Topics in Recreational Mathematics 4/2015

Charles Ashbacher

Artwork by Caytie Ribble

CONTENTS

CONCATENATION PROBLEMS

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Abstract

This study has been inspired by questions asked by Charles Ashbacher in the *Journal of Recreational Mathematics, vol. 29.2* concerning the Smarandache Deconstructive Sequence. This sequence is a special case of a more general concatenation and sequencing procedure which is the subject of this study. The properties of this kind of sequences are studied with particular emphasis on the divisibility of their terms by primes.

Introduction

In this article the concatenation of a and b is expressed by a_b or simply ab where there can be no misunderstanding. Multiple concatenations like abcabcabc will be expressed by 3(abc).

We consider n different elements (or n objects) arranged (concatenated) one after the other in the following way to form:

$A=a_1a_2 \dots a_n$.

Infinitely many objects A, which will be referred to as cycles, are concatenated to form the chain:

$B= a_1a_2 \dots a_n\, a_1a_2 \dots a_n\, a_1a_2 \dots a_n \dots$

B contains identical elements which are at equidistant positions in the chain. Let's write B as

$B=b_1b_2b_3, \dots b_k \dots$.. where $b_k=a_j$ when $j\equiv k \pmod n$, $1\leq j\leq n$.

An infinite sequence $C_1, C_2, C_3, \dots C_k, \dots$. is formed by sequentially selecting 1, 2, 3, …k, … elements from the chain B:

$C_1=b_1=a_1$

$C_2=b_2b_3=a_2a_3$

$C_3=b_4b_5b_6=a_4a_5a_6$ (if $n\leq 6$, if n=5 we would have $C_3=a_4a_5a_1$)

The number of elements from the chain B used to form the first k-1 terms of the sequence C is 1+2+3+ … +k-1=(k-1)k/2. Hence

$$C_k = b_{\frac{(k-1)k}{2}+1}\, b_{\frac{(k-1)k}{2}+2} \cdots b_{\frac{k(k+1)}{2}}$$

However, what is interesting to see is how C_k is expressed in terms of $a_1,\dots,a_n$. For sufficiently large values of k, C_k will be composed of three parts:

The first part $F(k)=a_u \dots a_n$

The middle part $M(k)=AA\dots A$. The number of concatenated A's depends on k.

The last part $L(k)=a_1a_2 \dots a_w$

Hence

(1) $C_k=F(k)M(k)L(k)$.

The number of elements used to form $C_1, C_2, \dots C_{k-1}$ is ((k-1)k)/2. Since the number of elements in A is finite there will be infinitely many terms C_k which have the same first element a_u. u can be determined from

$$\frac{(k-1)k}{2}+1 \equiv u(\bmod n)$$

There can be at most n^2 different combinations to form F(k) and L(k).

Let C_j and C_i be two different terms for which F(i)=F(j) and L(i)=L(j). They will then be separated by a number m of complete cycles of length n, i.e.

$$\frac{(j-1)j}{2}-\frac{(i-1)i}{2}=mn$$

Let's write j=i+p and see if p exists so that there is a solution for p which is independent of i.

$$(i+p-1)(i+p)-(i-1)i=2mn$$

$$p^2+p(2i-1)=2mn$$

If n is odd we will put p=n to obtain n+2i-1=2m, or

$$m=\frac{n+2i-1}{2}$$

If n is even we put p=2n to obtain m=2n+2i-1. From this we see that the terms C_k have a peculiar periodic behaviour. The periodicity is p=n for odd n and p=2n for even n. Let's illustrate this for n=4 and n=5 for which the periodicity will be p=8 and p=5 respectively. It is seen from table 1 that the periodicity starts for i=3.

Numerals are chosen as elements to illustrate the case n=5 (table 2). Let's write i=s+k+pj , where s is the index of the term preceding the first periodical term, k=1,2,…,p is the index of members of the period and j is the number of the period (for convenience the first period is numbered 0). The first part of C_i is denoted B(k) and the last part E(k). C_i is now given by the following expression, where q is the number of cycles concatenated between the first part B(k) and the last part E(k).

(2) C_i=B(k)_qA_E(k), where k is determined from i-s≡k (mod p)

Table 1

n=4. A=abcd. B= abcdabcdabcdabcdabcd……

	C_I	Period #	F(i)	M(i)	L(i)
1	a		a		
2	bc		bc		
3	dab	1	d		ab
4	cdab	1	cd		ab
5	cdabc	1	cd		abc
6	dabcda	1	d	abcd	a
7	bcdabcd	1	bcd	abcd	
8	abcdabcd	1		2(abcd)	
9	abcdabcda	1		2(abcd)	a
10	bcdabcdabc	1	bcd	abcd	abc
11	dabcdabcdab	2	d	2(abcd)	ab
12	cdabcdabcdab	2	cd	2(abcd)	ab
13	cdabcdabcdabc	2	cd	2(abcd)	abc
14	dabcdabcdabcda	2	d	3(abcd)	a
15	bcdabcdabcdabcd	2	bcd	3(abcd)	
16	abcdabcdabcdabcd	2		4(abcd)	
17	abcdabcdabcdabcda	2		4(abcd)	a
18	bcdabcdabcdabcdabc	2	bcd	3(abcd)	abc
19	dabcdabcdabcdabcdab	3	d	4(abcd)	ab
20	cdabcdabcdabcdabcda b	3	cd	4(abcd)	ab

Table 2

n=5. A=12345. B= 123451234512345.........

I	C_I	k	q	F(i)/B(k)	M(I)	L(i)/E(k)
1	1			1		
s=2	23			23		
	j=0					
3	451	1	0	45		1
4	2345	2	0	2345		
5	12345	3	1		12345	
6	123451	4	1		12345	1
7	2345123	5	0	2345		123
	j=1					
3+5j	45123451	1	j	45	12345	1
4+5j	234512345	2	j	2345	12345	
5+5j	1234512345	3	j+1		2(12345)	
6+5j	12345123451	4	j+1		2(12345)	1
7+5j	234512345123	5	j	2345	12345	123
	j=2					
3+5j	45123451234512	1	j	45	2(12345)	1
4+5j	234512345123456	2	j	2345	2(12345)	
...						

The Smarandache Deconstructive Sequence

The Smarandache Deconstructive Sequence of integers [1] is constructed by sequentially repeating the digits 1-9 in the following way:

1,23,456,789123,4567891,23456789,123456789,1234567891, ...

The sequence was studied in a booklet by Kashihara [2] and a number of questions on this sequence were posed by Ashbacher [3]. In thinking about these questions two observations led to this study.

1. Why did Smarandache exclude 0 from the integers used to create the sequence? After all 0 is indispensible in all arithmetics most of which can be done using 0 and 1 only.
2. The process used to create the Deconstructive Sequence is a process which applies to any set of objects as has been shown in the introduction.

The periodicity and the general expression for terms in the "generalized deconstructive sequence" shown in the introduction may be the most important results of this study. These results will now be used to examine the questions raised by Ashbacher. It is worth noting that these divisibility questions are dealt with in base 10 although only the nine digits 1,2,3,4,5,6,7,8,9 are used to express numbers. In the last part of this article questions on divisibility will be posed for a deconstructive sequence generated from A="0123456789".

For $i > 5$ $(s = 5)$ any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A="123456789" and a last part E(k), where $i = 5 + k + 9j$, $k = 1,2,\ldots 9$, $j \geq 0$, as expressed in (2) and $q = j$ or $j + 1$ as shown in table 3.

Members of the Smarandache Deconstructive Sequence are now interpreted as decimal integers. The factorization of B(k) and E(k) is shown in table 3. The last two columns of this table will be useful later in this article.

Together with the factorization of the cycle $A=123456789=3^2 *3607 * 3803$ it is now possible to study some divisibility properties of the sequence. We will first find expressions for C_i for each of the 9 values of k. In cases where E(k) exists let's introduce $u = 1 + [\log_{10}E(k)]$. We also define the function $\delta(j)$ so that $\delta(j) = 0$ for $j = 0$ and $\delta(j) = 1$ for $j > 0$. It is possible to construct one algorithm to cover all the nine cases but more functions like $\delta(j)$ would have to be introduced to distinguish between the numerical values of the strings "" (empty string) and "0" which are both evaluated as 0 in computer applications. In order to avoid this four formulas are used:

Table 3

Factorization of Smarandache Deconstructive Sequence

i	k	B(k)	q	E(k)	Digit sum	$3\|C_i$?
6+9j	1	789=3·263	j	123=3·41	30+j·45	3
7+9j	2	456789=3·43·3541	j	1	40+j·45	No
8+9j	3	23456789	j		44+j·45	No
9+9j	4		j+1		(j+1)·45	$9 \cdot 3^z$ *
10+9j	5		j+1	1	1+(j+1)·45	No
11+9j	6	23456789	j	123=341	50+j·45	No
12+9j	7	456789=3·43·3541	j	$123456=2^6 \cdot 3 \cdot 643$	60+j·45	3
13+9j	8	789=3·263	j+1	1	25+(j+1)·45	No
14+9j	9	23456789	j	$123456=2^6 \cdot 3 \cdot 643$	65+j·45	No

*) where z depends on j.

For k=1, 2, 6, 7 and 9:

$$(3)\quad C_{5+k+9j} = E(k) + \delta(j) * A * \sum_{r=0}^{j-1} 10^{9r} + B(k) * 10^{9j+u}.$$

For k=3:

$$(4)\quad C_{5+k+9j} = \delta(j) * A * \sum_{r=0}^{j-1} 10^{9r} + B(k) * 10^{9j}.$$

For k=4:

(5) $$C_{5+k+9j} = A * \sum_{r=0}^{j} 10^{9r}.$$

For k=5 and 8:

(6) $$C_{5+k+9j} = E(k) + A * 10^{u} * \sum_{r=0}^{j} 10^{9r} + B(k) * 10^{9(j+1)+u}.$$

Before dealing with the questions posed by Ashbacher we recall the familiar rules: An even number is divisible by 2; a number whose last two digit form a number which is divisible by 4 is divisible by 4.

In general we have the following:

Theorem. Let N be an n-digit integer such that $N > 2^{\alpha}$, then N is divisible by 2^{α} if and only if the number formed by the α last digits of N is divisible by 2^{α}.

Proof. To begin with we note that:

If x divides a and x divides b then x divides (a+b).

If x divides one but not the other of a and b then x does not divide (a+b).

If x does not divides neither a nor b then x may or may not divide (a+b).

Let's write the n-digit number in the form $a * 10^{\alpha} + b$. We then see from the following that

$a * 10^{\alpha}$ is divisible by 2^{α}.

$$10 \equiv 0 \pmod 2$$

$$100 \equiv 0 \pmod 4$$

$$1000 = 2^3 * 5^3 \equiv 0 \pmod{2^3}$$

...

$$10^{\alpha} \equiv 0 \pmod{2^{\alpha}}$$

and then

$a \cdot 10^{\alpha} \equiv 0 \pmod{2^{\alpha}}$ independent of a.

Now let b be the number formed by the α last digits of N. We see from the introductory remark that N is divisible by 2^α if and only if the number formed by the α last digits is divisibele by 2^α.

Question 1. Does every even element of the Smarandache Deconstructive Sequence contain at least three instances of the prime 2 as a factor?

Question 2. If we form a sequence from the elements of the Smarandache Deconstructive Sequence that end in a 6, do the powers of 2 that divide them form a montonically increasing sequence?

These two questions are related and are dealt with together. From the previous analysis we know that all even elements of the Smarandache Deconstructive Sequence end in a 6. For $i \le 5$ they are:

$C_3 = 456 = 57 * 2^3$

$C_5 = 23456 = 733*2^5$

For $i > 5$ they are of the forms:

C_{12+9j} and C_{14+9j} which both end in …789123456.

Examining the numbers formed by the 6, 7 and 8 last digits for divisibility by 2^6, 2^7 and 2^8 respectively we have:

$123456=2^6 * 3 * 643$

$9123456=2^7 * 149 * 4673,$

89123456 is not divisible by 2^8.

From this we conclude that all even Smarandache Deconstructive Sequence elements for $i \ge 12$ are divisible by 2^7 and that no elements in the sequence are divisible by higher powers of 2 than 7.

Answer to question 1. Yes.

Answer to question 2. The sequence is monotonically increasing for $i \le 12$. For $i \ge 12$ the powers of 2 that divide even elements remain constant=2^7.

Question 3. Let x be the largest integer such that $3^x \mid i$ and y the largest integer such that $3^y \mid C_i$. Is it true that x is always equal to y?

From table 3 we se that the only elements C_i of the Smarandache Deconstructive Sequence which are divisible by powers of 3 correspond to $i = 6 + 9j, 9 + 9j,$ or $12 + 9j$. Furthermore, we see that $i = 6 + 9j$ and C_{6+9j} are divisible by 3 no more no less. The same is true for $i = 12 + 9j$ and C_{12+9j}. So the statement holds in these cases.

From the conguences

$$9 + 9j \equiv 0 \pmod{3^x} \text{ for the index of the element}$$

and

$$45(1 + j) \equiv 0 \pmod{3^y} \text{ for the corresponding element.}$$

we conclude that $x = y$.

Answer to question 3: The statement is true. It is interesting to note that, for example the 729 digit number C_{729} is divisible by 729.

Question 4. Are there other patterns of divisibility in this sequence?

A search for other patterns would continue by examining divisibility by the next lower primes 5, 7, 11, ... It is obvious from table 3 and the periodicity of the sequence that there are no elements divisible by 5. The algorithms will prove very useful. For each value of k the value of C_i depends on j only. The divisibilty by a prime p is therefore determined by finding out for which values of j and k the congruence $C_i \equiv 0 \pmod p$ holds. We evaluate

$$\sum_{r=0}^{j-1} 10^{9r} = \frac{10^{9j} - 1}{10^9 - 1}.$$

and introduce $G = 10^9 - 1$. We note that $G = 3^4 * 37 * 333667$. From formulas (3) to (6) we now obtain:

For k=1,2,6,7 and 9:

(3′) $C_i * G = 10^u * (\delta(j) * A + B(k) * G) * 10^{9j} + E(k) * G - 10^u * \delta(j) * A.$

For k=3:

(4′). $C_i * G = ((\delta(j) * A + B(k) * G) * 10^{9j} - \delta(j) * A.$

For k=4:

(5′) $C_i * G = A * 10^{9j} - A.$

For k=5 and 8:

(6′) $C_i * G = 10^{u+9}(A + B(k) * G) * 10^{9j} + E(k) * G - 10^u * A.$

The divisibility of C_i by a prime p other than 3, 37 and 333667 is therefore determined by solutions for j to the congruences $C_iG \equiv 0 \pmod p$ which are of the form

(7) $\quad a * (10^9)^j + b \equiv 0 \pmod p.$

Table 4 shows the results from computer implementation of the congruences. The appearance of elements divisible by a prime p is periodic, the periodicity is given by $j = j_1 + m * d$, $m = 1, 2, 3, \ldots$. The first element divisible by p appears for i_1 corresponding to j_1. In general the terms C_i divisible by p are $C_{5+k+9(j_1 + md)}$, where d is specific to the prime p and m=1, 2, 3,... .We note from table 4 that d is either equal to p - 1 or a divisor of p - 1 except for the case p=37 which as we have noted is a factor of A. Indeed this periodicity follows from Euler's extension of Fermat's little theorem because if we write (mod p):

$$a * (10^9)^j + b = a * (10^9)^{j_1+md} + b \equiv a * (10^9)^{j_1} + b \text{ for } d = p - 1 \text{ or a divisor of } p - 1.$$

Finally we note that the periodicity for p=37 is d=37.

Question: Table 4 indicates some interesting patterns. For instance, the primes 19, 43 and 53 only divide elements corresponding to k=1, 4 or 7 for j < 150 which was set as an upper limit for this study. Similarly, the primes 7,11, 41, 73, 79 and 91 only divides elements corresponding to k = 4. Is 5 the only prime that cannot divide an element of the Smarandache Deconstructive Sequence?

Table 4

Smarandache Deconstructive Sequence elements divisible by p

p=7	k	4
d=2	i_1	18
	j_1	1

p=11	k	4
d=2	i_1	18
	j_1	1

p=13	k	4	8	9
d=2	i_1	18	22	14
	j_1	1	1	0

p=17 d=16	k	1	2	3	4	5	6	7	8	9
	i_1	6	43	44	144	100	101	138	49	95
	j_1	0	4	4	15	10	10	14	4	9

p=19 d=2	k	1	4	7
	i_1	15	18	21
	j_1	1	1	1

p=23 d=22	k	1	2	3	4	5	6	7	8	9
	i_1	186	196	80	198	118	200	12	184	14
	j_1	20	21	8	21	12	21	0	19	0

p=29 d=28	k	1	2	3	4	5	6	7	8	9
	i_1	24	115	197	252	55	137	228	139	113
	j_1	2	12	21	27	5	14	24	14	11

p=31 d=5	k	3	4	5
	i_1	26	45	19
	j_1	2	4	1

p=37 d=37	k	1	2	3	4	5	6	7	8	9
	i_1	222	124	98	333	235	209	111	13	320
	j_1	24	13	10	36	25	22	11	0	34

p=41	k	4
d=5	i_1	45
	j_1	4

P=43	1	4	7
d=7	33	63	30
	3	6	2

p=47	k	1	2	3	4	5	6	7	8	9
d=46	i_1	150	250	368	414	46	164	264	400	14
	j_1	16	27	40	45	4	17	28	43	0

p=53	k	1	4	7
d=13	i_1	24	117	12
	j_1	2	12	9

p=59	k	1	3	5	6	7	8	9
d=58	i_1	267	413	109	11	255	256	266
	j_1	29	45	11	0	27	27	28

p=61	k	2	4	6
d=20	i_1	79	180	101
	j_1	8	19	10

p=67	k	4	8	9
d=11	i_1	99	67	32
	j_1	10	6	2

p=71	k	1	3	4	5	7
d=35	i_1	114	53	315	262	201
	j_1	12	5	34	28	21

p=73	k	4
d=8	i_1	72
	j_1	7

p=79	k	4
d=13	i_1	117
	j_1	12

p=83	k	1	2	4	6	7	8	9
d=41	i_1	348	133	369	236	21	112	257
	j_1	38	14	40	25	1	11	27

p=89	k	2	4	6
d=44	i_1	97	396	299
	j_1	10	43	32

p=97	k	1	2	3	4	5	6	7	8	9
d=32	i_1	87	115	107	288	181	173	201	202	86
	j_1	9	12	11	31	19	18	21	21	8

3. A Deconstructive Sequence generated by the cycle A=0123456789.

Instead of sequentially repeating the digits 1-9 as in the case of the Smarandache Deconstructive Sequence we will use the digits 0-9 to form the corresponding sequence:

0,12,345,6789,01234,567890,1234567,89012345,678901234,678901234,56789012345,678901234567, …

In this case the cycle has n = 10 elements. As we have seen in the introduction the sequence then has a period = 2n = 20. The periodicity starts for i = 8. Table 5 shows how for i > 7 any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A="0123456789" and a last part E(k), where i = 7 + k + 20j, k = 1,2,…20, j ≥ 0, as expressed in (2) and q = 2j, 2j + 1 or 2j + 2. In the analysis of the sequence it is important to distinguish between the cases where E(k) = 0, k = 6,11,14,19 and cases where E(k) does not exist, i.e. k = 8,12,13,14. In order to cope with this problem we introduce a function u(k) which will at the same time replace the functions δ(j) and $u=1+[\log_{10}E(k)]$ used previously. u(k) is defined as shown in table 5. It is now possible to express C_i in a single formula

$$(8)\quad C_i = C_{7+k+20j} = E(k) + A * \sum_{r=0}^{q(k)+2j-1} (10^{10})^r + B(k) * (10^{10})^{q(k)+2j}) * 10^{u(k)} .$$

The formula for C_i was implemented modulus prime numbers less then 100. The result is shown in table 6 for p ≤ 41. Again we note that the divisibility by a prime p is periodic with a period d which is equal to p - 1 or a divisor of p - 1, except for p =11 and p = 41 which are factors of 10^{10} - 1. The cases p = 3 and 5 have very simple answers and are not included in table 6.

Table 5

n=10, A=0123456789

i	k	B(k)	q	E(k)	u(k)
8+20j	1	89	2j	012345=3·5·823	6
9+20j	2	6789=3·31·73	2j	01234=2·617	5
10+20j	3	56789=109·521	2j	01234=2·617	5
11+20j	4	56789=109·521	2j	012345=3·5·823	6
12+20j	5	6789=3·31·73	2j	01234567=127·9721	8
13+20j	6	89	2j+1	0	1
14+20j	7	123456789=3^2·3607·3803	2j	01234=2·617	5
15+20j	8	56789=109·521	2j+1		0
16+20j	9		2j+1	012345=3·5·823	6

17+20j	10	6789=3·31·73	2j+1	012=2²·3	3
18+20j	11	3456789=3·7·97·1697	2j+1	0	1
19+20j	12	123456789=3²·3607·3803	2j+1		0
20+20j	13		2j+2		0
21+20j	14		2j+2	0	1
22+20j	15	123456789=3²·3607·3803	2j+1	012=2²·3	3
23+20j	16	3456789=3·7·97·1697	2j+1	012345=3·5·823	6
24+20j	17	6789=3·31·73	2j+2		0
25+ 20j	18		2j+2	01234=2·617	5
26+20j	19	56789=109·521	2j+2	0	1
27+20j	20	123456789=3²·3607·3803	2j+1	01234567=127·9721	8

Table 6

Divisibility of the 10-cycle destructive sequence by primes $7 \leq p \leq 41$

p=7 d=3	k	3	6	7	8	11	12	13	14	15	18	19	20
	i_1	30	13	14	15	38	59	60	61	22	45	46	47
	j_1	1	0	0	0	1	2	2	2	0	1	1	1

p=11 d=11	k	1	2	3	4	5	6	7	8	9	10
	i_1	88	9	110	211	132	133	74	35	176	137
	j_1	4	0	5	10	6	6	3	1	8	6
	k	11	12	13	14	15	16	17	18	19	20
	i_1	18	219	220	221	202	83	44	185	146	87
	j_1	0	10	10	10	9	3	1	8	6	3

p=13	k	2	3	4	12	13	14
d=3	i_1	49	30	11	59	60	61
	j_1	2	1	0	2	2	2

p=17	k	1	5	10	12	13	14	16
d=4	i_1	48	32	37	79	80	81	43
	j_1	2	1	1	3	3	3	1

p=19	k	1	2	3	4	5	10	12	13	14	16
d=9	i_1	128	149	90	31	52	117	179	180	181	63
	j_1	6	7	4	1	2	5	8	8	8	2

p=23	k	1	2	3	4	5	10	12	13	14	16
d=11	i_1	168	149	110	71	52	217	219	220	221	223
	j_1	8	7	5	3	2	10	10	10	10	10

p=29	k	2	4	10	12	13	14	16
d=7	i_1	129	11	97	139	140	141	43
	j_1	6	0	4	6	6	6	1

p=31	k	3	9	12	13	14	17
d=3	i_1	30	56	59	60	61	64
	j_1	1	2	2	2	2	2

p=37	k	2	3	4	12	13	14
d=3	i_1	9	30	51	59	60	61
	j_1	0	1	2	2	2	2

p=41	k	1	2	3	4	5	6	7	8	9	10
d=41	i_1	788	589	410	231	32	353	614	615	436	117
	j_1	39	29	20	11	1	17	30	30	21	5
	k	11	12	13	14	15	16	17	18	19	20
	i_1	678	819	820	821	142	703	384	205	206	467
	j_1	33	40	40	40	6	34	10	9	9	22

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