The quintic: $z^5 + z^3 + z^2 - 1 = 0$

Edgar Valdebenito

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abstract

The quintic: $p(z) = z^5 + z^3 + z^2 - 1 = 0$, the number pi , and fractals.

1. Introducción.

La ecuación :

$$p(z) = z^5 + z^3 + z^2 - 1 = 0 \tag{1}$$

posee una raíz real y cuatro complejas $\left(i = \sqrt{-1} \right)$:

$$r = 0.69973702 \ldots \tag{2}$$

$$s = 0.38012261 \ldots + i\, 1.29055465 \ldots \tag{3}$$

$$s^* = 0.38012261 \ldots - i\, 1.29055465 \ldots \tag{4}$$

$$t = -0.72999112 \ldots + i\, 0.50662103 \ldots \tag{5}$$

$$t^* = -0.72999112 \ldots - i\, 0.50662103 \ldots \tag{6}$$

poniendo : $u + i\, v = p(x + i\, y)$, se tiene :

$$u = \mathrm{Re}(p(x + i\, y)) = -1 + x^2 + x^3 + x^5 - y^2 - 3\, x\, y^2 - 10\, x^3\, y^2 + 5\, x\, y^4 \tag{7}$$

$$v = \mathrm{Im}(p(x + i\, y)) = 2\, x\, y + 3\, x^2\, y + 5\, x^4\, y - y^3 - 10\, x^2\, y^3 + y^5 \tag{8}$$

sea x_n , $n \in \mathbb{N}$, la sucesión definida por :

$$x_{n+1} = \frac{1 + x_n^2 + 2\, x_n^3 + 4\, x_n^5}{2\, x_n + 3\, x_n^2 + 5\, x_n^4} \ , \ x_1 =. .. \tag{9}$$

se tiene :

$$x_1 = 1/2 \Longrightarrow x_n \longrightarrow r \tag{10}$$

$$x_1 = \frac{1}{2} + i \Longrightarrow x_n \longrightarrow s \tag{11}$$

$$x_1 = \frac{1}{2} - i \Longrightarrow x_n \longrightarrow s^* \tag{12}$$

$$x_1 = -\frac{1}{2} + \frac{i}{2} \Longrightarrow x_n \longrightarrow t \tag{13}$$

$$x_1 = -\frac{1}{2} - \frac{i}{2} \Longrightarrow x_n \longrightarrow t^* \tag{14}$$

otras recurrencias para r , s , s^* , t , t^* :

$$x_{n+1} = \frac{1 + 4\,x_n - x_n^2 - x_n^3 - x_n^5}{4} \ , \ x_1 = 0 \Longrightarrow x_n \longrightarrow r \tag{15}$$

$$x_{n+1} = \frac{1 + (3 - 9\,i)\,x_n - x_n^2 - x_n^3 - x_n^5}{3 - 9\,i} \ , \ x_1 = 0 \Longrightarrow x_n \longrightarrow s \tag{16}$$

$$x_{n+1} = \frac{1 + (3 + 9\,i)\,x_n - x_n^2 - x_n^3 - x_n^5}{3 + 9\,i} \ , \ x_1 = 0 \Longrightarrow x_n \longrightarrow s^* \tag{17}$$

$$x_{n+1} = \frac{x_n^5 + x_n^3 + x_n^2 + (3 + 3\,i)\,x_n - 1}{3 + 3\,i} \ , \ x_1 = 0 \Longrightarrow x_n \longrightarrow t \tag{18}$$

$$x_{n+1} = \frac{x_n^5 + x_n^3 + x_n^2 + (3 - 3\,i)\,x_n - 1}{3 - 3\,i} \ , \ x_1 = 0 \Longrightarrow x_n \longrightarrow t^* \tag{19}$$

fórmulas para pi :

$$\pi = 4\tan^{-1}\left(r^2\right) + 4\tan^{-1}\left(r^3\right) \tag{20}$$

$$\pi = -4\tan^{-1}\left(1/s^2\right) - 4\tan^{-1}\left(1/s^3\right) \tag{21}$$

$$\pi = 4\tan^{-1}\left(t^2\right) + 4\tan^{-1}\left(t^3\right) \tag{22}$$

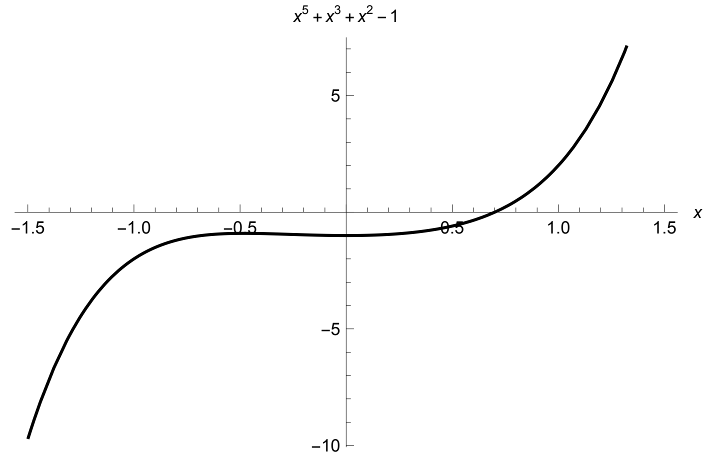
$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1}\,r^{n+1} = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1}\,t^{n+1} = -4\sum_{n=0}^{\infty}\frac{c_n}{n+1}\,(1/s)^{n+1} \tag{23}$$

donde

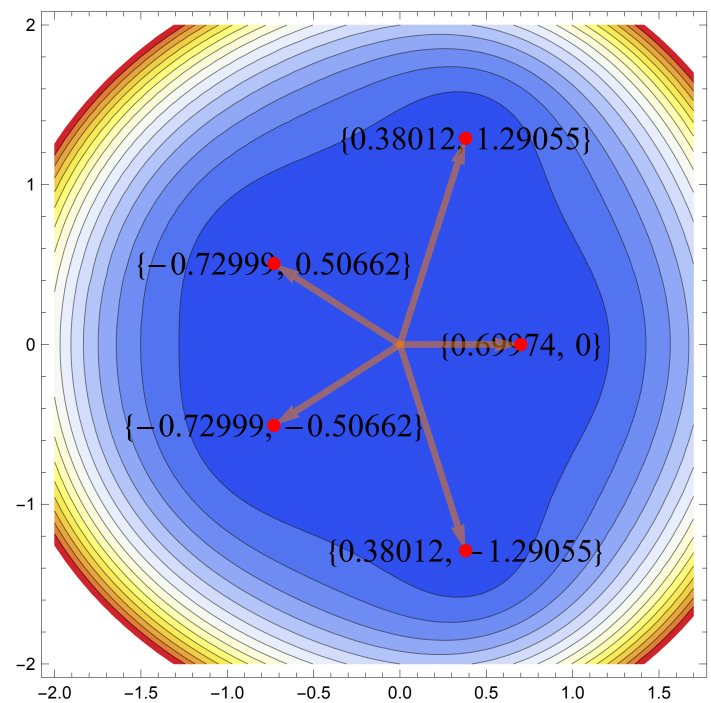
$$c_{n+10} = -(c_{n+6} + c_{n+4} + c_n) \tag{24}$$

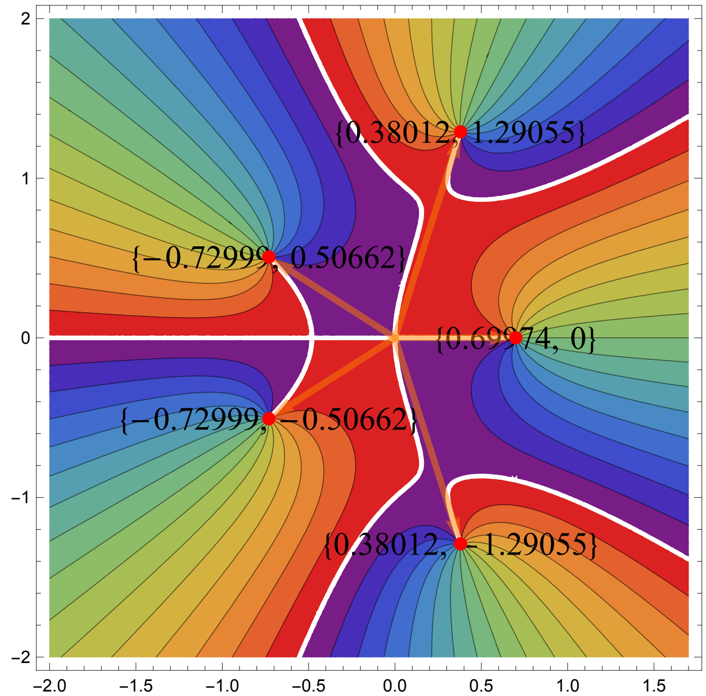
$$c_0 = 0\ ,\ c_1 = 2\ ,\ c_2 = 3\ ,\ c_3 = 0\ ,\ c_4 = 0\ ,\ c_5 = -2\ ,\ c_6 = 0\ ,\ c_7 = 0\ ,\ c_8 = -3\ ,\ c_9 = 2 \tag{25}$$

2. Gráfico de p(x).

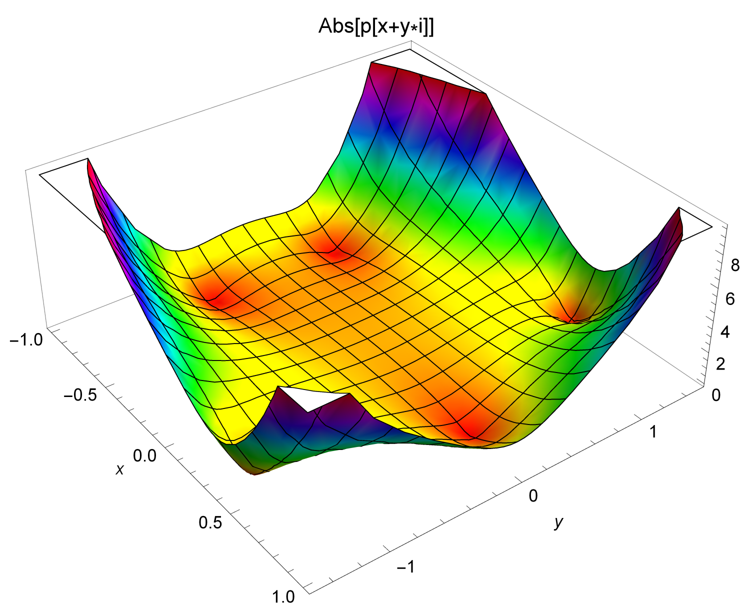


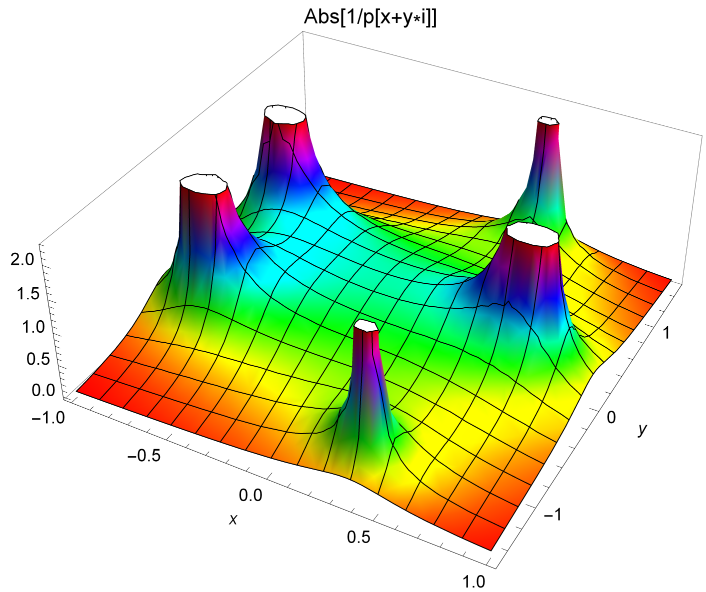
3. Gráfico de raíces complejas.





4. Gráficos para p(x).

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1.0
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x
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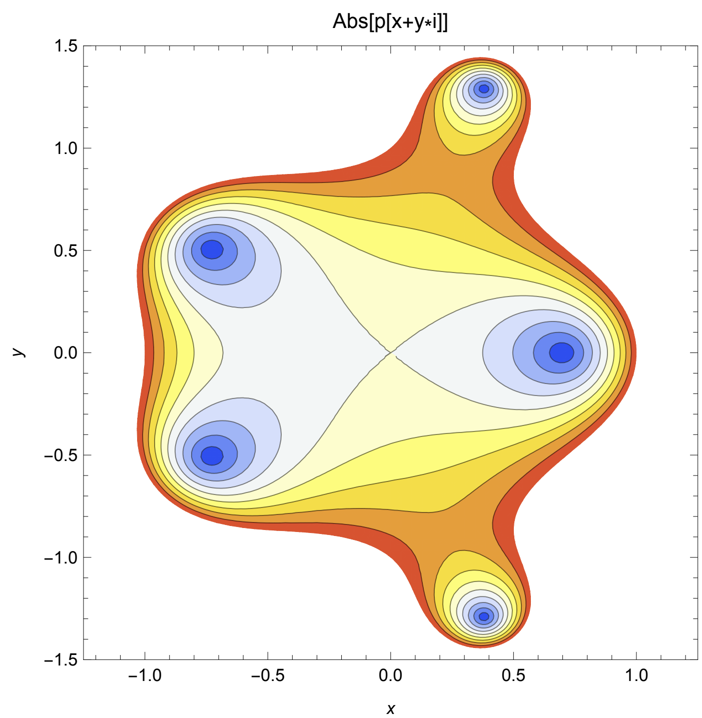
1.0
1
0
y
−1


Abs[p[x+y*i]]
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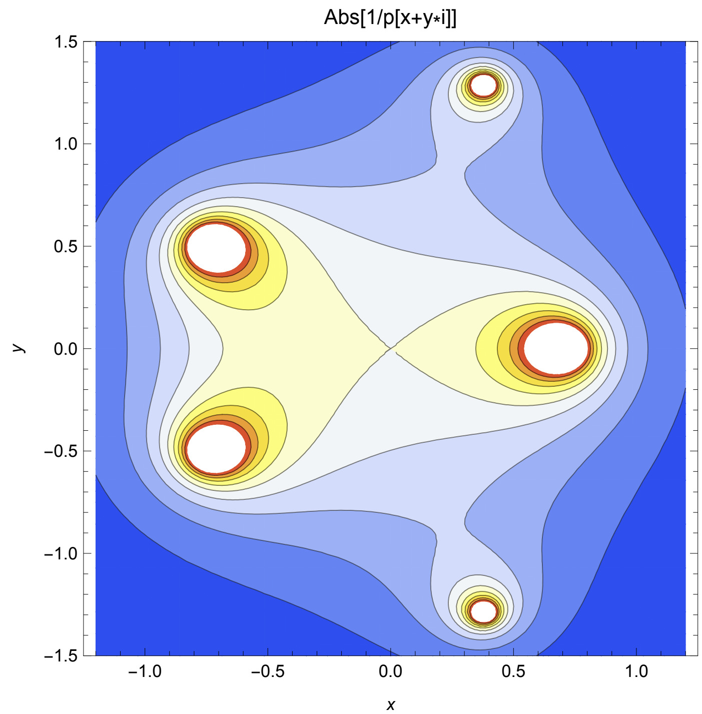
0.0
0.5
1.0
x


Abs[1/p[x+y*i]]
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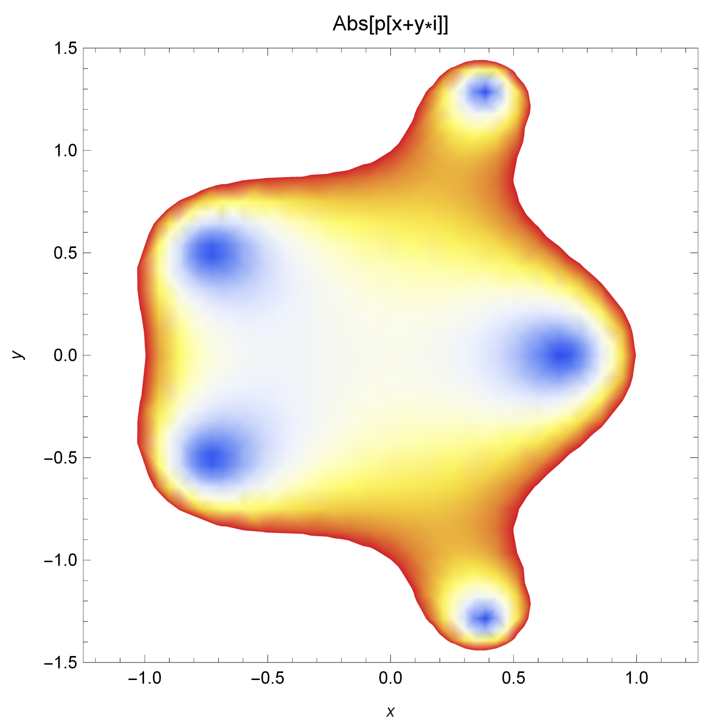
−0.5
0.0
0.5
1.0
x


Abs[p[x+y*i]]
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y
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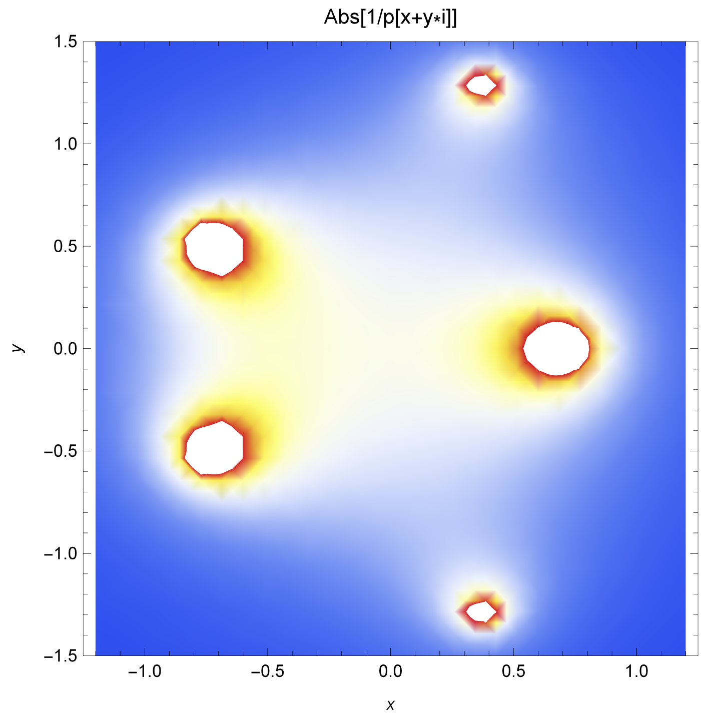
0.0
0.5
1.0
x


Abs[1/p[x+y∗i]]
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y
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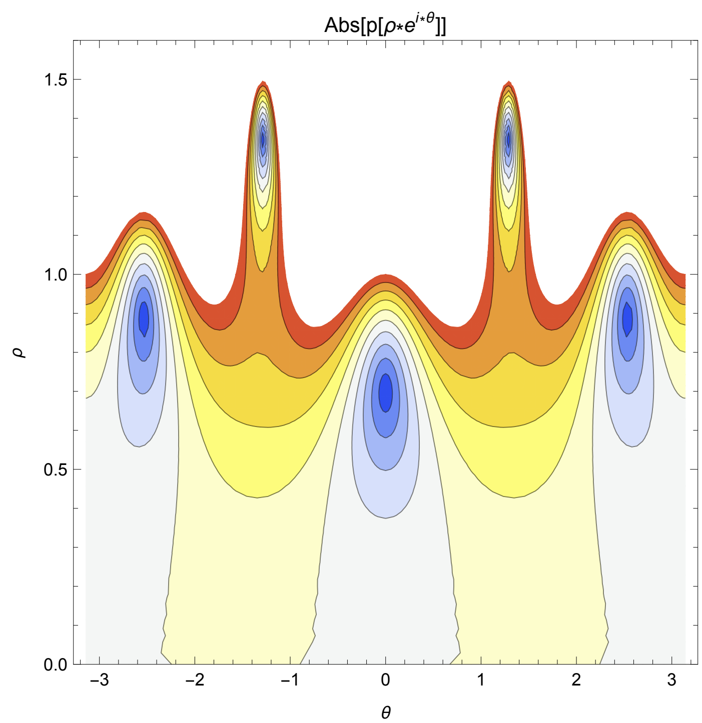
0.0
0.5
1.0
x


Abs[p[ρ∗e^{i∗θ}]]
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0.5
0.0
ρ
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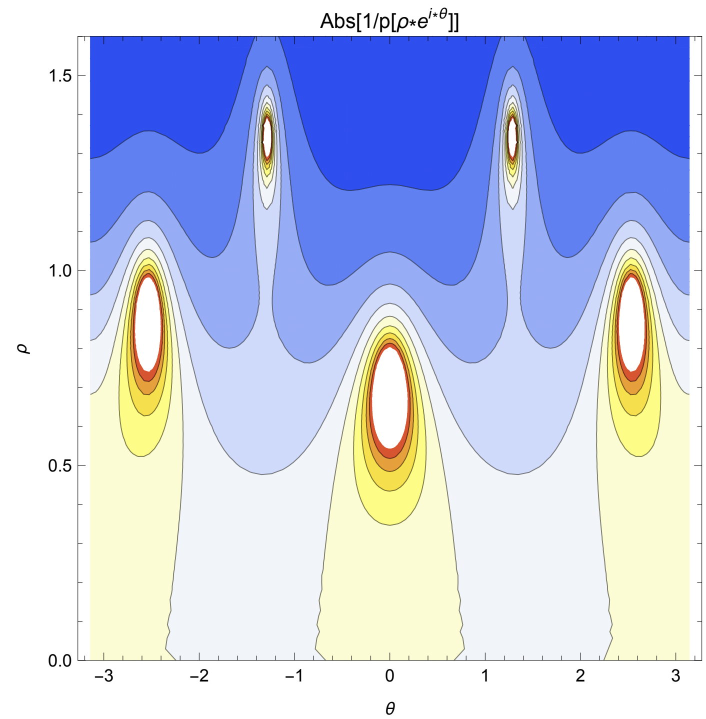
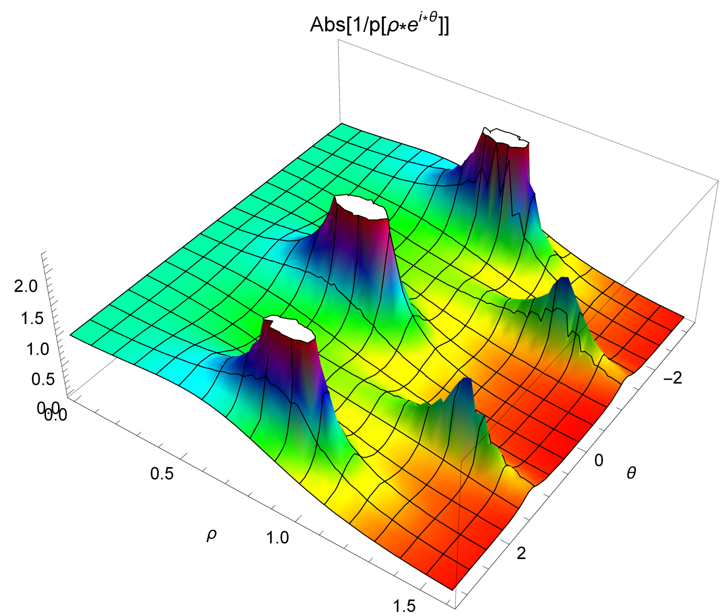
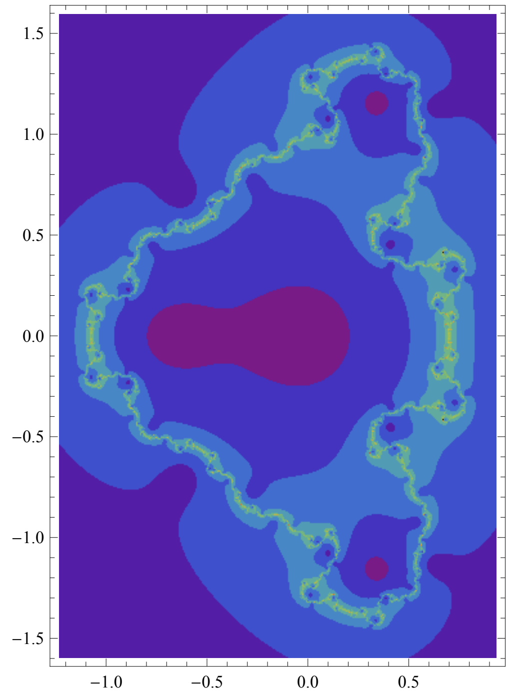
−2
−1
0
1
2
3
θ





5. Fractales relacionados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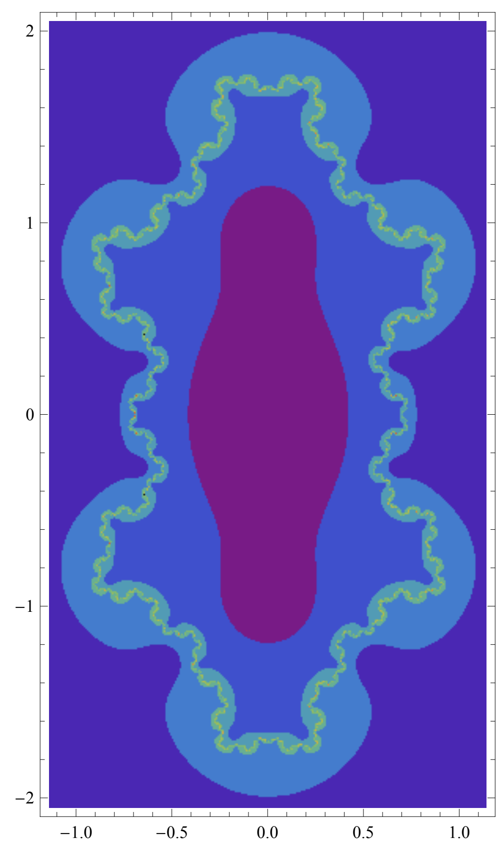
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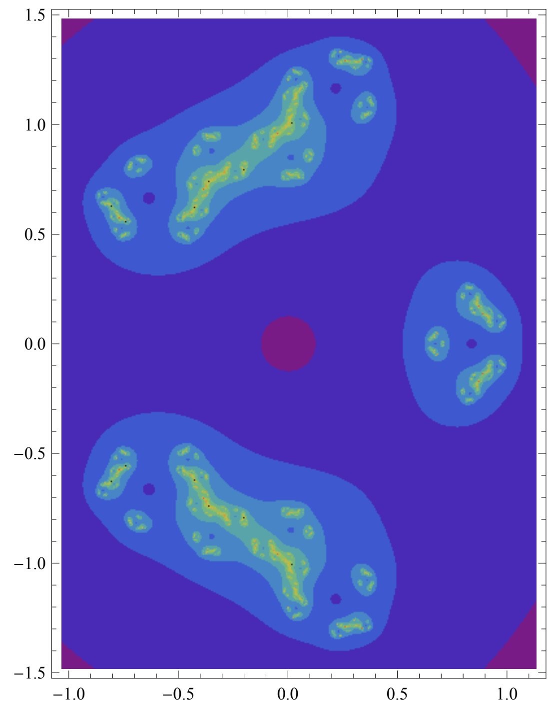
1.0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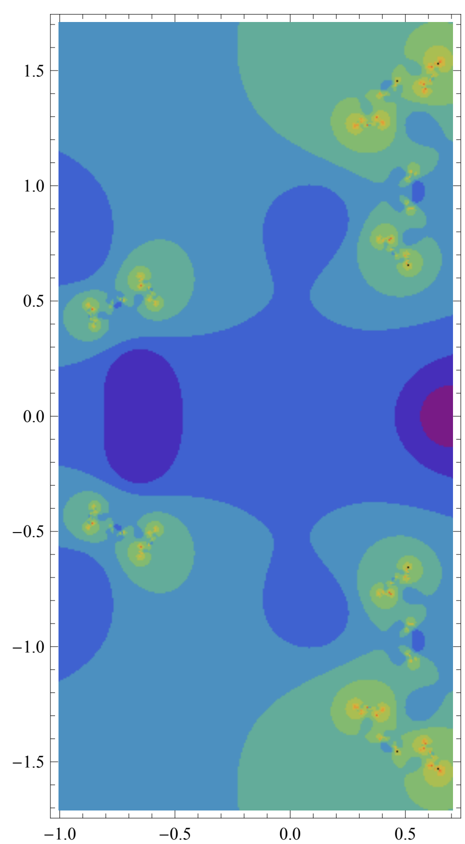
0.5
1.0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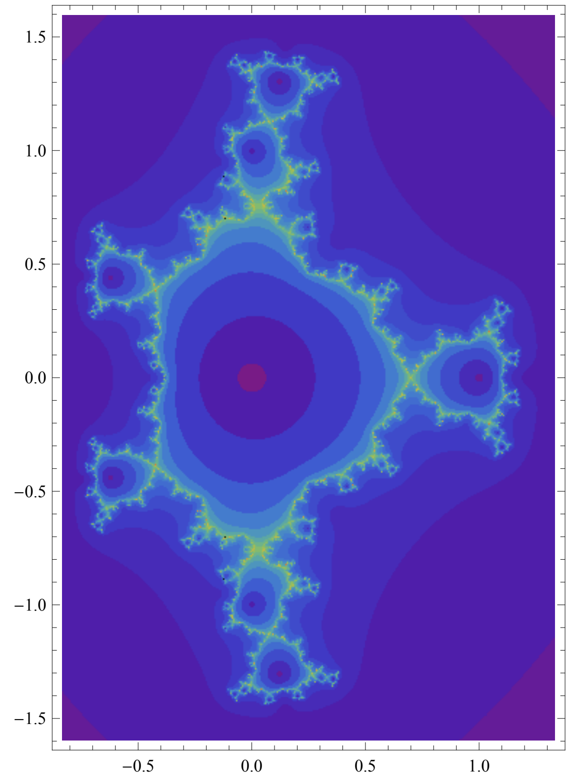
0.5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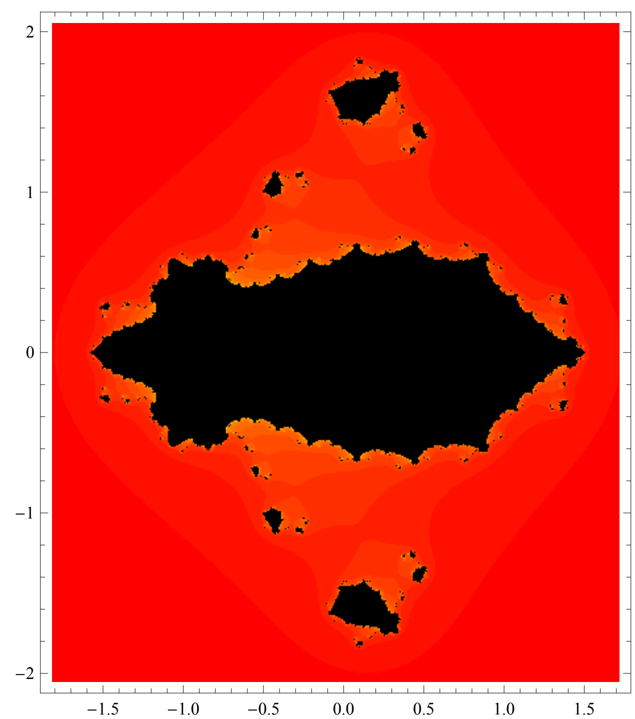
1.0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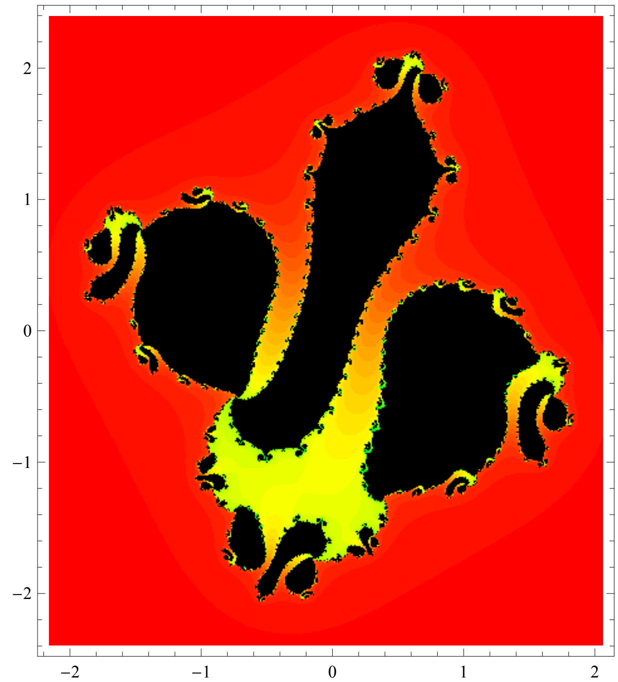
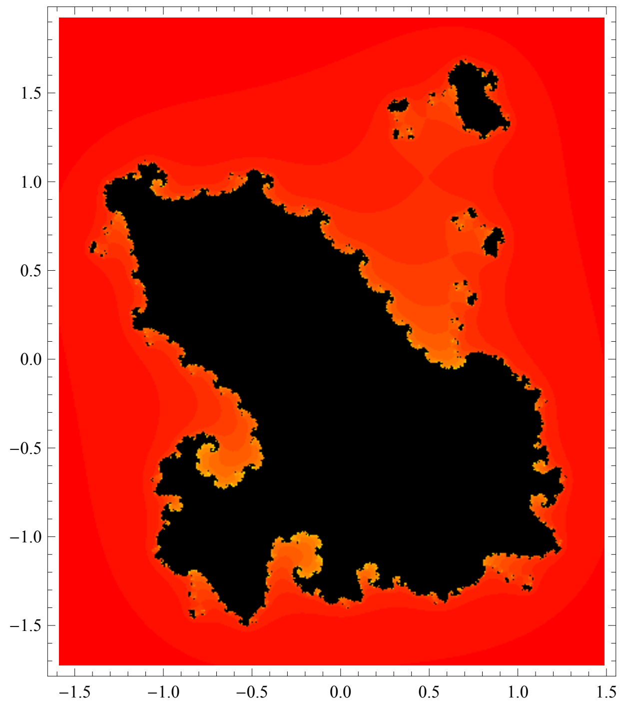
1.5





Referencia

A. Valdebenito , E.: The quintic $z^5 + z^3 + z^2 - 1 = 0$, and the pi number, unpublished note , 2016.