Article

Dirac Wavefunction as a 4×4 Component Function

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Abstract

Since it was discovered some 88 years ago, the Dirac equation is understood to admit 4×1 component wavefunctions. We demonstrate here that this same equation does admit 4×4 component wavefunctions as-well.

"My religion consists of a humble admiration of the illimitable superior spirit who reveals himself in the slight details we are able to perceive with our frail and feeble mind."

– **Albert Einstein** (***1879 – 1955***)

1 Introduction

THE Dirac (1928*a*,*b*) equation is known to admit as a solution, a 4×1 component wavefunction. We demonstrate herein that this same equation does admit as a solution, a 4×4 component wavefunction. This reading divorces itself from trying to find a meaning to the 4×4 component wavefunction but merely sets the record straight – that such a solution is possible. For instructive and self-containment purposes, we present in §(2) the Dirac equation. Thereafter, we present in §(3 the 4×1 component free particle solutions of the Dirac equation. In §(4), we present an addendum to the reading Nyambuya (2016); we show that the Dirac equation written in a different irreducible basis can be written not just in 24 irreducible representations as has been done in Nyambuya (2016), but in 92 irreducible representations. What is relevant in this addendum to this reading are the, 92, 4×4, unitary hermitian matrices presented therein §(4). In §(5), the 4×4 component wavefunction solution is presented and thereafter in §(6), a brief discussion is given.

2 Dirac Equation

Written in its covariant form, the Dirac equation is given by:

$$[\imath\hbar\gamma^{\mu}\partial_{\mu} - \mathrm{m}_0 c]\,\psi = 0, \tag{2.1}$$

where:

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.2}$$

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is the Dirac four component wavefunction and the left and right handed bispinors ψ_L and ψ_R are such that:

$$\psi_L = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \text{ and } \psi_R = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}, \tag{2.3}$$

and:

$$\gamma^0 = \begin{pmatrix} \mathcal{I}_2 & 0 \\ 0 & -\mathcal{I}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.4}$$

are the 4×4 Dirac gamma matrices where $\mathcal{I}_2$ and 0 are the 2×2 identity and null matrices respectively. Throughout this reading, the Greek indices will be understood to mean $\mu, \nu, ... = 0, 1, 2, 3$ and lower case English alphabet indices $i, j, k... = 1, 2, 3$.

3 Free Particle Solutions of the Dirac Equation

The free particle solutions of the Dirac equation are obtained by assuming a wavefunction of the form ($\psi = ue^{+\imath p_\mu x^\mu/\hbar}$) where u is a four component object, *i.e.*:

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{3.1}$$

Substituting this free particle solution ($\psi = ue^{+\imath p_\mu x^\mu/\hbar}$) into (2.1), one is led to the following set of simultaneous equations:

$$\begin{array}{l} (E - \mathrm{m}_0c^2)u_0 - c(p_x - \imath p_y)u_3 - cp_zu_2 = 0 \\ (E - \mathrm{m}_0c^2)u_1 - c(p_x + \imath p_y)u_2 + cp_zu_3 = 0 \\ (E + \mathrm{m}_0c^2)u_2 - c(p_x - \imath p_y)u_1 - cp_zu_0 = 0 \\ (E + \mathrm{m}_0c^2)u_3 - c(p_x + \imath p_y)u_0 + cp_zu_1 = 0 \end{array}. \tag{3.2}$$

From this – one obtains the following two solutions:

$$\psi(1) = \sqrt{\frac{E + \mathrm{m}_0c^2}{2E}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+\mathrm{m}_0c^2} \\ \frac{c(p_x+\imath p_y)}{E+\mathrm{m}_0c^2} \end{pmatrix} e^{+\imath p_\mu x^\mu/\hbar} \; \ldots\text{and}\; \ldots \psi(2) = \sqrt{\frac{E + \mathrm{m}_0c^2}{2E}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-\imath p_y)}{E+\mathrm{m}_0c^2} \\ -\frac{cp_z}{E+\mathrm{m}_0c^2} \end{pmatrix} e^{+\imath p_\mu x^\mu/\hbar}, \tag{3.3}$$

The factor $\sqrt{(E + \mathrm{m}_0c^2)/2E}$ has been inserted as a normalization constant. These solution $\psi(1)$ is obtained by setting ($u_0 = 1; u_1 = 0$) and then solving for u_2 and u_3 and the solution $\psi(2)$ is obtained by setting ($u_0 = 0; u_1 = 1$) and then solving for u_2 and u_3. These two solutions $\psi(1)$ and $\psi(2)$ are all positive energy solutions and $\psi(1)$ is a spin-up particle while $\psi(2)$ is a spin down particle.

The second set of solutions is obtained by assuming a wavefunction of the form ($\psi = ue^{-\imath p_\mu x^\mu/\hbar}$). Substituting this free particle solution ($\psi = ue^{-\imath p_\mu x^\mu/\hbar}$) into (2.1), one is led to the following set of simultaneous equations:

$$\begin{array}{l} (E + \mathrm{m}_0c^2)u_0 - c(p_x - \imath p_y)u_3 - cp_zu_2 = 0 \\ (E + \mathrm{m}_0c^2)u_1 - c(p_x + \imath p_y)u_2 + cp_zu_3 = 0 \\ (E - \mathrm{m}_0c^2)u_2 - c(p_x - \imath p_y)u_1 - cp_zu_0 = 0 \\ (E - \mathrm{m}_0c^2)u_3 - c(p_x + \imath p_y)u_0 + cp_zu_1 = 0 \end{array}. \tag{3.4}$$

From this – one obtains the following two solutions:

$$\psi(3)=\sqrt{\frac{E+\mathrm{m}_0c^2}{2E}}\left(\begin{array}{c}\frac{cp_z}{E+\mathrm{m}_0c^2}\\ \frac{c(p_x+\imath p_y)}{E+\mathrm{m}_0c^2}\\ 1\\ 0\end{array}\right)e^{-\imath p_\mu x^\mu/\hbar}\;\ldots\text{and}\;\ldots\psi(4)=\sqrt{\frac{E+\mathrm{m}_0c^2}{2E}}\left(\begin{array}{c}\frac{c(p_x-\imath p_y)}{E+\mathrm{m}_0c^2}\\ -\frac{cp_z}{E+\mathrm{m}_0c^2}\\ 0\\ 1\end{array}\right)e^{-\imath p_\mu x^\mu/\hbar}. \tag{3.5}$$

Again, the factor $\sqrt{(E+\mathrm{m}_0c^2)/2E}$ has been inserted as a normalization constant. These solutions $\psi(3)$ have obtained by setting $(u_2=1;u_3=0)$ and then solving for u_0 and u_1 and the solution $\psi(4)$ is obtained by setting $(u_2=0;u_3=1)$ and then solving for u_0 and u_1. These two solutions $\psi(3)$ and $\psi(4)$ are all negative energy solutions and $\psi(3)$ is a spin-up particle while $\psi(3)$ is a spin down particle.

4 Dirac Equation in 92 Representations

Before we go into the Dirac wavefunction as a 4×4 component function, we shall here make an addendum to the reading Nyambuya (2016) where the Dirac equation has been presented in an irreducible representation of 24 equations. These 24 Dirac equations are:

$$i\hbar\tilde{\gamma}^\mu\partial_\mu\psi=\mathcal{U}_\ell\mathrm{m}_0c\psi \tag{4.1}$$

where the matrices $\tilde{\gamma}^\mu$ satisfy the following *Dirac Algebra*:

$$\tilde{\gamma}^{\mu\dagger}\tilde{\gamma}^\nu+\tilde{\gamma}^{\nu\dagger}\tilde{\gamma}^\mu=-2\mathcal{I}_4\eta^{\mu\nu}, \tag{4.2}$$

and $\mathcal{I}_4$ is the 4×4 identity matrix. These $\tilde{\gamma}$-matrices are defined such that:

$$\tilde{\gamma}^0=\left(\begin{array}{cc}0&\mathcal{I}_2\\ -\mathcal{I}_2&0\end{array}\right)\text{ and }\tilde{\gamma}^k=\left(\begin{array}{cc}\sigma^k&0\\ 0&\sigma^k\end{array}\right). \tag{4.3}$$

The number 24 arises because at the time, we only found 24 unitary hermitian matrices $\mathcal{U}_\ell:[\ell=(1-24)]$ that satisfy the requirements to generate the Dirac equation. In this addendum, we improve on this and show that in actual fact, there are 96 such matrices and not 24 as initially suggested. These 96 matrices are listed in Table (1).

5 Dirac Wavefunction as a 4 × 4 Component Function

Apart from the 4×1 wavefunction, the Dirac equation does admit solutions for which the wavefunction is a 4×4 matrix, that is, a wavefunction of the form:

$$\psi=\left(\begin{array}{cccc}\psi_{00}&\psi_{01}&\psi_{02}&\psi_{03}\\ \psi_{10}&\psi_{11}&\psi_{12}&\psi_{13}\\ \psi_{20}&\psi_{21}&\psi_{22}&\psi_{23}\\ \psi_{30}&\psi_{31}&\psi_{32}&\psi_{33}\end{array}\right). \tag{5.1}$$

The 4×1 free particle solutions $(\psi=ue^{\pm\imath p_\mu x^\mu/\hbar})$ of the Dirac equation are "normalised" such that $(\psi^\dagger\psi=1)$. If ψ is now a 4×4 matrix, then, this normalisation will have to be such that $(\psi^\dagger\psi=\mathcal{I}_4)$. In this free particle solution $(\psi=ue^{\pm\imath p_\mu x^\mu/\hbar})$, the object u is a 4×4 unitary and hermitian matrix because $(u^\dagger u=\mathcal{I}_4)$. As demonstrated in the previous section, there are ninety six 4×4 unitary and hermitian matrices satisfying this condition $(u^\dagger u=\mathcal{I}_4)$. Therefore:

$$\psi=\mathcal{U}_\ell e^{\pm\imath p_\mu x^\mu/\hbar}. \tag{5.2}$$

Because $\mathcal{U}_\ell$ can be written in block form as 2×2 matrix of 2×2 block matrices, let us write:

$$\mathcal{U}_\ell = \begin{pmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{pmatrix}. \tag{5.3}$$

Therefore:

$$\psi = \begin{pmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{pmatrix} e^{\pm \imath p_\mu x^\mu / \hbar}. \tag{5.4}$$

Now, substituting the wavefunction (5.4) into (2.1) and then evaluating the derivatives and thereafter reducing the equation to its simplest form, one will obtain:

$$\overbrace{\begin{pmatrix} (\pm E/c - \mathrm{m}_0 c)\,\mathcal{I}_2 & \pm \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \mp \boldsymbol{\sigma} \cdot \boldsymbol{p} & (\mp E/c - \mathrm{m}_0 c)\,\mathcal{I}_2 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{pmatrix}}^{u} = 0. \tag{5.5}$$

Since ($\psi \neq 0$) and its determinant is not equal to zero, a solution to (5.5) exists *if and only if* the determinant of the matrix A as defined in (5.5) is zero. Having the determinant of A being equal to zero implies that:

$$E^2 = p^2 c^2 + \mathrm{m}_0^2 c^4, \tag{5.6}$$

which is the usual Einstein energy-momentum equation. The meaning of this is that the particle ψ (just the Dirac particle that we are used to) satisfies the Einstein momentum equation. Equation (5.5) can be written as a set of four equations, as:

$$\left(E \pm \mathrm{m}_0 c^2\right) a_\ell + c_\ell \boldsymbol{\sigma} \cdot \boldsymbol{p} c = 0. \tag{5.7}$$

$$\left(E \pm \mathrm{m}_0 c^2\right) b_\ell + d_\ell \boldsymbol{\sigma} \cdot \boldsymbol{p} c = 0. \tag{5.8}$$

5.1 Solution (I)

For wavefunctions whose ℓ-indices are: $\ell = (1-64, 65, 68-77, 80-82, 84-93, 96)$, we have as a solution, the following:

$$\left(E \pm \mathrm{m}_0 c^2 = 0\right) \text{ and } p_x = p_y = p_z = 0. \tag{5.9}$$

These particles whose wavefunctions are represented by these indices [*i.e.*, $\ell = (1-64, 65, 68-77, 80-82, 84-93, 96)$] are in a state of rest. These particles can however attain a non-rest state *via* a Lorentz boost.

5.2 Solution (II)

For wavefunctions whose ℓ-indices are: $\ell = (66, 67, 78, 79; 82, 83, 94, 95)$, we have as a solution, the following:

$$(E \pm \mathrm{m}_0 c^2 \neq 0), (p_x \neq 0)\ (p_y \neq 0) \text{ and } (p_z = 0). \tag{5.10}$$

These particles whose wavefunctions are represented by these indices $\ell = (66, 67, 78, 79; 82, 83, 94, 95)$ are confined to travel only on the xy-plane. It follows that such particles must have an orbital angular momentum along the z-axis.

6 General Discussion

We have shown that apart from the 4×1 component wavefunctions, the usual Dirac (1928*a*,*b*) equation does admit 4×4 component wavefunctions as-well. It may be asked therefore: "*What use are these 4 × 4 component wavefunctions . . . ?*" Apart from the intellectual curiosity and need for completeness, *i.e.*, the need to understand every solution of an accepted equation even if this solution has no relation to reality, these new 4×4 component wavefunctions are interesting and may lead to a different way of doing physics. For example:

1. The Dirac equation has the concept of spin-up and spin-down and as-well the concept of a left and right-handed spinor. These concepts do not have a place in the 4×4 component wavefunction solution. This implies that if we are to do physics with these 4×4 component wavefunctions, we are going to have to rework our understanding of physics since all our present physics insofar as the Dirac equation is concerned – is understand in-terms of spin-up, spin-down, left and right-handed spinors.
2. It would be interesting to apply, for example, the chiral representation and other known represations of the Dirac equation.
3. We have performed calculations here of the 4×4 Dirac wavefunction for free particle solutions, there surely is need to study the 4×4 Dirac wavefunction under some interaction.
4. It is important to search for some possible physical meaning of these 4×4 component wavefunctions, for example, they might have some connection with the quaternionic version of Dirac equation (see *e.g.* Colladay et al. 2010, for the quaternionic version of Dirac equation).

In-closing, we should say that, it is not the scope of this reading to explore the emerge physics from the 4×4 component wavefunction solution of the Dirac equation. Actually, this can not be accomplished in a single reading but will have to be an effort of the physics community in general. What this reading merely conveys is the message to the effect that, there is – apart from the 4×1 component wavefunction solutions of the Dirac equation; are 4×4 component wavefunction solutions as-well.

References

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A Tables

In Tables (2) to (7), we present the 96 unitary hermitian matrices $\mathcal{U}_\ell$. The order that we have chosen for the ℓ-index is our choice. One is free to order these matrices in an order of their choice. Table (1) lists these 96 matrices in compact form while in Tables (2) to (7), they are written out explicitly.

Table 1: **List of the 96 $\mathcal{U}_\ell$-Matrices**

		Matrix-$\mathcal{U}_\ell$	**ℓ-index**	**Category**
$\mathcal{U}_\ell$	=	$\begin{pmatrix} \sigma^\mu & 0 \\ 0 & \sigma^\nu \end{pmatrix}$	$\ell = (1-16)$	Group (I)
$\mathcal{U}_\ell$	=	$\begin{pmatrix} \sigma^\mu & 0 \\ 0 & -\sigma^\nu \end{pmatrix}$	$\ell = (17-32)$	Group (II)
$\mathcal{U}_\ell$	=	$\begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\nu & 0 \end{pmatrix}$	$\ell = (32-48)$	Group (III)
$\mathcal{U}_\ell$	=	$i\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\nu & 0 \end{pmatrix}$	$\ell = (48-64)$	Group (IV)
$\mathcal{U}_\ell$	=	$\frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^\mu & \sigma^\nu \\ \sigma^\nu & \sigma^\mu \end{pmatrix}$	$\ell = (65-80)$	Group (V)
$\mathcal{U}_\ell$	=	$\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^\mu & \sigma^\nu \\ \sigma^\nu & -\sigma^\mu \end{pmatrix}$	$\ell = (81-96)$	Group (VI)

The sigma-matrices $\sigma^\mu : [\mu = (0,1,2,3)]$, are 2×2 matrices where σ^0 is the 2×2 identity matrix and $\sigma^k : [k = (1,2,3)]$ are the usual 2×2 Pauli matrices. Each group has 16 matrices and one can off-cause order the ℓ-index in any manner of their choice. In an order our choice, the 96 $\mathcal{U}$-matrices are listed in Tables (2) to (7) below.

Table 2: **List of the 16 Group (I) Matrices** [$\ell = (\mathbf{1}-\mathbf{16})$]

$\mathcal{U}_1 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_2 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^1 \end{pmatrix}$	$\mathcal{U}_3 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_4 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^3 \end{pmatrix}$
$\mathcal{U}_5 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_6 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$	$\mathcal{U}_7 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_8 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^3 \end{pmatrix}$
$\mathcal{U}_9 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_{10} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{11} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{12} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^3 \end{pmatrix}$
$\mathcal{U}_{13} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_{14} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{15} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{16} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$

Table 3: **List of the 16 Group (II) Matrices** $[\ell = (\mathbf{17} - \mathbf{32})]$

$\mathcal{U}_{17} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{18} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^1 \end{pmatrix}$	$\mathcal{U}_{19} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{20} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$
$\mathcal{U}_{21} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{22} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}$	$\mathcal{U}_{23} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{24} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$
$\mathcal{U}_{25} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{26} = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{27} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{28} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$
$\mathcal{U}_{29} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{30} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{31} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{32} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$

Table 4: **List of the 16 Group (III) Matrices** $[\ell = (\mathbf{33} - \mathbf{48})]$

$\mathcal{U}_{33} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{34} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^1 & 0 \end{pmatrix}$	$\mathcal{U}_{35} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{36} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{37} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{38} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$	$\mathcal{U}_{39} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{40} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{41} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{42} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{43} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{44} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{45} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{46} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{47} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{48} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$

Table 5: **List of the 16 Group (IV) Matrices** $[\ell = (\mathbf{49} - \mathbf{64})]$

$\mathcal{U}_{33} = \imath \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{34} = \imath \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^1 & 0 \end{pmatrix}$	$\mathcal{U}_{35} = \imath \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{36} = \imath \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{37} = \imath \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{38} = \imath \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$	$\mathcal{U}_{39} = \imath \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{40} = \imath \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{41} = \imath \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{42} = \imath \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{43} = \imath \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{44} = \imath \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^3 & 0 \end{pmatrix}$
$\mathcal{U}_{45} = \imath \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^0 & 0 \end{pmatrix}$	$\mathcal{U}_{46} = \imath \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{47} = \imath \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^2 & 0 \end{pmatrix}$	$\mathcal{U}_{48} = \imath \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$

Table 6: **List of the 16 Group (V) Matrices** [$\ell = (\mathbf{65} - \mathbf{80})$]

$\mathcal{U}_{65} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^0 & \sigma^0 \\ \sigma^0 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_{66} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^0 & \sigma^1 \\ \sigma^1 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_{67} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^0 & \sigma^2 \\ \sigma^2 & \sigma^0 \end{pmatrix}$	$\mathcal{U}_{68} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^0 & \sigma^3 \\ \sigma^3 & \sigma^0 \end{pmatrix}$
$\mathcal{U}_{69} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^1 & \sigma^0 \\ \sigma^0 & \sigma^1 \end{pmatrix}$	$\mathcal{U}_{70} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^1 & \sigma^1 \\ \sigma^1 & \sigma^1 \end{pmatrix}$	$\mathcal{U}_{71} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^1 & \sigma^2 \\ \sigma^2 & \sigma^1 \end{pmatrix}$	$\mathcal{U}_{72} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^1 & \sigma^3 \\ \sigma^3 & \sigma^1 \end{pmatrix}$
$\mathcal{U}_{73} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^2 & \sigma^0 \\ \sigma^0 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{74} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^2 & \sigma^1 \\ \sigma^1 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{75} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{pmatrix}$	$\mathcal{U}_{76} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^2 & \sigma^3 \\ \sigma^3 & \sigma^2 \end{pmatrix}$
$\mathcal{U}_{77} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^3 & \sigma^0 \\ \sigma^0 & \sigma^3 \end{pmatrix}$	$\mathcal{U}_{78} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^3 & \sigma^1 \\ \sigma^1 & \sigma^3 \end{pmatrix}$	$\mathcal{U}_{79} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^3 & \sigma^2 \\ \sigma^2 & \sigma^3 \end{pmatrix}$	$\mathcal{U}_{80} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma^3 & \sigma^3 \\ \sigma^3 & \sigma^3 \end{pmatrix}$

Table 7: **List of the 16 Group (VI) Matrices** $[\ell = (\mathbf{81} - \mathbf{96})]$

$\mathcal{U}_{81} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^0 \\ \sigma^0 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{82} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^1 \\ \sigma^1 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{83} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^2 \\ \sigma^2 & -\sigma^0 \end{pmatrix}$	$\mathcal{U}_{84} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^3 \\ \sigma^3 & -\sigma^0 \end{pmatrix}$
$\mathcal{U}_{85} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^1 & \sigma^0 \\ \sigma^0 & -\sigma^1 \end{pmatrix}$	$\mathcal{U}_{86} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^1 & \sigma^1 \\ \sigma^1 & -\sigma^1 \end{pmatrix}$	$\mathcal{U}_{87} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^1 & \sigma^2 \\ \sigma^2 & -\sigma^1 \end{pmatrix}$	$\mathcal{U}_{88} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^1 & \sigma^3 \\ \sigma^3 & -\sigma^1 \end{pmatrix}$
$\mathcal{U}_{89} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^2 & \sigma^0 \\ \sigma^0 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{90} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^2 & \sigma^1 \\ \sigma^1 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{91} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & -\sigma^2 \end{pmatrix}$	$\mathcal{U}_{92} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^2 & \sigma^3 \\ \sigma^3 & -\sigma^2 \end{pmatrix}$
$\mathcal{U}_{93} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 & \sigma^0 \\ \sigma^0 & -\sigma^3 \end{pmatrix}$	$\mathcal{U}_{94} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 & \sigma^1 \\ \sigma^1 & -\sigma^3 \end{pmatrix}$	$\mathcal{U}_{95} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 & \sigma^2 \\ \sigma^2 & -\sigma^3 \end{pmatrix}$	$\mathcal{U}_{96} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^3 & \sigma^3 \\ \sigma^3 & -\sigma^3 \end{pmatrix}$