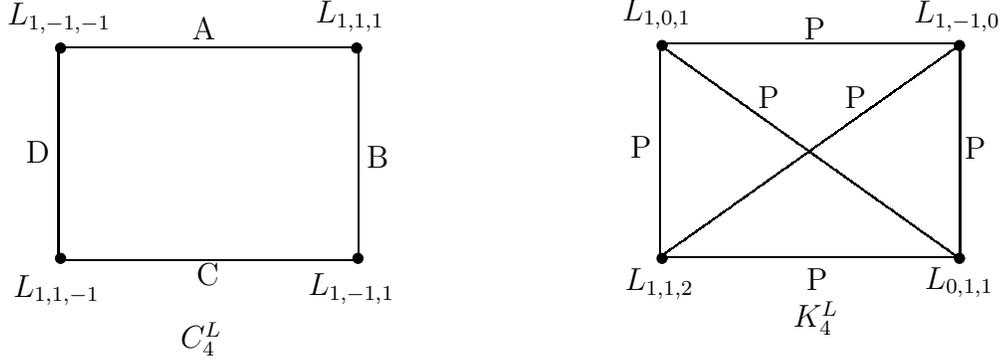
Fig.7

Theorem 2.4 *A system* (2.4) *of equations is* $\vee$-*solvable if* $F_i \in \mathbb{C}^1$ *and* $F_i|_{(x_1^0, x_2^0, \cdots, x_n^0)}$
$= 0$ *but* $\left.\dfrac{\partial F_i}{\partial x_i}\right|_{(x_1^0, x_2^0, \cdots, x_n^0)} \neq 0$ *for any integer* i, $1 \leq i \leq n$.

Proof Applying the implicity function theorem, the proof is completed by definition. □

Theorem 2.5 *A system* (1.1) *of differential equations on food web* $\overrightarrow{G}^L$ *is uniquely* $\vee$-*solvable if* f_{ij}, $f'_{ij} \in \mathbb{C}^1$ *for integers* $1 \leq i, j \leq n$ *and* $(x_1(0), x_2(0), \cdots, x_n(0)) = (x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$.

Proof Applying the existence and uniqueness theorem on the Cauchy problem of differential equations,

$$\dot{x}_i = x_i \left(\sum_{v_k \in N^-(v_i)} f'_{k1}(x_k, x_i) - \sum_{v_l \in N^+(v_i)} f_{1l}(x_i, x_l) \right)$$

with $(x_1(0), x_2(0), \cdots, x_n(0)) = (x_1^0, x_2^0, \cdots, x_n^0) \in \mathbb{R}^n$, it is uniquely solvable for any integer $1 \leq i \leq n$. Consequently, the system (1.1) is uniquely $\vee$-solvable in $\mathbb{R}^n$ by definition. □

2.3 Equilibrium Sets of Linear Equations

Certainly, the Lotka-Volterra model on biological 2-system is a system of linear growth rates. Generally, if all f_{ij}, f'_{ij} are linear for integers $1 \leq i, j \leq n$, then it is a generalization of Lotka-Volterra model on biological n-system. We can further characterize the equilibrium sets of linear system (2.4) by linear algebra.

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Definition 2.6 *For any positive integers i, j, $i \neq j$, the linear equations*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

are called parallel if there exists a constant c such that

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

The following criterion is known in [8].

Theorem 2.7([8]) *For any integers i, j, $i \neq j$, the linear equation system*

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j \end{cases}$$

is non-solvable if and only if they are parallel.

By Theorem 2.7, we divide all linear equations L_i, $1 \leq i \leq n$ in (2.4) into *parallel families*

$$\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$$

by the property that all equations in a family $\mathscr{C}_i$ are parallel and there are no other equations parallel to equations in $\mathscr{C}_i$ for integers $1 \leq i \leq s$. Denoted by $|\mathscr{C}_i| = n_i$, $1 \leq i \leq s$. Then we can characterize equilibrium sets of linear system (2.1) by Theorem 2.6 in [8] following.

Theorem 2.8([8]) *The equilibrium sets of system* (2.1) *with linear growth rates f_{ij}, f'_{ij}, $1 \leq i, j \leq n$ can be classified into 3 classes following:*

(LC1) there is only point $(0, 0, \cdots, 0) \in \mathbb{R}^n$ holding with linear system (2.1), i.e., its $\vee$-solution

$$G^L \simeq K^L_{n_1, n_2, \cdots, n_s}$$

with $n_1 + n + 2 + \cdots + n_s = n$, where $n_i = |\mathscr{C}_i|$ and $\mathscr{C}_i$ is the parallel family for integers $1 \leq i \leq s$, $s \geq 2$.

(LC2) there is only point $(0, \cdots, 0, c_1, 0, \cdots, 0, c_2, 0, \cdots, 0, c_{n-l}, 0, \cdots, 0) \in \mathbb{R}^n$ holding system (2.1) with numbers $c_1, c_2, \cdots, c_{n-l} > 0$ respectively on columns $i_1, i_2, \cdots, i_{n-l}$ for $1 \leq l < n$, i.e., its $\vee$-solution

$$G^L \simeq K^L_{n_1, n_2, \cdots, n_t}$$

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with $n_1 + n + 2 + \cdots + n_t = l$, where $n_i = |\mathscr{C}_i|$ and $\mathscr{C}_i$ is the parallel family for integers $1 \leq i \leq t$, $s \geq 2$.

$(LC3)$ there is an unique point $(c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$ holding linear system (2.1) with constant $c_i > 0$ for integers $1 \leq i \leq n$.

§3. Biology Over Equations

Classically, a solvable system (1.1) of differential equations characterizes dynamical behaviors of a food web in area. However, the solvable systems are individual but non-solvable systems are universal. Then *what about biology over those of non-solvable systems* (1.1)? *Are there no biological significance*? The answer is negative.

Firstly, let us think about a food web how to run. Certainly, a food chain only follows a direct, linear pathway of one animal at a time, and different thing T has his own food chain for living, even for the same kind of things.
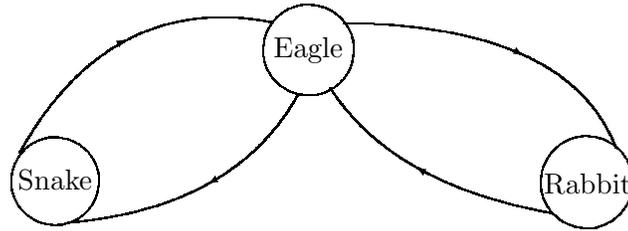
Fig.8

For example, the eagle can lives respectively on the rabbit, on the snake or on the both via its food chains snake→eagle or rabbit→eagle with interactions in Fig.1, i.e., although the eagle preys on the snake and the rabbit but it is also dependent on the 2 populations such as those shown in Fig.8, and its living web should be consisted of circuits eagle→snake→eagle, eagle→rabbit→eagle or eagle→snake and rabbit→eagle, Eulerian subgraphs.

Generally, a predator P preys on a living thing T, i.e., P action on T and there are also T reacts on P at the same time, which implies the interaction between living things, the in and out action exist together. We therefore know a biological fact following.

Fact 3.1 *A living thing must live in an Eulerian subgraph of bi-digraph of a food web* $\overrightarrow{G}^L$.

The following result characterizes action flows on Eulerian subgraphs with that of solvable subsystems of equations (1.1).

Theorem 3.2 *Let $\overrightarrow{G}^L$ be a food web with solvable or non-solvable conservative equations (1.1) on initial value $(x_1(0), x_2(0), \cdots, x_n(0)) = (x_1^0, x_2^0, \cdots, x_n^0)$ and $\overrightarrow{H}^L \prec \left(\overrightarrow{G}^L \bigcup \overleftarrow{G}\right)^{\widehat{L}}$, a food web containing species T with solvable conservation equations*

$$\dot{x}_{i_0} = x_{i_0} \left(\sum_{v_k \in N^-(v_{i_0})} f'_{ki_0}(x_k, x_{i_0}) - \sum_{v_l \in N^+(v_{i_0})} f_{i_0 l}(x_{i_0}, x_l) \right), \quad 1 \le i_0 \le |H| \quad (3.2)$$

in $\overrightarrow{H}^L$ where $L(v_{i_0}) = \dot{x}_{i_0}$. Then $\overrightarrow{H}$ is an Eulerian digraph and H^L is an action flow.

Proof If $\overrightarrow{H}^L$ is a food web, by Fact 3.1 $\overrightarrow{H}$ must be an Eulerian digraph.

Now let $x_{i_0} = f(x_1, x_2, \cdots, x_n), 1 \le i \le |H|$ be the solution of (2.5). Notice that in solution x_{i_0}, x_i can be any chosen constant c, particularly, $x_i = 0$ if $i \notin \{i_0, \ 1 \le i \le |H|\}$ in (3.2), i.e.,

$$x_{i_0} = f(0, \cdots, 0, x_{1_0}, 0, \cdots, 0, x_{2_0}, 0, \cdots, 0, x_n, 0 \cdots, 0), 1 \le i \le |H|$$

is also the solution of (2.5) with $f_{ik}(x_i, x_k) = 0, \ f'_{ki}(x_k, x_i) = 0$, which implies that $\overrightarrow{H}^L$ is an action flow with conservation laws at each vertex. $\qquad \square$

Let $\overrightarrow{H}_1, \overrightarrow{H}_2, \cdots, \overrightarrow{H}_s$ be subgraphs of digraph $\overrightarrow{G}$ with $\overrightarrow{H}_i \ne \overrightarrow{H}_j$ $\overrightarrow{H}_i \bigcap \overrightarrow{H}_j = \emptyset$ or $\ne \emptyset$ for integers $1 \le \ne j \le s$. If $\overrightarrow{G} = \bigcup_{i=1}^{s} \overrightarrow{H}_i$, they are called a *subgraph multi-decomposition* of $\overrightarrow{G}$. Furthermore, if each $\overrightarrow{H}_i$ is Eulearian, such a decomposition is called an *Eulerian multi-decomposition*, denoted by $\overrightarrow{G} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i$. For example, an Eulerian multi-decomposition of the graph on the left is shown on the right in Fig.9.
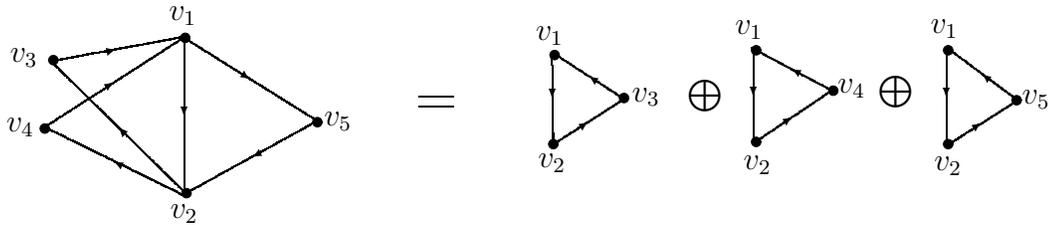
Fig.9

Particularly, if $E\left(\overrightarrow{H}_i\right) \bigcap E\left(\overrightarrow{H}_j\right) = \emptyset$ for integers $1 \leq i \neq j \leq s$, such a decomposition on $\overrightarrow{G}$ is called an *Eulearian decomposition* of $\overrightarrow{G}$.

The next result characterizes food webs by Eulerian multi-decomposition.

Theorem 3.3 *If there are Eulerian subgraphs $\overrightarrow{H}_i^L$, $1 \leq i \leq s$ with solvable conservative equations, i.e., food webs such that $\overrightarrow{G}^{\widehat{L}} = \bigoplus\limits_{i=1}^{s} \overrightarrow{H}_i^L$ with*

$$\widehat{L} : v \rightarrow \sum_{i=1}^{l} \dot{x}_{v^i}, \quad \forall v \in V(\overrightarrow{G})$$

if $v \in \bigcap\limits_{i_0=1}^{l} V(\overrightarrow{H}_{i_0})$ with $L(v) = \dot{x}_{v^i}$ in $\overrightarrow{H}_{i_0}^L$ and

$$\widehat{L} : (u,v) \rightarrow \left(\sum_{l=1}^{s} F'_{i_l}(u \rightarrow v), (x,y), \sum_{l=1}^{s} F_{i_l}(v \rightarrow u) \right), \quad \forall (u,v) \in E(\overrightarrow{G}^L)$$

if $(u,v) \in \bigcap\limits_{j_0=1}^{s} E(\overrightarrow{H}_{j_0})$, then $\overrightarrow{G}^{\widehat{L}}$ is also a food web, i.e., an action flow.

Proof Clearly, $\overrightarrow{G}^{\widehat{L}}$ is a labeling graph holding with conservative law on each vertex $v \in V(\overrightarrow{G}^L)$, i.e., an action flow. □

§4. **Global Stability and Extinction**

In biology, the generation is the necessary condition for the continuation of species in a food web constraint on the interaction, i.e., the stability in dynamics with small perturbations on initial values. Usually, the dynamical behavior is characterized by differential equations, which maybe solvable or not and can not immediately apply to the stability of food web $\overrightarrow{G}^L$ for $n \geq 3$ by Theorems 3.2 and 3.3. Generalizing the classical stability enables one to define the stability of food web following.

Definition 4.1 *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$, where $L(v) = \dot{x}_v$, $L_0(v) = \dot{x}_v^0$ for $v \in V(\overrightarrow{G}^L)$ is globally stable or asymptotically stable for any initial value $\overrightarrow{G}^{L'_0}$, where $L'_0(v) = \dot{y}_v^0$ for $v \in V(\overrightarrow{G}^L)$ and a number $\varepsilon_v > 0$, there is always a number $\delta_v > 0$ such that if $|y_v^0 - x_v^0| < \delta_v$ exists for all $t \geq 0$, then*

$$|y_v(t) - x_v(t)| < \varepsilon_v, \quad \forall v \in V(\overrightarrow{G}^L),$$

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or furthermore,

$$\lim_{t \to 0} |y_v(t) - x_v(t)| = 0, \quad \forall v \in V(\overrightarrow{G}^L).$$

Certainly, we need new criterions on the classic for discussing the stability of species in biology.

Theorem 4.2 *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally stable or asymptotically stable if and only if there is an Eulerian multi-decomposition*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$$

with solvable stable or asymptotically stable conservative equations on labeling Eulerian subgraphs $\overrightarrow{H}_i^L$ for integers $1 \leq i \leq s$.

Proof The necessary is obvious because if $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally stable or asymptotically stable, then $\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}}$ is Eulerian itself by Fact 3.1.

Now if there is an Eulerian multi-decomposition

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$$

on labeling bi-digraph $\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}}$ with stable or asymptotically stable conservative equations on labeling Eulerian subgraphs $\overrightarrow{H}_i^L$, i.e., for any number $\varepsilon_v > 0$, there is a number $\delta_v > 0$ such that if $|y_v^0 - x_v^0| < \delta_v$ exists for all $t \geq 0$, then

$$|y_v(t) - x_v(t)| < \varepsilon_v, \quad \forall v \in V(\overrightarrow{H}_i^L),$$

or furthermore,

$$\lim_{t \to 0} |y_v(t) - x_v(t)| = 0, \quad \forall v \in V(\overrightarrow{H}_i^L)$$

for integers $1 \leq i \leq s$, let λ_v be the multiple of vertex $v \in V(\overrightarrow{G}^L)$ appeared in subgraphs $\overrightarrow{H}_i^L, 1 \leq i \leq s$, we then know that

$$|y_v(t) - x_v(t)| < \varepsilon_v^i$$

for $v \in V(\overrightarrow{H}_i^L)$ if $|y_v^0 - x_v^0| < \delta_v^i$ for integers $1 \leq i \leq \lambda_v$.

Define

$$\delta_v = \min\{\delta_v^i, 1 \leq i \leq \lambda_v\} \quad \text{and} \quad \varepsilon_v = \max\{\varepsilon_v^i, 1 \leq i \leq \lambda_v\}.$$

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We therefore know that

$$|y_v(t) - x_v(t)| < \varepsilon_v,$$

i.e., the species on vertex v is stable if the conservative equations of $\overrightarrow{H}_i^L$ are stable for integers $1 \leq i \leq \lambda_v$ and $\overrightarrow{G}^L$ is globally stable.

Now if furthermore, x_v is asymptotically stable, i.e.,

$$\lim_{t \to 0} |y_v(t) - x_v(t)| = 0$$

in food web $\overrightarrow{H}_i^L$, $1 \leq i \leq \lambda_v$, it is clear that

$$\lim_{t \to 0} |y_v(t) - x_v(t)| = 0$$

in $\overrightarrow{G}^L$ also, i.e., $\overrightarrow{G}^L$ is globally asymptotically stable. This completes the proof. $\square$

Corollary 4.3 *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally stable or asymptotically stable if there is an Eulerian decomposition*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$$

with solvable stable or asymptotically stable conservative equations on labeling Eulerian subgraphs $\overrightarrow{H}_i^L$ for integers $1 \leq i \leq s$.

Clearly, the bi-digraph $\overrightarrow{G} \bigcup \overleftarrow{G}$ has an Eulerian decomposition, called *parallel decomposition*

$$\overrightarrow{G} \bigcup \overleftarrow{G} = \bigoplus_{(u,v) \in E(\overrightarrow{G})} \left((u,v) \bigcup (v,u) \right).$$

We get the next conclusion.

Corollary 4.4 *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally stable or asymptotically stable if it is parallel stable or asymptotically stable, i.e., $((u,v) \bigcup (v,u))^{\widehat{L}}$ is an action flow for $\forall(u,v) \in E(\overrightarrow{G}^L)$.*

For an equilibrium point $\overrightarrow{G}^{L_0}$ of (2.1), we can also linearize $F(v,u)$, $F'(v,u)$ at (x_0, y_0) for $\forall(v,u) \in E(\overrightarrow{G}^L)$ and know the stable behavior of $\overrightarrow{G}^L$ in neighborhood of $\overrightarrow{G}^{L_0}$ by applying the following well-known result.

Theorem 4.5([3]) *If an n-dimensional system $\dot{X} = F(X)$ has an equilibrium point X_0 that is hyperbolic, i.e., all of the eigenvalues of DF_{X_0} have nonzero real parts,*

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then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of X_0.

The next result on the stability of food webs is an immediate application of Theorem 4.5.

Theorem 4.6 *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally asymptotically stable if there is an Eulerian multi-decomposition*

$$\left(\overrightarrow{G}\bigcup\overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{k=1}^{s} \overrightarrow{H}_k^L$$

with solvable conservative equations such that $Re\lambda_i < 0$ for characteristic roots λ_i of A_v in the linearization $A_v X_v = 0_{h_v \times h_v}$ of conservative equations at an equilibrium point $\overrightarrow{H}_k^{L_0}$ in $\overrightarrow{H}_k^L$ for integers $1 \leq i \leq h_v$ and $v \in V(\overrightarrow{H}_k^L)$, where $V(\overrightarrow{H}_k^L) = \{v_1, v_2, \cdots, v_{h_v}\}$,

$$A_v = \begin{pmatrix} a_{11}^v & a_{12}^v & \cdots & a_{1h_v}^v \\ a_{21}^v & a_{22}^v & \cdots & a_{2h_v}^v \\ a_{h1}^v & a_{h2}^v & \cdots & a_{hh_v}^v \end{pmatrix}$$

a constant matrix and $X_k = (x_{v_1}, x_{v_2}, \cdots, x_{v_{h_v}})^T$ for integers $1 \leq k \leq l$.

Proof Applying the theory of linear ordinary differential equations, we are easily know the species

$$x_v(t) = \sum_{i=1}^{h_v} c_i \overline{\beta}_i(t) e^{\lambda_i t},$$

where, c_i is a constant, $\overline{\beta}_i(t)$ is an h_v-dimensional vector consisting of polynomials in t determined as follows

$$\overline{\beta}_1(t) = \begin{bmatrix} t_{11} \\ t_{21} \\ \cdots \\ t_{h_v 1} \end{bmatrix}, \quad \overline{\beta}_2(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots \cdots \cdots \\ t_{n1}t + t_{h_v 2} \end{bmatrix},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots ,$$

$$\overline{\beta}_{k_1}(t) = \begin{bmatrix} \frac{t_{11}}{(k_1-1)!}t^{k_1-1} + \frac{t_{12}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{1k_1} \\ \frac{t_{21}}{(k_1-1)!}t^{k_1-1} + \frac{t_{22}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{2k_1} \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \frac{t_{h_v 1}}{(k_1-1)!}t^{k_1-1} + \frac{t_{h_v 2}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{h_v k_1} \end{bmatrix},$$

$$\overline{\beta}_{k_1+1}(t) = \begin{bmatrix} t_{1(k_1+1)} \\ t_{2(k_1+1)} \\ \cdots \cdots \\ t_{h_v(k_1+1)} \end{bmatrix}, \quad \overline{\beta}_{k_1+2}(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots \cdots \cdots \\ t_{n1}t + t_{h_v 2} \end{bmatrix},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots,$$

$$\overline{\beta}_{h_v}(t) = \begin{bmatrix} \frac{t_{1(h_v - k_s + 1)}}{(k_s - 1)!} t^{k_s - 1} + \frac{t_{1(h_v - k_s + 2)}}{(k_s - 2)!} t^{k_s - 2} + \cdots + t_{1h_v} \\ \frac{t_{2(h_v - k_s + 1)}}{(k_s - 1)!} t^{k_s - 1} + \frac{t_{2(h_v - k_s + 2)}}{(k_s - 2)!} t^{k_s - 2} + \cdots + t_{2h_v} \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \frac{t_{h_v(h_v - k_s + 1)}}{(k_s - 1)!} t^{k_s - 1} + \frac{t_{h_v(h_v - k_s + 2)}}{(k_s - 2)!} t^{k_s - 2} + \cdots + t_{h_v h_v} \end{bmatrix}$$

with each t_{ij} a real number for $1 \leq i, j \leq h_v$ such that $\det([t_{ij}]_{h_v \times h_v}) \neq 0$,

$$\alpha_i = \begin{cases} \lambda_1, & if \ 1 \leq i \leq k_1; \\ \lambda_2, & if \ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots \cdots \cdots \cdots \cdots \cdots; \\ \lambda_s, & if \ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq h_v. \end{cases}$$

If $Re\lambda_i < 0$ for integers $1 \leq i \leq h_v$, it is clear that

$$\lim_{t \to \infty} x_v(t) = 0$$

for vertex $v \in E(\overrightarrow{H}_k^L)$, i.e., each linearized conservative equation $A_v X_v = 0_{h_v \times h_v}$ is stable for $1 \leq k \leq s$. Applying Theorems 4.2 and 4.5, we therefore know that $\overrightarrow{G}^L$ is globally asymptotically stable. $\qquad \square$

Comparatively, we also get the next conclusion on the unstable of a species by Theorem 4.2 following.

Corollary 4.7 *A species T is unstable in a food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ if and only if the subgraph containing T in all Eulerian multi-decompositions*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$$

of $\overrightarrow{G}^L$ is unstable.

A unstable behavior of species T will causes the redistribution of flows and makes for a stable situation on the food web $\overrightarrow{G}^L$. If it established, the food web

works in order again. Otherwise, a few species will evolve finally to extinction, i.e., ceases to exist in that area. If all species in a food web $\overrightarrow{G}^L$ vanished on that area, there must be a series of species $x_1, x_2, \cdots, x_s$ successively died out on the time, the stability of the web is repeatedly broken, established and broken, and finally, all species become extinct. In this case there must be vertices $v_1, v_2, \cdots, v_s \in V(\overrightarrow{G}^L)$ and a series of action flows

$$\overrightarrow{G}^L \to \overrightarrow{G}^L - v_1 \to \left(\overrightarrow{G}^L - v_1\right) - v_2 \to \cdots \to \overrightarrow{G}^L - \{v_1, v_2, \cdots, v_s\}$$

such that there are no flows in $\overrightarrow{G}^L - \{v_1, v_2, \cdots, v_s\}$, i.e., $\overrightarrow{G}^L - \{v_1, v_2, \cdots, v_s\} \simeq \overline{K}_l$, where $l = |\overrightarrow{G}| - s$.

Notice that if species x is extinct, there must be $\lim\limits_{t \to +\infty} x(t) = 0$. Let $f(t)$ be a differentiable function on populations of a species x. If $f(t) = O(t^{-\alpha})$, $\alpha > 1$, i.e., there are constants $A > 0$ such that $|f(t)| \le At^{-\alpha}$ holds with $t \in (0, +\infty)$, then f is said to be α-declined and x a species extinct in rate α. Furthermore, if $f(t) = O(e^{-\beta t})$ for $\beta > 0$, because

$$e^{\beta t} = 1 + \beta t + \frac{\beta^2}{2!}t^2 + \cdots + \frac{\beta^n}{n!}t^n + \cdots,$$

we are easily know that there is a constant $A > 0$ such that $|f(t)| \le At^n$ for any integer $n \ge 1$. In this case, f is said to be ∞-declined and x a species extinct in rate ∞.

The results following characterize the extinct behavior of species in a food web.

Theorem 4.8 *Let $\overrightarrow{G}^L$ be a food web hold with labeling $L: v_i \to \dot{x}_i$ on vertices v_i, $L: (v_i, v_j) \to \{F_{ij}, (x_i, x_j), F'_{ij}\}$ on edges for integers $1 \le i, j \le n$, $V \subset V(\overrightarrow{G}^L)$. If*

$$V(t) = \sum_{v \in V} \left(\sum_{v' \in N^-(v)} F'_{v'v}(v' \to v) - \sum_{v' \in N^+(v)} F_{vv'}(v \to v') \right)$$

is α-declined, then all species $X = \sum\limits_{v \in V} x_v$ in V is extinct in at least rates $\alpha - 1$ and particularly, if $V = \{v\}$, the species x_v is extinct in at least rates $\alpha - 1$ on t.

Proof Notice that the conservative equation at vertex $v \in V$ is

$$\dot{x}_v = \sum_{v' \in N^-(v)} F'_{v'v}(x_{v'} \to x_v) - \sum_{v' \in N^+(v)} F_{vv'}(x_v \to x_{v'})$$

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and

$$\dot{X} = \frac{d\left(\sum\limits_{v \in V} x_v\right)}{dt} = \sum_{v \in V} \dot{x}_v$$

$$= \sum_{v \in V} \left(\sum_{v' \in N^-(v)} F'_{v'v}(v' \to v) - \sum_{v' \in N^+(v)} F_{vv'}(v \to v') \right).$$

Now, if $V(t)$ is α-declined, there must be constant $A > 0$ such that

$$-\frac{A}{t^\alpha} \leq \dot{X} = \sum_{v \in V} \left(\sum_{v' \in N^-(v)} F'_{v'v}(v' \to v) - \sum_{v' \in N^+(v)} F_{vv'}(v \to v') \right) = V(t) \leq \frac{A}{t^\alpha}.$$

Consequently,

$$|X| \leq \int\limits_0^{+\infty} \frac{A}{t^\alpha} dt = A \int\limits_0^{+\infty} \frac{1}{t^\alpha} dt = \frac{A}{(\alpha - 1)t^{\alpha-1}} = O(t^{-\alpha+1}).$$

Therefore, all species X in V is extinct in at least rates $\alpha - 1$ on t, and particularly, it holds with the case of $V = \{v\}$. $\qquad\square$

Theorem 4.9 *Let $\overrightarrow{G}^L$ be a food web hold with labeling $L : v_i \to \dot{x}_i$ on vertices v_i, $L : (v_i, v_j) \to \{F_{ij}, (x_i, x_j), F'_{ij}\}$ on edges for integers $1 \leq i, j \leq n$, and $V \subset V(\overrightarrow{G}^L)$ a cut set with components $C_1, C_2, \cdots, C_l$ in $\overrightarrow{G}^L \setminus V$, where $l \geq 2$. If*

$$f_v(t) = \sum_{v' \in N^-(v)} F'_{v'v}(v' \to v) - \sum_{v' \in N^+(v)} F_{vv'}(v \to v')$$

is α_v-declined for $\forall v \in V$ with $\alpha = \min\limits_{v \in V} \alpha_v$, then

(1) $\overrightarrow{G}^L$ turns to l food webs $\overrightarrow{C}_1^L, \overrightarrow{C}_2^L, \cdots, \overrightarrow{C}_l^L$ finally;

(2) the species $X_V = \sum\limits_{v \in V} x_v$, particularly, x_v is extinct in at least rates $\alpha - 1$ on t for $\forall v \in V$.

Proof Applying Theorem 4.8, all species X in V is extinct in at least rates $\alpha - 1$ on t, and finally, extinction if $t \to \infty$. In this case, there are only left components $C_1, C_2, \cdots, C_l$, and each of them is a food web because if $x_v = 0$, there must be $F(v \to u) = 0$ and $F'(u \to v) = 0$ for $\forall v \in V$ and $u \in V(\overrightarrow{G}^L) \setminus V$. Therefore, the conservative laws

$$\dot{x}_u = \sum_{v \in N^-(u)} F'_{vu}(x_u \to x_v) - \sum_{v \in N^+(u)} F_{uv}(x_u \to x_v)$$

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in $\overrightarrow{G}^L$ turns to

$$\dot{x}_u = \sum_{v \in N^-(u) \bigcap V(C_i)} F'_{vu}(x_v \to x_u) - \sum_{v \in N^+(u) \bigcap V(C_i)} F_{uv}(x_u \to x_v),$$

i.e., it holds also with vertex u in $\overrightarrow{C}_i^L$ for integers $1 \le i \le l$, the assertion (1).

For (2), by the proof of Theorem 4.8 there is a number $A > 0$ such that

$$-\int_0^{+\infty} \kappa(\overrightarrow{G}^L) \frac{A}{t^\alpha} dt \le X_V(t) = \sum_{v \in V} x_v(t) \le \int_0^{+\infty} \kappa(\overrightarrow{G}^L) \frac{A}{t^\alpha} dt$$

by definition, where $\kappa(\overrightarrow{G}^L)$ is the connectivity of $\overrightarrow{G}^L$. Whence, $X_V(t) = O(t^{-\alpha+1})$, and particularly, $x_v(t) = O(t^{-\alpha+1})$ for $v \in V$. $\qquad\square$

Finally, there are indeed the case of extinction of species in rate ∞. For example, the proof of Theorem 4.6 implies the case of extinction in rate ∞ on t following.

Theorem 4.10 *Let $\overrightarrow{G}^L$ be food web with an Eulerian multi-decomposition*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{k=1}^s \overrightarrow{H}_k^L$$

and all conservative equations on $\overrightarrow{H}_k^L$ are solvable for integers $1 \le k \le l$. For a vertex $v \in V(\overrightarrow{G}^L)$ including repeatedly in $\overrightarrow{H}_{i_1}^L, \overrightarrow{H}_{i_2}^L, \cdots, \overrightarrow{H}_{i_l}^L$, if $Re\lambda_i < 0$ for characteristic roots λ_i of A_k in the linearization

$$A_k X_k = 0_{h_k \times h_k}$$

of conservative equation at an equilibrium point $\overrightarrow{H}_k^{L_0}$, $v \in V(\overrightarrow{H}_k^L)$ for integers $1 \le i \le h_k$, then the species x_v is simultaneously extinct in rate ∞ on time t and asymptotically stable, where $V(\overrightarrow{H}_k^L) = \{v_1, v_2, \cdots, v_{h_k}\}$, A_k and X_k are as the same in Theorem 4.6 for integers $1 \le k \le l$.

Proof By the proof of Theorem 4.6, we know that $x_v(t)$ is asymptotically stable with

$$x_v(t) = \sum_{i=1}^{h_k} c_i \overline{\beta}_i(t) e^{\lambda_i t}.$$

Define $\beta = \min\limits_{1 \le i \le h_k} \lambda_i$. If $\lambda_i < 0$ for integers $1 \le i \le h_k$, then $x_v(t)$ is clearly an ∞-declined function and species x_v is extinct in rate ∞ on time t. $\qquad\square$

§5. Algorithm

Let $\mathscr{G} = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_m\}$ be solvable Eulerian multi-decompositions of bi-digraph $(\overrightarrow{G} \bigcup \overleftarrow{G})^{\widehat{L}}$ of a food web $\overrightarrow{G}^L$ with conservation equations (1.1) solvable or not, where $\mathscr{C}_1$ and $\mathscr{C}_m$ are respectively a parallel decomposition, $\overrightarrow{G}^L$ itself of $\overrightarrow{G}^L$. Theorems 4.2 and 4.6 conclude the following algorithm on the global stability of $\overrightarrow{G}^L$.

Algorithm 5.1 *The stability of a food web $\overrightarrow{G}^L$ can be determined by programming following:*

STEP 1.   *Input $X_i = \mathscr{C}_i$ and $i = 1, 2, \cdots, m$;*

STEP 2.   *Determine Eulerian circuits in X_i is globally stable or not;*

STEP 3.   *If X_i is globally stable, go to STEP 6; Otherwise, go to STEP 4;*

STEP 4.   *Replace X_i by X_{i+1}, return to STEP 2;*

STEP 5.   *If X_{i+1} is globally stable, go to STEP 6; Otherwise, go to STEP 4 if $i < m$, or go to STEP 7 if $i = m$;*

STEP 6.   *$\overrightarrow{G}^L$ is globally stable, the algorithm is terminated;*

STEP 7.   *$\overrightarrow{G}^L$ is globally non-stable, the algorithm is terminated.*

This algorithm certainly enables one to determine the stability of a food web $\overrightarrow{G}^L$ regardless of whether its conservation equations solvable or not, and get stability of food webs with more species than 3 by conclusions on 2 or 3 species.

Example 5.2 Determine the stability of a biological 5-system $\overrightarrow{G}^L$ shown in Fig.10,
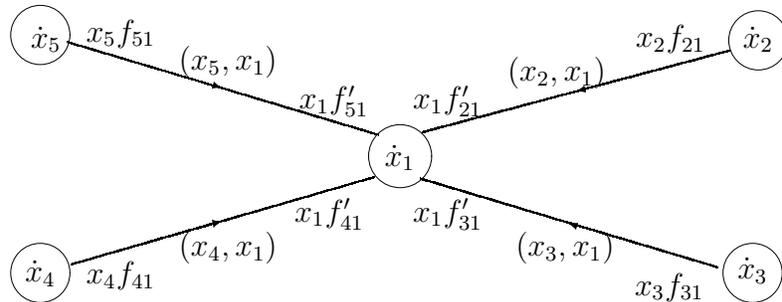
Fig.10

where, f_{ij} and f'_{ij}, $1 \le i, j \le 5$ are defined by

$$f_{21}(x_2, x_1) = 1 - x_2 - \lambda_1 x_1, \quad f'_{21}(x_2, x_1) = 1 - x_1 - \lambda_2 x_2,$$

$$f_{31}(x_3, x_1) = 1 - x_3 - \lambda_1 x_1, \quad f'_{31}(x_3, x_1) = 1 - x_1 - \lambda_3 x_3,$$

$$f_{41}(x_4, x_1) = 1 - x_4 - \lambda_1 x_1, \quad f'_{41}(x_4, x_1) = 1 - x_1 - \lambda_4 x_4,$$

$$f_{51}(x_5, x_1) = 1 - x_5 - \lambda_1 x_1, \quad f'_{51}(x_5, x_1) = 1 - x_1 - \lambda_5 x_5$$

with conservative equations

$$\begin{cases} \dot{x}_1 = x_1 \left(4 - 4x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 x_4 - \lambda_5 x_5 \right) \\ \dot{x}_2 = -x_2 \left(1 - x_2 - \lambda_1 x_1 \right) \\ \dot{x}_3 = -x_3 \left(1 - x_3 - \lambda_1 x_1 \right) \\ \dot{x}_4 = -x_4 \left(1 - x_4 - \lambda_1 x_1 \right) \\ \dot{x}_5 = -x_5 \left(1 - x_5 - \lambda_1 x_1 \right) \end{cases} \tag{5.1}$$

Let $(x_1^0, x_2^0, x_3^0, x_4^0, x_5^0)$ be an equilibrium point of (5.1). Calculation shows the linearization of (5.1) is

$$\begin{cases} \dot{x}_1 = A x_1 - \lambda_2 x_1^0 x_2 - \lambda_3 x_1^0 x_3 - \lambda_4 x_1^0 x_4 - \lambda_5 x_1^0 x_5 \\ \dot{x}_2 = \lambda_1 x_2^0 x_1 + \left(-1 + 2x_2^0 + \lambda_1 x_1^0 \right) x_2 \\ \dot{x}_3 = \lambda_1 x_3^0 x_1 + \left(-1 + 2x_3^0 + \lambda_1 x_1^0 \right) x_3 \\ \dot{x}_4 = \lambda_1 x_4^0 x_1 + \left(-1 + 2x_4^0 + \lambda_1 x_1^0 \right) x_4 \\ \dot{x}_5 = \lambda_1 x_5^0 x_1 + \left(-1 + 2x_5^0 + \lambda_1 x_1^0 \right) x_5 \end{cases} \quad , \tag{5.2}$$

where $A = 4 - 8x_1^0 - \lambda_2 x_2^0 - \lambda_3 x_3^0 - \lambda_4 x_4^0 - \lambda_5 x_5^0$.
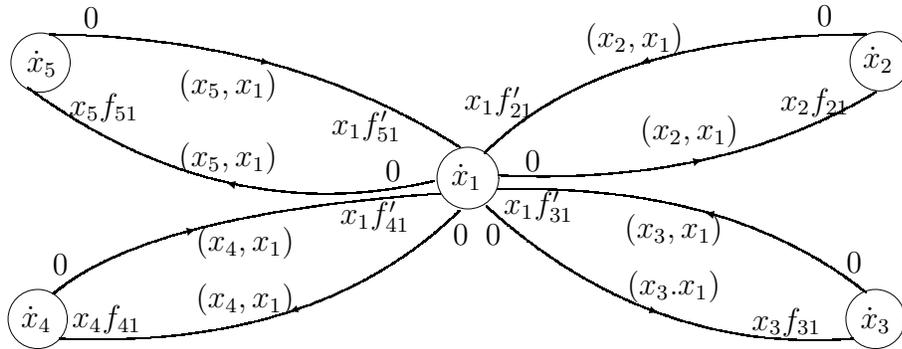
Fig.11

As usual, we can hold on the stability of system (5.2) of linear equations and then, the stability of (5.1) by Theorem 4.6 on equilibrium points with tedious calculation. However, we apply Algorithm 5.1 for the objective.

Notice that bi-digraph $(\overrightarrow{G} \bigcup \overleftarrow{G})^{\widehat{L}}$ of $\overrightarrow{G}^L$ in Fig.11 has a parallel decomposition such as those shown in Fig.12,
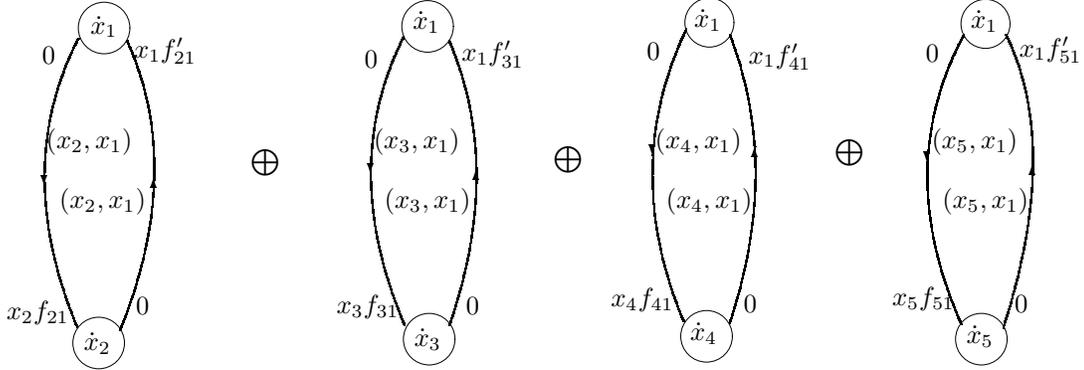
Fig.12

and the conservation equations on these parallel edges are respectively

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - \lambda_2 x_2) \\ \dot{x}_2 = x_2(1 - x_2 - \lambda_1 x_1) \end{cases} \tag{5.3}$$

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - \lambda_3 x_3) \\ \dot{x}_3 = x_3(1 - x_3 - \lambda_1 x_1) \end{cases} \tag{5.4}$$

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - \lambda_4 x_4) \\ \dot{x}_4 = x_4(1 - x_4 - \lambda_1 x_1) \end{cases} \tag{5.5}$$

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1 - \lambda_5 x_5) \\ \dot{x}_5 = x_5(1 - x_5 - \lambda_1 x_1) \end{cases} \tag{5.6}$$

We have known the stability of equations (5.3)-(5.6) by their linearizations following ([20]):

(1) *the equilibrium point* $(x_1, x_i) = (0, 0)$ *is unstable for equations (5.3)-(5.6), where* $i = 2, 3, 4, 5$;

(2) *the equilibrium point* $(x_1, x_i) = (1, 0)$ *is stable if* $\lambda_1 > 1$ *for* $i = 2, 3, 4, 5$;

(3) *the equilibrium point* $(x_1, x_i) = (0, 1)$ *is stable if* $\lambda_i > 1$ *for equations (5.3)-(5.6), where* $i = 2, 3, 4, 5$;

(4) *the equilibrium point* $\left(\dfrac{\lambda_i - 1}{\lambda_1 \lambda_i - 1}, \dfrac{\lambda_1 - 1}{\lambda_1 \lambda_i - 1} \right)$ *is asymptotically stable if* $\lambda_1 > 1$
and $\lambda_i > 1$ *for equations (5.3)-(5.6), where* $i = 2, 3, 4, 5$.

Therefore, we know this biological 5-system is unstable on the equilibrium point $(0, 0, 0, 0, 0)$ but stable on the equilibrium points $(0, 1, 1, 1, 1)$ and $(1, 0, 0, 0, 0)$, and asymptotically stable on the equilibrium point

$$\left(\frac{\lambda - 1}{\lambda_1 \lambda - 1}, \frac{\lambda_1 - 1}{\lambda_1 \lambda - 1}, \frac{\lambda_1 - 1}{\lambda_1 \lambda - 1}, \frac{\lambda_1 - 1}{\lambda_1 \lambda - 1}, \frac{\lambda_1 - 1}{\lambda_1 \lambda - 1} \right)$$

of system (5.1) if $\lambda > 1$ and $\lambda_1 > 1$ by Theorem 4.2, where $\lambda = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$.

§6. Conclusion

Today, we have many mathematical theories but we are still helpless on opening the mystery of the nature as Einstein's complaint, i.e., *as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality* because the multiple nature, or contradiction is universal on things, particularly, living things different from rigid bodies. Usually, we establish differential equations for characterizing things T and holds on their behavior by solutions, which is only hold on those of coherent behaviors of things, not the individual. Thus, we encounter non-solvable cases in biology, and even if it is solvable, finding the exact solution is nearly impossible in most cases. In fact, the solvable equation is individual but the non-solvable is universal for knowing the nature. This fact implies that we should also research those of non-solvable equations for revealing reality of things in mathematics, which finally brings about the mathematics over topological graphs, i.e., *action flows*, or *mathematical combinatorics*, and only which is the practicable way for understanding things, particularly, living things in the world.

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