A two-dimensional vector space algebra with identity
2x2 matrix basis matrix multiplication homomorphism

Theorem I.1: There exists a homomorphism between any two-dimensional
vector space algebra with identity and a 2x2 matrix basis under ordinary matrix multiplication

proof:

Let the set:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \quad , \quad a,b,c,d \in \mathbb{F}$$

be a 2-dimensional vector basis spanning a number field:

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A+Ba & Bb \\ Bc & A+Bd \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{array}{|c|c|} \hline a^2+bc = A+Ba & ab+bd = Bb \\ \hline cb+d^2 = A+Bd & ca+dc = Bc \\ \hline \end{array} \right\} \Rightarrow \left\{ \begin{array}{|c|c|} \hline a^2+bc = A+Ba & a+d = B \\ \hline cb+d^2 = A+Bd & a+d = B \\ \hline \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{|c|} \hline a^2+bc = A+Ba \\ \hline cb+(B-a)^2 = A+B(B-a) \\ \hline \end{array} \right\} \Rightarrow \left\{ \begin{array}{|c|} \hline a^2-Ba+(bc-A) = 0 \\ \hline (B-a)^2 - B(B-a) + (cb-A) = 0 \\ \hline \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{|c|} \hline a = \dfrac{B \pm \sqrt{B^2-4(bc-A)}}{2} \\ \hline (B-a) = \dfrac{B \pm \sqrt{B^2-4(cb-A)}}{2} = a \\ \hline \end{array} \right\} \Rightarrow a = \frac{1}{2}B = d \Rightarrow 0 = B^2 - 4(bc-A)$$

$$\Rightarrow B^2 + 4A = 4bc \Rightarrow c = \frac{1}{b}\left(\frac{1}{4}B^2 + A\right)$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} \right\} \quad , \quad (b \neq 0)$$

$$\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix}\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} = \begin{pmatrix} \frac{1}{4}B^2 + \frac{1}{4}B^2 + A & \frac{1}{2}Bb + \frac{1}{2}Bb \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right)\frac{1}{2}B + \frac{1}{2}B\frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{4}B^2 + A + \frac{1}{4}B^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}B^2 + A & Bb \\ B\frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B^2 + A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} \quad , \quad (b \neq 0)$$

the basis is linearly independent:

$$0 = C1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} C1 + aC2 & bC2 \\ cC2 & C1 + dC2 \end{pmatrix} \quad , \quad (b \neq 0)$$

$$\Rightarrow C2 = 0 \Rightarrow C1 = 0$$

□

This is a statement of constructive existence of an algebra [1].
Unfortunately, the vector space of the algebra must be known to be 2-dimensional.
Given that the vector space of the algebra is known to be 2-dimensional, the algebra product determines the constants: A,B,b ; determining the basis of the algebra, as follows:

$$u = u_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix}$$

$$v = v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_2\begin{pmatrix} \frac{1}{2} & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow uv = \left[u_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} \right]\left[v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} \right]$$

$$= u_1v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [u_1v_2 + u_2v_1]\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix} +$$

$$+ u_2v_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix}\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2+A\right) & \frac{1}{2}B \end{pmatrix}$$

$$= u_1v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [u_1v_2 + u_2v_1]\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix} +$$

$$+ u_2v_2\begin{pmatrix} \frac{1}{4}B^2 + \left(\frac{1}{4}B^2 + A\right) & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \left(\frac{1}{4}B^2 + A\right) + \frac{1}{4}B^2 \end{pmatrix}$$

$$= u_1v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [u_1v_2 + u_2v_1]\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix} + u_2v_2\begin{pmatrix} \frac{1}{2}B^2 + A & 0 \\ 0 & \frac{1}{2}B^2 + A \end{pmatrix}$$

$$= \left[u_1v_1 + u_2v_2\left(\frac{1}{2}B^2 + A\right)\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [u_1v_2 + u_2v_1]\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix}$$

For an agebra homomorphism h :

$h(u) = U \quad ; \quad u \in \mathbb{V} \quad , \quad U \in \mathbb{W}$

$$\Rightarrow \begin{cases} h(au + bv) = ah(u) + bh(v) \quad ; \quad u,v \in \mathbb{V} \quad , \quad h(u),h(v),h(au+bv) \in \mathbb{W} \quad ; \quad a,b \in \mathbb{F} \\ h(auv) = ah(u)h(v) \quad ; \quad u,v \in \mathbb{V} \quad , \quad h(u),h(v),h(auv) \in \mathbb{W} \quad ; \quad a \in \mathbb{F} \end{cases}$$

So, the identity element transforms to the identity element, and if it is a base vector, then in a vector space of dimension greater than 1 , there is another base vector of the vector space. Under the above conditions, let this basis be denoted by $\{\mathbf{e},\mathbf{f}\}$

$$\Rightarrow h(\mathbf{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad h(\mathbf{f}) = \begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix}$$

$$u = u_1\mathbf{e} + u_2\mathbf{f} \Leftrightarrow h(u) = u_1h(\mathbf{e}) + u_2h(\mathbf{f}) = u_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix}$$

$$v = v_1\mathbf{e} + v_2\mathbf{f} \Leftrightarrow h(v) = v_1h(\mathbf{e}) + v_2h(\mathbf{f}) = v_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_2\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix}$$

$$\Rightarrow uv = u_1v_1\mathbf{e} + (u_1v_2 + u_2v_1)\mathbf{f} + u_2v_2\mathbf{f}^2$$

$$h\left(\mathbf{f}^2\right) = h(\mathbf{f})h(\mathbf{f}) = h(\mathbf{f})^2$$

$$\Rightarrow h(uv) = u_1v_1h(\mathbf{e}) + (u_1v_2 + u_2v_1)h(\mathbf{f}) + u_2v_2h(\mathbf{f})^2$$

$$= \left[u_1v_1 + u_2v_2\left(\frac{1}{4}B^2 + A\right)\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + [u_1v_2 + u_2v_1]\begin{pmatrix} \frac{1}{2}B & b \\ \frac{1}{b}\left(\frac{1}{4}B^2 + A\right) & \frac{1}{2}B \end{pmatrix}$$

$$\Rightarrow h(uv) = u_1v_1h(\mathbf{e}) + (u_1v_2 + u_2v_1)h(\mathbf{f}) + u_2v_2h(\mathbf{f})^2$$

$$\Rightarrow u_2v_2h(\mathbf{f})^2 = u_2v_2\left(\frac{1}{4}B^2 + A\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow h(\mathbf{f})^2 = \left(\frac{1}{4}B^2 + A\right)h(\mathbf{e})$$

$$\Rightarrow h\left(\mathbf{f}^2 + \left(\frac{1}{4}B^2 + A\right)\mathbf{e}\right) = 0$$

$$\Rightarrow \mathbf{f}^2 = \left(\frac{1}{4}B^2 + A\right)\mathbf{e}$$

is an equation the algebra non-identity base vector must satisfy.

(thus, the basis of a two-dimensional vector space unitary algebra is a cyclic group [3] of order 2)

NOTE:

$$B = 0 \quad , \quad A = -1 \quad , \quad b = 1 \Rightarrow h(\mathbf{f}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is isomorphic to the complex number vector space

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -h(\mathbf{e})$$

References

[1] https://en.wikipedia.org/wiki/Algebra_over_a_field

[2] https://en.wikipedia.org/wiki/Algebra_homomorphism

[3] https://en.wikipedia.org/wiki/Cyclic_group

[4] Algebra ; Thomas W. Hungerford ; Springer New York ; 1974
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