THE THEORETICAL VALUE OF THE COSMOLOGICAL CONSTANT

Ru-Jiao Zhang

Zhang Inc., P.O.Box 16606, Sugar Land, Texas 77496, USA
rzhang868@gmail.com

Abstract

This paper presents a derivation of the theoretical value of the cosmological constant. The approach was based on Einstein's gravitational field equations, Hubble's law, and Friedmann-Robertson-Walker model of the universe. The theoretical value of the cosmological constant Λ was found to be: $\Lambda = \frac{2H_0^2}{3}$, here H_0 is the Hubble constant. The theoretical value is very close to the observational value. Open space (k=-1) and closed space (k=+1) most likely coexist in our universe. The study implies that the expansion of the universe is an inherent property of vacuum space, not by dark matter and dark energy.

Keywords: theoretical cosmological constant, General relativity, Hubble law, Freidmann metric, closed and open space, vacuum space, negative gravity.

1. Introduction

The Gravitational field equation was a theoretical equation established by Einstein in 1915. The original field equation did not include the cosmological constant Λ.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad [1] \tag{1}$$

In 1916, after Einstein published his general relativity theory, Karl Schwarzschild soon obtained the exact solution to Einstein's field equations for the gravitational field outside a non-rotating, static spherically symmetric body. The Schwarzschild solution to Einstein's field equations was

$$ds^2 = -\left(1 - \frac{r_G}{r}\right)c^2dt^2 + \left(1 - \frac{r_G}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \; [2] \tag{2}$$

Here, $r_G = \frac{2GM}{c^2}$ is the Schwarzschild radius.

The Schwarzschild solution is for the metric of empty space (also known as "vacuum") surrounding a spherically-symmetric massive object, such as a black hole; it is a vacuum solution to Einstein's field equation.

In 1922 Friedmann solved the Einstein gravitational field equation and obtained non static cosmological solutions presenting an expanding universe. Einstein thought that the universe should be static and unchanged forever, then he modified his original field equations by adding the so-called cosmological term Λ which can stop the expansion.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ [3] \tag{3}$$

In 1929 Hubble found experimentally that the distant galaxies are receding from us, and the farther the galaxy the bigger its velocity as determined by its redshift. Hubble's initial value for the expansion rate is the Hubble constant H_0. This observation caused Einstein to abandon the cosmological constant, and calling it the '"biggest blunder" of his life.

However, recent discover indicate that the expansion of the universe has been accelerating. Einstein's modified equation may be the apposite form after all. Scientists have revived Einstein's cosmological constant to explain a mysterious force called dark energy that seems to be counteracting gravity, and causing the universe to expand at an accelerating speed. "Einstein may not have blundered after all". (Carroll)[3]

Recalling the past of one hundred years of development of cosmology, the cosmological constant had been added in and taken out from the gravitational field equations in few times by Einstein and scientists. What is the actual meaning for the cosmological constant?

This paper attempt to derivate the theoretical value of the cosmological constant based on the Einstein's gravitational field equation and the Hubble's law with the Friedmann universe models and the Schwarzschild model. The derivation doesn't involve any dark energy and dark matter. The derivation is pure geometric method. It try to find out the relation between the cosmological constant and the Hubble constant: $\Lambda \propto H_0$.

The Einstein's gravitational field equation with the cosmological constant Λ is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{4}$$

Where
$R_{\mu\nu}$ is the Ricci tensor.
R is the curvature scalar.
Λ is the cosmological constant.
$g_{\mu\nu}$ is the metric tensor.
G is the gravitational constant.
$T_{\mu\nu}$ is the energy–momentum tensor.

The Hubble's Law [4] is

$$\dot{D} = H_0 D \tag{5}$$

Where
$\dot{D}$ is the recessional velocity, typically expressed in km/s.
H_0 is the Hubble constant. $H_0 = 67.6^{+0.7}_{-0.6} km\ s^{-1} Mpc^{-1}$ [5]
D is the proper distance, measured in mega parsecs (Mpc).

The Friedmann-Robertson-Walker metric of the Einstein's field equations (1) is

$$ds^2 = -c^2dt^2 + a^2[(1 - kr^2)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)] \ [6] \qquad (6)$$

2. The calculation for ΔS of the sphere on Friedmann-Robertson-Walker model

The Universe is homogeneous and isotropic. Isotropy means that the metric must be diagonal. Because, as we shall see, space is allowed to be curved, it will turn out to be useful to use spherical coordinates (r, θ, φ) for describing the metric. The center of the spherical coordinate system is us (the observers) as we look out into the cosmos. Let us focus on the spatial part of the metric.

We will calculate the expansion shell layer ΔS of an arbitrarily selected sphere with the radius D in the space by using the Einstein's gravitational field equation (4), and the Hubble's law (5). The radius D shall be a large cosmological distance measured in mega parsecs (Mpc).

Both approaches should give the same result of the volume of the expansion shell layer ΔS as shown in yellow in Figure 1.
Figure 1 is not to scale, the ΔS actually is very thin, such as D = 10Mpc = 3.0857 × 10^{20}km, thus $\dot{D} \approx 675 km/s$.

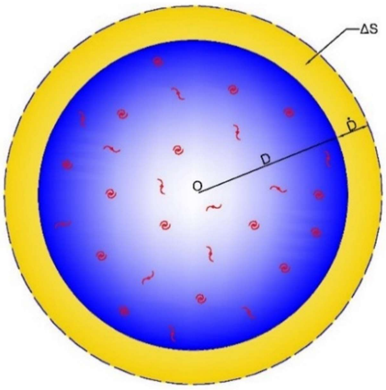

Figure 1: (not to scale) An expanding sphere with the radius D; ΔS is the increased shell layer with the expansion speed $\dot{D}$ on the surface of the sphere.

What we are concerned with is the cosmological constant, therefore we only investigate the third item $\Lambda g_{\mu\nu}$ on the left side of the Einstein's gravitational field equation (4). By using the third item $\Lambda g_{\mu\nu}$ as the integrand function in the spherical coordinates, we shall get the expansion shell layer ΔS from sphere volume formula

$$V = \iiint \Lambda \sqrt{g}\, dr d\theta d\varphi \ [7] \qquad (7)$$

here $g = g^{\mu}_{\mu}; \quad \mu = 1,2,3,4$

The Robertson-Walker metric of the Einstein's field equations (1) is

$$ds^2 = -c^2dt^2 + a^2[(1 - kr^2)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)] \ [8] \qquad (8)$$

$g_{\mu\nu}$ are the metric tensors of the Robertson-Walker metric

$$g_{\mu\nu} = a^2 \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \tag{9}$$

Here $g_{00} = -1 \quad g_{11} = \frac{1}{1-kr^2}; \; g_{22} = r^2; \, g_{33} = r^2 sin^2\theta.$

There are three cases of interest: k = - 1, k = 0, and k = + 1.
The k = - 1 case corresponds to constant negative curvature on, and is called open;
The k = 0 case corresponds to no curvature on, and is called flat;
The k = + 1 case corresponds to positive curvature on, and is called closed.

It is still a mystery that our universe belong to which geometry space; open, flat or closed. Our derivation of the cosmological constant could answer this question. Followings are calculations of the cosmological constant for three cases of the Friedmann-Robertson-Walker model.

2.1 The flat space $k = 0$

For the k = 0 case, the absolute value of the determinant of Robertson-Walker metric is

$$\begin{aligned} |g| &= -g_{00} \cdot \; g_{11} \cdot \; g_{22} \cdot g_{33} \\ &= a^2 \cdot (1 - kr^2)^{-1} \cdot \; r^2 \cdot r^2 sin^2\theta \\ &= a^2 \cdot \; (1 - kr^2)^{-1} r^4 sin^2\theta \end{aligned}$$

Then $\sqrt{|g|} = a(1 - kr^2)^{-1/2} \, r^2 \sin\theta$

$$\begin{aligned} \mathit{\Delta} S &= \mathit{\Lambda} D \int_D^{D+\dot{D}} (1 - kr^2)^{-1/2} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \mathit{\Lambda} D \int_D^{D+\dot{D}} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \mathit{\Lambda} 4\pi D^4 H_0 \end{aligned} \tag{10}$$

2.2 The negative curvature space $k = -1$

For the k = -1 case, the absolute value of the determinant of Robertson-Walker metric is

$$\begin{aligned} \mathit{\Delta} S &= \mathit{\Lambda} D \int_D^{D+\dot{D}} (1 - kr^2)^{-1/2} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \mathit{\Lambda} D \int_D^{D+\dot{D}} (1 + kr^2)^{-1/2} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \mathit{\Lambda} 2\pi D^3 H_0 \end{aligned} \tag{11}$$

Note: $\int(kr^2+1)^{-1/2}r^2dr=\frac{r}{2}\sqrt{r^2+1}-\frac{1}{2}\ln\left(r+\sqrt{r^2+1}\right)$

2.2 The positive curvature space $k=+1$

For the k = +1 case, $1-kr^2<0$ then $(1-kr^2)^{-1/2}$ shall be changed to $-(1-kr^2)^{-1/2}=(r^2-1)^{-1/2}$, because $|g|$ in (7) need to be absolute number.

$$\begin{aligned}\varDelta S &= \varLambda D\int_D^{D+\dot{D}}(1-kr^2)^{-1/2}r^2dr\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi}d\varphi\\ &= \varLambda D\int_D^{D+\dot{D}}(kr^2-1)^{-1/2}r^2dr\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi}d\varphi\\ &= \varLambda 2\pi D^3H_0 \qquad (12)\end{aligned}$$

Note: $\int(kr^2-1)^{-1/2}r^2dr=\frac{r}{2}\sqrt{r^2-1}+\frac{1}{2}\ln\left(r+\sqrt{r^2-1}\right)$

3. The calculation for $\varDelta S$ of the sphere on Schwarzschild model

"It has been widely suggested that the study of cosmology has close similarity with the study of the interior of black holes. In this paper we will outline a framework for cosmology which is partly inspired by the analysis of AdS/Schwarzschild black holes. This system, although unrealistic as a cosmology, is extremely well understood through the AdS/CFT duality." Figure 2. [9]

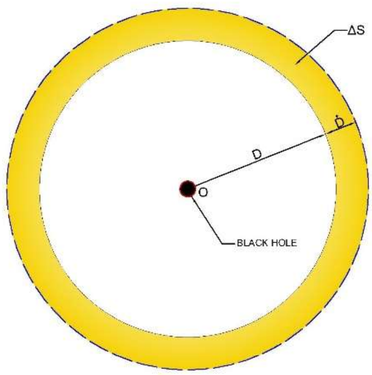

Figure 2: (not to scale) Schwarzschild black holes model. An expanding sphere with the radius D; $\varDelta S$ is the increased shell layer with the expansion speed $\dot{D}$ on the surface of the sphere.

The Schwarzschild solution of the Einstein's field equations (1) is

$$ds^2=-\left(1-\frac{r_G}{r}\right)c^2dt^2+\left(1-\frac{r_G}{r}\right)^{-1}dr^2+r^2(d\theta^2+sin^2\theta d\varphi^2) \qquad (13)$$

Here, $r_G=\frac{2GM}{c^2}$ is the Schwarzschild radius.

$g_{\mu\nu}$ are the metric tensors of the Schwarzschild geometry

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \tag{14}$$

Here $g_{00} = -\left(1 - \frac{r_G}{r}\right)$; $g_{11} = \left(1 - \frac{r_G}{r}\right)^{-1}$; $g_{22} = r^2$; $g_{33} = r^2 sin^2\theta$.

The absolute value of the determinant of Schwarzschild metric is

$$\begin{aligned} |g| &= -g_{00} \cdot g_{11} \cdot g_{22} \cdot g_{33} \\ &= \left(1 - \frac{r_G}{r}\right) \cdot \left(1 - \frac{r_G}{r}\right)^{-1} \cdot r^2 \cdot r^2 sin^2\theta \\ &= r^4 sin^2\theta \end{aligned} \tag{15}$$

Then $\sqrt{|g|} = r^2 \sin\theta$

For integrating the shell layer ΔS, the volume element shall be

$$\sqrt{|g|}dV = r^2 \sin\theta \, cdtdrd\theta d\varphi \tag{16}$$

The Schwarzschild metric is static and spherically symmetric, it is time independent, therefore

$$g_{\mu\nu} = g_{\mu\nu}(x^i),\ g_{0i} = 0$$

We calculate the expansion shell layer ΔS of the sphere at an instantaneous time t_o with the expansion speed $\dot{D}$, therefore the Schwarzschild metric does not include the the time metric component cdt.

Hence, for integrating shell layer ΔS, the volume element becomes

$$\sqrt{|g|}dV = r^2 \sin\theta \, drd\theta d\varphi \tag{17}$$

The calculation of the integrating shell layer ΔS of the sphere is shown as followings.

$$\begin{aligned} \Delta S &= \iiint \Lambda g_{\mu\nu} \, dV \\ &= \Lambda \iiint \sqrt{|g|} \, dV \\ &= \Lambda \iiint r^2 \sin\theta \, drd\theta d\varphi \\ &= \Lambda \int_D^{D+\dot{D}} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \Lambda 4\pi D^3 H_0 \end{aligned} \tag{18}$$

4. The calculation for ΔS of the sphere by Hubble's law

We use the Hubble's law to calculate the volume of the expansion shell layer ΔS of the sphere with the radius *D*. The volume of the sphere with the radius D is

$$S = \frac{4}{3}\pi D^3 \tag{19}$$

Multiplying the Hubble constant H_0 to the radius *D* in equation (19), we get the expansion shell layer ΔS shown in yellow in Figure 1.

$$\Delta S = \frac{4}{3}\pi D^3 H_0^3 \tag{20}$$

5. The evaluations of the results for the cosmological constant

We have previously calculated the ΔS for five cases. The evaluations of the results are shown as followings.

1) Combining both equations (10) and (20), we obtained the following equation

$$\Lambda 4\pi D^4 H_0 = \frac{4}{3}\pi D^3 H_0^3 \tag{21}$$

The value of the cosmological constant is

$$\Lambda = \frac{H_0^2}{3D} \tag{22}$$

This is the case of k = 0, corresponds to no curvature on, or the flat space of the Friedmann model of universe. It indicates that the cosmological constant has been increasing with time. Earth-bound observers look back in time as they look out in distance D. However the value of the cosmological constant is not match the observational value, therefore the k = 0 of flat space unlikely be the model of our universe.

2) Combining both equations (11) and (20), we obtained the following equation

$$\Lambda 2\pi D^3 H_0 = \frac{4}{3}\pi D^3 H_0^3 \tag{23}$$

The value of the cosmological constant is

$$\Lambda = \frac{2H_0^2}{3} \tag{24}$$

$$\Lambda = 3.1996 \times 10^{-36} \mathrm{S}^{-2}$$

This is the case of k = -1, corresponds to negative curvature on, or the open space of the Friedmann model of universe. The value of the cosmological constant is close to the observational value.

3) Combining both equations (12) and (20), we obtained the following equation

$$\Lambda 2\pi D^3 H_0 = \frac{4}{3}\pi D^3 H_0^3 \tag{25}$$

The value of the cosmological constant is

$$\Lambda = \frac{2H_0^2}{3} \tag{26}$$

$$\Lambda = 3.1996 \times 10^{-36} \mathrm{S}^{-2}$$

This is the case of k = +1, corresponds to positive curvature on, or the closed space of the Friedmann model of universe. The value of the cosmological constant is close to the observational value.

4) Combining both equations (18) and (20), we obtained the following equation

$$\Lambda 4\pi D^3 H_0 = \frac{4}{3}\pi D^3 H_0^3 \tag{27}$$

The value of the cosmological constant is

$$\Lambda = \frac{H_0^2}{3} \tag{28}$$

$$\Lambda = 1.5998 \times 10^{-36} \mathrm{S}^{-2}$$

This is the case corresponds to the vacuum space of the Schwarzschild model of universe.

The observational value of the cosmological constant "is often expressed as 10^{-52} m^{-2}, 10^{-35} s^{-2}, 10^{-47} GeV^4, 10^{-29} g/cm^3.[12] In terms of Planck units, and as a natural dimensionless value, the cosmological constant, Λ, is on the order of 10^{-122}.Modern calculations considering the vacuum energy of all known scalar and vector fields leads to 10^{-54} orders of magnitude smaller than the prediction." [10]

One article suggest the cosmological constant could be $\Lambda = 3H_0^2$.[11]

One observational value of the density of the dark energy is $\Lambda = \frac{0.21H_0^2}{c^2}$, for $c = 1$ then

$$\Lambda = 0.21H_0^2 \approx 0.67H_0^2 = \frac{2H_0^2}{3}$$

The comparisons of the cosmological constant values are listed in following table 1.

Model of the universe	Friedmann-Robertson-Walker model			Schwarzschild model	Observational value	
	$k = 0$ flat space	$k = -1$ open space	$k = +1$ closed space			
Λ	$\frac{H_0^2}{3D}$	$\frac{2H_0^2}{3}$	$\frac{2H_0^2}{3}$	$\frac{H_0^2}{3}$	10^{-35} s^{-2}	$0.21H_0^2$

Table 1. Comparisons of the value of the cosmological constant Λ

The comparisons Table.1 indicates that our universe likely to be the two cases of spaces: open space (k=−1) or closed space (k=+1) of the Friedmann universe. Both spaces have same value of the cosmological constant $\Lambda = \frac{2H_0^2}{3}$ which close to the observational value of $0.21H_0^2$. Our universe likely is the coexistence of both closed space and open space. "Because the thermodynamic arrow refers ideally to closed systems while the historical arrow manifests itself only in open systems, their coexistence presents no problem." [12]

Our pure geometry derivations of the cosmological constant do not include any dark energy and dark matter, however two of them get same value of the cosmological constant which so close to the observational value. The results may imply that the expansion of the universe is the inherent properties of the vacuum space. The expansion of the universe are not caused by the dark energy and the dark matter at all. The expansion of the vacuum space creates energy and mass, not the energy and the mass generate the expansion of the universe. The expansion of the vacuum space promote the motions of planets, stars, and galaxies.

6. Conclusion: the relation of mass, energy and vacuum space

A growing number of researchers claim a mysterious "dark energy," which most cosmologists believe fills space, might not exist. "the current accelerated expansion of the Universe can be explained without resorting to dark energy," wrote Olga Mena of Fermi National Accelerator Laboratory in Batavia, Ill. [13]

The vacuum space, the mass and the energy should be the same things but in different appearances in the Nature. The mass is the condensed vacuum space; and the vacuum space is the diluted matter. Einstein's famous formula $E = mc^2$ shows the relationship between the energy and the mass. The critical density $\rho_c = \frac{3H_0^2}{8\pi G}$ in Friedmann equation could revealed the relation between mass and the vacuum space. If M denoted as mass (kg), and V denoted vacuum space (meter^3), the following formula could show the relation between the mass and vacuum space, here G is the gravitation constant, H_0 is the Hubble constant.

$$V = \frac{8\pi G}{3H_0^2} M \tag{29}$$

The mass (ordinary matter) of our universe is at least 10^{53} kg, and its diameter is about 91 billion light-year. The ordinary (baryonic) matter is 4.9%, the dark matter is (26.8%), and the dark matter is (68.3%) in our universe.[14] Based on those percentage, the dark matter is $5.47 \times 10^{53} kg$, and the dark energy is $1.4 \times 10^{54} kg$. The total mass of dark matter and dark energy is $1.94 \times 10^{54} kg$.

Let us calculate the equivalent mass for the vacuum space, or said "vacuum space mass" by using formula (29). The diameter of the vacuum space is 91 billion light-year. The result of the "vacuum space mass" is about $2.86 \times 10^{54} kg$. $2.86 \times 10^{54} kg \approx 1.94 \times 10^{54} kg$. Is this a coincidence! The "vacuum space mass" equal to the total mass of dark matter and dark energy.

The ordinary vacuum space contain mass, therefore the vacuum space also have gravity, but is a negative gravity to the ordinary matter. This negative gravity should be the expansion force.

The alternate actions of the gravity and the expansion force (negative gravity) in the vacuum space create everything in the universe. The alternate oscillation is in extremely high frequency.

The phenomenon of conversion between the mass and the vacuum space are not strange in the Nature. In the quantum mechanics, the phenomenon of the electron self-energy, the virtual particles and the creation and annihilation of particles demonstrate the conversion between the mass and the vacuum space. In the cosmological scale, the supernovas are explicitly examples of the mass convert to the space. The black holes are the giant "machine" which could compressed the vacuum space to form the mass, later release them as the black hole jets. "According to the Swift team, these jets appear to be made of protons and electrons,…They are a primary means of redistributing mass and energy in the universe." [15]

The process of the mass converting to the vacuum space presents of the "closed space model", in another hand the process of the vacuum space converting to mass presents of the "open space model". Both open and closed space models coexist in our universe. The cosmological constant Λ demonstrates that the "open space model" is slightly greater than the "closed space model", as the results, the universe has been expanding in a tiny scale of Λ at present age.

References

[1] A. Einstein, Sitz. Preuss. Akad. d. Wiss. Phys.-Math 142 (1917)
[2] K. Schwarzschild (arXiv:physics/9905030 [physics.hist-ph] 12 May 1999)
[3] Sean M. Carroll (arXiv:astro-ph/0004075v2 8 Apr. 2000)
[4] E.P. Hubble, Proc. Natl. Acad. Sci. 15, 168 (1929)
[5] Grieb, Jan N.; Sanchez, Ariel G.; Salazar-Albornoz (arXiv: 1607.03143[astro-ph.CO] 11 Jul 2016)
[6] A. Friedmann, Z. Phys. 10, 377 (1922); 21, 326 (1924)
[7] 费保俊. 相对论与非欧几何. 北京:科学出版社, 2005
(Fei, Bao-jun. Relativity and Non-Euclidean Geometry. Beijing: Science Press, 2005)
[8] Sean M. Carroll Lecture Notes on General Relativity (arXiv:gr-qc/9712019v1 3 Dec. 2016)
[9] Lisa Dyson, Matthew Kleban, Leonard Susskind (arXiv:hep-th/0208013 14 Nov 2002)
[10] https://en.wikipedia.org/wiki/Cosmological_constant
[11] https://en.wikipedia.org/wiki/Scale_factor_(cosmology)
[12] David Layzer (arXiv:1008.1229 [quant-ph])
[13] Olga Mena and Jos´e Santiago (arXiv:astro-ph/0510453 v1 15 Oct 2005)
[14] https://en.wikipedia.org/wiki/Universe
[15] https://www.nasa.gov/centers/goddard/news/topstory/2006/swift_blazars.html