Fluid Flow at Branching Junctions

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Abstract

The flow of fluids at branching junctions plays important kinematic and dynamic roles in most biological and industrial flow systems. The present paper highlights some key issues related to the flow of fluids at these junctions with special emphasis on the biological flow networks particularly blood transportation vasculature.

Keywords: fluid dynamics; branch flow; blood flow; bifurcation; trifurcation; branching junction; branching radius; branching angle; Murray's law.

1 Introduction

The flow at branching junctions is a common feature of most fluid dynamics systems both in the natural and synthetic worlds. In particular, it is one of the distinctive features of the biological flow networks such as blood circulation and air respiration systems. There are many studies related to the various aspects of branching flow; the majority of these studies come from the biological and biomedical literature particularly blood flow. The obvious reason is the vitality of this field and the importance of the role that branching flow plays in biological systems both in normal and pathological conditions. In fact branching flow is at the foundation of most biological systems where its importance and common occurrence can hardly be matched by any non-biological system.

It is customary to classify the flow ducts connected to the branching junctions using the labels 'parents' and 'daughters' or other synonymous words. However the definition of parent and daughter may be rather artificial, especially for highly symmetric branching trees or chaotic flow systems, and can be based on a geometric criterion such as the size of the vessels, where the parent is identified as the bigger in size, or based on a flow dynamics criterion such as the flow direction where the parent is identified as the source of flow that injects fluid into the junction. One or both of these labeling criteria may not be applicable in some circumstances.

For example, the flow dynamics criterion for defining the parent and daughter will be the only available one for the wholly symmetric branching where all branches are identical in size and shape with a constant angle in between. In fact even this dynamically-based definition will not be applicable when the flow direction is time-dependent or chaotic or ambiguous. The arbitrariness of these labels may be highlighted by comparing the arterial part to the venous part of the blood vasculature where the role of the parent and daughter in one of these parts according to one criterion becomes the opposite in the other part.

For the fractal-type networks and the networks with a high degree of regularity and hence have a strong similarity with the fractal networks, the geometric designation of parent and daughter is reasonably clear and natural, but this becomes arbitrary in the highly irregular networks. Anyway, there are no natural physical principles associated with these labels as such and hence the labeling can be flexible and dependent on the context and convenience although the size criterion seems to be more suitable in most cases. To avoid ambiguity and unnecessary phrasing complexities, the parent and daughter in this paper will follow the commonly-used labeling which is generally based on a branching configuration with a large parent and small daughters where the flow is in a diverging (i.e. parent to daughters) rather than merging (i.e. daughters to parent) state.

Various types of branching occur in biological and non-biological flow networks. These types include one-to-two branching (bifurcation) and one-to-many branching (e.g. trifurcation, quadfurcation and so on). Branching types should also be extended technically to include more than one parent and less than two daughters although some of these types cannot be strictly called ‘branching’ considering the primary meaning of the word although they have all the fluid dynamics features of branching flow. Branching types could also be extended to include one-to-one ‘branching’, which occurs through an abrupt change in the shape or size (expansion or contraction) of the flow duct, since this extension is sensible and useful in some circumstances. For example, this extension enables flexible modeling of discontinuous transitions between two neighboring flow ducts with different shape and/or size of their cross sectional area [1]. A bifurcation-type flow may occur at such one-to-one transition junctions, such as that of the traditional benchmark problem of the 1:4 expansion, where symmetric and asymmetric branching flow patterns occur in various flow regimes some of which are notably observed with non-Newtonian fluids [2–4].

Most branching flow studies are dedicated to the bifurcation branching in the diverging flow state. Although the bifurcation geometry is the most common type of branching in the fluid systems, especially in the living organisms, the branching flow in the merging state is as common as the diverging state, e.g. the venous flow opposite to the arterial flow in the blood vasculature, and exhalation versus inhalation in the lung airways. The geometric analogy in these two branching flow types does not imply an analogy in the flow patterns as these two flow types can be very different.

The branching could also be symmetric or asymmetric with respect to the branching angle or radius or both. A proposed metric for quantifying the degree of branching asymmetry in the bifurcation case with regard to the radius size may be defined as the ratio of the radius of the small daughter to the radius of the large daughter [5]. This ‘asymmetry index’ which approaches zero for highly asymmetric bifurcation and takes the value unity for the symmetric bifurcation is yet to be matched by a similar asymmetry measure with regard to the branching angle. This can be simply done by replacing the radius with the angle in the previous definition, that is the asymmetry index with regard to the bifurcation angle is the ratio of the small to the large angles of the two daughters with respect to the extended axis of the parent. Extensions are also required for the symmetry and asymmetry of branching with regard to the radius and angle for non-bifurcation branching types.

In the biological flow systems, most branching is either symmetric or quasi-symmetric with regard to the daughter radius and branching angle; or at least it does not deviate from symmetry excessively except in exceptional cases [5]. In fact considerable parts of the biological flow networks, such as blood vasculature and lung wind pipes, closely match fractal-type networks [6, 7]. This may be more evident in the lung wind pipes than in the blood vasculature due to the high symmetry of the lung compared to other organs and porous tissue. In fact radius branching rules, as exemplified by the Murray-type laws, as given by the forthcoming Equation 3 for a parametric branching exponent, are generally based on the fractal nature of the branching trees. Fractal-type fluid transportation branching networks, which are very common in nature, may be favored and naturally selected for their property of minimizing energy consumption in comparison to other branching patterns [8, 9]. The branching can also vary in shape such as T-junction, Y-junction, cross junction, and star junction for regular branching as well as many other shapes for irregular branching. Extensive studies about the branching flow related to most of these branching shapes can be found in the literature (e.g. [10–25]).

It is obvious that branching networks are not limited to those with cylindrically shaped tubes and hence they include other flow duct geometries, such as ducts with square cross sectional shape [26], and even free surface open channel networks [21]. They should also include mixed branching shapes where some branches are circular in shape while others are triangular or square for instance. However, the analysis of those types of branching is usually more difficult. Branching flow studies in the literature are mostly dedicated to the common type of branching, that is confined flow networks with cylindrically shaped tubes over the whole network, due to their common occurrence, especially in the biological flow systems, and relative ease of modeling and analysis compared to other branching types.

Although the scope of this paper is restricted to the flow of fluids, the branching flow rules can be extended to other types of flow, such as the electric current in the electric power networks and electronic circuits, as there are many similarities between the two flows. In fact most of the established branching flow laws in these two types of network have the same form and hence they are mathematically equivalent. For example Poiseuille equation is mathematically identical in form to the Ohm's law, while the Kirchhoff's current law is identical to the continuity equation [27, 28]. This is reflected by the exploitation of the analogy between fluid flow and electric current in many fluid mechanical studies through the use of electric models to describe and simulate the fluid systems as typified by the common use of the Windkessel model in the hemodynamic investigations or the use of Kirchhoff's laws in the fluid dynamics simulations.

With regard to the blood circulation system, particularly in large mammals, which is one of the main investigation fields of branching flow and is one of the principal subjects of the present paper, the flow of blood in large vessels is essentially laminar with possible superposition of minor secondary flows at branching, bending and curving zones [29]. In small vessels and capillaries the flow generally slows down and hence it steadily approaches a creeping condition in the direction of branching of the vascular tree, i.e. in the flow direction in the arterial system and opposite to the flow in the venous system. In fact this is a consequence of the increase in the cross sectional area at the transition from parent to daughters. The area increase is one of the implications of Murray-type laws with the exponent being greater than 2 [30, 31], as will be discussed in the forthcoming sections. However, a Murray-type law is a sufficient condition for the area increase but not a necessary one. A consequence of this area increase and subsequent flow slowing down is that non-Newtonian effects associated with low and medium deformation rates become increasingly important in the direction of branching [32, 33].

2 Branch Flow Modeling

The flow at the branching zones is normally assumed incompressible with no tangential or normal fluid velocity at the vessel wall and with a Poiseuille-type parabolic flow profile for Newtonian fluids [10, 34–39]. With regard to the blood flow, the latter condition can only be justified for the large and medium size vessels because non-Newtonian effects become significant in the small vessels and porous tissue resulting in a more flattened velocity profile due to the shear-thinning nature of blood [24, 40–42].

Simplifying conditions such as fully developed flow [43] at the entrance of the outflow vessels (daughters in diverging flow and parent in merging flow) may also be assumed although this is not strictly valid in most cases. Less severe edge effects are also expected to occur at the exit of inflow vessels (parent in diverging flow and daughters in merging flow). The boundary layer at the branching zone is found to be thinner than that at the vessel wall in a fully developed flow [44].

The fully developed flow assumption could suffer further violation when complex flow patterns, such as vortices, are induced near the junction with possible propagation by viscous diffusion; moreover it becomes less realistic for short tubes as the edge effects become more significant in such tubes. The assumption therefore can be justified for long tubes when the flow is laminar at relatively low Reynolds numbers as the flow settles to its fully developed state over a short distance [45]. A Forchheimer correction term may be added to the flow equation to account for the deviation from the parabolic profile [46]. With regard to the short tubes, the fractal nature of the biological flow networks, as typified by the circulatory and respiratory systems, observes a persistent proportionality between the length and radius of the flow ducts making this an exception.

Other assumptions used to simplify the branching flow analysis in the biological flow networks include the shape of the vessels as being cylindrical and the type of the flow as being laminar uniaxial with no energy losses. Deviation from most of these assumptions are more grave at the vessels periphery in the immediate neighborhood of the branching junctions.

In most fluid dynamics flow models, which include Poiseuille and one-dimensional Navier-Stokes, the branching junctions are assumed, explicitly or implicitly, as connecting geometric points with no volume to store fluid or entail additional pressure loss [47–50]. This assumption simplifies the analytical and numerical treatment of the flow in the network and splits it into a flow in tubes associated with tuning conditions at the branching points. However, this is a gross approximation that discards many flow features associated with the flow patterns at the branching zones which significantly contribute to the flow in the network and define its overall behavior [51].

The Poiseuille and one-dimensional models in their basic forms are indifferent to the branching angle, and this is one of their major limitations, although angle-dependent dissipation effects have been included in some one-dimensional models to account for this deficiency [28]. In fact these models lack most geometric features, such as directionality, that affect the flow in real three-dimensional networks [1]. This may be acceptable for the flow at low Reynolds numbers where the process is essentially viscous, but it becomes less accurate at high Reynolds numbers where inertial effects become increasingly important. However, in most cases of biological flow this approximation is not far from reality and hence is generally acceptable. It should be remarked that in this context, Poiseuille flow network model should be extended to include Poiseuille-like non-Newtonian flow network models which are based on extending Poiseuille flow network formulation to include non-Newtonian viscous rheology for generalized Newtonian fluids [52–58].

Network flow models should provide coupling conditions at the branching junctions for the flow to be consistent. The purpose of these conditions is to force the flow in the individual ducts, which is subject to the assumed tube flow model, to comply with the flow state in the network as a whole according to certain coordination rules. The nature of the coupling conditions may be related to the flow model used to describe the flow in the individual tubes that comprise the network, although in most cases it is derived from general conservation principles such as mass and energy conservation. The number of the required coupling conditions depends on the number of variables used to describe the flow at the branching junctions.

More specifically, there are two prominent and widely used prototypes in the network flow investigations, and hemodynamic studies in particular, for modeling and simulating the flow in the fluid transportation networks: the Poiseuille model for the flow in rigid tubes, and the one-dimensional Navier-Stokes model for the flow in distensible tubes [37, 59–62]. These models have different versions with different implementations and flavors; the following description mostly applies to these models as described in references [1, 62]. For the Poiseuille flow network model, each junction has a single pressure variable and hence a single coupling condition, which is normally derived from the continuity of the volumetric flow rate, as given by Equation 1, and is based on the conservation of mass is employed. For the commonly used form of the one-dimensional Navier-Stokes biological flow network model [1, 59, 60, 62], at each branching junction connecting N vessels there are N pressure variables and N flux variables and hence $2N$ coupling constraints are required to match the flow at the branching nodes. These $2N$ constraints are usually provided by N compatibility conditions, derived from Riemann's method of characteristics and arise from projecting the differential equations of the flow model in the direction of the outgoing characteristics [1], and N matching conditions based on the flow continuity, which is derived from the conservation of mass for incompressible flow, and the Bernoulli condition, which is based on the conservation of energy, as summarized in the following relations

$$\sum_{i=1}^{n} q_i = 0 \tag{1}$$

and

$$p_k + \frac{1}{2}\rho v_k^2 - p_l - \frac{1}{2}\rho v_l^2 = 0 \tag{2}$$

In the last two equations, q is the volumetric flow rate which is signed $(+/-)$ according to its direction (toward the junction or away from it), i is a dummy index that runs over all the n tubes connected to the junction, p is the local pressure, ρ is the fluid mass density, v $(= \frac{|q|}{A})$ is the fluid speed averaged over the vessel cross section, and k and l are indices of two distinct branching tubes. More details about the branching coupling conditions related to the Poiseuille and one-dimensional Navier-Stokes models can be found in references [1, 62].

The use of the Bernoulli equation, which is based on the conservation of mechanical energy, is justified by the fact that the energy losses at the junctions are normally negligible [37, 63] although this assumption may not be applicable in some situations with the involvement of complex flow patterns such as the setting of vortices and turbulence at the junctions. The energy losses associated with the flow patterns at the branching junctions depend on several factors such as the Reynolds number and the angle of branching [37]. A compensation term may be added to the Bernoulli equation to account for these losses [63].

The Bernoulli condition, in the form given by Equation 2, is based on the assumption of negligible gravitational body forces at the branching scale relative to the other forces involved in the flow and hence the gravitational term in the general Bernoulli equation is dropped. An even more reduced form of the Bernoulli equation may be used as a matching condition where only the pressure term is maintained. The problem with the latter form is that it is valid only when the velocity term is negligible compared to the pressure term, i.e. in the creeping flow condition, which is not legitimate in most cases related to the biological flows where this reduced form has been employed. A pressure continuity condition, equivalent to the latter Bernoulli condition, has been used for the branch flow coupling in the one-dimensional flow model for networks of compliant vessels [63, 64]. However, the theoretical justification of this condition may not be based on the Bernoulli principle in its more reduced form although they are practically equivalent. The more reduced form of the Bernoulli condition should receive more justification when the total area of the daughters approaches the area of the parent, which is the case for instance in the Murray-type laws with the branching exponent approaching the area-preserving value of 2.

Anyway, there is a general problem in using Bernoulli as a coupling condition in the one-dimensional flow model because Bernoulli equation is based on an inviscid flow assumption which contradicts the viscous assumption that the one-dimensional model relies upon. However, apart from this basically conceptual contradiction the viscous effects at the junction in the network flow are relatively small in most circumstances and hence the use of the inviscid Bernoulli condition can be justified. The fact that the boundary layer at the branching zone is thinner, as indicated earlier, could provide a further justification for the use of the Bernoulli equation.

The Bernoulli equation is based on other simplifying assumptions such as steady, laminar, adiabatic flow in straight, rigid tubes [63]. With regard to the biological systems, and blood distribution networks in particular where the one-dimensional model is commonly used, some of these assumptions are not far from reality as applied in the close proximity of the branching zones. Anyway, most of the simplifications related to the coupling conditions at the branching zones are diluted by the more significant simplifications which are normally employed in the principal flow models that are used to describe the flow in the vessels themselves. For example, Poiseuille equation is based on several simplifying assumptions when applied to the blood circulation, such as laminar steady flow in straight rigid tubes. The errors associated with these simplifications, most of which are similar to the previous coupling simplifications, are normally more important than the errors of the coupling simplifications. Similarly the one-dimensional Navier-Stokes distensible model disregards most of the phenomena associated with the three-dimensional flow such as directionality, vortices, turbulence and flow separation. Moreover, it is normally based on a pure elastic pressure-area constitutive relation instead of the more elaborate and realistic viscoelastic characteristics although elastic models are acceptable for describing the wall behavior in large vessels.

Another potential inconsistency in the one-dimensional flow model at the branching junctions, which also originates from the use of the Bernoulli equation, is that according to some coupling strategies as outlined earlier there is no unique pressure value at each junction, because the pressure at the junctions is vessel dependent and hence there is a number of pressure values matching the number of vessels connected to that junction. This may contradict the assumption of zero-volume junctions and negligible viscous and non-viscous losses at the branching zones as these assumptions are mostly based on negligible branching volume. In general, it may be argued that while some of these assumptions are based on negligible size of the junction space, such as negligible viscous dissipation as implied by Bernoulli, others, like non-unique pressure at the junctions, may only be justified if the junction space is assumed sizeable.

Concerning blood perfusion in porous tissue in the microcirculation system which consists of strongly bifurcating networks, the distensible Darcy model [65–67] is widely used to describe the flow. Because this lumped porous media model lacks the necessary details to account for the branching effects, no branching flow features are considered explicitly or implicitly in this model. However, very generic branching effects may still be present within the porous medium mechanisms through intrinsic lumped parameters like permeability; otherwise the use of these models should only be justified if minimal branching effects are assumed in the flow regimes to which these porous models apply.

It should be remarked that reliable description of the branching flow requires three-dimensional rather than one-dimensional models to account for the flow features that can only be accessed through three-dimensional models. However, this may not be affordable in most circumstances due to practical restrictions on the computational resources, especially when dealing with extensive flow networks, as well as mathematical and numerical difficulties. The one-dimensional models then require improvement to incorporate the essential flow features at the branching junctions to avoid the widely adopted approach in the one-dimensional modeling where these models are centered on the flow in the network ducts while the branching flow conditions are used only to couple the flow in the individual tubes to orchestrate the flow in the network as a whole.

With regard to the branching in the blood vasculature, modeling and analysis of branching flow is generally more difficult in the venous system compared to the arterial system due to the fact that the pressure is low and hence the cross sectional area of the venous vessels could strongly deviate from cylindrical to elliptical shape or even a collapsed state especially in the low pressure phase of the cardiac cycle. Moreover, the presence of valves in the veins introduces more complications on the velocity profile and flow patterns.

3 Branching Radius

A large number of studies are dedicated to the effect and optimal design of the radius branching ratio especially in the biological flow systems. There seems to be a widespread consensus that branching morphology in the biological systems is subject to optimization principles which may be justified by evolutionary morphogenetic arguments based on natural selection [30, 31, 68–73]. Radius branching rules, similar to those derived from the optimization principles, have been observed in various biological flow systems across the animal and plant kingdoms such as lung airways in mammals, air diffusion systems in insects, and sap transport networks in trees [70, 72]. This may originate from their close resemblance to the fractal structures which are widespread in nature. Similar optimization arguments can also be employed to justify the optimal design of branching networks in non-biological flow systems although simpler physical rules can be used in the latter case. For example, biologically inspired arguments, based on Murray's law, have been used to justify the optimal design of artificial microfluidic networks [73, 74].

In fact there have been proposals [69] that Murray's law and the argument on which it is based hold for any branching flow system, living or non-living, that is subject to the flow resistance minimization objective within a specified volume. Some biological arguments cannot be extended automatically to non-biological systems due to the involvement of biologically-specific parameters such as metabolism although similar parameters like manufacturing cost may be applied.

The most prominent biological radius branching model is the Murray's law [30, 31] which is based on an optimization principle related to minimizing the energy consumption of flow systems in living organisms [75]. The roots of Murray's law can be traced back to Thomas Young and other scientists in the 19th century and the early 20th century [9, 69, 76–79]. The essence of Murray argument is that the radius branching morphology in the blood circulation network is subject to an energy optimization principle where two energy consumption factors do compete: the metabolic energy required to maintain the volume of the blood that fills the vessels, and the mechanical energy required to pump the blood throughout the network. The energy consumption of the first factor is directly proportional to the volume and hence to the radius squared, whereas the consumption of the second factor is inversely proportional to the radius fourth power as a consequence of Poiseuille law [69]. While minimizing the energy consumption according to the first factor requires diminishing the size of the blood vessels to reduce the maintained blood volume, the second factor requires increasing the size of the vessels to reduce the flow resistance and hence the energy of pumping. The final radius branching geometry is then determined so that the total energy consumption required by these two factors is minimal. Murray's law has also been explained by geometric arguments, based on the capacity of the living body for controlling blood distribution, without resorting to the energy optimization principles [9].

The pumping cost in the energy minimization argument is based on a purely viscous flow, as implied by the use of Poiseuille condition, and hence it does not account for non-viscous pumping losses. The condition, anyway, is generally accepted in the biological networks where the flow is at relatively low Reynolds numbers. The volume maintenance cost may also include the volume cost of the vessels as well as the pumped fluid [9, 69]. Although Murray's law, and the cost argument on which it is based, is originally derived for the blood flow where the fluid is living and hence has a metabolic cost, it can be extended to the biological flow networks where the fluid is inert with no metabolic cost such as the air in the respiratory system [69]. This extension is justified by the metabolic cost associated with maintaining the living pipe network, as indicated already, even if the extension of Murray's law to non-living flow systems is rejected.

Formally, the Murray's law is given by

$$R_p^g = \sum_i^n R_{d_i}^g \tag{3}$$

where R_p and R_{d_i} are the radius of the parent and the ith daughter vessel respectively, n is the number of daughter vessels which is 2 in most cases in the biological flow networks, and g is the branching exponent which according to Murray is 3, but other values like 2.1-2.2, 7/3, 2.6, 2.3-2.7, 2.75 and 2.0-3.0 are also theoretically derived or experimentally observed as reported in the literature [50, 69, 78, 80–83].

In Figure 1 the difference between the total cross sectional area of the daughter vessels and the cross sectional area of the parent vessel as a function of the branching exponent for symmetric bifurcation, trifurcation and quadfurcation branching is plotted. As seen, $g = 2$ is the break-even area-preserving value where the parent cross sectional area is equal to the sum of the daughters cross sectional area. Above this value, the total area of daughters exceeds the area of parent and this difference increases as g and the number of daughters increase. In Figure 2 the contours of the difference between the total cross sectional area of the daughter vessels and the cross sectional area of the parent vessel as a function of the branching exponent and the radius of one of the daughter vessels for non-symmetric bifurcation is plotted. As seen in these figures, the difference in area, in favor of the daughters total area, increases as the branching exponent increases for all the symmetric and asymmetric cases.

One of the important implications of the Murray-type laws with the branching exponent being greater than 2 is that the total cross sectional area increases in going from one parent generation of vessels to the next daughter generation at the branching junctions. A consequence of this increase in the total cross sectional area is that the incompressible blood flow will slow down in the direction from large to small vessels in the vascular network, i.e. in the flow direction in the flow diverging networks (e.g. arterial) and opposite to this direction in the flow merging networks (e.g. venous). This has a direct impact on several phenomena that depend on the flow speed and rate of deformation such as pressure and non-Newtonian rheology. A consequence of this on the blood circulation, for example, is the steady increase of the significance of shear-dependent non-Newtonian effects, which are associated with low and medium deformation rate regimes, in the branching direction [33].

It should be remarked that the derivation of Murray's law is based, explicitly or implicitly, on several simplifying assumptions which include steady, laminar, Newtonian flow with a parabolic flow profile in straight, rigid, cylindrically-shaped vessels with constant wall shear stress and possibly identical daughters [30, 50, 69, 84, 85]. The tubes in the biological flow systems, where these models are employed, are generally curved, tapered and distensible; moreover they are not perfectly circular in shape. The deviation from being circular is aggravated in the blood vessels by the pulsatility of pressure field, especially during the diastolic phase and in the venous system, where the vessels do not only deviate significantly from being circular but can even collapse in part of the cardiac cycle. However, most of

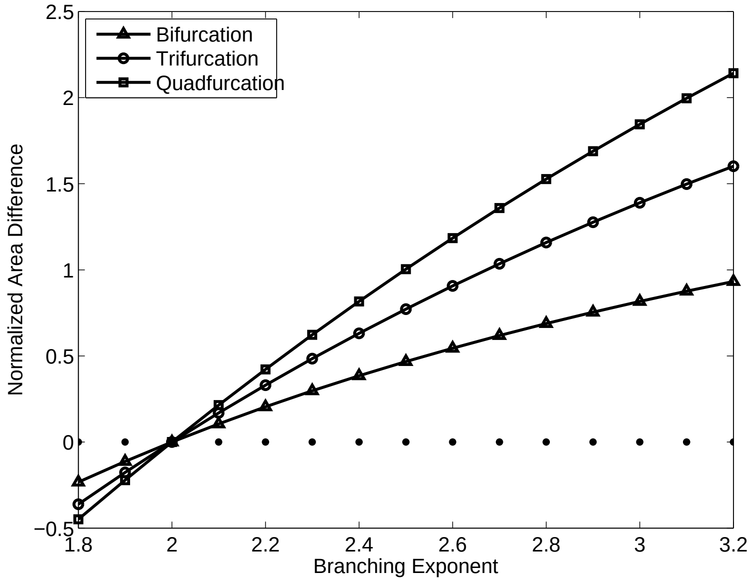

Figure 1: Difference between the total cross sectional area of the daughter vessels and the cross sectional area of the parent vessel as a function of the branching exponent for symmetric bifurcation, trifurcation and quadfurcation branching according to the Murray-type laws where the parent and daughter radii are normalized to the parent radius.

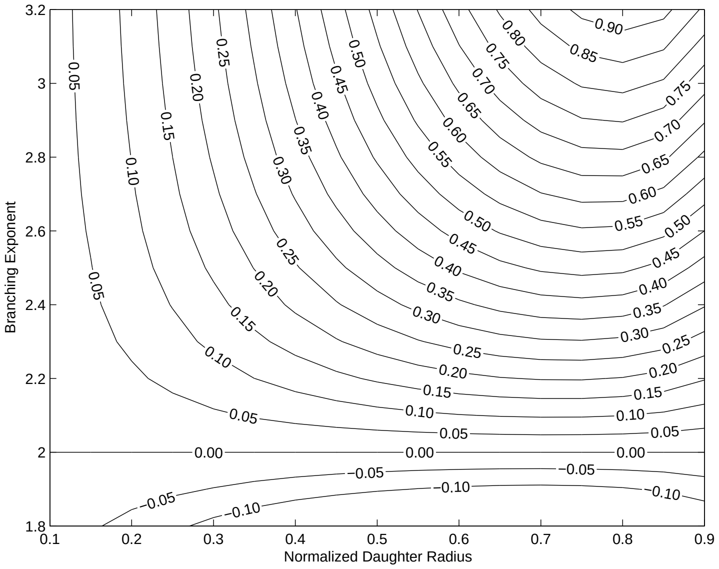

Figure 2: Contours of the difference between the total cross sectional area of the daughter vessels and the cross sectional area of the parent vessel as a function of the branching exponent and the radius of one of the daughter vessels for non-symmetric bifurcation branching according to the Murray-type laws where all the radii are normalized to the radius of parent. The radius of the other vessel is computed from the Murray equation 3.

these assumptions are widely accepted in the hemodynamic studies although some of which, such as rigidity, are not as good approximation as others like laminar; moreover the validity of some of these assumptions are questionable [69, 78, 81].

There are attempts to incorporate complex blood rheology, which includes radius dependent viscosity and hematocrit level effects, in the Murray's formulation [84]. Extensions of Murray's law to flow networks with non-cylindrical ducts have been proposed for the design of synthetic flow systems [73, 74]. Murray's law has also been extended to include steady-state turbulent flow with the branching exponent taking the value 7/3 [69, 80]. Other efforts in this context include the generalization of Murray's law to include non-Newtonian fluids of power law type [86, 87] and the extension to non-circular tubes with elliptical cross section [87].

A possible problem with the Murray-type laws, which is related to the assumption of rigidity of the branching vessels, is that due to the pulsatility of the pressure field and the flexibility of the biological vessels, the size of the vessels usually varies during the flow cycle (e.g. cardiac or respiratory cycle). Because this variation in general could be out of proportion to the reference pressure magnitude by scaling the branching vessels up or down by the same factor, the radius ratio could change throughout the cycle rendering the optimization principle, or any other principle on which the derivation is based, invalid. However, due to the close geometric and material similarities between the parent and daughter vessels the scaling should not be far from proportionality throughout the whole flow cycle. The effects of propagating pressure waves that distorts the radius ratio in selective areas and hence affecting the validity of Murray-type laws should be minimal.

The robustness of Murray-type laws may also be undermined in the small capillaries due to the sudden change in the blood viscosity at the junction transition caused by the Fåhræus-Lindqvist effect and plasma skimming which results in a lower effective viscosity in the smaller daughter vessels although this seems to have minor effect on the validity of the Murray-type laws in general [69]. Anyway, this only applies to a limited generation of branching vessels where Fåhræus-Lindqvist effect takes place.

As indicated earlier, some of the theoretical studies, which are based on different optimization or geometric or purely fluid mechanical arguments, have concluded radius branching laws similar in form to the Murray's law but with different values for the branching exponent, g. Area preservation principle with $g = 2$ has also been proposed as a radius branching law [9, 78, 83, 88]. Strange values ranging between 1.2-1.6 for the branching exponent, which undermine the principle of minimum work argument as well as some other arguments, has also been reported in relation to carotid bifurcation [85]. Despite the fact that an exponent value of less than 2 in the Murray-type models is possible in principle for certain types of branching in the blood network, it cannot be accepted in general over major parts of the vascular tree due to the existing evidence in support of the fact that the flow in general slows down as the size of the vessels converges toward the capillary networks and porous tissue.

Other studies have led to radius branching models different in form to the Murray's law. One of these is the empirical model proposed recently by Finet *et al.* [7] which, for a bifurcation, is given by

$$R_p = 0.678\,(R_{d_1} + R_{d_2}) \tag{4}$$

Assuming the validity of the Murray-type laws in their general form with a parametric branching exponent, several reasons can be proposed to explain the discrepancy in the reported values of the exponent as observed and measured in the experimental studies. These reasons include measurement and analysis errors associated with some vagueness in the definition of the radius of parent and daughters, especially in the branching neighborhood, as well as tapering which is a general feature in the biological flow networks. However, some of these discrepancies are too large to explain by random or systematic errors. The exponent could also vary in pathological cases due to cardiovascular diseases [75] which adjust the vessels cross sectional area with or without the deposition of foreign materials on the luminal surface. Some of these discrepancies may also be related to the preparation and measurement techniques, as well as differences in the applied procedural and analysis methods. Variations between species and individual subjects is another possible reason for some of these contradictions. The branching exponent may also vary depending on the location and rank in the vascular tree and the difference between the type of vasculature such as arterial versus venous [89].

Some studies suggested that Murray's law is good for large arteries and arterioles but not for microcirculation networks [84], while others seem to suggest an increasing exponent down the arterial tree [81, 85]. The convergence of the branching exponent to the theoretical Murray value of 3 with decreasing vessels size has also been reported in one study with the explanation that Poiseuille law, which most Murray-type models are based upon, is better approximated in the small vessels [90].

In summary, the reported results in the literature are not only infested by significant discrepancies in the value of the branching exponent between different studies, but also by large fluctuations and error margins in some of these studies, and this subjects the proposed Murray-type laws to many uncertainties and question marks. Similarly, the other empirical and theoretical radius branching models, which are different in form to the Murray-type models, are not far away from controversies and uncertainties.

4 Branching Angle

The geometry of the branching zones plays an important role in the distribution and collection of fluids and their ingredients. In the blood circulation network, the shape of branching, which includes branching angle, has an obvious impact on the flow of blood and the movement of its constituents, such as red blood cells, in the vessels. The branching angle of the daughter tube is normally measured between the daughter axis and the extended orientation of the parent axis. Some ambiguity in the definition of branching angle especially in the biological systems may arise due, for example, to irregular shape of the parent and daughter vessels and the deviation from the optimal cylindrical shape as well as curvature of the vessels at the branching zone. Pulsatility of biological flow, as seen in circulation and respiration, associated with possible change in the branch orientation and apex position [71] can lead to a time-dependent alteration in the branching angle.

The definition of the branching angle in the literature is normally based on a two-dimensional branching configuration where the axes of the branching vessels are coplanar; moreover, these definitions are generally based on a bifurcation branching type and hence some ambiguity may arise with branching in three-dimensional networks and branching orders higher than bifurcation. However, the previous definition of the branching angle should apply to branching vessels that do not share a common plane. Similarly, non-bifurcation branching can be accommodated with some flexibility in the definition of branching angle.

Work minimization arguments, similar to the ones proposed for the radius branching ratio, have been proposed for the optimal design of branching angle [31, 70]. However, it has been suggested that branching angles are irrelevant to the minimization of energy consumption, due to their large variations, and hence they are generally determined by other factors [71, 72, 75, 91]. In fact a potential optimization principle in the branching angle design does not necessarily require a fixed branching angle, similar to the fixed radius ratio of Murray's law for instance, since optimization of any physical parameter like energy in the flow system may require variable angle depending on other factors. Anyway, the most important factor that determines the branching angle in the living tissue should be the optimal delivery of the fluid to the target tissue and this seems to be the most appropriate design principle for the branching angle.

Great variations in the branching angle are also observed in non-living natural and synthetic flow networks, such as geological structures and manufactured porous media which are subject to spontaneous sedimentation and flow formation processes. Although this may not be significant due to the absence of obvious optimization principle in the formation of these systems, these processes could be subject to certain optimization rules that include the branching angle of the flow ducts. In fact optimization principles are at the heart of many physical phenomena in the natural and synthetic worlds.

Branching junctions are recognized to lower the critical threshold of Reynolds number for setting turbulence. The angle of branching has an influence on the transition from laminar to turbulent flow where the threshold limit decreases with increasing angle [71]. This is due to inertial effects where the increase in the branching angle entails larger and more abrupt change in momentum that promotes the setting of turbulence. The bifurcation angle also has a strong influence on the secondary flows and recirculation zones at the branching regions [35, 92]. On the other hand, branching angle seems to have a minor influence on plasma skimming, and phase separation in general. One possible reason is that these effects occur in the minute capillaries where creeping flow of viscous nature is the norm.

Concerning the relation between the radius and angle of branching, there is no correlation between the branching angle and the size, i.e. calibre, of the branching tree in the biological flow networks [75]. This is in concordance with the approximate fractal nature of these networks. However, there is a tentative relation between the parent-to-daughter size ratio and the branching angle which increases as the ratio increases; that is the smaller daughter vessel branches at a larger angle than the larger one [27].

Despite the fact that both radius branching ratio and branching angle are influential in determining the flow patterns at the branching regions, one of these factors may be more influential than the other for certain phenomena or in certain circumstances. There are no general rules about this due to the complexity of these issues and the involvement of many factors; such as the type of branching, Reynolds regime, and fluid rheology; although some definite conclusions have been reported in the literature in relation to some of these issues [93].

With regard to the experimental methods used to investigate the radius and angle branching laws, various imaging and measurement techniques, such as computed tomography scan, magnetic resonance imaging and confocal laser microscopy, as well as other acquisition techniques like polymer casting and cryo-microtome sectioning have been used to obtain geometric and topological information from biological branching networks [5, 61, 82, 94–107]. However, the majority of these techniques and their complementary analysis methods are susceptible to significant errors and hence cannot provide accurate geometrical data to test the validity of the branching rules such as Murray's law. For instance, there are technical difficulties in making precise measurements of the radius around the complexly shaped branching region due partly to technical difficulties and partly to some vagueness in its definition at the branching zone. Similarly, there is an ambiguity in the definition of the vessels axes, due to curvatures and shape irregularities, that prevents precise determination of the branching angle. Moreover, most branching measurements are carried out assuming the branching trees are contained in a two-dimensional plane, whereas in reality these trees are three-dimensional entities.

5 Branching Effects

There are various flow effects that occur at the branching junctions; some of which are briefly discussed in this section. In general, branching flow effects complicate the flow patterns and hence flow modeling and analysis at the branching regions. These effects are either triggered or exacerbated by the branching flow. One of these effects is the non-Newtonian rheology [33, 108] which can have a major contribution to the flow patterns at the branching junctions. Non-Newtonian effects are supported by experimental and numerical investigations [2, 4, 12, 16, 24, 41, 43, 109–114] where significant differences between the behavior of Newtonian and non-Newtonian fluids at the branching junctions have been widely observed. Although non-Newtonian effects are originally related to the rheology of the fluid and hence are not specific or limited to the flow at the branching regions, the complex geometric factors at the branching zone can stimulate or aggravate these effects. For example the converging-diverging and tortuous nature of the flow paths in these zones can activate ceratin viscoelastic modes associated with the fluid rheology.

Another type of branching flow effects is time-dependency. There are two main reasons for the time dependent effects to occur in the branching flow: the non-

Newtonian rheology [33] such as thixotropy and viscoelasticity, and the pulsatile nature of the flow associated with the pulsatility of the vessels size, pressure field and velocity profiles [115] as seen for example in the blood circulation. Although these causes are not specific to the flow at the branching junctions, some of which may be stimulated or exacerbated by the flow types in the branching zones. The time dependent features of the branching flow can be very complex and include recirculation zones, skewed velocity profiles, flow separation regions and secondary flows. Some of these features also exist in the time independent non-pulsatile branching flow [4, 24, 109, 116, 117]. These features and the contributing factors complicate the flow patterns at the branching junctions and make the analysis more difficult.

Another effect, which is a distinctive feature of the branching flow, is phase separation where a complex multi-phase fluid disintegrates into its components at the diverging junction due to the fact that the daughter pathway is more favorable to the passage of a particular fluid phase than to the other phases. In blood circulation, phase separation demonstrates itself in plasma skimming in the minute capillaries with a considerable drop in the fluid viscosity due to a low level of hematocrit in the daughter vessels which results from the reduction in the vessels size as it becomes comparable to the size of the red blood cells.

Branching flow effects also include the occurrence of several complex flow patterns such as the formation of turbulence zones and vortices at the branching regions with possible energy losses. The presence of constrictions and wall deformability, due for example to a stenosis or aneurysm, at or near the branching junction can cause the development or exacerbation of these complex flow patterns. However in biological systems, such as blood flow in arteries and air flow in the lung air pipes, such complex flow patterns are the exception rather than the norm due to the laminar nature of the flow over predominant parts of the flow network. Moreover when these exceptional flow patterns occur they normally disappear quickly due to the time-dependent pulsatile nature of the flow although in some circumstances they could be persistent and follow the periodic pattern of the primary flow. Other factors, such as loops, in the branching flow network can exacerbate the situation and introduce more complications on the flow patterns at the branching regions.

6 Circulatory Branching Lesions

In this section, we outline some of the branching lesions which are normally found in the blood circulation system. Despite the fact that the causes of the arterial diseases are common to all parts of the flow system, whether branching zones or not, the branching zones are generally favored for the development of arterial lesions, such as plaque depositions and aneurysms. This is due to the localized complex flow and shear stress patterns and the involvement of composite hemorheologic and hemodynamic factors; such as pressure, fluid velocity, and particle residence time; which can modify the geometric and material properties of the vessels wall at the neighborhood of these sites through sedimentation of foreign materials, alteration of the physical characteristics of the original wall material, and changing the shape of the walls. In this regard, the endothelial layer, which is highly sensitive to the magnitude and fluctuations of the wall shear stress, plays a significant role in the genesis and progression of lesions at the branching zones and in blood vessels in general.

The pulsatility of flow, which results in a cyclic change in the geometry of branching zones such as the apex position and branching angle as well as the radius size, can also influence the formation and exacerbation of branching lesions through the imposition of persistent abnormal flow and stress patterns and fatigue zones by creating, for example, turbulence regions, or inflicting high or low shear stress, or stretching or shrinking certain spots around the branching region. These problems can be worsened by the involvement of other supporting factors like arterial hypertension and aging. The rheology of blood, which essentially behaves as a Newtonian fluid in large vessels and non-Newtonian in small vessels, can also have a positive or negative impact on the branching lesions. For example, non-Newtonian effects can contribute to the complication of the flow and pressure patterns at the branching zones affecting, directly or indirectly, positively or negatively, the genesis and progression of these lesions [70–72, 92, 104, 118–123].

Several geometric and hemodynamic factors contribute to the development and progression of lesions at the branching zones. The contribution of most of these factors is based on effects related to fluid-structure interaction. The flow pattern at the branching junctions, especially if associated with complex flow phenomena such as vortices, may create time- and space-dependent wall shear stress that contributes in the long term to the development of lesions such as the deposition of plaque, wall thickening and reduced distensibility [110, 124, 125]. Plaques seem to develop mostly in the low endothelial wall shear stress areas at the branching junctions and hence these areas are more susceptible to plaque-related lesions like atherosclerosis which commonly occurs near the branching zones such as lateral bifurcation walls [116, 126–131]. Low endothelial shear stress may also play a role in the formation of thrombosis and re-stenosis following stenting operation of atherosclerotic lesions [132].

7 Wave Propagation at Junctions

In general, there are four main factors that affect the transmission and reflection of flow and pressure waves in the fluid-filled tubes: the physical properties of the fluid such as mass density, the physical properties of the tubes material such as Young's modulus, the geometry of the tubes such as tube radius, and the fluid-structure interaction as represented for instance by the interactive relation between the pressure and cross sectional area. The significance and contribution of these factors vary in different circumstances; moreover the contribution could depend on other aspects as well.

These factors are common to the branching junctions and other zones in the fluid transportation networks. What is specific and particularly relevant to the branching zones, especially in the context of distensible biological flow networks such as blood vasculature, is the geometry of the junction zone and the physical characteristics of the wall material, such as the elasticity or viscoelasticity of the wall, since these characteristics are subject to alterations due, for example, to plaque depositions and atherogenic processes [133]. As discussed in section 6, the branching zones are highly susceptible to such adaptations and characteristics-changing developments. As indicated earlier, some of the geometric and material factors that affect wave transmission and reflection are partly reflected in the employed constitutive relation that correlates the transmural pressure to the tube cross sectional area for modeling the branching flow; which in essence is a fluid-structure interaction influence.

Partial reflection of the pressure wave takes place at the points of abrupt change in the vessel shape and its total cross sectional area; the most obvious example of these reflection points are the branching junctions. In the blood transportation vasculature, stents, stenoses and aneurysms can also be points for wave reflection; moreover they normally affect the material properties, like elasticity or viscoelasticity, and geometry, such as stenotic constriction or aneurysmal dilation, of the walls resulting in alteration of the wave characteristics like the speed of propagation

[123].

8 Conclusions

Branching flow is commonplace in fluid dynamics systems in general and biological flow networks in particular. The most prominent example of the biological models that govern radius branching is the Murray's law and its variations. However, there are many controversies in the literature about the validity of Murray's law as well as inconsistencies related to the value of the branching exponent assuming the validity of the law in its generic form. While some studies categorically support the Murray's law, others agree with this law in form only with a different value for the branching exponent. Yet other investigations totally disagree with Murray's law since no exponent value within acceptable error margin was found to satisfy the general form of Murray's law according to these studies.

Assuming the physical reality of the design rules that govern radius branching ratio in the biological and naturally-occurring systems, the problem with the proposed theoretical models, as exemplified by Murray's law, is that there is no unique argument that can be instated to derive and substantiate these relations due to the existence of various possibilities for the principle that can be used to justify such relations. Therefore, these relations can only be accepted if there is an overwhelming evidence from experimental and observational data over major parts of the branching flow network in support of these models. As there are many inconsistencies and controversies in the literature about the parametric values in these relations as well as their general form, these models should be treated with caution. Regarding the empirical models, more thorough investigations are required to establish these models. Possible variations in the form and parametric values in the proposed radius branching models between species, individuals, and even position and rank in the flow networks should also be considered.

Despite all these controversial issues, there is one thing that seems to be established about the radius branching in predominant parts of the blood circulatory system, that is the total area increases at the branching junctions in the transition from large to small vessels with an obvious consequence that the flow generally slows down in the branching direction. However, this does not imply a Murray-type law or any other theoretical or empirical model due to the existence of various alternatives with regard to the form and parametric values as well as the possibility of the absence of a persistent radius branching pattern over the whole or even major parts of these networks. In the absence of thorough studies that cover all the biological flow networks in different species considering possible variations in the individual subjects and their health status, no general law can be definitely concluded.

Concerning the branching lesions in the circulation system, complex flow patterns at the branching zones; as well as material, geometric and hemodynamic factors such as pressure, fluid velocity, and wall shear stress; play a significant role in the genesis and progression of these lesions. The branching zone is normally a low shear stress area, and since low endothelial shear stress is a stimulus for plaque formation and atherogenesis the branching zones are strong candidates for lesions like atherosclerosis and stenoses. Clinical intervention such as bifurcation stenting can also introduce changes on the wall shear stress patterns at the branching zones with long term consequences on the development and progression of branching lesions. Periodical change in the shape of the branching region, such as the angle of branching and apex position, due to the flow pulsatility can also have an impact on the development of lesions originating, for instance, from fatigue and aneurysmal dilation. Recirculation zones in the branching region may also act as a stimulus for the development of lesions like atherosclerosis and thrombosis. In summary, the branching zones are exceptionally susceptible to the genesis and progression of arterial defects due largely to the complex fluid and solid dynamics involved in these zones as well as fluid-structure interaction factors.

Non-Newtonian rheology, whose importance increases in the small blood vessels, introduces more complex branching flow patterns such as setting or increasing the magnitude or widening the separation areas of vortices and turbulence zones. Non-Newtonian rheology and its ensuing effects can also influence the localization, distribution and magnitude of the wall shear stress around the branching junctions with possible long term impact on facilitating or hindering lesions such as stroke. Although energy losses at the branching junctions are generally negligible in the biological flow systems, the development of complex flow patterns, such as turbulent fluctuations and transient and steady-state vortices, can increase their significance.

Wave reflection generally occurs at points of sudden change in the geometry of the transmission route. Branching points therefore have a major contribution to the transmission and reflection of the pressure waves in the distensible fluid-filled networks with obvious impact on the fluid transportation and long term evolution of these flow systems.

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