Proposed modifications to tidal accelerations and implications for geodesic deviation in general relativity

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An approach to deriving tidal accelerations is presented using a modified solution approach based on velocity dependence of acceleration for a weak field. Further explorations of this approach are investigated using general relativity to see how it might fit into its framework and whether any modifications may need to be suggested.

I. INTRODUCTION

In a recent paper it was proposed that

$$a' = \frac{GM}{r^2}\left(1-\frac{v^2}{c^2}\right)^{3/2}, \tag{1}$$

will hold in a weak field. Applicable to strong fields the Schwarzschild result is

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}\left(1-\frac{2\mu}{r}\right)\left(3\left(1-\frac{2\mu}{r}\right)\left(1-\frac{v^2}{c^2}\right)-2\right). \tag{2}$$

where $\mu = GM/c^2$ and r is measured from the center and outside the mass. This equation follows from a solution to the field equations of general relativity[1] for a static, non-rotating, spherical mass

$$c^2d\tau^2 = \left(1-\frac{2\mu}{r}\right)c^2dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 \tag{3}$$
$$-r^2d\theta^2 - r^2\cos^2\theta d\phi^2.$$

II. ANALYSIS

The Newtonian equation for tidal accelerations is

$$a_{\text{tidal}} = da = \frac{2GM}{r^3}dr, \tag{4}$$

where dr is the radial separation distance of two points in a gravitational field, and objects at an outward radial distance dr accelerate away from the observer at a distance r.

Alternatively, we can derive the expected tidal forces consistent with Eq. (1). We have

$$-da = \frac{GM}{r^2}\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} - \frac{GM}{(r+dr)^2}\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}. \tag{5}$$

Now, letting $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ and finding a common denominator we find

$$\begin{aligned} da &= \frac{GM((r+dr)^2-r^2)}{\gamma^3r^2(r+dr)^2} \\ &= \frac{2GMdr}{\gamma^3r^3}, \end{aligned} \tag{6}$$

where we have eliminated vanishing differential terms as $dr \ll r$. The velocity dependent term γ implies that tidal forces will tend to reduce for higher velocities.

III. SCHWARZSCHILD METRIC

Now, beginning from Eq. (2), but letting $r_s = 2\mu$, we have

$$\begin{aligned} da &= \frac{GM}{r^2}\left(1-\frac{r_s}{r}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r}\right)-2\right) \\ &\quad - \frac{GM}{(r+dr)^2}\left(1-\frac{r_s}{r+dr}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r+dr}\right)-2\right) \\ &= \frac{2GMdr}{\gamma^2r^3}\left(3\left(1-\frac{r_s}{r}\right)\left(1-\frac{2r_s}{r}\right)-\gamma^2\left(2-\frac{3r_s}{r}\right)\right) \end{aligned} \tag{7}$$

for the tidal acceleration for the Schwarzschild metric, see Appendix A for a detailed derivation.

For the case where $r \gg r_s$ the Schwarzschild radius, we have

$$da = \frac{2GMdr}{r^3}\frac{(3-2\gamma^2)}{\gamma^2}. \tag{8}$$

Now, if we let $\gamma = 1+\epsilon$ then $\gamma^n \approx 1+n\epsilon$ and so, to first order, $\frac{(3-2\gamma^2)}{\gamma^2} \approx (1-4\epsilon)(1-2\epsilon) \approx 1-6\epsilon \approx \frac{1}{\gamma^6}$ and so twice the relativistic effect as the weak field formula in Eq. (5). Also, we can see that for relativistic velocities the tidal forces become compressive when $\gamma^2 > 3/2$ or about 58% of the speed of light.

For the non-relativistic limit when $\gamma \to 1$, we have

$$da = \frac{2GMdr}{r^3}, \tag{9}$$

which agrees with the Newtonian formula, in Eq. (4), as expected.

For the ultra-relativistic case with $\gamma \to \infty$ we find

$$da = -\frac{4GMdr}{r^3}, \tag{10}$$

so that nearby objects now accelerate towards the observer at twice the separation rate of the Newtonian case.

IV. DISCUSSION

The tidal acceleration between point objects separated by a distance dr is shown by the method above to also be dependent on the initial velocities of the falling objects, as shown in Eq. (6). This is only a first weak field approximation. A more exact solution will be acquired by looking at the General Relativity case. The first attempt will be to see how this fits into either Schwarzschild or the geodesic equations to see where and how General Relativity deals with the velocity dependence.

Appendix A: Schwarzschild metric

We have the tidal acceleration

$$da = \frac{GM}{r^2}\left(1-\frac{r_s}{r}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r}\right)-2\right) - \frac{GM}{(r+dr)^2}\left(1-\frac{r_s}{r+dr}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r+dr}\right)-2\right). \tag{A1}$$

Using a common denominator we produce

$$da = \frac{GM}{r^2(r+dr)^2}\Bigg(\left(1-\frac{r_s}{r}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r}\right)-2\right)(r+dr)^2 - \left(1-\frac{r_s}{r+dr}\right)\left(\frac{3}{\gamma^2}\left(1-\frac{r_s}{r+dr}\right)-2\right)r^2\Bigg).$$

Expanding the brackets we find

$$\begin{aligned} da = \frac{GM}{\gamma^2 r^4 (r+dr)^4}\Big(& dr^3(r_s-r)(3r_s-(3-2\gamma^2)r) \\ &+4dr^2(r_s-r)r(3r_s-(3-2\gamma^2)r) \\ &+drr^2(18r_s^2-12(3-\gamma^2)r_s r+5(3-2\gamma^2)r^2) \\ &+2r^3(6r_s^2-3(3-\gamma^2)r_s r+(3-2\gamma^2)r^2)\Big). \end{aligned}$$

Now, as we let $dr \to 0$ only the last term will be significant leaving

$$\begin{aligned} da &= \frac{GM}{\gamma^2 r^8}\left(2r^3(6r_s^2-3(3-\gamma^2)r_s r+(3-2\gamma^2)r^2)\right) \\ &= \frac{2GM}{\gamma^2 r^3}\left(\frac{6r_s^2}{r^2}-3(3-\gamma^2)\frac{r_s}{r}+3-2\gamma^2\right) \\ &= \frac{2GM}{r^3}\left(\frac{3}{\gamma^2}\left(\frac{2r_s^2}{r^2}+1-\frac{3r_s}{r}\right)-\left(2-\frac{3r_s}{r}\right)\right) \\ &= \frac{2GM}{r^3}\left(\frac{3}{\gamma^2}\left(1-\frac{2r_s}{r}\right)\left(1-\frac{r_s}{r}\right)-\left(2-\frac{3r_s}{r}\right)\right). \end{aligned}$$

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman and Company, San Francisco, 1973).