Surface Formulations of the Electromagnetic-Power-based Characteristic Mode Theory for Material Bodies — Part II

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***Abstract*—Both the previous Part I and this Part II focus on the linear electromagnetic system constructed by a Single Homogeneous Material body (SHM), and the SHM can be electric and/or magnetic. The studies for the system constructed by Multiple Homogeneous Material bodies (MHM) will be finished in Part III.**

It is indispensable for the Surface formulations of the ElectroMagnetic-Power-based SHM Characteristic Mode Theory (Surf-SHM-EMP-CMT) to relate the surface equivalent electric and magnetic currents, and a boundary condition based method for establishing the relation has been provided in the Part I. In this Part II, some further studies for the boundary condition based method are done (such as the revelation for physical essence, the numerical analysis, and the improvement), and a new conservation law based method is given.

As a supplement to the Part I, some new surface formulations for the output power of a SHM are developed in this Part II, and then some new surface formulations for constructing the Output power Characteristic Mode (OutCM) set and some new variational formulations for the scattering problem of a SHM are established.

In addition, the power relation contained in the PMCHWT formulation for a SHM is analyzed. Then, it is clearly revealed that the physical essence of the PMCHWT formulation for the SHM scattering problem is the conservation law of energy; the power character of the CM set derived from the PMCHWT-based CMT is not always identical to the OutCM set derived from the Surf-SHM-EMP-CMT; the PMCHWT-based CMT can be viewed as a special case of the object-oriented EMP-CMT.

***Index Terms*—Characteristic Mode (CM), Electromagnetic Power, Extinction Theorem, Material Body, PMCHWT, Surface Equivalent Principle.**

I. Introduction

For a certain linear ElectroMagnetic (EM) system, all possible operating states constitute a linear space. To construct the basis sets of the space is very valuable for both theoretical researches and engineering applications, and there have existed many different theories and methods for constructing the basis sets, such as the separation of variables [1]-[2], the Eigen-Mode Theory (EMT) [3]-[4], and the Characteristic Mode Theory (CMT) [5]-[12], etc. The basis derived from the separation of variables are usually called as special functions; the basis derived from the EMT are called as eigen-modes; the basis derived from the CMT are called as Characteristic Modes (CMs), and this basis set is correspondingly called as CM set.

For the CMT, especially the ElectroMagnetic-Power-based CMT (EMP-CMT) [8]-[12], the procedure to select a series of complete and independent EM quantities (called as *basic variables* in [9]-[12]) and to express all of the EM quantities in terms of the selected basic variables is an indispensable preprocessing step, because the absence of this preprocessing may lead to some unphysical CMs, as explained in [9]-[12], and this procedure can be specifically called as "*to unify variables*".

In the previous Part I [10], a boundary condition based method to unify variables was provided for a Single Homogeneous Material body (SHM), such that all EM quantities related to the SHM were expressed in terms of the surface equivalent electric *or* magnetic current on the boundary of SHM, and three different surface formulations for the output power of a SHM were given, and then the Surface formulations of the EMP-CMT for a SHM (Surf-SHM-EMP-CMT) was established.

In this Part II, the physical essence revelation, numerical analysis, and improvement for the precious boundary condition based method are done, and a new conservation law based method to unify variables is developed; some new surface formulations for the output power of a SHM are constructed, and then some new surface formulations for deriving the Output power CM (OutCM) set and some new variational formulations for the scattering problem of a SHM are given.

Just like the previous discussions on the power characters of the EFIE for perfect electric conductors [8] and the VIE for material bodies [9], the power character of the PMCHWT (Poggio, Miller, Chang, Harrington, Wu, and Tsai) formulation for a SHM is carefully analyzed in this paper. Then, it is clearly revealed that the physical essence of the Galerkin-based PMCHWT-MoM for the SHM scattering problem is the conservation law of energy; the power character of the CM set derived from the PMCHWT-based CMT is not always identical to the OutCM set derived from the Surf-SHM-EMP-CMT; the PMCHWT-based CMT can be viewed as a special case of the object-oriented EMP-CMT.

In what follows, the $e^{j\omega t}$ convention is used throughout.

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II. Source-Field Relationships

When an external field $\bar{F}^{inc}$ incidents on a material body V, the scattering sources will be excited on the V, and then the scattering field $\bar{F}^{sca}$ is generated, here $F = E,H$. The summation of $\bar{F}^{inc}$ and $\bar{F}^{sca}$ is the total field, and it is denoted as $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$.

A. Physical scattering sources on V

When the conductivity of V is not infinity, the scattering sources on V include the volume ohmic electric current $\bar{J}^{vo}$ and the related electric charge, the volume polarized electric current $\bar{J}^{vp}$ and the related electric charge, and the volume magnetized magnetic current $\bar{M}^{vm}$ and the related magnetic charge [13]-[14]. The various charges are related to the corresponding currents by current continuity equations, so it is sufficient to only use the scattering currents to determine the scattering field [13]-[14]. In addition, the summation of $\bar{J}^{vo}$ and $\bar{J}^{vp}$ is denoted as $\bar{J}^{vop}$, i.e., $\bar{J}^{vop} \triangleq \bar{J}^{vo} + \bar{J}^{vp}$.

B. Surface equivalent currents on ∂V

For a certain V, the surface equivalent currents on its boundary ∂V are defined as follows [10]

$$\bar{J}_{\pm}^{SE}(\bar{r}) \triangleq \left[\hat{n}_{\pm}(\bar{r}) \times \bar{H}^{tot}(\bar{r}_{\pm})\right]_{\bar{r}_{\pm} \to \bar{r}} \quad , \quad (\bar{r} \in \partial V) \tag{1}$$

$$\bar{M}_{\pm}^{SE}(\bar{r}) \triangleq \left[\bar{E}^{tot}(\bar{r}_{\pm}) \times \hat{n}_{\pm}(\bar{r})\right]_{\bar{r}_{\pm} \to \bar{r}} \quad , \quad (\bar{r} \in \partial V) \tag{2}$$

here $\hat{n}_{+}$ and $\hat{n}_{-}$ are respectively the external and internal normal directions of ∂V; $\bar{r}_{+} \in \mathrm{ext}V \triangleq \mathbb{R}^3 \setminus \mathrm{cl}V$, and $\bar{r}_{-} \in \mathrm{int}V$, here the $\mathbb{R}^3$ is three-dimensional Euclidean space, and the symbols $\mathrm{cl}V$ and $\mathrm{int}V$ are respectively the closer and interior of point set V [15]; as a companion to the $\mathrm{int}V$, the $\mathrm{ext}V$ can be called as the exterior of V. From a purely mathematical viewpoint, $\mathrm{int}V \subseteq V \subseteq \mathrm{cl}V$ for any V [15], but it is restricted in this paper that $V = \mathrm{cl}V$, and then $V = \mathrm{int}V \bigcup \partial V$.

When the magnetized magnetic current model is employed to depict the magnetization phenomenon, there doesn't exist the surface physical scattering current on ∂V [13]-[14], so the tangential component of $\bar{F}^{sca}$ is continuous on ∂V. In fact, the tangential component of $\bar{F}^{inc}$ is also continuous on ∂V, because it has been assumed that the source of $\bar{F}^{inc}$ doesn't distribute on V [10]. Then, the tangential component of $\bar{F}^{tot}$ is continuous on ∂V. In addition, $\hat{n}_{+}(\bar{r}) = -\hat{n}_{-}(\bar{r})$ for any $\bar{r} \in \partial V$, so a unified expression for $\bar{C}_{-}^{SE}$ and $\bar{C}_{+}^{SE}$ can be introduced as follows [10]

$$\bar{C}^{SE}(\bar{r}) \triangleq \bar{C}_{-}^{SE}(\bar{r}) = -\bar{C}_{+}^{SE}(\bar{r}) \quad , \quad (\bar{r} \in \partial V) \tag{3}$$

here $C = J,M$.

The various fields, physical scattering currents, and surface equivalent currents are illustrated in Fig. 1.

C. To express the various fields and physical scattering currents in terms of the surface equivalent currents

The various fields and physical scattering currents can be expressed in terms of the surface equivalent currents as follows [10]

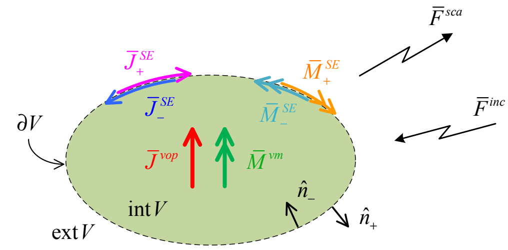

Fig. 1. The various fields, physical scattering currents, and surface equivalent currents on a single homogeneous material body.

$$\begin{aligned}\bar{F}^{sca}(\bar{r}) &= \mathcal{F}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\\ &= \begin{cases}\mathcal{F}_{-}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , & (\bar{r} \in \mathrm{int}V)\\ \mathcal{F}_{+}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , & (\bar{r} \in \mathrm{ext}V)\end{cases}\end{aligned} \tag{4}$$

$$\bar{F}_{-}^{tot}(\bar{r}) = \mathcal{F}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{5}$$

$$\bar{F}_{-}^{inc}(\bar{r}) = \mathcal{F}_{-}^{inc}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{6}$$

and

$$\bar{J}^{vo}(\bar{r}) = \mathcal{J}^{vo}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{7.1}$$

$$\bar{J}^{vp}(\bar{r}) = \mathcal{J}^{vp}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{7.2}$$

$$\bar{J}^{vop}(\bar{r}) = \mathcal{J}^{vop}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{7.3}$$

$$\bar{M}^{vm}(\bar{r}) = \mathcal{M}^{vm}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \quad , \quad (\bar{r} \in \mathrm{int}V) \tag{8}$$

here $F = E,H$, and correspondingly $\mathcal{F} = \mathcal{E},\mathcal{H}$; to utilize the subscript "$-$" in $\bar{F}_{-}^{tot}$ and $\bar{F}_{-}^{inc}$ is to emphasize that they are respectively the total and incident fields in the interior of V.

The specific mathematical expressions of the operators in above (4)-(8) can be found in Appendix.

III. Variable Unification

It has been pointed out in [10] that the $\bar{J}^{SE}$ and $\bar{M}^{SE}$ are not independent, and it is indispensable for Surf-SHM-EMP-CMT to establish the relation between $\bar{J}^{SE}$ and $\bar{M}^{SE}$. The procedure to establish the relation is specifically called as "*to unify variables*" or equivalently as *variable unification*, and it is done in this section.

A. Some fundamental concepts related to variable unification (Not restricted to a single homogeneous material body)

Although this paper mainly focuses on the EM system constructed by a SHM, it must be clearly claimed here that the concepts mentioned in this subsection are suitable for any linear EM system, as illustrated in [9]-[12].

For a certain EM system, any one of its possible operating states is called as a *mode*, and this mode is not restricted to the eigen-mode or CM. Because of the linear superposition principle [16], all of the modes constitute a linear space, and the space is called as *modal space*, and denoted as $\{\mathrm{mode}_m\}_{m=1}^{\infty}$, here the subscript "$m$" is the modal index. To quantitively depict the modal EM character from different aspects, scientists defined a series of EM quantities, such as the modal current $\bar{C}_m$, the modal charge ρ_m, the modal field $\bar{F}_m$ generated by $\bar{C}_m$ and ρ_m, and the modal power P_m generated by $\bar{C}_m$ and ρ_m, etc. For any mode, all of these modal EM quantities constitute a set $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}$. Obviously, the spaces $\{\text{mode}_m\}_{m=1}^{\infty}$ and $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}_{m=1}^{\infty}$ are isomorphic to each other, and the latter is called as *EM quantity space*.

For some quantities $q_m^1,q_m^2,\cdots,q_m^n$ which are the elements in set $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}$, if there is a quantity q_m^i such that it can be determined by other $n-1$ quantities, the quantities $q_m^1,q_m^2,\cdots,q_m^n$ are called as being *dependent*, otherwise they are called as being *independent*; if all of the elements in set $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}$ can be determined by $q_m^1,q_m^2,\cdots,q_m^n$, the quantities $q_m^1,q_m^2,\cdots,q_m^n$ are called as being *complete*; if the quantities $q_m^1,q_m^2,\cdots,q_m^n$ are both independent and complete, they are called as *basic variables* [9]-[12], and the set $\{q_m^1,q_m^2,\cdots,q_m^n\}$ is called as *basic set*. In general, the basic set is not unique, as illustrated in [9]-[12].

For a selected basic set $\{q_m^1,q_m^2,\cdots,q_m^n\}$, the ordered array $[q_m^1,q_m^2,\cdots,q_m^n]$ can be regarded as a vector called as *basic vector*, and denoted as $\bar{q}_m$. If the basic set can guarantee that all of possible values of basic vector constitute a linear space, this space is called as *basic vector space*, and denoted as $\{\bar{q}_m\}_{m=1}^{\infty}$. It is easy to find out that the $\{\bar{q}_m\}_{m=1}^{\infty}$ and $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}_{m=1}^{\infty}$ are isomorphic, and then the $\{\bar{q}_m\}_{m=1}^{\infty}$ and $\{\text{mode}_m\}_{m=1}^{\infty}$ are isomorphic too.

If the $\{\bar{b}_\xi\}_{\xi=1}^{\Xi}$ is a complete and independent basis function set of space $\{\bar{q}_m\}_{m=1}^{\infty}$, any vector $\bar{q}_m$ can be uniquely expanded in terms of $\{\bar{b}_\xi\}_{\xi=1}^{\Xi}$ as $\bar{q}_m=\sum_{\xi=1}^{\Xi}a_{m;\xi}\bar{b}_\xi$, and then a one-to-one mapping between the space $\{\bar{q}_m\}_{m=1}^{\infty}$ and the *expansion vector space* $\{\bar{a}_m\}_{m=1}^{\infty}$ is automatically established, here $\bar{a}_m=[a_{m;1},a_{m;2},\cdots,a_{m;\Xi}]^T$. At the same time, the one-to-one mappings from the spaces $\{\text{mode}_m\}_{m=1}^{\infty}$ and $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}_{m=1}^{\infty}$ to the space $\{\bar{a}_m\}_{m=1}^{\infty}$ are also established, as illustrated in Fig. 2.

Above these imply that the researches for the any one of spaces $\{\text{mode}_m\}_{m=1}^{\infty}$, $\{\bar{C}_m,\rho_m,\bar{F}_m,P_m,\cdots\}_{m=1}^{\infty}$, $\{\bar{q}_m\}_{m=1}^{\infty}$, and $\{\bar{a}_m\}_{m=1}^{\infty}$ can be accomplished by researching the any one of the other three spaces. Based on this, these four spaces are collectively referred to as *space* to simplify the terminological system of this paper. Generally speaking, it is relatively easy to research the space $\{\bar{a}_m\}_{m=1}^{\infty}$, because this kind of research belongs to the linear algebra problem, which has had a very complete theoretical system [17].

In the following parts of this section, the mapping from modal space to expansion vector space is established at first. Then, the method to unify variables is provided in the expansion vector space.

B. From modal space to expansion vector space

The $\bar{C}^{SE}$ is expanded in terms of the basis function set $\{\bar{b}_\xi^C\}_{\xi=1}^{\Xi^C}$ as follows

$$\bar{C}^{SE}(\bar{r}) = \sum_{\xi=1}^{\Xi^C} a_\xi^C \bar{b}_\xi^C(\bar{r}) = \bar{B}^C \cdot \bar{a}^C \quad , \quad (\bar{r}\in\partial V) \tag{9}$$

here $C=J,M$, and

$$\bar{B}^C = \left[\bar{b}_1^C(\bar{r}) \;,\; \bar{b}_2^C(\bar{r}) \;,\; \cdots \;,\; \bar{b}_{\Xi^C}^C(\bar{r})\right] \tag{10.1}$$

$$\bar{a}^C = \left[\; a_1^C \;,\; a_2^C \;,\; \cdots \;,\; a_{\Xi^C}^C \;\right]^T \tag{10.2}$$

The superscript "T" in (10.2) represents matrix transposition.

C. To unify variables — Methodology I: The EM boundary condition viewpoint and its physical essence, numerical analysis, and improvement

Four EM boundary condition based methods to relate $\bar{a}^J$ with $\bar{a}^M$ have been developed in [10]. In this subsection, their physical essence is revealed at first, and then the numerical analysis and improvement for them are given.

To select $\bar{a}^J$ as the basic variable: The (ME) and (MH) schemes given in [10] and their another implementation ways

When the $\bar{a}^J$ is selected as basic variable, the following two schemes can be utilized to construct the transformation matrix $\bar{\bar{T}}^{J\to M}$ from $\bar{a}^J$ to $\bar{a}^M$, such that $\bar{a}^M=\bar{\bar{T}}^{J\to M}\cdot\bar{a}^J$.

The (ME) scheme: To match the tangential boundary condition of $\bar{E}^{sca}$ on ∂V, and to test the boundary condition by employing the basis functions used to expand $\bar{M}^{SE}$.

The transformation matrix $\bar{\bar{T}}^{J\to M}$ derived from this scheme is correspondingly denoted as $\bar{\bar{T}}^{J\to M}_{\bar{M}^{SE}\cdot\bar{E}^{sca}}$, and it is as follows [10]

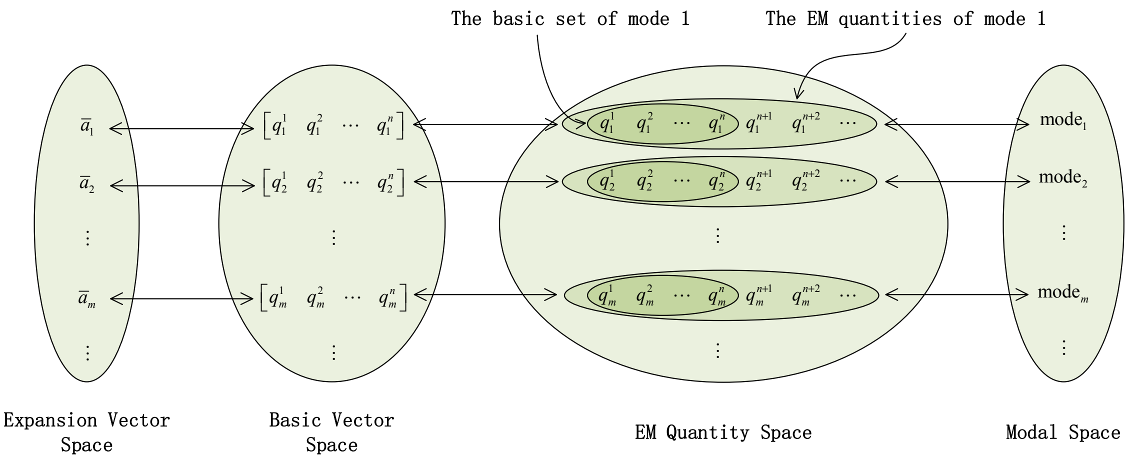

Fig. 2. The one-to-one mappings among modal space, EM quantity space, basic vector space, and expansion vector space.

$$\overline{\overline{T}}{}^{J\to M}_{\bar{M}^{SE}\cdot\bar{E}^{sca}} = \left(\overline{\overline{E}}{}^{sca}_{-;\,MM} - \overline{\overline{E}}{}^{sca}_{+;\,MM}\right)^{-1}\cdot\left(\overline{\overline{E}}{}^{sca}_{+;\,MJ} - \overline{\overline{E}}{}^{sca}_{-;\,MJ}\right) \tag{11}$$

In [10], the matrices $\overline{\overline{E}}{}^{sca}_{-;\,MM}$ and $\overline{\overline{E}}{}^{sca}_{+;\,MM}$ were obtained by respectively discretizing the following two integral operators, based on the Galerkin-based MoM

$$\begin{aligned}\mathcal{K}^{ME}_{\Delta-}\left(\bar{M}^{SE}\right) &= \mathcal{E}^{sca}_{-}\left(0,\bar{M}^{SE}\right)\\ &= +\oint\!\!\!\oint_{\partial V}\Delta\overline{\overline{G}}{}^{ME}_{-}(\bar{r}_-,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{K}^{ME}_{0+}\left(\bar{M}^{SE}\right) &= \mathcal{E}^{sca}_{+}\left(0,\bar{M}^{SE}\right)\\ &= -\oint\!\!\!\oint_{\partial V}\overline{\overline{G}}{}^{ME}_{0}(\bar{r}_+,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{13}$$

and the expansion and testing functions are both $\{\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$. In (12)-(13), $\bar{r}_-\in\operatorname{int}V$, and $\bar{r}_+\in\operatorname{ext}V$.

In fact, the matrix $\overline{\overline{E}}{}^{sca}_{-;\,MM} - \overline{\overline{E}}{}^{sca}_{+;\,MM}$ as a whole can also be directly derived from discretizing the following integral operator (14) by using expansion functions $\{\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$ and testing functions $\{\hat{n}_-\times\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$.

$$\begin{aligned}\mathcal{T}^{ME}_{\tan}\left(\bar{M}^{SE}\right) &= \hat{n}_-(\bar{r})\times\left[\mathcal{K}^{ME}_{\Delta-}\left(\bar{M}^{SE}\right)-\mathcal{K}^{ME}_{0+}\left(\bar{M}^{SE}\right)\right]\\ &= \bar{M}^{SE}(\bar{r})+\hat{n}_-(\bar{r})\times\mathcal{K}^{ME}_{\mathrm{m}-}\left(\bar{M}^{SE}\right)\end{aligned} \tag{14}$$

here $\bar{r}\in\partial V$, and

$$\begin{aligned}\mathcal{K}^{ME}_{\mathrm{m}-}\left(\bar{M}^{SE}\right) &= \mathcal{E}^{tot}_{-}\left(0,\bar{M}^{SE}\right)\\ &= \oint\!\!\!\oint_{\partial V}\overline{\overline{G}}{}^{ME}_{-}(\bar{r}_-,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{15}$$

and the second equality in (14) originates from the (A-7.2) and the tangential boundary condition corresponding to the electric field generated by a surface magnetic current.

Of course, the matrix $\overline{\overline{E}}{}^{sca}_{+;\,MJ} - \overline{\overline{E}}{}^{sca}_{-;\,MJ}$ in (11) can also be derived as a whole, i.e., to discretize the operator (22) by employing expansion functions $\{\bar{b}^J_\xi\}_{\xi=1}^{\Xi^J}$ and testing functions $\{\hat{n}_+\times\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$.

The (MH) scheme: To match the tangential boundary condition of $\bar{H}^{sca}$ on ∂V, and to test the boundary condition by employing the basis functions used to expand $\bar{M}^{SE}$.

The transformation matrix $\overline{\overline{T}}{}^{J\to M}$ derived from this scheme is correspondingly denoted as $\overline{\overline{T}}{}^{J\to M}_{\bar{M}^{SE}\cdot\bar{H}^{sca}}$, and it is as follows [10]

$$\overline{\overline{T}}{}^{J\to M}_{\bar{M}^{SE}\cdot\bar{H}^{sca}} = \left(\overline{\overline{H}}{}^{sca}_{-;\,MM} - \overline{\overline{H}}{}^{sca}_{+;\,MM}\right)^{-1}\cdot\left(\overline{\overline{H}}{}^{sca}_{+;\,MJ} - \overline{\overline{H}}{}^{sca}_{-;\,MJ}\right) \tag{16}$$

In [10], the matrices $\overline{\overline{H}}{}^{sca}_{-;\,MM}$ and $\overline{\overline{H}}{}^{sca}_{+;\,MM}$ were obtained by respectively discretizing the following two integral operators, based on the Galerkin-based MoM

$$\begin{aligned}\mathcal{L}^{MH}_{\Delta-}\left(\bar{M}^{SE}\right) &= \mathcal{H}^{sca}_{-}\left(0,\bar{M}^{SE}\right)\\ &= +\oint\!\!\!\oint_{\partial V}\Delta\overline{\overline{G}}{}^{MH}_{-}(\bar{r}_-,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{17}$$

$$\begin{aligned}\mathcal{L}^{MH}_{0+}\left(\bar{M}^{SE}\right) &= \mathcal{H}^{sca}_{+}\left(0,\bar{M}^{SE}\right)\\ &= -\oint\!\!\!\oint_{\partial V}\overline{\overline{G}}{}^{MH}_{0}(\bar{r}_+,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{18}$$

and the expansion and testing functions are both $\{\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$.

In fact, the matrix $\overline{\overline{H}}{}^{sca}_{-;\,MM} - \overline{\overline{H}}{}^{sca}_{+;\,MM}$ as a whole can also be directly derived from discretizing the following integral operator (19) by employing expansion functions $\{\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$ and testing functions $\{\hat{n}_-\times\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$.

$$\begin{aligned}\mathcal{T}^{MH}_{\tan}\left(\bar{M}^{SE}\right) &= \hat{n}_-(\bar{r})\times\left[\mathcal{L}^{MH}_{\Delta-}\left(\bar{M}^{SE}\right)-\mathcal{L}^{MH}_{0+}\left(\bar{M}^{SE}\right)\right]\\ &= \hat{n}_-(\bar{r})\times\mathcal{L}^{MH}_{\mathrm{m}-}\left(\bar{M}^{SE}\right)\end{aligned} \tag{19}$$

here $\bar{r}\in\partial V$, and

$$\begin{aligned}\mathcal{L}^{MH}_{\mathrm{m}-}\left(\bar{M}^{SE}\right) &= \mathcal{H}^{tot}_{-}\left(0,\bar{M}^{SE}\right)\\ &= \oint\!\!\!\oint_{\partial V}\overline{\overline{G}}{}^{MH}_{-}(\bar{r}_-,\bar{r}')\cdot\bar{M}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{20}$$

and the second equality in (19) originates from the (A-7.4) and the tangential boundary condition corresponding to the magnetic field generated by a surface magnetic current.

Of course, the matrix $\overline{\overline{H}}{}^{sca}_{+;\,MJ} - \overline{\overline{H}}{}^{sca}_{-;\,MJ}$ in (16) can also be directly obtained as a whole, i.e., to discretize the operator (27) by employing expansion functions $\{\bar{b}^J_\xi\}_{\xi=1}^{\Xi^J}$ and testing functions $\{\hat{n}_+\times\bar{b}^M_\xi\}_{\xi=1}^{\Xi^M}$.

To select $\bar{a}^M$ as the basic variable: The (JE) and (JH) schemes given in [10] and their another implementation ways

When the $\bar{a}^M$ is selected as basic variable, the following two schemes can be utilized to construct the transformation matrix $\overline{\overline{T}}{}^{M\to J}$ from $\bar{a}^M$ to $\bar{a}^J$, such that $\bar{a}^J=\overline{\overline{T}}{}^{M\to J}\cdot\bar{a}^M$.

The (JE) scheme: To match the tangential boundary condition of $\bar{E}^{sca}$ on ∂V, and to test the boundary condition by employing the basis functions used to expand $\bar{J}^{SE}$.

The transformation matrix $\overline{\overline{T}}{}^{M\to J}$ derived from this scheme is correspondingly denoted as $\overline{\overline{T}}{}^{M\to J}_{\bar{J}^{SE}\cdot\bar{E}^{sca}}$, and it is as follows [10]

$$\overline{\overline{T}}{}^{M\to J}_{\bar{J}^{SE}\cdot\bar{E}^{sca}} = \left(\overline{\overline{E}}{}^{sca}_{-;\,JJ} - \overline{\overline{E}}{}^{sca}_{+;\,JJ}\right)^{-1}\cdot\left(\overline{\overline{E}}{}^{sca}_{+;\,JM} - \overline{\overline{E}}{}^{sca}_{-;\,JM}\right) \tag{21}$$

Similarly to the (MH) scheme, the matrix $\overline{\overline{E}}{}^{sca}_{-;\,JJ} - \overline{\overline{E}}{}^{sca}_{+;\,JJ}$ as a whole can be derived from discretizing the following integral operator (22) by employing expansion functions $\{\bar{b}^J_\xi\}_{\xi=1}^{\Xi^J}$ and testing functions $\{\hat{n}_-\times\bar{b}^J_\xi\}_{\xi=1}^{\Xi^J}$.

$$\begin{aligned}\mathcal{T}^{JE}_{\tan}\left(\bar{J}^{SE}\right) &= \hat{n}_-(\bar{r})\times\left[\mathcal{L}^{JE}_{\Delta-}\left(\bar{J}^{SE}\right)-\mathcal{L}^{JE}_{0+}\left(\bar{J}^{SE}\right)\right]\\ &= \hat{n}_-(\bar{r})\times\mathcal{L}^{JE}_{\mathrm{m}-}\left(\bar{J}^{SE}\right)\end{aligned} \tag{22}$$

here $\bar{r}\in\partial V$, and

$$\begin{aligned}\mathcal{L}^{JE}_{\Delta-}\left(\bar{J}^{SE}\right) &= \mathcal{E}^{sca}_{-}\left(\bar{J}^{SE},0\right)\\ &= +\oint\!\!\!\oint_{\partial V}\Delta\overline{\overline{G}}{}^{JE}_{-}(\bar{r}_-,\bar{r}')\cdot\bar{J}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{L}^{JE}_{0+}\left(\bar{J}^{SE}\right) &= \mathcal{E}^{sca}_{+}\left(\bar{J}^{SE},0\right)\\ &= -\oint\!\!\!\oint_{\partial V}\overline{\overline{G}}{}^{JE}_{0}(\bar{r}_+,\bar{r}')\cdot\bar{J}^{SE}(\bar{r}')\,dS'\end{aligned} \tag{24}$$

and

$$\begin{aligned}\mathcal{L}_{\text{m}-}^{JE}\left(\bar{J}^{SE}\right) &= \mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},0\right)\\ &= \oiint_{\partial V}\bar{\bar{G}}_{-}^{JE}\left(\bar{r}_{-},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{25}$$

The second equality in (22) is due to the (A-7.1) and the tangential boundary condition corresponding to the electric field generated by a surface electric current.

In addition, the matrix $\bar{\bar{E}}_{+;JM}^{sca}-\bar{\bar{E}}_{-;JM}^{sca}$ in (21) can be derived from discretizing the operator (14) by employing expansion functions $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$ and testing functions $\{\hat{n}_{+}\times\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$.

The (JH) scheme: To match the tangential boundary condition of $\bar{H}^{sca}$ on ∂V, and to test the boundary condition by employing the basis functions used to expand $\bar{J}^{SE}$.

The transformation matrix $\bar{\bar{T}}^{M\to J}$ derived from this scheme is correspondingly denoted as $\bar{\bar{T}}_{\bar{J}^{SE}\cdot\bar{H}^{sca}}^{M\to J}$, and it is as follows [10]

$$\bar{\bar{T}}_{\bar{J}^{SE}\cdot\bar{H}^{sca}}^{M\to J} = \left(\bar{\bar{H}}_{-;JJ}^{sca}-\bar{\bar{H}}_{+;JJ}^{sca}\right)^{-1}\cdot\left(\bar{\bar{H}}_{+;JM}^{sca}-\bar{\bar{H}}_{-;JM}^{sca}\right) \tag{26}$$

Similarly to the (ME) scheme, the matrix $\bar{\bar{H}}_{-;JJ}^{sca}-\bar{\bar{H}}_{+;JJ}^{sca}$ as a whole can be derived from discretizing the following integral operator (27) by employing expansion functions $\{\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$ and testing functions $\{\hat{n}_{-}\times\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$.

$$\begin{aligned}\mathcal{T}_{\tan}^{JH}\left(\bar{J}^{SE}\right) &= \hat{n}_{-}\left(\bar{r}\right)\times\left[\mathcal{K}_{\Delta-}^{JH}\left(\bar{J}^{SE}\right)-\mathcal{K}_{0+}^{JH}\left(\bar{J}^{SE}\right)\right]\\ &= -\bar{J}^{SE}\left(\bar{r}\right)+\hat{n}_{-}\left(\bar{r}\right)\times\mathcal{K}_{\text{m}-}^{JH}\left(\bar{J}^{SE}\right)\end{aligned} \tag{27}$$

here $\bar{r}\in\partial V$, and

$$\begin{aligned}\mathcal{K}_{\Delta-}^{JH}\left(\bar{J}^{SE}\right) &= \mathcal{H}_{-}^{sca}\left(\bar{J}^{SE},0\right)\\ &= +\oiint_{\partial V}\Delta\bar{\bar{G}}_{-}^{JH}\left(\bar{r}_{-},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{28}$$

$$\begin{aligned}\mathcal{K}_{0+}^{JH}\left(\bar{J}^{SE}\right) &= \mathcal{H}_{+}^{sca}\left(\bar{J}^{SE},0\right)\\ &= -\oiint_{\partial V}\bar{\bar{G}}_{0}^{JH}\left(\bar{r}_{+},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{29}$$

and

$$\begin{aligned}\mathcal{K}_{\text{m}-}^{JH}\left(\bar{J}^{SE}\right) &= \mathcal{H}_{-}^{tot}\left(\bar{J}^{SE},0\right)\\ &= \oiint_{\partial V}\bar{\bar{G}}_{-}^{JH}\left(\bar{r}_{-},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{30}$$

The second equality in (27) is due to the (A-7.3) and the tangential boundary condition corresponding to the magnetic field generated by a surface electric current.

In addition, the matrix $\bar{\bar{H}}_{+;JM}^{sca}-\bar{\bar{H}}_{-;JM}^{sca}$ in (26) can be derived from discretizing the operator (19) by employing expansion functions $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$ and testing functions $\{\hat{n}_{+}\times\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$.

The physical essence of the above four schemes

Based on the above observations, it can be concluded that the $\bar{J}^{SE}$ and $\bar{M}^{SE}$ are related to each other by the following integral operators.

$$\mathcal{T}_{\tan}^{ME}\left(\bar{M}^{SE}\right) = -\mathcal{T}_{\tan}^{JE}\left(\bar{J}^{SE}\right) \tag{31}$$

$$\mathcal{T}_{\tan}^{JH}\left(\bar{J}^{SE}\right) = -\mathcal{T}_{\tan}^{MH}\left(\bar{M}^{SE}\right) \tag{32}$$

By inserting the (14) and (22) into the (31), it is found out that the (31) is equivalent to say that

$$\begin{aligned}\bar{M}^{SE}\left(\bar{r}\right) &= \left[\mathcal{L}_{\text{m}-}^{JE}\left(\bar{J}^{SE}\right)+\mathcal{K}_{\text{m}-}^{ME}\left(\bar{M}^{SE}\right)\right]_{\bar{r}_{-}\to\bar{r}}\times\hat{n}_{-}\left(\bar{r}\right)\\ &= \left[\mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\times\hat{n}_{-}\left(\bar{r}\right)\right]_{\bar{r}_{-}\to\bar{r}}\quad,\quad\left(\bar{r}\in\partial V\right)\end{aligned} \tag{33}$$

By inserting (19) and (27) into the (32), it is found out that the (32) is equivalent to say that

$$\begin{aligned}\bar{J}^{SE}\left(\bar{r}\right) &= \hat{n}_{-}\left(\bar{r}\right)\times\left[\mathcal{K}_{\text{m}-}^{JH}\left(\bar{J}^{SE}\right)+\mathcal{L}_{\text{m}-}^{MH}\left(\bar{M}^{SE}\right)\right]_{\bar{r}_{-}\to\bar{r}}\\ &= \left[\hat{n}_{-}\left(\bar{r}\right)\times\mathcal{H}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\right]_{\bar{r}_{-}\to\bar{r}}\quad,\quad\left(\bar{r}\in\partial V\right)\end{aligned} \tag{34}$$

Obviously, the above (33) and (34) are just the extinction theorems for total electric and magnetic fields in an inside-out manner.

The numerical analysis for above four schemes and the improvements for (ME) and (JH) schemes

In the electric extinction theorem (33), the $\mathcal{K}_{\text{m}-}^{ME}$ is a compact operator, and the $\mathcal{L}_{\text{m}-}^{JE}$ is a unbounded operator [18]; the inhomogeneous term $\bar{M}^{SE}$ corresponds to an identity operator, if the expansion and testing functions are both $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$. Then, the eigen-values of operator $\bar{M}^{SE}-\mathcal{K}_{\text{m}-}^{ME}\times\hat{n}_{-}$ will be clustered around a non-zero point [18], so its matrix is usually well-conditioned, if the expansion and testing functions are both $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$; the matrix corresponding to operator $\mathcal{L}_{\text{m}-}^{JE}\times\hat{n}_{-}$ is usually bad-conditioned, because the matrix corresponding to a unbounded operator usually has a relatively large condition number [18].

These above imply that: If the expansion and testing functions are both $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$ instead of using the testing functions $\{\hat{n}_{-}\times\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^{M}}$ like the original (ME) scheme, the electric extinction theorem (33) is more suitable for constructing the transformation from $\bar{J}^{SE}$ to $\bar{M}^{SE}$, and it is called as improved (ME) scheme.

Similarly, it can be concluded that: If the expansion and testing functions are both $\{\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$ instead of using the testing functions $\{\hat{n}_{-}\times\bar{b}_{\xi}^{J}\}_{\xi=1}^{\Xi^{J}}$ like the original (JH) scheme, the magnetic extinction theorem (34) is more suitable for constructing the transformation from $\bar{M}^{SE}$ to $\bar{J}^{SE}$, and it is called as improved (JH) scheme.

In addition, if a purely mathematical viewpoint, such as the EM boundary condition, is directly utilized to established the relation between $\bar{J}^{SE}$ and $\bar{M}^{SE}$ just like the paper [10] did, the matrix $\bar{\bar{E}}_{-;MM}^{sca}-\bar{\bar{E}}_{+;MM}^{sca}$ in (11) can be obtained by respectively computing the matrices $\bar{\bar{E}}_{-;MM}^{sca}$ and $\bar{\bar{E}}_{+;MM}^{sca}$. However this scheme may lead to some numerical problems, for example, the electric field generated by $\bar{M}^{SE}$ is very singular near ∂V [19], and then it is sometimes difficult to accurately compute the matrices $\bar{\bar{E}}_{-;MM}^{sca}$ and $\bar{\bar{E}}_{+;MM}^{sca}$; the inhomogeneous term $\bar{M}^{SE}$ in (33) cannot be well tested, because the expansion and testing functions are respectively selected as the $\{\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^M}$ and $\{\hat{n}_{-}\times\bar{b}_{\xi}^{M}\}_{\xi=1}^{\Xi^M}$, and then the eigen-values of $\bar{\bar{E}}_{-;MM}^{sca}-\bar{\bar{E}}_{+;MM}^{sca}$ may be clustered around zero point, and this will lead to the difficulty for inversing $\bar{\bar{E}}_{-;MM}^{sca}-\bar{\bar{E}}_{+;MM}^{sca}$.

From above analysis, it is found out that it is more desirable for establishing the relation between $\bar{J}^{SE}$ and $\bar{M}^{SE}$ to utilize some physical methodologies instead of some purely mathematical methodologies. To follow this philosophy, another physics-based method to relate $\bar{J}^{SE}$ with $\bar{M}^{SE}$ is provided below, based on the conservation law of energy.

D. To unify variables — Methodology II: The conservation law viewpoint and its essential relationship with boundary condition viewpoint

At first, the following functionals are defined.

$$\mathfrak{F}^{E}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \triangleq \mathcal{S}_{-}^{E}\left(\bar{J}^{SE},\bar{M}^{SE}\right)-\mathcal{S}_{+}^{E}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{35.1}$$

$$\mathfrak{F}^{H}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \triangleq \mathcal{S}_{-}^{H}\left(\bar{J}^{SE},\bar{M}^{SE}\right)-\mathcal{S}_{+}^{H}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{35.2}$$

in which

$$\begin{aligned}
&\mathcal{S}_{\pm}^{E}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\\
&= \frac{1}{2}\oiint_{\partial V_{\pm}}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^{*}\right]\cdot d\bar{S}\\
&= \frac{1}{2}\oiint_{\partial V}\left\{\mathcal{E}_{\pm}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\times\left[\mathcal{H}_{+}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\right]^{*}\right\}\cdot d\bar{S}
\end{aligned} \tag{36.1}$$

and

$$\begin{aligned}
&\mathcal{S}_{\pm}^{H}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\\
&= \frac{1}{2}\oiint_{\partial V_{\pm}}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^{*}\right]\cdot d\bar{S}\\
&= \frac{1}{2}\oiint_{\partial V}\left\{\mathcal{E}_{+}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\times\left[\mathcal{H}_{\pm}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right)\right]^{*}\right\}\cdot d\bar{S}
\end{aligned} \tag{36.2}$$

here superscript " $*$ " represents the complex conjugate of relevant quantity. The second equalities in (36.1) and (36.2) are based on the continuity of the tangential component of the $\bar{F}^{sca}$ on ∂V.

For any physically possible $\{\bar{J}^{SE},\bar{M}^{SE}\}$, the $\mathfrak{F}^{F}\left(\bar{J}^{SE},\bar{M}^{SE}\right)$ must be zero, because

$$\begin{aligned}
0 &= j\,2\omega\left[\frac{1}{4}\left\langle\bar{H}^{sca},\mu_0\bar{H}^{sca}\right\rangle_{\Delta\Omega}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\right\rangle_{\Delta\Omega}\right]\\
&= \mathcal{S}_{-}^{F}\left(\bar{J}^{SE},\bar{M}^{SE}\right)-\mathcal{S}_{+}^{F}\left(\bar{J}^{SE},\bar{M}^{SE}\right)
\end{aligned} \tag{37}$$

In (37), the inner product is defined as $<\bar{f},\bar{g}>_{\Omega}\triangleq\int_{\Omega}\bar{f}^{*}\cdot\bar{g}\,d\Omega$; integral domain $\Delta\Omega$ is the region enclosed by ∂V_{+} and ∂V_{-}; the second equality is based on the formulation (A-7) in [10], and its physical explanation is the conservation law of energy [16], i.e., the radiative power flux passing through ∂V_{-} will totally pass through ∂V_{+}, and the difference between the non-radiative power fluxes passing through ∂V_{-} and ∂V_{+} is totally stored in $\Delta\Omega$ (because of the absence of surface physical scattering current on ∂V); the first equality is due to that the volume of $\Delta\Omega$ is zero, and that the $\bar{F}^{sca}(\bar{r})$ is finite for any $\bar{r}\in\Delta\Omega$ (because of the absence of surface physical scattering current on ∂V).

Inserting $\bar{J}^{SE}=\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\bar{b}_{\zeta}^{J}=\bar{B}^{J}\cdot\bar{a}^{J}$ and $\bar{M}^{SE}=\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\bar{b}_{\zeta}^{M}=\bar{B}^{M}\cdot\bar{a}^{M}$ into (35), the $\mathfrak{F}^{F}$ can be rewritten as the version (38). In (38), $\mathcal{F}_{\pm}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\triangleq\mathcal{F}_{\pm}^{sca}\left(\bar{b}_{\zeta}^{J},0\right)$, and $\mathcal{F}_{\pm}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\triangleq\mathcal{F}_{\pm}^{sca}\left(0,\bar{b}_{\zeta}^{M}\right)$.

To select $\bar{a}^{J}$ as the basic variable

For a certain $\bar{a}^{J}$, the $\bar{a}^{M}$ will make the $\mathfrak{F}^{E}$ be zero and stationary, because of the conservation law of energy [16]. If the Rayleigh-Ritz scheme [20] is applied to $\mathfrak{F}^{E}$, the following simultaneous equations can be derived.

$$\begin{aligned}
0 = &\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\oiint_{\partial V}\left\{\left[\mathcal{E}_{-}^{sca}\left(\bar{b}_{\zeta}^{J}\right)-\mathcal{E}_{+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]\times\left[\mathcal{H}_{+}^{sca}\left(\bar{b}_{\xi}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}\\
&+\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\oiint_{\partial V}\left\{\left[\mathcal{E}_{-}^{sca}\left(\bar{b}_{\zeta}^{M}\right)-\mathcal{E}_{+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]\times\left[\mathcal{H}_{+}^{sca}\left(\bar{b}_{\xi}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}
\end{aligned} \tag{39}$$

for any $\xi=1,2,\cdots,\Xi^{M}$. In fact, the (39) can also be rewritten as the following matrix form.

$$0 = \bar{\bar{S}}^{JM}\cdot\bar{a}^{J}+\bar{\bar{S}}^{MM}\cdot\bar{a}^{M} \tag{40}$$

here

$$\bar{\bar{S}}^{JM} = \left[s_{\xi\zeta}^{JM}\right]_{\Xi^M\times\Xi^J} \tag{41.1}$$

$$\bar{\bar{S}}^{MM} = \left[s_{\xi\zeta}^{MM}\right]_{\Xi^M\times\Xi^M} \tag{41.2}$$

in which

$$\begin{aligned}
\mathfrak{F}^{E/H}\left(\bar{B}^{J}\cdot\bar{a}^{J},\bar{B}^{M}\cdot\bar{a}^{M}\right) = &+\frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\mathcal{E}_{-/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]\times\sum_{\zeta=1}^{\Xi^J}\left[a_{\zeta}^{J}\mathcal{H}_{+/-}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]^{*}\right\}\cdot d\bar{S} + \frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\mathcal{E}_{-/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]\times\sum_{\zeta=1}^{\Xi^M}\left[a_{\zeta}^{M}\mathcal{H}_{+/-}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}\\
&+\frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\mathcal{E}_{-/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]\times\sum_{\zeta=1}^{\Xi^J}\left[a_{\zeta}^{J}\mathcal{H}_{+/-}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]^{*}\right\}\cdot d\bar{S} + \frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\mathcal{E}_{-/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]\times\sum_{\zeta=1}^{\Xi^M}\left[a_{\zeta}^{M}\mathcal{H}_{+/-}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}\\
&-\frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\mathcal{E}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]\times\sum_{\zeta=1}^{\Xi^J}\left[a_{\zeta}^{J}\mathcal{H}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]^{*}\right\}\cdot d\bar{S} - \frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^J}a_{\zeta}^{J}\mathcal{E}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]\times\sum_{\zeta=1}^{\Xi^M}\left[a_{\zeta}^{M}\mathcal{H}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}\\
&-\frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\mathcal{E}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]\times\sum_{\zeta=1}^{\Xi^J}\left[a_{\zeta}^{J}\mathcal{H}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{J}\right)\right]^{*}\right\}\cdot d\bar{S} - \frac{1}{2}\oiint_{\partial V}\left\{\left[\sum_{\zeta=1}^{\Xi^M}a_{\zeta}^{M}\mathcal{E}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]\times\sum_{\zeta=1}^{\Xi^M}\left[a_{\zeta}^{M}\mathcal{H}_{+/+}^{sca}\left(\bar{b}_{\zeta}^{M}\right)\right]^{*}\right\}\cdot d\bar{S}
\end{aligned} \tag{38}$$

$$s_{\xi\zeta}^{JM} = \oiint_{\partial V} \Big\{ \big[\mathcal{E}_{-}^{sca}(\bar{b}_{\zeta}^{J}) - \mathcal{E}_{+}^{sca}(\bar{b}_{\zeta}^{J}) \big] \times \big[\mathcal{H}_{+}^{sca}(\bar{b}_{\xi}^{M}) \big]^{*} \Big\} \cdot d\bar{S} \tag{42.1}$$

$$s_{\xi\zeta}^{MM} = \oiint_{\partial V} \Big\{ \big[\mathcal{E}_{-}^{sca}(\bar{b}_{\zeta}^{M}) - \mathcal{E}_{+}^{sca}(\bar{b}_{\zeta}^{M}) \big] \times \big[\mathcal{H}_{+}^{sca}(\bar{b}_{\xi}^{M}) \big]^{*} \Big\} \cdot d\bar{S} \tag{42.2}$$

By solving the (40), it is obtained that

$$\bar{a}^{M} = \bar{\bar{T}}^{J\to M} \cdot \bar{a}^{J} \tag{43}$$

here

$$\bar{\bar{T}}^{J\to M} = -\left(\bar{\bar{S}}^{MM}\right)^{-1} \cdot \bar{\bar{S}}^{JM} \tag{44}$$

The superscript “ -1 ” in (44) represents the matrix inversion.

To select $\bar{a}^{M}$ as the basic variable

For a certain $\bar{a}^{M}$, the $\bar{a}^{J}$ will make the $\mathfrak{F}^{H}$ be zero and stationary. If the Rayleigh-Ritz scheme is applied to $\mathfrak{F}^{H}$, the following simultaneous equations are derived.

$$\begin{aligned} 0 = &\sum_{\zeta=1}^{\Xi^{J}} \left(a_{\zeta}^{J}\right)^{*} \oiint_{\partial V} \Big\{ \mathcal{E}_{+}^{sca}(\bar{b}_{\xi}^{J}) \times \big[\mathcal{H}_{-}^{sca}(\bar{b}_{\zeta}^{J}) - \mathcal{H}_{+}^{sca}(\bar{b}_{\zeta}^{J}) \big]^{*} \Big\} \cdot d\bar{S} \\ &+ \sum_{\zeta=1}^{\Xi^{M}} \left(a_{\zeta}^{M}\right)^{*} \oiint_{\partial V} \Big\{ \mathcal{E}_{+}^{sca}(\bar{b}_{\xi}^{J}) \times \big[\mathcal{H}_{-}^{sca}(\bar{b}_{\zeta}^{M}) - \mathcal{H}_{+}^{sca}(\bar{b}_{\zeta}^{M}) \big]^{*} \Big\} \cdot d\bar{S} \end{aligned} \tag{45}$$

for any $\xi = 1,2,\cdots,\Xi^{J}$. In fact, the (45) can also be rewritten as the following matrix form.

$$0 = \bar{\bar{S}}^{JJ} \cdot \bar{a}^{J} + \bar{\bar{S}}^{JM} \cdot \bar{a}^{M} \tag{46}$$

in which

$$\bar{\bar{S}}^{JJ} = \left[s_{\xi\zeta}^{JJ} \right]_{\Xi^{J}\times\Xi^{J}} \tag{47.1}$$

$$\bar{\bar{S}}^{JM} = \left[s_{\xi\zeta}^{JM} \right]_{\Xi^{J}\times\Xi^{M}} \tag{47.2}$$

here

$$s_{\xi\zeta}^{JJ} = \oiint_{\partial V} \Big\{ \big[\mathcal{E}_{+}^{sca}(\bar{b}_{\xi}^{J}) \big]^{*} \times \big[\mathcal{H}_{-}^{sca}(\bar{b}_{\zeta}^{J}) - \mathcal{H}_{+}^{sca}(\bar{b}_{\zeta}^{J}) \big] \Big\} \cdot d\bar{S} \tag{48.1}$$

$$s_{\xi\zeta}^{JM} = \oiint_{\partial V} \Big\{ \big[\mathcal{E}_{+}^{sca}(\bar{b}_{\xi}^{J}) \big]^{*} \times \big[\mathcal{H}_{-}^{sca}(\bar{b}_{\zeta}^{M}) - \mathcal{H}_{+}^{sca}(\bar{b}_{\zeta}^{M}) \big] \Big\} \cdot d\bar{S} \tag{48.2}$$

By solving the (46), it is obtained that

$$\bar{a}^{J} = \bar{\bar{T}}^{M\to J} \cdot \bar{a}^{M} \tag{49}$$

here

$$\bar{\bar{T}}^{M\to J} = -\left(\bar{\bar{S}}^{JJ}\right)^{-1} \cdot \bar{\bar{S}}^{JM} \tag{50}$$

In fact, the (43) and (49) can be uniformly written as follows

$$\bar{a}^{\Psi} = \bar{\bar{T}}^{\Phi\to\Psi} \cdot \bar{a}^{\Phi} \tag{51}$$

here $(\Phi,\Psi) = (J,M)$, if the $\bar{a}^{J}$ is selected as basic variable; $(\Phi,\Psi) = (M,J)$, if the $\bar{a}^{M}$ is selected as basic variable. So far, the variable unification in the expansion vector space is done.

The essential relationship between the conservation law based method and the boundary condition based method

It is obvious that the (39) can also be derived from testing the (31) by using $\{\mathcal{H}_{+}^{sca}(\bar{b}_{\xi}^{M})\}_{\xi=1}^{\Xi^{M}}$ (based on the inner product), and that the (45) can also be derived from testing the conjugate of (32) by using $\{\mathcal{E}_{+}^{sca}(\bar{b}_{\xi}^{J})\}_{\xi=1}^{\Xi^{J}}$ (based on the symmetrical product instead of the inner product). Then, it can be concluded that the conservation law based methodology and the boundary condition based methodology have the same physical essence.

IV. Surface Formulation of Output Power

In [10], three different surface formulations for the output power of a SHM have been established, and another new surface formulation is developed in this section, and some other new surface formulations are provided in Sec. V-C.

A. New surface formulation of output power

For a SHM, it can be proven that

$$\begin{aligned} &(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} \\ &= \frac{\mu\,\Delta\varepsilon_{c}^{*}}{\mu_{0}\varepsilon_{0} - \mu\varepsilon_{c}^{*}} \left[\frac{1}{2}\left\langle \bar{J}^{SE}, \bar{E}^{inc} \right\rangle_{\partial V} + \frac{\mu_{0}}{\mu}\frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V}^{*} \right] \end{aligned} \tag{52.1}$$

$$\begin{aligned} &(1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V} \\ &= \frac{\varepsilon_{c}\,\Delta\mu}{\varepsilon_{0}\mu_{0} - \varepsilon_{c}\mu} \left[\frac{\varepsilon_{0}}{\varepsilon_{c}}\frac{1}{2}\left\langle \bar{J}^{SE}, \bar{E}^{inc} \right\rangle_{\partial V}^{*} + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V} \right] \end{aligned} \tag{52.2}$$

if $\mu_{0}\varepsilon_{0} - \mu\varepsilon_{c}^{*}, \varepsilon_{0}\mu_{0} - \varepsilon_{c}\mu \neq 0$. Then, the output power P^{out} and the input power P^{inp} can be evaluated as follows

$$\begin{aligned} P^{out/inp} &= (1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} + (1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V} \\ &= \frac{\mu\,\Delta\varepsilon_{c}^{*}}{\mu_{0}\varepsilon_{0} - \mu\varepsilon_{c}^{*}} \left[\frac{1}{2}\left\langle \bar{J}^{SE}, \bar{E}_{-}^{inc} \right\rangle_{\partial V} + \frac{\mu_{0}}{\mu}\frac{1}{2}\left\langle \bar{H}_{-}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V}^{*} \right] \\ &\quad + \frac{\varepsilon_{c}\,\Delta\mu}{\varepsilon_{0}\mu_{0} - \varepsilon_{c}\mu} \left[\frac{\varepsilon_{0}}{\varepsilon_{c}}\frac{1}{2}\left\langle \bar{J}^{SE}, \bar{E}_{-}^{inc} \right\rangle_{\partial V}^{*} + \frac{1}{2}\left\langle \bar{H}_{-}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V} \right] \end{aligned} \tag{53.1}$$

here $\Delta\varepsilon_{c}^{*} = \varepsilon_{c}^{*} - \varepsilon_{0}$, and $\varepsilon_{c} = \varepsilon + \sigma/j\omega$, and $\Delta\mu = \mu - \mu_{0}$. In (53.1), the second equality is for the utilization of formulation (6), and based on the continuity of the tangential component of the $\bar{F}^{inc}$ on ∂V.

Inserting the (6) into the (53.1), the P^{inp} and P^{out} can be uniformly written as the following operator form.

$$P^{inp} = P^{out} = \mathcal{P}_{4}^{out}\left(\bar{J}^{SE}, \bar{M}^{SE}\right) \tag{53.2}$$

The subscript “ 4 ” in operator $\mathcal{P}_{4}^{out}$ is to emphasize that this operator is different from another three operators given in [10].

B. The matrix form of new surface formulation (53)

In this subsection, the operator (53) is transformed into its matrix form for the convenience to construct the OutCM set in the expansion vector space.

The matrix forms of two building-block powers

Inserting the (6) and (9) into the $(1/2) < \bar{J}^{SE}, \bar{E}_{-}^{inc} >_{\partial V}$ and $(1/2) < \bar{H}_{-}^{inc}, \bar{M}^{SE} >_{\partial V}$, these two building-block powers can be transformed into the following matrix forms.

$$(1/2)\left\langle \bar{J}^{SE}, \bar{E}_{-}^{inc} \right\rangle_{\partial V} = \bar{a}^{H} \cdot \bar{\bar{P}}_{JE} \cdot \bar{a} \tag{54.1}$$

$$(1/2)\left\langle \bar{H}_{-}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V} = \bar{a}^{H} \cdot \bar{\bar{P}}_{HM} \cdot \bar{a} \tag{54.2}$$

here superscript "H" represents the transpose conjugate of related matrix, and

$$\bar{\bar{P}}_{JE} = \begin{bmatrix} \bar{\bar{P}}_{JE}^{JJ} & \bar{\bar{P}}_{JE}^{JM} \\ 0 & 0 \end{bmatrix} \tag{55.1}$$

$$\bar{\bar{P}}_{HM} = \begin{bmatrix} 0 & \bar{\bar{P}}_{HM}^{JM} \\ 0 & \bar{\bar{P}}_{HM}^{MM} \end{bmatrix} \tag{55.2}$$

and

$$\bar{a} = \begin{bmatrix} \bar{a}^{J} \\ \bar{a}^{M} \end{bmatrix} \tag{56}$$

In (55), the submatrices are as follows

$$\bar{\bar{P}}_{JE}^{JC'} = \left[p_{JE;\xi\zeta}^{JC'} \right]_{\Xi^{J} \times \Xi^{C'}} \tag{57.1}$$

$$\bar{\bar{P}}_{HM}^{C'M} = \left[p_{HM;\xi\zeta}^{C'M} \right]_{\Xi^{C'} \times \Xi^{M}} \tag{57.2}$$

here $C' = J, M$, and

$$p_{JE;\xi\zeta}^{JC'} = (1/2)\left\langle \bar{b}_{\xi}^{J}, \mathcal{E}_{-}^{inc}\left(\bar{b}_{\zeta}^{C'}\right) \right\rangle_{\partial V} \tag{58.1}$$

$$p_{HM;\xi\zeta}^{C'M} = (1/2)\left\langle \mathcal{H}_{-}^{inc}\left(\bar{b}_{\xi}^{C'}\right), \bar{b}_{\zeta}^{M} \right\rangle_{\partial V} \tag{58.2}$$

In (58.1), $\xi = 1, 2, \cdots, \Xi^{J}$, and $\zeta = 1, 2, \cdots, \Xi^{C'}$. In (58.2), $\xi = 1, 2, \cdots, \Xi^{C'}$, and $\zeta = 1, 2, \cdots, \Xi^{M}$.

The matrix form of operator (53)

Inserting (54) into (53), the P^{inp} and P^{out} can be written as the following matrix form.

$$P^{inp} = P^{out} = \bar{a}^{H} \cdot \bar{\bar{P}} \cdot \bar{a} \tag{59}$$

here

$$\bar{\bar{P}} = \frac{\mu\, \Delta\varepsilon_{c}^{*}}{\mu_{0}\varepsilon_{0} - \mu\varepsilon_{c}^{*}} \left(\bar{\bar{P}}_{JE} + \frac{\mu_{0}}{\mu} \bar{\bar{P}}_{HM}^{H} \right) + \frac{\varepsilon_{c}\, \Delta\mu}{\varepsilon_{0}\mu_{0} - \varepsilon_{c}\mu} \left(\frac{\varepsilon_{0}}{\varepsilon_{c}} \bar{\bar{P}}_{JE}^{H} + \bar{\bar{P}}_{HM} \right) \tag{60}$$

Inserting the (51) and (56) into (59), the (59) can be rewritten as follows

$$P^{inp} = P^{out} = \left(\bar{a}^{\Phi} \right)^{H} \cdot \bar{\bar{P}}^{\Phi} \cdot \bar{a}^{\Phi} \tag{61}$$

here $(\Phi, \Psi) = (J, M)$, if the $\bar{a}^{J}$ is selected as basic variable; $(\Phi, \Psi) = (M, J)$, if the $\bar{a}^{M}$ is selected as basic variable. In (61),

$$\bar{\bar{P}}^{\Phi} = \bar{\bar{P}}^{J} = \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J \to M} \end{bmatrix}^{H} \cdot \bar{\bar{P}} \cdot \begin{bmatrix} \bar{\bar{I}} \\ \bar{\bar{T}}^{J \to M} \end{bmatrix} \tag{62.1}$$

for the case $\Phi = J$, and

$$\bar{\bar{P}}^{\Phi} = \bar{\bar{P}}^{M} = \begin{bmatrix} \bar{\bar{T}}^{M \to J} \\ \bar{\bar{I}} \end{bmatrix}^{H} \cdot \bar{\bar{P}} \cdot \begin{bmatrix} \bar{\bar{T}}^{M \to J} \\ \bar{\bar{I}} \end{bmatrix} \tag{62.2}$$

for the case $\Phi = M$. The $\bar{\bar{I}}$ in (62) is identity matrix.

The procedure for constructing the OutCM set is completely similar to [10], and it will not be repeated in this paper.

V. Discussions

For the surface formulations of both CMT and scattering problem of a SHM, many topics are worthy to be discussed in detail, but it was not done in the previous Part I to make the Part I be more compact, and it is finished in this section.

A. On the physical essence of the symmetrical PMCHWT formulation for the scattering problem of a single homogeneous material body

Based on the (1)-(3) and the continuity of the tangential component of $\bar{F}^{inc}$ on ∂V and the Maxwell's equations for $\bar{F}^{inc}$ and $\bar{F}^{tot}$, it is easy to prove that

$$\begin{aligned} (1/2)\left\langle \bar{J}^{SE}, \bar{E}_{+}^{inc} \right\rangle_{\partial V} &= -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} \\ &\quad - j\left(P_{V}^{si,\,react,\,vac} + P_{V}^{inc,\,react,\,vac} \right) \end{aligned} \tag{63.1}$$

$$\begin{aligned} (1/2)\left\langle \bar{H}_{+}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V} &= -(1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V} \\ &\quad - j\left(P_{V}^{is,\,react,\,vac} + P_{V}^{inc,\,react,\,vac} \right) \end{aligned} \tag{63.2}$$

here $\bar{F}_{+}^{inc}$ is the incident field in region $\operatorname{ext} V$, and

$$P_{V}^{si,\,react,\,vac} = 2\omega \left[\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_{0}\bar{H}^{inc} \right\rangle_{V} - \frac{1}{4}\left\langle \varepsilon_{0}\bar{E}^{sca}, \bar{E}^{inc} \right\rangle_{V} \right] \tag{64.1}$$

$$P_{V}^{is,\,react,\,vac} = 2\omega \left[\frac{1}{4}\left\langle \bar{H}^{inc}, \mu_{0}\bar{H}^{sca} \right\rangle_{V} - \frac{1}{4}\left\langle \varepsilon_{0}\bar{E}^{inc}, \bar{E}^{sca} \right\rangle_{V} \right] \tag{64.2}$$

and

$$P_{V}^{inc,\,react,\,vac} = 2\omega \left[\frac{1}{4}\left\langle \bar{H}^{inc}, \mu_{0}\bar{H}^{inc} \right\rangle_{V} - \frac{1}{4}\left\langle \varepsilon_{0}\bar{E}^{inc}, \bar{E}^{inc} \right\rangle_{V} \right] \tag{65}$$

Based on the (1)-(3) and the continuity of the tangential component of $\bar{F}^{sca}$ on ∂V and the Maxwell's equations for $\bar{F}^{sca}$ and $\bar{F}^{tot}$, it is easy to prove that

$$(1/2)\left\langle \bar{J}^{SE}, \bar{E}_{+}^{sca} \right\rangle_{\partial V} = -(1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V} + P^{sca,rad} + j\left(P_{\mathbb{R}^3\backslash V}^{sca,react,vac} - P_{V}^{is,react,vac}\right) \tag{66.1}$$

$$(1/2)\left\langle \bar{H}_{+}^{sca}, \bar{M}^{SE} \right\rangle_{\partial V} = -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} + P^{sca,rad} + j\left(P_{\mathbb{R}^3\backslash V}^{sca,react,vac} - P_{V}^{si,react,vac}\right) \tag{66.2}$$

here $\bar{F}_{+}^{sca}$ is the scattering field in region $\mathrm{ext}V$, and

$$P^{sca,rad} = \frac{1}{2}\oiint_{S_\infty}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^{*}\right]\cdot d\bar{S} \tag{67}$$

$$P_{\mathbb{R}^3\backslash V}^{sca,react,vac} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_0\bar{H}^{sca} \right\rangle_{\mathbb{R}^3\backslash V} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca}, \bar{E}^{sca} \right\rangle_{\mathbb{R}^3\backslash V}\right] \tag{68}$$

Based on the (1)-(3) and the Maxwell's equations for $\bar{F}^{tot}$, it is easy to prove that

$$(1/2)\left\langle \bar{J}^{SE}, \bar{E}_{-}^{tot} \right\rangle_{\partial V} = -P^{tot,loss} - j\,P_{V}^{tot,react} \tag{69.1}$$

$$(1/2)\left\langle \bar{H}_{-}^{tot}, \bar{M}^{SE} \right\rangle_{\partial V} = -P^{tot,loss} - j\,P_{V}^{tot,react} \tag{69.2}$$

here $\bar{F}_{-}^{tot}$ is the total field in region $\mathrm{int}V$, and

$$P^{tot,loss} = (1/2)\left\langle \sigma\bar{E}^{tot}, \bar{E}^{tot} \right\rangle_{V} \tag{70}$$

$$P_{V}^{tot,react} = P_{V}^{tot,react,vac} + P^{tot,react,mat} \tag{71}$$

In (71),

$$P_{V}^{tot,react,vac} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{tot}, \mu_0\bar{H}^{tot} \right\rangle_{V} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{tot}, \bar{E}^{tot} \right\rangle_{V}\right] \tag{72}$$

$$P^{tot,react,mat} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{tot}, \Delta\mu\bar{H}^{tot} \right\rangle_{V} - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^{tot}, \bar{E}^{tot} \right\rangle_{V}\right] \tag{73}$$

In addition, the following relation is evident.

$$P_{V}^{tot,react,vac} = P_{V}^{sca,react,vac} + P_{V}^{coup,react,vac} + P_{V}^{inc,react,vac} \tag{74}$$

here

$$P_{V}^{sca,react,vac} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_0\bar{H}^{sca} \right\rangle_{V} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca}, \bar{E}^{sca} \right\rangle_{V}\right] \tag{75}$$

$$P_{V}^{coup,react,vac} = P_{V}^{si,react,vac} + P_{V}^{is,react,vac} \tag{76}$$

In fact, the $P_{\mathbb{R}^3\backslash V}^{sca,react,vac}$ in (68) and the $P_{V}^{sca,react,vac}$ in (75) satisfy the following relation [10].

$$P^{sca,react,vac} = P_{\mathbb{R}^3\backslash V}^{sca,react,vac} + P_{V}^{sca,react,vac} \tag{77}$$

here

$$P^{sca,react,vac} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_0\bar{H}^{sca} \right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}^{sca}, \bar{E}^{sca} \right\rangle_{\mathbb{R}^3}\right] \tag{78}$$

Obviously, the summation of the left-hand sides of (63) and (66) equals to the left-hand side of (69), viz.,

$$\begin{aligned}&(1/2)\left\langle \bar{J}^{SE}, \bar{E}_{+}^{inc} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{+}^{inc}, \bar{M}^{SE} \right\rangle_{\partial V} \\ &+ (1/2)\left\langle \bar{J}^{SE}, \bar{E}_{+}^{sca} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{+}^{sca}, \bar{M}^{SE} \right\rangle_{\partial V} \\ &= (1/2)\left\langle \bar{J}^{SE}, \bar{E}_{-}^{tot} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{-}^{tot}, \bar{M}^{SE} \right\rangle_{\partial V}\end{aligned} \tag{79}$$

because of the following boundary condition [7]

$$\hat{n}_{+}(\bar{r})\times\left[\bar{F}^{inc}(\bar{r}_{+}) + \bar{F}^{sca}(\bar{r}_{+})\right]_{\bar{r}_{+}\to\bar{r}} = \hat{n}_{+}(\bar{r})\times\left[\bar{F}^{tot}(\bar{r}_{-})\right]_{\bar{r}_{-}\to\bar{r}} \tag{80}$$

for any $\bar{r}\in\partial V$, here $\bar{r}_{+}\in\mathrm{ext}V$ and $\bar{r}_{-}\in\mathrm{int}V$. For the convenience of the following discussions, the (79) can be equivalently rewritten as the following (81), based on (3).

$$\begin{aligned}&(1/2)\left\langle \bar{J}_{+}^{SE}, \bar{E}_{+}^{inc} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{+}^{inc}, \bar{M}_{+}^{SE} \right\rangle_{\partial V} \\ &+ (1/2)\left\langle \bar{J}_{+}^{SE}, \bar{E}_{+}^{sca} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{+}^{sca}, \bar{M}_{+}^{SE} \right\rangle_{\partial V} \\ &= (1/2)\left\langle \bar{J}_{+}^{SE}, \bar{E}_{-}^{tot} \right\rangle_{\partial V} + (1/2)\left\langle \bar{H}_{-}^{tot}, \bar{M}_{+}^{SE} \right\rangle_{\partial V}\end{aligned} \tag{81}$$

Because of above (79)/(81), the summation of the right-hand sides of (63) and (66) must equal to the right-hand side of (69), i.e.,

$$\begin{aligned}&\left[\frac{1}{2}\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V}\right] + j\left(P_{V}^{coup,react,vac} + 2P_{V}^{inc,react,vac}\right) \\ &+ \left[\frac{1}{2}\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V} + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V}\right] - 2P^{sca,rad} + j\left(P_{V}^{coup,react,vac} - 2P_{\mathbb{R}^3\backslash V}^{sca,react,vac}\right) \\ &= 2P^{tot,loss} + j\,2\left(P_{V}^{sca,react,vac} + P_{V}^{inc,react,vac} + P_{V}^{coup,react,vac} + P^{tot,react,mat}\right)\end{aligned} \tag{82}$$

here the first line corresponds to the (63), and the second line corresponds to the (66), and the third line corresponds to the (69).

From the (82) and the above-mentioned relationships among various powers, it can be found out that the physical essence of (79)/(81) is just the conservation law of energy, because the (82) is equivalent to say that $P^{inp} = P^{out}$, here $P^{out} = P^{tot,loss} + P^{sca,rad} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right)$ [9]. Then, it can be concluded that the physical essence of the *symmetrical* Galerkin-based PMCHWT-MoM (79)/(81) for the scattering problem of a SHM is also the conservation law of energy, just like the Galerkin-based EFIE-MoM for PEC scattering problem [8], the Galerkin-based MFIE-MoM for PMC scattering problem, and the *symmetrical* Galerkin-based VIE-MoM for material scattering problem [9]. The reason to use the adjective "symmetrical" can be found in [9], and generally speaking the *traditional* PMCHWT-MoM (or *non-symmetrical* Galerkin-based PMCHWT-MoM) [7] and the *traditional* VIE-MoM (or *non-symmetrical* Galerkin-based VIE-MoM) [6] cannot be completely consistent with the conservation law of energy, though the consistence may be existed for some special cases, as explained in [9].

B. On the symmetrical PMCHWT based surface formulation of the CMT for a single homogeneous material body

In fact, the (81) can be equivalently rewritten as follows

$$\begin{aligned} &(1/2)\langle \bar{J}_+^{SE}, \bar{E}^{inc} \rangle_{\partial V} + (1/2)\langle \bar{H}^{inc}, \bar{M}_+^{SE} \rangle_{\partial V} \\ = \;& (1/2)\langle \bar{J}_+^{SE}, \bar{E}_-^{tot} \rangle_{\partial V} + (1/2)\langle \bar{H}_-^{tot}, \bar{M}_+^{SE} \rangle_{\partial V} \\ & -(1/2)\langle \bar{J}_+^{SE}, \bar{E}_+^{sca} \rangle_{\partial V} + (1/2)\langle \bar{H}_+^{sca}, \bar{M}_+^{SE} \rangle_{\partial V} \end{aligned} \tag{81'}$$

here the relation $\lim_{\bar{r}_+ \to \bar{r} \in \partial V} \bar{F}_+^{inc}(\bar{r}_+) = \bar{F}^{inc}(\bar{r})$ has been utilized in the first line. If the left-hand side of (81') is viewed as the power done by incident fields $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ on equivalent currents $\{\bar{J}_+^{SE}, \bar{M}_+^{SE}\}$, and denoted as $P^{inc \to equ}$, it is easy to find out from the (63) that

$$\begin{aligned} P^{inc\to equ} &= P^{inp/out} + j\left(P_V^{coup,react,vac} + 2P_V^{inc,react,vac}\right) \\ &= P^{tot,loss} + P^{sca,rad} + j\left(P_{\mathbb{R}^3\setminus V}^{sca,react,vac} + P_V^{inc,react,vac} + P_V^{tot,react}\right) \end{aligned} \tag{83}$$

It is obvious from (83) that

$$\begin{aligned} \operatorname{Re}\{P^{inc\to equ}\} &= P^{tot,loss} + P^{sca,rad} \\ &= \operatorname{Re}\{P^{inp/out}\} \end{aligned} \tag{84.1}$$

$$\begin{aligned} \operatorname{Im}\{P^{inc\to equ}\} &= P_{\mathbb{R}^3\setminus V}^{sca,react,vac} + P_V^{inc,react,vac} + P_V^{tot,react} \\ &= \operatorname{Im}\{P^{inp/out}\} + P_V^{coup,react,vac} + 2P_V^{inc,react,vac} \end{aligned} \tag{84.2}$$

because the $P^{tot,loss}$, $P^{sca,rad}$, $P_{\mathbb{R}^3\setminus V}^{sca,react,vac}$, $P_V^{inc,react,vac}$, $P_V^{coup,react,vac}$, and $P_V^{tot,react}$ are real numbers. Then, it is found out that the active power done by $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ on $\{\bar{J}_+^{SE}, \bar{M}_+^{SE}\}$ equals to the active power done by $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ on $\{\bar{J}^{vop}, \bar{M}^{vm}\}$, but the reactive power done by $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ on $\{\bar{J}_+^{SE}, \bar{M}_+^{SE}\}$ is not always the same as the reactive power done by $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ on $\{\bar{J}^{vop}, \bar{M}^{vm}\}$. This implies that the power character of the CMs derived from the symmetrical PMCHWT based CMT is not always identical to the OutCMs derived from Surf-SHM-EMP-CMT, especially for the resonant modes.

However, this above doesn't imply that the symmetrical PMCHWT based CMT is worthless, because the EMP-CMT is an object-oriented theory [8]-[9], and any physically reasonable power can be treated as the objective power to be optimized by EMP-CMT. In fact, the symmetrical PMCHWT based CMT may have its values in some aspects, and it can be viewed as a special case of the object-oriented EMP-CMT.

In addition, it is obvious that $P_V^{si,react,vac} = (P_V^{is,react,vac})^*$ and $P_V^{inc,react,vac} \in \mathbb{R}$, so $(1/2) < \bar{J}_+^{SE}, \bar{E}^{inc} >_{\partial V} + (1/2) < \bar{H}^{inc}, \bar{M}_+^{SE} >_{\partial V}^* = (1/2) < \bar{J}^{vop}, \bar{E}^{inc} >_V + (1/2) < \bar{H}^{inc}, \bar{M}^{vm} >_{\partial V}^*$ because of the (63), and then generally speaking the power character of the CM set derived from the non-symmetrical PMCHWT based CMT is also different from the OutCM set, because the power character of the CM set derived from the non-symmetrical VIE based CMT [6] is not always identical to the OutCM set as explained in [8].

C. On the surface formulations of the EMP-CMT for a single homogeneous material body

Three equivalent surface formulations for the output power of a SHM were established in [10], and another equivalent formulation is developed in the Sec. IV-A of this paper. In fact, there also exist many other equivalent surface formulations for output power, such as the following formulations V and VI.

Surface formulation V for output power

Based on the Maxwell's equations of $\bar{F}^{sca}$ and $\bar{F}^{inc}$, it is easy to prove that

$$\begin{aligned} P^{inp} = P^{out} &= -\frac{1}{2}\oint\!\!\!\oint_{\partial V}\left[\bar{E}^{sca}\times(\bar{H}^{inc})^* + \bar{E}^{inc}\times(\bar{H}^{sca})^*\right]\cdot d\bar{S} \\ &\quad - j\,P_V^{coup,react,vac} \\ &= -\frac{1}{2}\oint\!\!\!\oint_{\partial V}\left[\bar{E}_+^{sca}\times(\bar{H}_-^{inc})^* + \bar{E}_-^{inc}\times(\bar{H}_+^{sca})^*\right]\cdot d\bar{S} \\ &\quad - j\,P_V^{coup,react,vac} \end{aligned} \tag{85}$$

here the third equality is due to the continuity of the tangential components of $\bar{F}^{sca}$ and $\bar{F}^{inc}$ on ∂V. Inserting the (4) and (6) into the third equality of (85), the (85) can be written as the following operator form V.

$$P^{inp} = P^{out} = \mathcal{P}_5^{out}\left(\bar{J}^{SE}, \bar{M}^{SE}\right) \tag{86}$$

Surface formulation VI for output power

In fact, the (63) is equivalent to say that

$$\begin{aligned} P^{inp} = P^{out} &= -(1/2)\langle \bar{J}^{SE}, \bar{E}_+^{inc} \rangle_{\partial V} - (1/2)\langle \bar{H}_+^{inc}, \bar{M}^{SE} \rangle_{\partial V} \\ &\quad - j\left(P_V^{coup,react,vac} + 2P_V^{inc,react,vac}\right) \\ &= -(1/2)\langle \bar{J}^{SE}, \bar{E}_-^{inc} \rangle_{\partial V} - (1/2)\langle \bar{H}_-^{inc}, \bar{M}^{SE} \rangle_{\partial V} \\ &\quad - j\left(P_V^{coup,react,vac} + 2P_V^{inc,react,vac}\right) \end{aligned} \tag{87}$$

here the third equality is due to the continuity of the tangential component of $\bar{F}^{inc}$ on ∂V. Inserting the (4) and (6) into the third equality of (87), the (87) can be written as the following operator form VI.

$$P^{inp} = P^{out} = \mathcal{P}_6^{out}\left(\bar{J}^{SE}, \bar{M}^{SE}\right) \tag{88}$$

A brief discussion for various surface formulations

Various surface formulations given in [10] and this paper have their own merits respectively, and three typical ones are briefly discussed as below.

1) The operator form I given in the formulation (14) in [10] is very suitable for the multi-body system as illustrated in [12], because the scattering field explicitly appears, and then the coupling among different bodies can be easily expressed.

2) The operator form III given in the formulation (16) in [10] clearly expresses the interaction between incident field and the physical scattering currents, and then it has a clearer physical picture than other forms.

3) The calculation effort of the operator form IV given in the formulation (53) of this paper is smaller than other forms. Specifically, the unknowns of all forms are the same, and all forms (except the form IV) need to do some volume integrals, but the form IV need only to do two surface integrals.

D. Some new variational formulations for the scattering problem of a single homogeneous material body

When a certain incident field $\{\vec{E}^{inc},\vec{H}^{inc}\}$ is applied to a homogeneous material body, the surface equivalent currents $\{\vec{J}^{SE},\vec{M}^{SE}\}$ will make the following functional (89) be zero and stationary [8]-[12], because of the conservation law of energy [16].

$$\begin{aligned}
&\mathfrak{F}\left(\vec{J}^{SE},\vec{M}^{SE}\right)\\
\triangleq\;& \frac{\mu\,\Delta\varepsilon_c^*}{\mu_0\varepsilon_0-\mu\varepsilon_c^*}\left[\frac{1}{2}\left\langle \vec{J}^{SE},\vec{E}^{inc}\right\rangle_{\partial V}+\frac{\mu_0}{\mu}\frac{1}{2}\left\langle \vec{H}^{inc},\vec{M}^{SE}\right\rangle_{\partial V}^*\right]\\
&+\frac{\varepsilon_c\,\Delta\mu}{\varepsilon_0\mu_0-\varepsilon_c\mu}\left[\frac{\varepsilon_0}{\varepsilon_c}\frac{1}{2}\left\langle \vec{J}^{SE},\vec{E}^{inc}\right\rangle_{\partial V}^*+\frac{1}{2}\left\langle \vec{H}^{inc},\vec{M}^{SE}\right\rangle_{\partial V}\right]\\
&-\mathcal{P}_4^{out}\left(\vec{J}^{SE},\vec{M}^{SE}\right)
\end{aligned}\tag{89}$$

Inserting the (9) into (89) and employing the Rayleigh-Ritz scheme [20], a simultaneous equations for the expansion coefficients in (9) can be derived. By solving the equations, the expansion coefficients and then the $\{\vec{J}^{SE},\vec{M}^{SE}\}$ can be obtained. Inserting the derived $\{\vec{J}^{SE},\vec{M}^{SE}\}$ into (4)-(8), various fields and physical scattering currents can be obtained.

A similar and detailed procedure of above method can be found in [8]-[12], and it will not be repeated here. In addition, the variantional formulations corresponding to (86) and (88) can be similarly constructed, and they are not specifically given here.

VI. CONCLUSIONS

As a supplement to the Surf-SHM-EMP-CMT established in the previous Part I, some further studies are done in this Part II. For example:

The physical essence revelation and the numerical analysis for the boundary condition based method to unify variables are given, and then it is found out that its physical essence is so-called extinction theorem, and that the improved (ME) and (JH) schemes are more desirable for variable unification; a new conservation law based method for variable unification is developed, and it has the same physical essence as the boundary condition based method; some new surface formulations for the output power of a SHM are provided, and then some new surface formulations for constructing the OutCM set and some new variational formulations for the scattering problem of a SHM are established, and it is found out that the formulation IV is more advantageous than the other formulations to save computational resources.

The power relation contained in the symmetrical PMCHWT formulation is carefully researched, and it is found out that the physical essence of the symmetrical PMCHWT formulation for the SHM scattering problem is the conservation law of energy, just like the surface EFIE for PEC scattering problem and the symmetrical VIE for material scattering problem. Then, it is clearly revealed why the symmetrical Galerkin-based MoM for these integral equations is variational stationary. Moreover, it is also found out that the power character of the CM sets derived from the symmetrical and non-symmetrical PMCHWT based CMTs are not always identical to the OutCM set derived from the Surf-SHM-EMP-CMT; the symmetrical PMCHWT based CMT can be viewed as a special case of EMP-CMT, because of the object-oriented character of EMP-CMT.

Based on the above observations, it can be concluded that the modal theory for physicists and engineers should be established in a more physical framework instead of a purely mathematical framework, just like Arnold Sommerfeld (an outstanding educator and scientist) said in [21] that:

> "We do not really deal with mathematical physics, but with physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned 'prestabilized harmony' between what is mathematically interesting and what is physically important is met at each step and lends an esthetic – I should like to say metaphysical – attraction to our subject."

APPENDIX

In this appendix, the specific mathematical expressions of the operators in (4)-(8) are provided.

Because the relevant material parameters are constants in both $\text{int}V$ and $\text{ext}V$, the dyadic Green's functions used in [10] satisfy the following dual relationships

$$\overline{\overline{G}}_X^{JE}(\vec{r}',\vec{r}) \quad\leftrightarrow\quad \overline{\overline{G}}_X^{MH}(\vec{r}',\vec{r}) \tag{A-1.1}$$

$$\overline{\overline{G}}_X^{JH}(\vec{r}',\vec{r}) \quad\leftrightarrow\quad -\overline{\overline{G}}_X^{ME}(\vec{r}',\vec{r}) \tag{A-1.2}$$

$$\breve{\mu} \quad\leftrightarrow\quad \breve{\varepsilon} \tag{A-1.3}$$

$$\breve{\varepsilon} \quad\leftrightarrow\quad \breve{\mu} \tag{A-1.4}$$

here the subscript "X" can be "$-$" or "0"; $(\breve{\mu},\breve{\varepsilon})=(\mu,\varepsilon_c)$ for the region $\text{int}V$, and $(\breve{\mu},\breve{\varepsilon})=(\mu_0,\varepsilon_0)$ for the region $\text{ext}V$.

In addition, the Green's functions $\overline{\overline{G}}_X^{JE}(\vec{r}',\vec{r})$ and $\overline{\overline{G}}_X^{JH}(\vec{r}',\vec{r})$ satisfy the following symmetry [22].

$$\overline{\overline{G}}_X^{JE}(\vec{r},\vec{r}') = \left[\overline{\overline{G}}_X^{JE}(\vec{r}',\vec{r})\right]^T \tag{A-2.1}$$

$$\overline{\overline{G}}_X^{JH}(\vec{r},\vec{r}') = \left[\overline{\overline{G}}_X^{JH}(\vec{r}',\vec{r})\right]^T \tag{A-2.2}$$

here the superscript "T" represents the transpose operation for the relevant Green's function, as defined in [22].

Inserting the (1)-(3) and (A-1)-(A-2) into the formulations (C-7) and (C-13)-(C-15) in Part I [10], the followings integral expressions for various fields are obtained.

$$\begin{aligned}\mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{-}^{JE}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{-}^{ME}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-3.1}$$

$$\begin{aligned}\mathcal{H}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{-}^{JH}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{-}^{MH}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-3.2}$$

here $\bar{r},\bar{r}'\in \mathrm{int}V$, and

$$\begin{aligned}\mathcal{E}_{-}^{inc}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{0}^{JE}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{0}^{ME}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-4.1}$$

$$\begin{aligned}\mathcal{H}_{-}^{inc}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{0}^{JH}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{0}^{MH}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-4.2}$$

here $\bar{r},\bar{r}'\in \mathrm{int}V$, and

$$\begin{aligned}\mathcal{E}_{+}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{0}^{JE}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}_{+}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{0}^{ME}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}_{+}^{SE}\left(\bar{r}'\right)dS' \\ &= -\oiint_{\partial V}\bar{\bar{G}}_{0}^{JE}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad-\oiint_{\partial V}\bar{\bar{G}}_{0}^{ME}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-5.1}$$

$$\begin{aligned}\mathcal{H}_{+}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\bar{\bar{G}}_{0}^{JH}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}_{+}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\bar{\bar{G}}_{0}^{MH}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}_{+}^{SE}\left(\bar{r}'\right)dS' \\ &= -\oiint_{\partial V}\bar{\bar{G}}_{0}^{JH}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad-\oiint_{\partial V}\bar{\bar{G}}_{0}^{MH}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-5.2}$$

here $\bar{r},\bar{r}'\in \mathrm{ext}V$, and

$$\begin{aligned}\mathcal{E}_{-}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\Delta\bar{\bar{G}}_{-}^{JE}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\Delta\bar{\bar{G}}_{-}^{ME}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-6.1}$$

$$\begin{aligned}\mathcal{H}_{-}^{sca}\left(\bar{J}^{SE},\bar{M}^{SE}\right) &= +\oiint_{\partial V}\Delta\bar{\bar{G}}_{-}^{JH}\left(\bar{r},\bar{r}'\right)\cdot\bar{J}^{SE}\left(\bar{r}'\right)dS' \\ &\quad+\oiint_{\partial V}\Delta\bar{\bar{G}}_{-}^{MH}\left(\bar{r},\bar{r}'\right)\cdot\bar{M}^{SE}\left(\bar{r}'\right)dS'\end{aligned} \tag{A-6.2}$$

here $\bar{r},\bar{r}'\in \mathrm{int}V$, and

$$\Delta\bar{\bar{G}}_{-}^{JE}\left(\bar{r},\bar{r}'\right) = \bar{\bar{G}}_{-}^{JE}\left(\bar{r},\bar{r}'\right) - \bar{\bar{G}}_{0}^{JE}\left(\bar{r},\bar{r}'\right) \tag{A-7.1}$$

$$\Delta\bar{\bar{G}}_{-}^{ME}\left(\bar{r},\bar{r}'\right) = \bar{\bar{G}}_{-}^{ME}\left(\bar{r},\bar{r}'\right) - \bar{\bar{G}}_{0}^{ME}\left(\bar{r},\bar{r}'\right) \tag{A-7.2}$$

$$\Delta\bar{\bar{G}}_{-}^{JH}\left(\bar{r},\bar{r}'\right) = \bar{\bar{G}}_{-}^{JH}\left(\bar{r},\bar{r}'\right) - \bar{\bar{G}}_{0}^{JH}\left(\bar{r},\bar{r}'\right) \tag{A-7.3}$$

$$\Delta\bar{\bar{G}}_{-}^{MH}\left(\bar{r},\bar{r}'\right) = \bar{\bar{G}}_{-}^{MH}\left(\bar{r},\bar{r}'\right) - \bar{\bar{G}}_{0}^{MH}\left(\bar{r},\bar{r}'\right) \tag{A-7.4}$$

In (A-3)-(A-7), [13], [22]

$$\bar{\bar{G}}_{0}^{JE}\left(\bar{r},\bar{r}'\right) = \frac{1}{j\omega\varepsilon_0}\left(\nabla\nabla+k_0^2\bar{\bar{I}}\right)g_0\left(\bar{r},\bar{r}'\right) \tag{A-8.1}$$

$$\bar{\bar{G}}_{0}^{ME}\left(\bar{r},\bar{r}'\right) = -\nabla\times\left[\bar{\bar{I}}g_0\left(\bar{r},\bar{r}'\right)\right] \tag{A-8.2}$$

$$\bar{\bar{G}}_{0}^{JH}\left(\bar{r},\bar{r}'\right) = \nabla\times\left[\bar{\bar{I}}g_0\left(\bar{r},\bar{r}'\right)\right] \tag{A-8.3}$$

$$\bar{\bar{G}}_{0}^{MH}\left(\bar{r},\bar{r}'\right) = \frac{1}{j\omega\mu_0}\left(\nabla\nabla+k_0^2\bar{\bar{I}}\right)g_0\left(\bar{r},\bar{r}'\right) \tag{A-8.4}$$

here $\bar{\bar{I}}$ is the identity dyad, and

$$g_0\left(\bar{r},\bar{r}'\right) = \frac{1}{4\pi\left|\bar{r}-\bar{r}'\right|}e^{-jk_0\left|\bar{r}-\bar{r}'\right|} \tag{A-9}$$

and

$$\bar{\bar{G}}_{-}^{JE}\left(\bar{r},\bar{r}'\right) = \frac{1}{j\omega\varepsilon_c}\left(\nabla\nabla+k_c^2\bar{\bar{I}}\right)g_-\left(\bar{r},\bar{r}'\right) \tag{A-10.1}$$

$$\bar{\bar{G}}_{-}^{ME}\left(\bar{r},\bar{r}'\right) = -\nabla\times\left[\bar{\bar{I}}g_-\left(\bar{r},\bar{r}'\right)\right] \tag{A-10.2}$$

$$\bar{\bar{G}}_{-}^{JH}\left(\bar{r},\bar{r}'\right) = \nabla\times\left[\bar{\bar{I}}g_-\left(\bar{r},\bar{r}'\right)\right] \tag{A-10.3}$$

$$\bar{\bar{G}}_{-}^{MH}\left(\bar{r},\bar{r}'\right) = \frac{1}{j\omega\mu}\left(\nabla\nabla+k_c^2\bar{\bar{I}}\right)g_-\left(\bar{r},\bar{r}'\right) \tag{A-10.4}$$

here

$$g_-\left(\bar{r},\bar{r}'\right) = \frac{1}{4\pi\left|\bar{r}-\bar{r}'\right|}e^{-jk_c\left|\bar{r}-\bar{r}'\right|} \tag{A-11}$$

In (A-9), $k_0=\omega\sqrt{\mu_0\varepsilon_0}$. In (A-11), $k_c=\omega\sqrt{\mu\varepsilon_c}$.

The operator forms of physical scattering currents are as follows

$$\mathcal{J}^{vo}\left(\bar{J}^{SE},\bar{M}^{SE}\right) = \sigma\mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{A-12.1}$$

$$\mathcal{J}^{vp}\left(\bar{J}^{SE},\bar{M}^{SE}\right) = j\omega\Delta\varepsilon\mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{A-12.2}$$

$$\mathcal{J}^{vop}\left(\bar{J}^{SE},\bar{M}^{SE}\right) = j\omega\Delta\varepsilon_c\mathcal{E}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{A-12.3}$$

$$\mathcal{M}^{vm}\left(\bar{J}^{SE},\bar{M}^{SE}\right) = j\omega\Delta\mu\mathcal{H}_{-}^{tot}\left(\bar{J}^{SE},\bar{M}^{SE}\right) \tag{A-13}$$

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