Accounting for the Use of Different Length Scale Factors in x, y and z Directions

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Abstract: This short article presents a mathematical formula required for metric corrections in image extraction and processing when using different length scale factors in three-dimensional space which is normally encountered in cryomicrotome image construction techniques.

Keywords: image extraction; correction formula; cryomicrotome imaging.

In many scientific and industrial situations, the coordinates space is scaled by different length factors in the three spatial directions, x, y and z; which affect the metric relations. For instance, in the cryomicrotomic image extraction techniques, the thickness of slices may be subject to errors or variations making the voxel size in one direction larger or smaller than its standard size in the two other directions. Consequently, the geometric parameters obtained from these images, which are based on the standard units of image space of an assumed cubic voxel unit, will be contaminated with errors causing a distortion because of the missing scale factors required by the isotropy of the physical space.

In the following we present a simple case based on real-life cryomicrotomic image construction algorithms in biomedical applications where vasculature trees are obtained by computing the radius of each vessel in a number of rotational steps through a whole circle and the results are then averaged to obtain the final radius [1, 2]. As these rotational steps are oriented differently in the 3D space, the contribution of the length scale factors will vary from one orientation to the other and hence a scaling correction is required to obtain the correct radius.

There are several possible ways for deriving a formula for this correction; these include the use of rotational matrices, and circle projection on the three standard planes (i.e. xy, yz and zx) to obtain three ellipses from which the three spatial components can be computed. However, an easier and more efficient way is to find a parameterized form of an intersection circle between a plane perpendicular to the vessel axis and the vessel itself, which in essence is equivalent to a great circle intersection of this plane and a sphere having the same radius as the vessel [3]. This method of derivation is outlined below.

For a regular cylindrical straight vessel oriented arbitrarily in 3D space and defined by its two end points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ and radius r, a free vector oriented in its axial direction is given by

$$\boldsymbol{a} = (a_x, a_y, a_z) = (x_1 - x_2, y_1 - y_2, z_1 - z_2) \tag{1}$$

while a plane perpendicular to this vector and (for simplicity with no loss of generality) passing through the origin is given by

$$a_x x + a_y y + a_z z = 0 \tag{2}$$

Now, in 3D space a parameterized circle of radius r centered (with no loss of generality) at the origin and lying in a plane identified by two orthonormal vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by the equation

$$r\left[\cos(t)\boldsymbol{b} + \sin(t)\boldsymbol{c}\right] \qquad 0 \le t < 2\pi \tag{3}$$

that is

$$(r\left[\cos(t)b_x+\sin(t)c_x\right], r\left[\cos(t)b_y+\sin(t)c_y\right], r\left[\cos(t)b_z+\sin(t)c_z\right]) \qquad 0 \le t < 2\pi \tag{4}$$

To find $\boldsymbol{b}$ and $\boldsymbol{c}$, a formal orthogonalization process, such as Gram–Schmidt, with normalization can be followed where random vectors non-collinear to $\boldsymbol{a}$ can be used. However a more convenient way is to find an arbitrary non-trivial vector lying in the plane by inserting arbitrary values for two variables (e.g. $x = 1$ and $y = 1$) in the plane equation and solving for the other variable (z) followed by normalizing through the division by its norm. If this vector is considered $\boldsymbol{b}$, then vector $\boldsymbol{c}$ is found by taking the cross product $\boldsymbol{a} \times \boldsymbol{b}$ and normalizing.

If the following length scale factors: α, β and γ are introduced on the x, y and z directions respectively, then the distorted radius, r', at a random orientation $t = \theta$ is given by

$$r' = r\sqrt{\left(\alpha\left[\cos(\theta)b_x+\sin(\theta)c_x\right]\right)^2+\left(\beta\left[\cos(\theta)b_y+\sin(\theta)c_y\right]\right)^2+\left(\gamma\left[\cos(\theta)b_z+\sin(\theta)c_z\right]\right)^2} \tag{5}$$

and hence the actual radius, r, is given by

$$r = \frac{r'}{\sqrt{\left(\alpha\left[\cos(\theta)b_x+\sin(\theta)c_x\right]\right)^2+\left(\beta\left[\cos(\theta)b_y+\sin(\theta)c_y\right]\right)^2+\left(\gamma\left[\cos(\theta)b_z+\sin(\theta)c_z\right]\right)^2}} \tag{6}$$

As the image construction algorithm computes r' at N rotational steps (e.g. 360 steps corresponding to 360°) and averages the results, to restore the corrected radius r, this correction should be introduced at each one of these steps. In an ideal situation where rotational symmetry holds, only one quarter of these steps, $\frac{N}{4}$, is required, resulting in a substantial computational economy. However due to the measurements and algorithmic errors at each step, it may be safer to maintain the N steps as the errors are expected to level out or diminish by applying this process through the whole circle.

This correction can also be extended to use for post processing correction by applying the correction on the final averaged radius following a correction-free extraction process. For N rotational steps we have

$$\sum_i^N r'_i = r \sum_i^N \sqrt{(\alpha\left[\cos(\theta_i) b_x + \sin(\theta_i) c_x\right])^2 + (\beta\left[\cos(\theta_i) b_y + \sin(\theta_i) c_y\right])^2 + (\gamma\left[\cos(\theta_i) b_z + \sin(\theta_i) c_z\right])^2} \tag{7}$$

Since the averaged post processing radius is

$$R_{av} = \frac{\sum_i^N r'_i}{N} \tag{8}$$

the actual radius is then given by

$$r = \frac{N R_{av}}{\sum_i^N \sqrt{(\alpha\left[\cos(\theta_i) b_x + \sin(\theta_i) c_x\right])^2 + (\beta\left[\cos(\theta_i) b_y + \sin(\theta_i) c_y\right])^2 + (\gamma\left[\cos(\theta_i) b_z + \sin(\theta_i) c_z\right])^2}} \tag{9}$$

Although post processing correction may not result in computational efficiency, it may be more convenient and useful to use when the non-corrected data are already obtained with no requirement to repeat the extraction process.

It should be remarked that this correction can be applied in general to correct for this type of distortion regardless of the number of steps (single or multiple) and the shape of the object as long as the x, y and z components of the position vector can be obtained for each point in space required to trace the path of the distorted shape. This process can also be extended from discrete to continuous by substituting the summations with integrations with some other minor modifications to account for this correction in analytical contexts rather than numerical discrete processes.

References

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