Rigid body motion – Limits on Acceleration

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Abstract

It is well-known that, according to special relativity, there is an absolute “speed limit” on objects traveling in space-time: nothing can travel faster than light. It turns out that an object’s *acceleration* is also limited by the geometry of space-time, but in a more complex manner.

For objects viewed as points (negligible spatial extent), special relativity imposes no particular constraints on the magnitude of their acceleration. For objects that have spatial extent, however, it turns out that the geometry of space-time *does* impose limits.

The case we are considering here is what has been defined as “rigid motion” (Born [1], Franklin [2]). This is motion in which an object’s speed is changed in such a way that it is neither stretched nor compressed. All of our discussion is limited to a single spatial dimension plus time (a moving rod). We assume that acceleration is applied all along the rod’s length with no assumptions required about its rigidity. Nor do we include such dynamic physical effects as momentum or elasticity.

It turns out that speed changes cannot be uniform along the length of the rod if it is to remain in rigid motion. Franklin [2] derived a formula relating the required accelerations of various points along the rod. His derivation was for the special case in which acceleration is constant over time. Here we show that Franklin’s key formula (Equation 14 in [2]) applies to acceleration that is non-constant as well.

Franklin’s formula reveals an interesting property of space-time: If the rod’s acceleration exceeds a fixed, finite bound the rod *must* experience distortion -- stretching or compressing in the direction of the acceleration. Furthermore, if a rod is accelerated *at* this bound, in order to maintain rigid motion, its trailing end must accelerate instantaneously (infinite acceleration), while its leading end accelerates at a finite constant rate. The rod’s trailing end will acquire its new speed in zero time, while the leading end takes a finite time. That is, the leading end ages, during this acceleration, over the trailing end.

The Basic Idea

Figure 1 shows a typical space-time diagram for a one-dimensional space plus time (x is in light-years and t is in years). A point object is initially stationary (solid green line). At time = 0 its speed changes instantaneously to 0.6 times the speed of light going to the right (solid red line).

Figure 2 shows an extended body – a rod – which starts out stationary. The trajectories of its left and right ends in this stationary phase are shown as solid green lines. Suppose we naively apply an instantaneous speed change at time = 0 to all of the points of the rod. The solid red lines of Figure 2 show the trajectories for the left and right ends of the rod at its new speed.

But now, consider Figure 3, obtained by applying the Lorentz transformation to Figure 2. It shows that in its new inertial frame, the rod is *longer* than it was in its initial inertial frame. Clearly, the acceleration pattern shown in Figures 2 & 3 has resulted in more than just a speed change – not what we were trying to depict.

The problem in this example is that, simultaneity of events is not preserved as the rod is shifted to the new inertial frame. We assumed that all points in the rod would be accelerated (instantaneously) to the new speed at the same time, in its initial frame, in which it started out at rest. But, in its new frame, we see that the right side of the rod was accelerated earlier than the left and the rod got stretched out.

Suppose we now examine the trajectory of the rod shown in Figures 4 & 5 (same trajectory shown in two inertial frames). The green-solid-straight-line portion of the trajectories depicts the rod at rest in the initial frame -- seen most clearly in Figure 4. The red-solid-straight-line portion of the trajectories depicts the rod at rest in the new inertial frame – seen most clearly in Figure 5.

Between the two solid-straight-line trajectory segments of the right end of the rod as its motion goes from its initial velocity to its new one, we see a blue-curved-line portion of the trajectory. We might have been tempted to draw this portion of the trajectory as a continuation of the green-solid-straight-line trajectory, shown as a green-dotted-straight-line in Figures 4 & 5, until it intersects the red-dotted-straight-line. That is, to assume that the right end of the rod would still get accelerated instantaneously, but simply at a different time. This kind of trajectory would not be consistent with rigid motion, however, since its length would be changing during this portion of its trajectory.

The blue-curved-line portion of the trajectory is specifically constructed so that the rod will retain its length, consistent with rigid motion. Of course, this means that, although the left end of the rod changes speed instantaneously, the right end changes gradually (as do points between the ends).

To get at the shape of the curved-line portion of the trajectory, we look, in Figure 6, at the rod's trajectory in an inertial frame at an intermediate speed – i.e., same trajectory as Figures 4 & 5, but viewed from a different inertial frame. In this frame, the rod is initially traveling leftward. Then at time = 0, its left end changes instantaneously to moving rightward. At this point, where the rod's motion switches direction, it is momentarily at rest in this inertial frame. And for the rod to experience no stretching or compression forces, no points on the rod can have a non-zero velocity. So, by the definition of rigid motion, its length must be the proper length of the rod. In Figure 6 the horizontal dashed blue line shows the rod in this position. Its right end gives us one of the points on the curved-line portion of the trajectory. The rest of that trajectory can be generated in a similar manner using other values of the speed of the observation frame. After some straightforward application of Lorentz transforms, it can be shown that the curved-line portion of the trajectory of Figure 4 satisfies

$$x = D / \sqrt{(1 - u^2)} \quad and \quad t = uD / \sqrt{(1 - u^2)} \tag{1}$$

where $0 \leqq u \leqq S$, S being the instantaneous speed change of the rod's trailing end (in Figures 4-6, S =0.6). (This is for a rod travelling to the right -- a comparable result applies to a rod moving in this way in the opposite direction.)

We note that Franklin [2] generated this trajectory of the rod undergoing what he termed impulsive acceleration, using a different derivation.

General Acceleration of the Rod

Consider now Figure 7, in which the rod is momentarily at rest in the inertial frame whose simultaneity lines are shown in green, with the position of the rod shown as the heavier green line segment. The position of the left end of the rod is $(x, t) = (0, 0)$ and the right end is $(x, t) = (D, 0)$, where D is the proper length of the rod – length of the rod in its rest frame.

Imagine that the rod is accelerated a small amount, its ends moving on the trajectories shown as the curved blue lines. Some short time later the rod is as shown by the heavier red line. Its left end is at some point (x_L, t_L), and its right end is at (x_R, t_R). Its speed has changed by v. The red lines are the lines of simultaneity in the new inertial frame moving at speed v.

Let (x_L', t_L') and (x_R', t_R') be the coordinates of those points, expressed in the units of the new inertial frame. Then, since these points are on the same line of simultaneity,

$$t_R' = t_L' \tag{2}$$

And, since the rod is undergoing rigid motion,

$$x_R' = x_L' + D \tag{3}$$

Applying standard Lorentz transformations (with relative velocity $-v$ from the standpoint of the rod's new inertial frame):

$$x_L = (x_L' + v\, t_L') / \sqrt{(1 - v^2)} \tag{4}$$

$$x_R = (x_R' + v\, t_R') / \sqrt{(1 - v^2)} \tag{5}$$
$$t_R = (t_R' + v\, x_R') / \sqrt{(1 - v^2)} \tag{6}$$

Some algebraic rearrangement allows us to express these as

$$x_R = x_L + D / \sqrt{(1 - v^2)} \tag{7}$$
$$t_R = t_L + v\, D / \sqrt{(1 - v^2)} \tag{8}$$

Since v is small, we take

$$t_R \approx t_L + v\, D \tag{9}$$

Because we are considering a small movement, we take the accelerations of the left and right side of the rod to be

$$a_L \approx v/t_L \tag{10}$$
$$a_R \approx v/t_R \tag{11}$$

Combining, and letting v go to zero we get

$$\boxed{1/a_R - 1/a_L = D} \tag{12}$$

This formula relates acceleration of points along the rod in the rod's rest frame (frame in which the rod's speed is momentarily zero).

We note that this is the formula that Franklin derived for the special case in which acceleration is constant (Equation 14 in [2]). Here we have shown that the same formula applies generally, not just for the case in which acceleration is constant.

To avoid any issues related to the transmission of forces along the rod, we imagine that acceleration, in the pattern defined by Equation (12), is applied all along the rod.

Limits on Acceleration

An implication of Equation (12) is that it makes sense to speak of "the" acceleration of a rod undergoing rigid motion even though each point on the rod is actually experiencing a different value of acceleration. For a rod in rigid motion, we need only specify the acceleration of one of its points.

It will be convenient to think about the acceleration of the rod's geometric center. We recast Equation (12) as:

$$1/a_p - 1/a_0 = pD \tag{13}$$

where a_0 is the acceleration of the rod's center, and p is a fraction corresponding to a point on the rod:

$$-1/2 < p < 1/2 \tag{13a}$$

($p=0$ is the center, $p=-1/2$ is the left end, and $p=1/2$ is the right end of the rod).

From this we can see that the acceleration a rod in rigid motion can have is *limited*.

In terms of the acceleration of the rod's center, a_0,we consider first the case where a_0 is positive. Then, a_p must also be positive if the rod is not to experience stretching or compression. So

$$1/a_p = 1/a_0 + pD > 0 \tag{14}$$

This takes on an extreme at $p=1/2$, leading to

$$a_0 < 2/D \tag{15}$$

Similarly, considering negative a_0 gives

$$a_0 > -2/D \tag{16}$$

So,

$$\boxed{-2/D < a_0 < 2/D \qquad (17)}$$

To put this another way, the magnitude of the acceleration of the rod's center must be less than *2/D*. *If the rod is accelerated at a rate greater than this limit, rigid motion cannot be maintained* – the rod must necessarily stretch or compress. This observation is unrelated to physical properties of the rod, such as its elasticity or mass. It is a consequence of the fundamental geometry of space-time.

Acceleration at the Extreme

Consider, now a rod in rigid motion whose acceleration at some instant is at an extreme of Inequality (17). At that instant, the rod's trailing end will be changing speed instantaneously (infinite acceleration: $1/a_L = 0$ in Equation (12) for a rod moving to the right). The rod's leading end will, in its rest frame, experience acceleration whose magnitude is the reciprocal of its length ($1/a_R = D$) according to Equation (12). That is, the acceleration of the rod's leading end, in its rest frame, *will be constant*.

This acceleration scenario is, in fact, the situation illustrated in Figures 4-6. We can readily see that Equation (1) describes just such a trajectory as follows. From Equation (1) we can generate the speed and acceleration, V and A , of the rod's leading end as

$$V = dx/dt = [dx/du]/[dt/du] \ \ and \ \ A = dV/dt = [dV/du]/[dt/du] \tag{18}$$

Following some algebra, we get

$$V = u \ \ and \ \ A = (1/D)\sqrt{(1 - u^2)} \tag{19}$$

In its rest frame, u=0, and so

$$V = 0 \ \ and \ \ A = (1/D) \tag{20}$$

Also, because the rod's leading end achieves a speed change of S, moving at this constant acceleration, the duration of the acceleration, as experienced by the leading end, will be

$$t = S/A = SD \tag{21}$$

Thus we see that *if the rod is experiencing its maximum acceleration compatible with rigid motion, its trailing end will experience infinite acceleration, while its leading end experiences finite constant acceleration whose magnitude is given by Equation (20). The trailing end will acquire its new speed in zero time, while the leading end takes a time given by Equation (21) to reach the new speed.* Once the two ends of the rod are at the same speed, they will once more, experience identical passage of time. Figure 8 illustrates, showing several successive positions of the rod in its rest frame, for the example given in Figures 4-6.

References

[1] Born, M. (1909), Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips. Ann. Phys., 335: 1–56. doi:10.1002/andp.19093351102

[2] J. Franklin, Rigid body motion in special relativity, Phys. 43, 1489-1501 (2013) DOI:10.1007/s10701-013-9757-xCite as:arXiv:1105.3899v3 [physics.gen-ph]

Figures

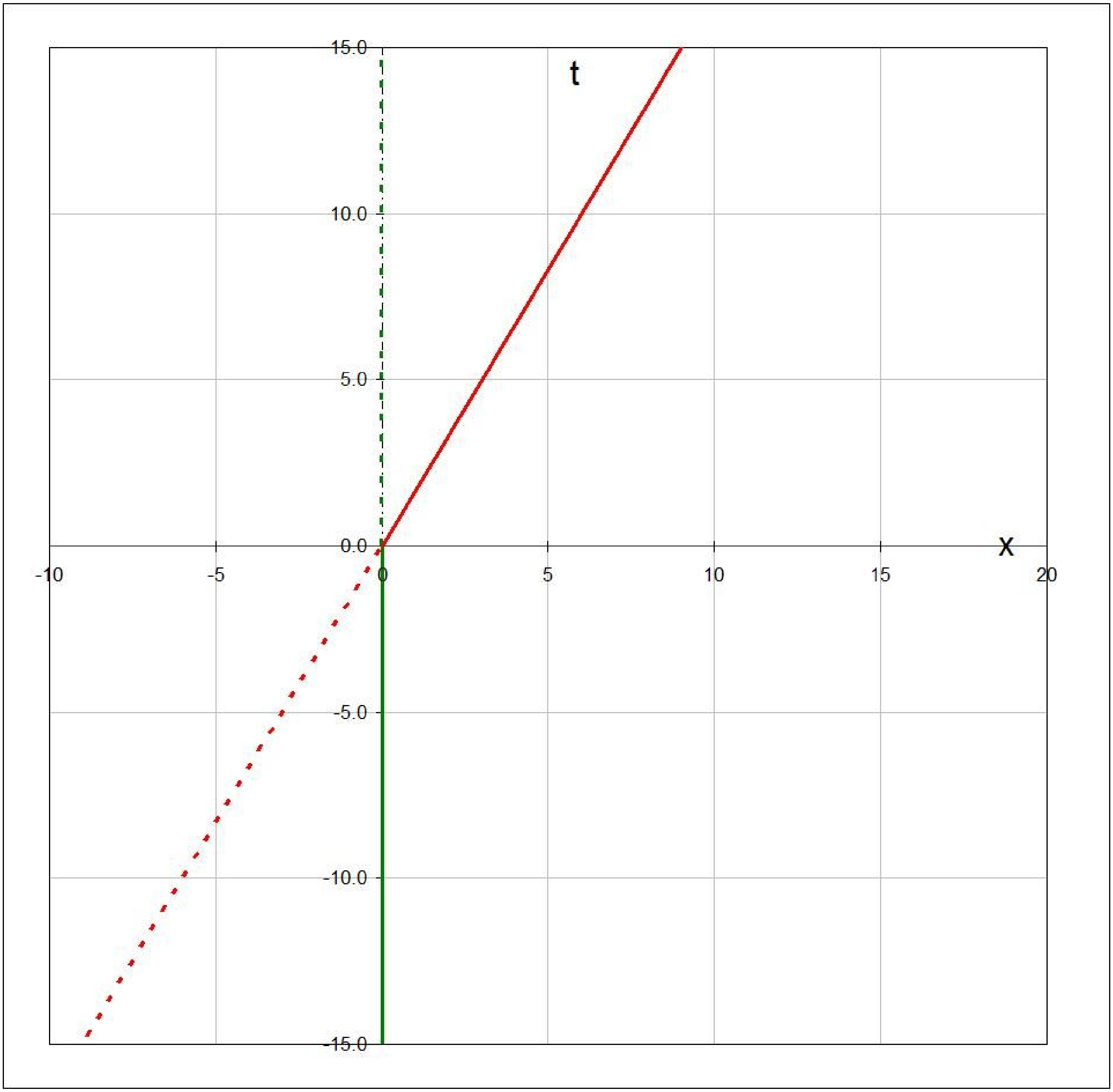

Figure 1. Space-time diagram for a point object initially stationary (green solid line) and accelerating instantaneously at time = 0 (solid red line)

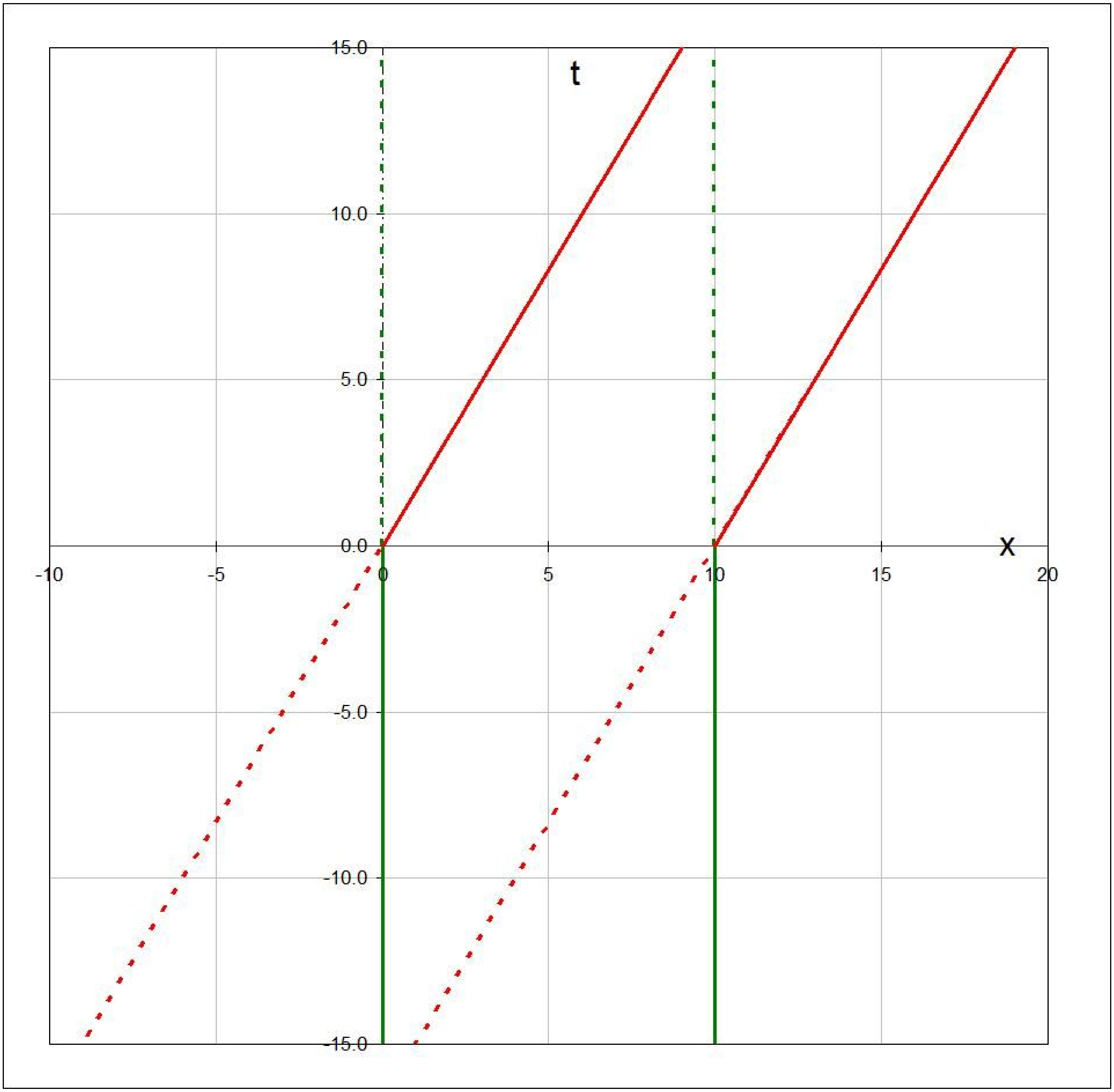

Figure 2. Naive application of instantaneous speed change: assume every point on the rod instantly changes speed at time = 0 in its rest frame.

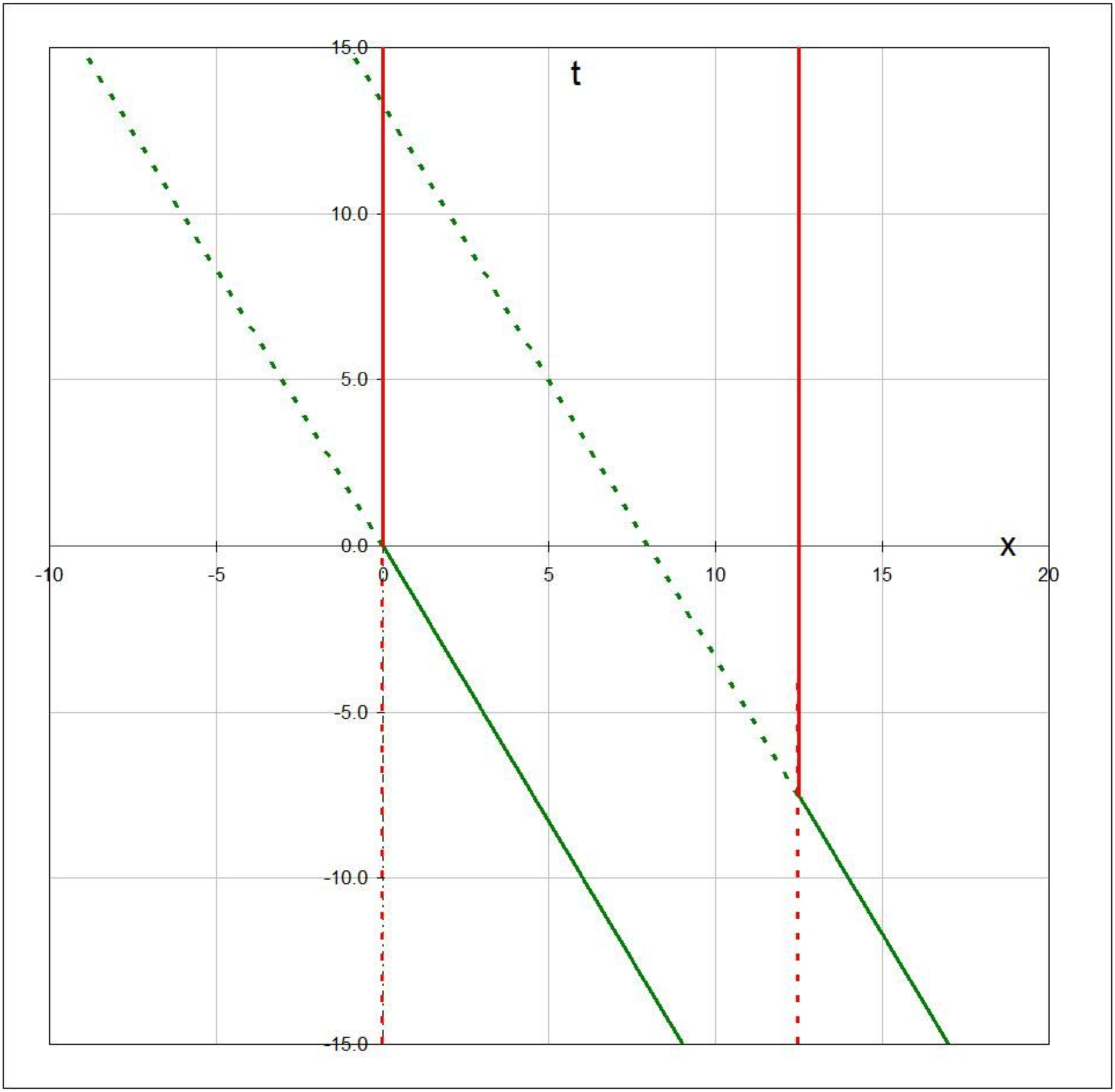

Figure 3. Lorentz transformation of Figure 2, showing the rod from the perspective of its new inertial frame. Note that the rod has been stretched in the process of the speed change.

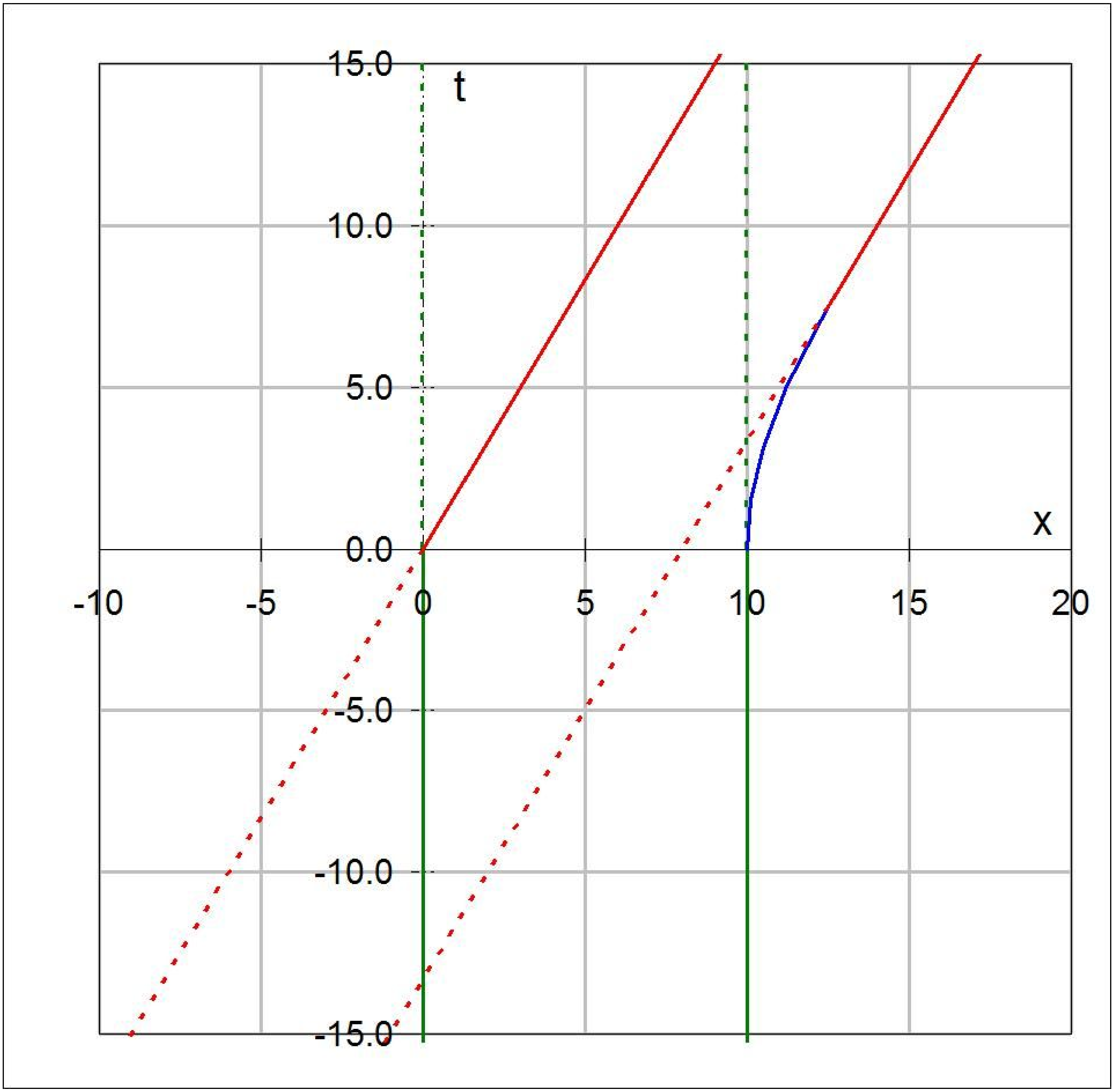

Figure 4. Left end of rod accelerates instantaneously; right end accelerates gradually.

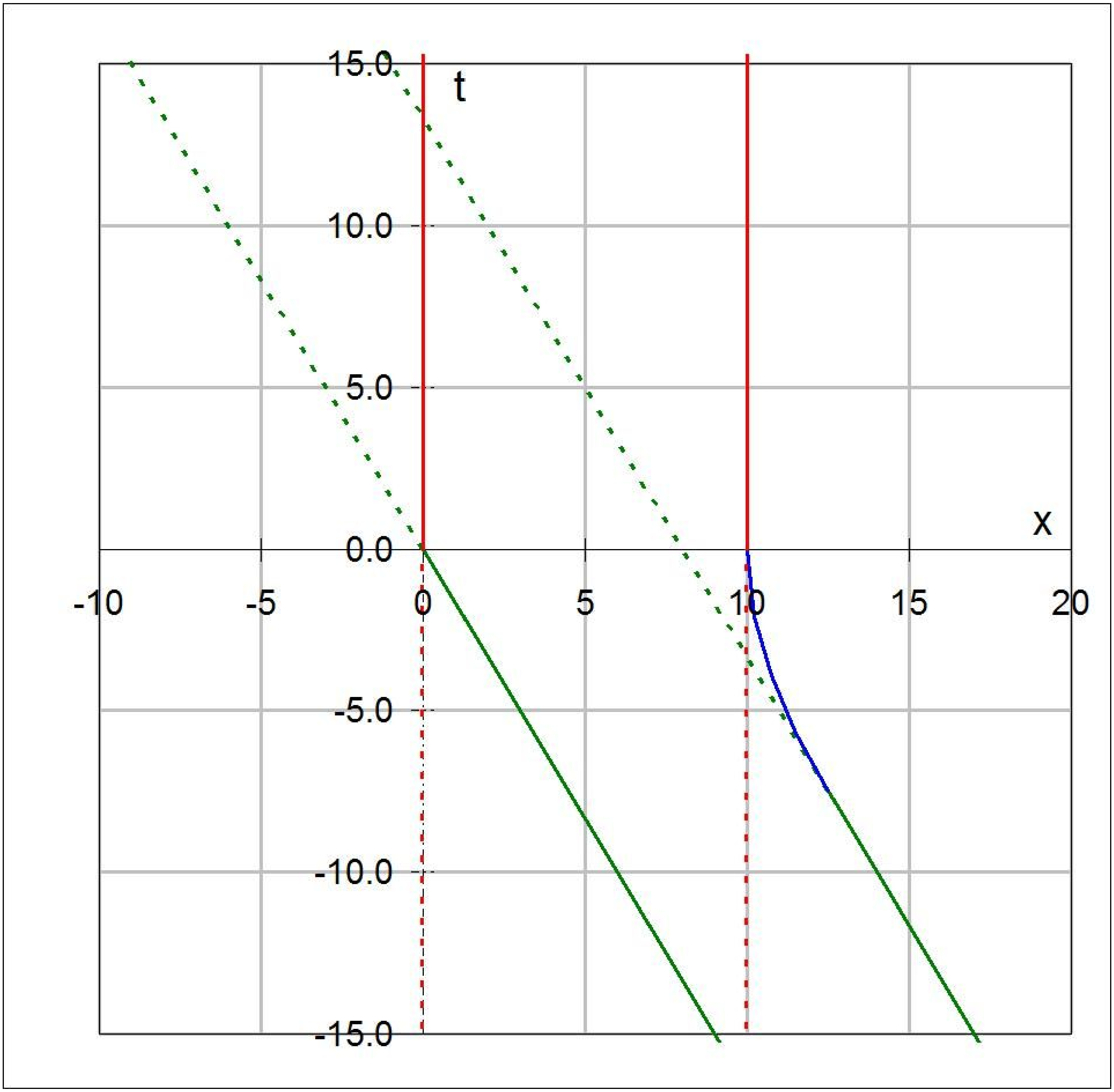

Figure 5. Lorentz transformation of Figure 4, showing the rod from the perspective of its new inertial frame.

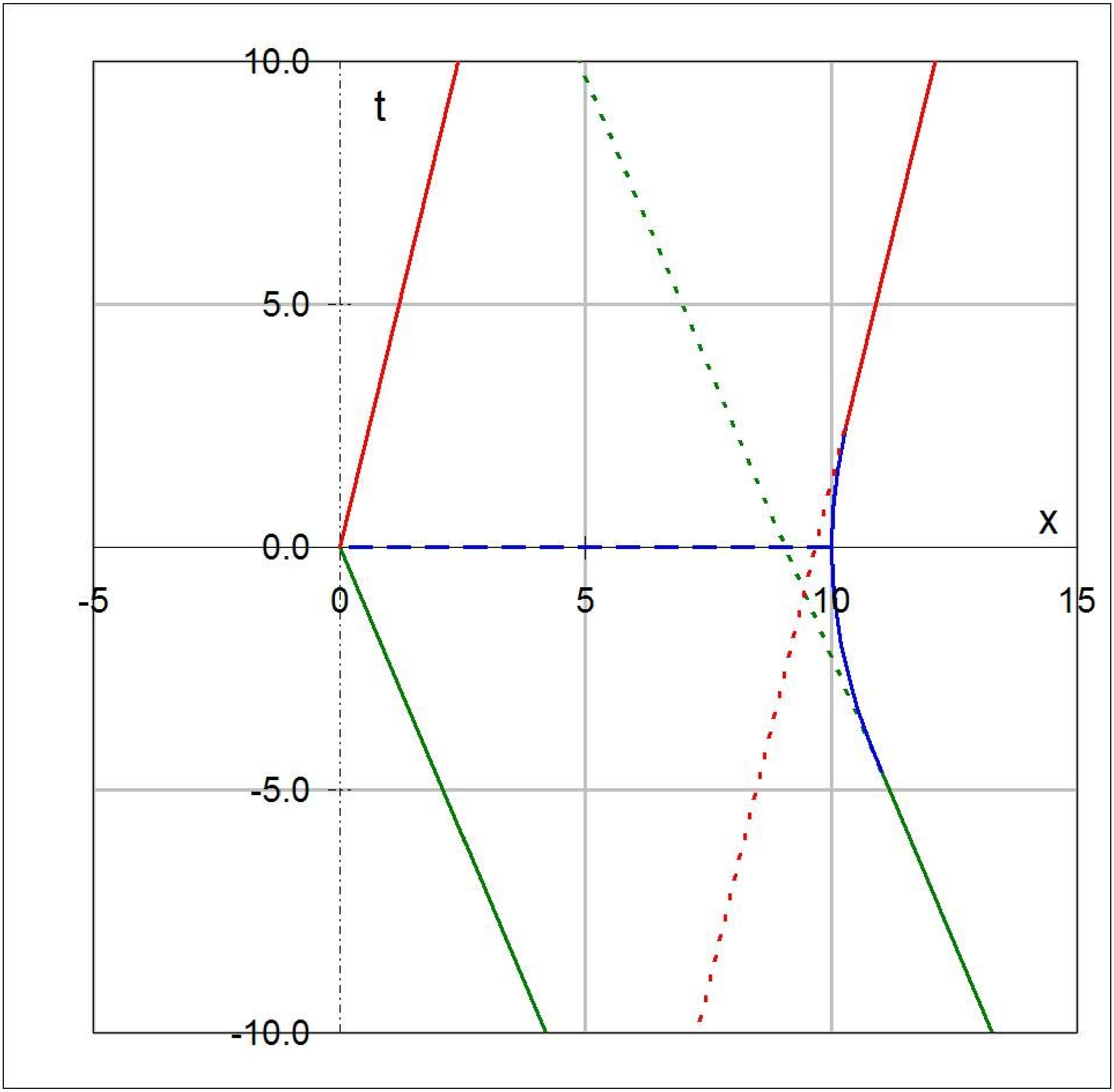

Figure 6. The rod in Figures 4 & 5, viewed from the perspective of an inertial frame intermediate between the two. Note that the rod (dashed blue line) extending from the origin to the curved blue line retains its original length.

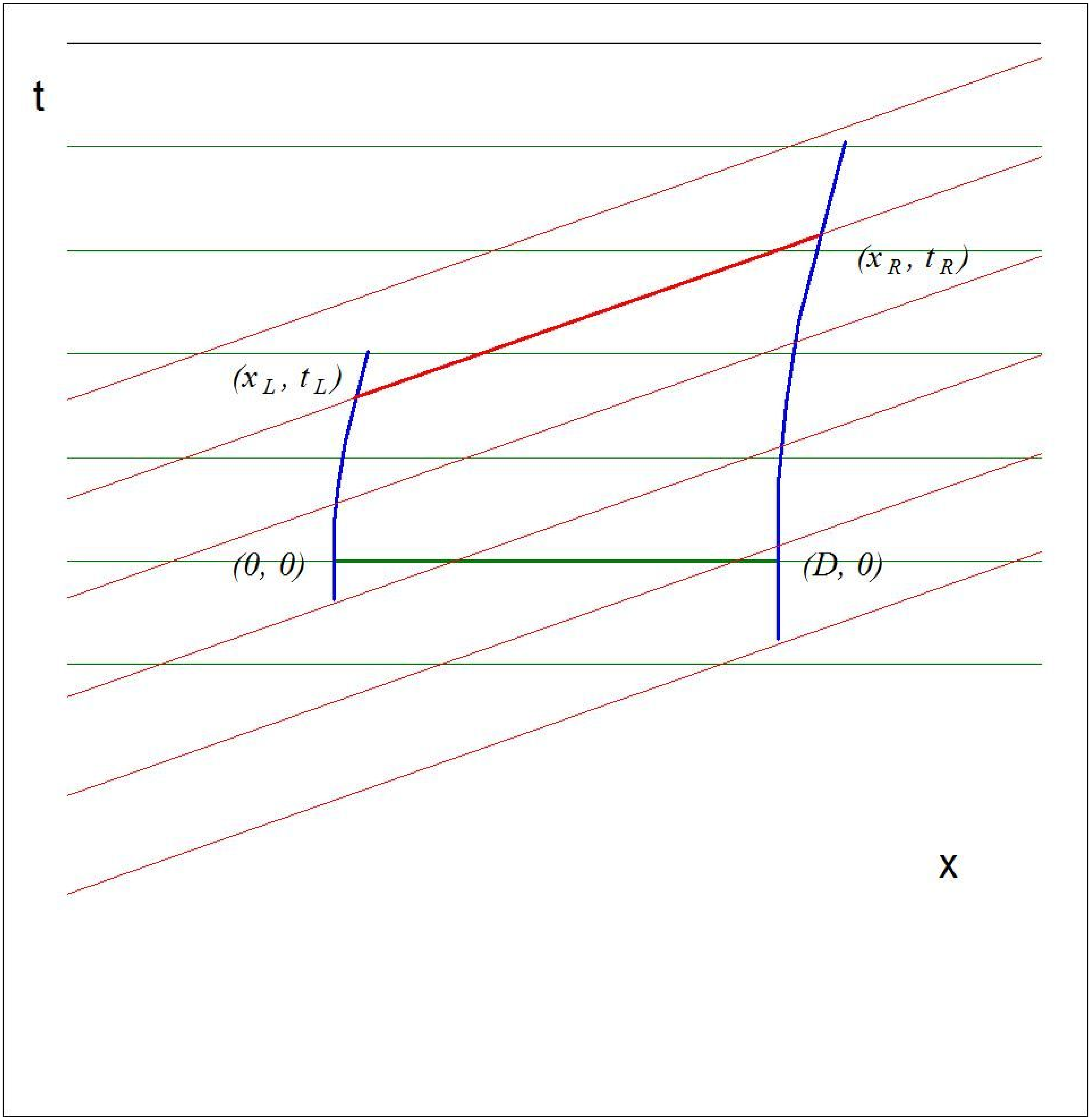

Figure 7. General (small) acceleration.

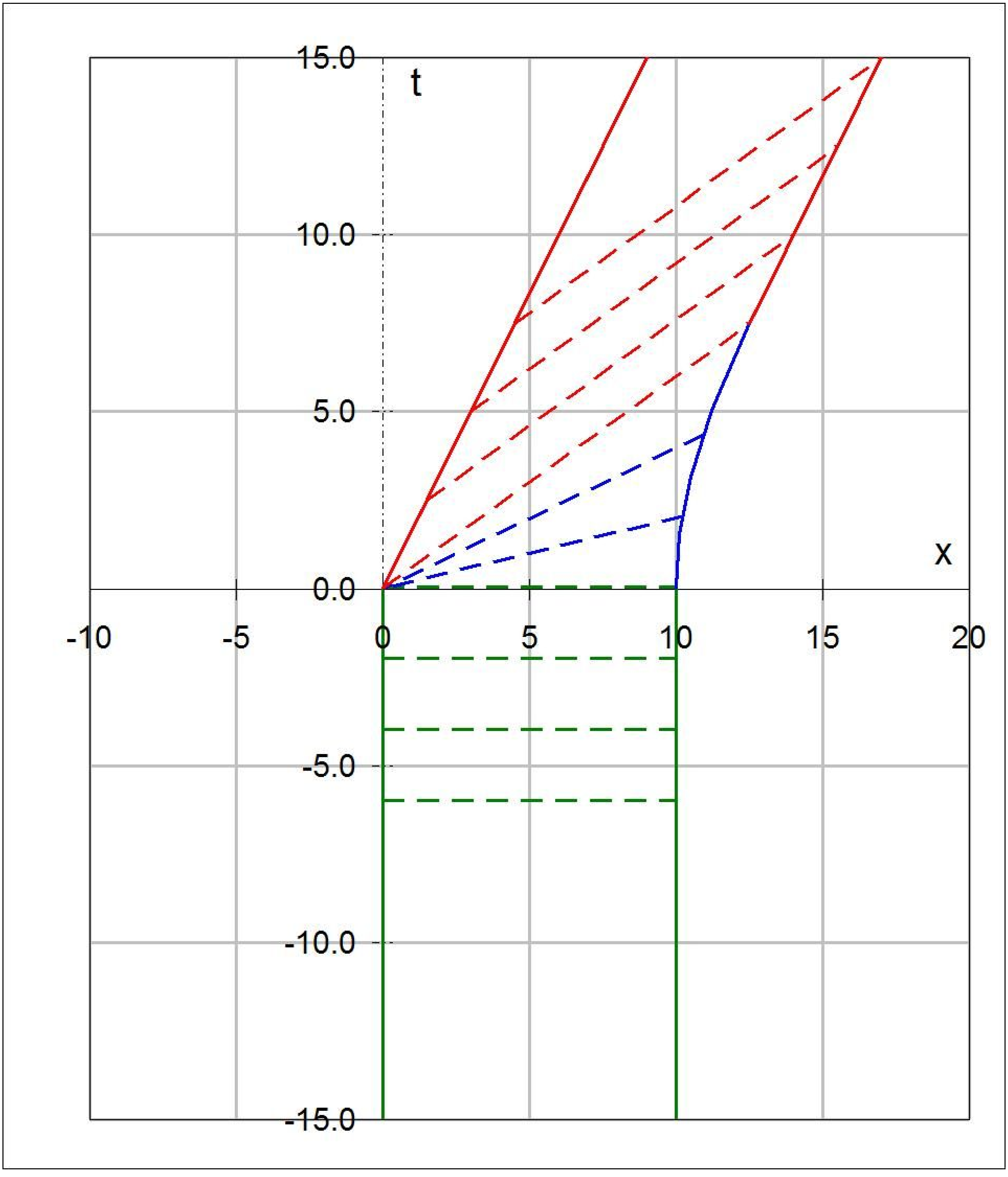

Figure 8. The rod in Figures 4-6, viewed in the inertial frame of Figure 4. Several successive positions of the rod are shown (dashed lines) in their rest frames as the rod is accelerated to its ultimate speed. The blue lines show the portion of its trajectory in which the left end is accelerated instantaneously while the right end is accelerated at the constant rate which is the maximum compatible with rigid motion.