Restatement and Extension of Various Spin Particle Equations

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Preface

This paper is based on my own previous articles [1–3]. I improve research methods and add some new contents in this paper. A more rigorous, more analytical, more complete and more organized mathematical physical method is adopted. And I am as far as possible to make the whole article have a sense of beauty.

Firstly, the mathematics foundation of constant tensors analysis methods [4–6] is established rigorously in Chapter One. Some wonderful mathematical properties are found. Many important constant tensors are proposed. Then in Chapter Two I use constant tensors as a mathematical tool to apply to physics. Some important physical quantities are defined by using constant tensors. All kinds of relationships between them are studied in detail. The canonical, analytical and strict mathematical physical sign system is established in this chapter. In Chapter Three, I use the mathematical tools in the previous two chapters to study spinorial formalism [4,5] of various spin particles classical equations [7–17] . And the equivalence between spinorial formalism and classical one is proved strictly. I focuse to study electromagnetic field, Yang-Mills field and gravitational field etc. Especially, a new spinorial formalism of the gravitational field identity [14–17] is proposed. In order to further explore, I study several important equations by contrast. Some new and interesting results are obtained.

The Chapter Four is the most important part of this thesis. It is also my original intention of writing this paper. In this chapter, I put forward a new form of particle equations: Spin Equation. The equation is directly constructed by spin and spin tensor. And I note that spin tensor is also the transformation matrix of corresponding field representation. So the physical meaning of this equation is very clear. The corresponding particle equation can be simply and directly written according to the transformation law of the particle field. It correctly describes neutrino [8], electromagnetic field [10,11], Yang-Mills field [9] and electron [7]etc. And it is found that it is completely equivalent to full symmetry Penrose equation [4,5]. A scalar field can be introduced naturally in this formalism. Thus, a more interesting equation is obtained: Switch Spin Equation. When the scalar field is zero, free particles can exist. When the scalar field is not zero, free particles can't exist. The scalar field acts as a switch. It can control particles generation and annihilation. This provides a new physical mechanism of particles generation and annihilation. At the same time, it can also answer the question: why the universe inflation period [18] can be completely described by the scalar fields. And the equation itself has an inherent limitation to the scalar field. So that the scalar will be quantized automatically. Each quantized value of the scalar is corresponding to different physical equations. That provides a new idea and an enlightenment for unity of five superstring theories [19].

Finally, in Chapter Five Bargmann-Wigner equation [20] is analyzed thoroughly. It is proved that it is equivalent to Rarita-Schwinger equation [21,22] in half integer spin case [23,24]. And it is equivalent to Klein-Gordon equation [22,25] in integer spin case [24]. The profound physical meanings of Bargmann-Wigner equation are revealed. By contrast, it is found that Bargmann-Wigner equation is suitable to describe massive particles, but not too suitable to describe massless particles. Penrose spinorial equation or Spin Equation is more suitable to describe massless particles.

Mathematics and physics of this paper have a stronger originality. Some mathematical and physical concepts, methods and contents also have a certain novelty. All of them are strictly calculated and established step by step by my own independent efforts. It takes me a lot of time and energy. I use spare time to finish the paper. Due to the limited time and my limited level, it is inevitable that there are a few mistakes. Comments and suggestions are welcome!

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Remark: This English version paper is translated from my own Chinese version paper [26]. But there are a few small adjustments and corrections.

Contents

Chapter 1

Constant tensors analysis

1 Mathematical preparation

1.1 Spin matrices $\sigma(s)$

$$[\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s), \sigma^2(s) = s(s+1), \varsigma = \pm 1 \tag{1.1}$$

1.2 Spin index definition

Symbol Convention:

$\sim$: Lorenz transform, $\prec$: Matrix expansion to components, $\succ$: Components contraction to a matrix

Spin index definition:

$$\begin{cases} {}^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)} & \Leftrightarrow {}^{A_+} \equiv {}^{A} \sim e^{(i\omega+\epsilon)\cdot\sigma(s)}, \quad {}^{A_-} \equiv {}_{A'} \sim e^{(i\omega-\epsilon)\cdot\sigma(s)} \\ {}_{A_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)} & \Leftrightarrow {}_{A_+} \equiv {}_{A} \sim e^{-(i\omega+\epsilon)\cdot\sigma^T(s)}, {}_{A_-} \equiv {}^{A'} \sim e^{-(i\omega-\epsilon)\cdot\sigma^T(s)} \end{cases} \tag{1.2a}$$

$$\begin{cases} {}_{A'_\varsigma} \sim e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)} & \Leftrightarrow {}_{A'_+} \equiv {}_{A'} \sim e^{(i\omega-\epsilon)\cdot\sigma(s)}, \quad {}_{A'_-} \equiv {}^{A} \sim e^{(i\omega+\epsilon)\cdot\sigma(s)} \\ {}^{A'_\varsigma} \sim e^{-(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)} & \Leftrightarrow {}^{A'_+} \equiv {}^{A'} \sim e^{-(i\omega-\epsilon)\cdot\sigma^T(s)}, {}^{A'_-} \equiv {}_{A} \sim e^{-(i\omega+\epsilon)\cdot\sigma^T(s)} \end{cases} \tag{1.2b}$$

Index relationships:

$$\begin{cases} {}^{A_\varsigma} \equiv {}_{A'_{-\varsigma}} \\ {}^{A_{-\varsigma}} \equiv {}_{A'_\varsigma} \end{cases}, \begin{cases} {}_{A_\varsigma} \equiv {}^{A'_{-\varsigma}} \\ {}_{A_{-\varsigma}} \equiv {}^{A'_\varsigma} \end{cases}, \begin{cases} {}^{A_+} \equiv {}_{A'_-} \equiv {}^{A} \\ {}^{A_-} \equiv {}_{A'_+} \equiv {}_{A'} \end{cases}, \begin{cases} {}_{A_+} \equiv {}^{A'_-} \equiv {}_{A} \\ {}_{A_-} \equiv {}^{A'_+} \equiv {}^{A'} \end{cases} \tag{1.2c}$$

Conjugate index:

$$\begin{cases} ({}^{A_\varsigma})^* \equiv {}^{A'_\varsigma} \\ ({}_{A_\varsigma})^* \equiv {}_{A'_\varsigma} \end{cases}, \begin{cases} ({}^{A'_\varsigma})^* \equiv {}^{A_\varsigma} \\ ({}_{A'_\varsigma})^* \equiv {}_{A_\varsigma} \end{cases}, \begin{cases} ({}^{A})^* \equiv {}^{A'} \\ ({}_{A})^* \equiv {}_{A'} \end{cases}, \begin{cases} ({}^{A'})^* \equiv {}^{A} \\ ({}_{A'})^* \equiv {}_{A} \end{cases} \tag{1.2d}$$

The corresponding measure of spin index: $\varepsilon\bar{\varepsilon} = \bar{\varepsilon}\varepsilon = I$

$$\begin{cases} \varepsilon_{A_\varsigma B_\varsigma} \succ \varepsilon \prec \varepsilon^{A'_\varsigma B'_\varsigma} \\ \bar{\varepsilon}_{A'_\varsigma B'_\varsigma} \succ \bar{\varepsilon} \prec \bar{\varepsilon}^{A_\varsigma B_\varsigma} \end{cases}, \begin{cases} \psi_{A_\varsigma} = \varepsilon_{A_\varsigma B_\varsigma}\psi^{B_\varsigma}, \psi^{A_\varsigma} = \bar{\varepsilon}^{A_\varsigma B_\varsigma}\psi_{B_\varsigma} \\ \psi^{A'_\varsigma} = \varepsilon^{A'_\varsigma B'_\varsigma}\psi_{B'_\varsigma}, \psi_{A'_\varsigma} = \bar{\varepsilon}_{A'_\varsigma B'_\varsigma}\psi^{B'_\varsigma} \end{cases} \tag{1.2e}$$

1.3 Electromagnetic spinorial index

Photon spin matrices:

$$\gamma = \{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \}, [\gamma_{\alpha_\varsigma}, \gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\gamma_{\gamma_\varsigma}, \gamma^2 = 1(1+1) \tag{1.3}$$

If we take a special representation $\sigma(1)=\gamma$, then there is the following electromagnetic spinorial index:

$$\alpha_\varsigma \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma} \Leftrightarrow \begin{cases} \alpha \sim e^{(i\omega+\epsilon)\cdot\gamma} \\ \alpha' \sim e^{(i\omega-\epsilon)\cdot\gamma} \end{cases}, \alpha'_\varsigma \sim e^{(i\omega-\varsigma\epsilon)\cdot\gamma} \Leftrightarrow \begin{cases} \alpha' \sim e^{(i\omega-\epsilon)\cdot\gamma} \\ \alpha \sim e^{(i\omega+\epsilon)\cdot\gamma} \end{cases} \tag{1.4}$$

$$\text{Index relationships:} \begin{cases} \alpha_\varsigma \equiv \alpha'_{-\varsigma} \\ \alpha_{-\varsigma} \equiv \alpha'_\varsigma \end{cases}, \begin{cases} \alpha_+ \equiv \alpha'_- \equiv \alpha \\ \alpha_- \equiv \alpha'_+ \equiv \alpha' \end{cases}, \text{Conjugate relationships:} \begin{cases} (\alpha_\varsigma)^* \equiv \alpha'_\varsigma \\ (\alpha'_\varsigma)^* \equiv \alpha_\varsigma \end{cases}, \begin{cases} (\alpha)^* \equiv \alpha' \\ (\alpha')^* \equiv \alpha \end{cases} \tag{1.5}$$

The corresponding metric tensor of the electromagnetic spinor: g

$$\begin{cases} g_{\alpha_\varsigma\beta_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma} \succ I, g^{\alpha_\varsigma\beta_\varsigma} = \delta^{\alpha_\varsigma\beta_\varsigma} \succ I \\ g_{\alpha'_\varsigma\beta'_\varsigma} = \delta_{\alpha'_\varsigma\beta'_\varsigma} \succ I, g^{\alpha'_\varsigma\beta'_\varsigma} = \delta^{\alpha'_\varsigma\beta'_\varsigma} \succ I \end{cases}, \begin{cases} \psi_{\alpha_\varsigma} = g_{\alpha_\varsigma\beta_\varsigma}\psi^{\beta_\varsigma}, \psi^{\alpha_\varsigma} = g^{\alpha_\varsigma\beta_\varsigma}\psi_{\beta_\varsigma} \\ \psi^{\alpha'_\varsigma} = g^{\alpha'_\varsigma\beta'_\varsigma}\psi_{\beta'_\varsigma}, \psi_{\alpha'_\varsigma} = g_{\alpha'_\varsigma\beta'_\varsigma}\psi^{\beta'_\varsigma} \end{cases} \tag{1.6}$$

The metric tensor is the unit matrix. You don't need to distinguish covariant and contravariant tensors. The superscript and subscript can be exchanged at will.

1.4 Lorenz representation index

$$\text{Lorenz representation index :}^{\mathcal{A}} \sim e^{\frac{1}{2}\vartheta^{ab}S_{ab}}, {}_{\mathcal{A}} \sim e^{-\frac{1}{2}\vartheta^{ab}S^T_{ab}} \tag{1.7}$$

1.5 A special representation of spin matrices

Starting from Lorenz transform properties of full symmetric two-component Weyl [8] spinorial tensors, a special representation of spin matrices can be obtained.

$$\sigma(s)=(\frac{1}{2}\begin{bmatrix} 0 & 2s & 0 & 0 & 0 \\ 1 & 0 & 2s-1 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 2s & 0 \end{bmatrix}, \frac{i}{2}\begin{bmatrix} 0 & -2s & 0 & 0 & 0 \\ 1 & 0 & -(2s-1) & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 2s & 0 \end{bmatrix}, \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix}) \tag{1.8a}$$

$$[\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s) \qquad \sigma^2(s) = s(s+1), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots \tag{1.8b}$$

$$\sigma_{\alpha_\varsigma}(s) \prec \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s), \alpha_\varsigma \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma}, {}^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}, {}_{B_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)} \tag{1.8c}$$

The corresponding metric tensor of the spin matrices:

$$\varepsilon_{A_\varsigma B_\varsigma}(s) \succ \varepsilon(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^0 C_n^0 \\ 0 & 0 & 0 & (-1)^1 C_n^1 & 0 \\ 0 & 0 & (-1)^2 C_n^2 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ (-1)^n C_n^n & 0 & 0 & 0 & 0 \end{bmatrix}, C_n^{-k} \equiv (C_n^k)^{-1} \tag{1.9a}$$

$$\bar{\varepsilon}^{A_\varsigma B_\varsigma}(s) \succ \bar{\varepsilon}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^n C_n^{-0} \\ 0 & 0 & 0 & (-1)^{n-1} C_n^{-1} & 0 \\ 0 & 0 & (-1)^{n-2} C_n^{-2} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ (-1)^0 C_n^{-n} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.9b}$$

$$\varepsilon(s)\bar{\varepsilon}(s) = \bar{\varepsilon}(s)\varepsilon(s) = I \tag{1.9c}$$

$$g_{\alpha_\varsigma\beta_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma} \succ I, g^{\alpha_\varsigma\beta_\varsigma} = \delta^{\alpha_\varsigma\beta_\varsigma} \succ I \tag{1.9d}$$

1.6 Common matrices

1.6.1 Pauli matrices

$$\sigma = \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}, tr(\sigma_{\alpha_\varsigma}) = 0, tr(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}) = 2\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.10}$$

$$[\sigma_{\alpha_\varsigma}, \sigma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}, \{\sigma_{\alpha_\varsigma}, \sigma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}, \sigma^2 = \frac{1}{2}(\frac{1}{2}+1) \tag{1.11}$$

$$\sigma_{\alpha_\varsigma} \prec \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}, \alpha_\varsigma \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma}, {}^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}, {}_{B_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T} \tag{1.12}$$

1.6.2 Photon matrices

$$\gamma=\{\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}\},tr(\gamma_{\alpha_\varsigma})=0,tr(\gamma_{\alpha_\varsigma}\gamma_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.13}$$

$$[\gamma_{\alpha_\varsigma},\gamma_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\gamma_{\gamma_\varsigma} \qquad \gamma^2=1(1+1) \tag{1.14}$$

$$\gamma_{\alpha_\varsigma}\prec\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}{}_{\gamma_\varsigma}\equiv-i\varepsilon_{\alpha_\varsigma}{}^{\beta_\varsigma}{}_{\gamma_\varsigma} \qquad \alpha_\varsigma,\beta_\varsigma,\gamma_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma} \tag{1.15}$$

1.6.3 Rotation generating element matrices

Spatial rotation generating element matrices:

$$R=\{\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\},tr(R_{\alpha_\varsigma})=0,tr(R_{\alpha_\varsigma}R_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.16a}$$

Lorenz boost generating element matrices:

$$L=\{\begin{bmatrix}0&0&0&i\\0&0&0&0\\0&0&0&0\\-i&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\},tr(L_{\alpha_\varsigma})=0,tr(L_{\alpha_\varsigma}L_{\beta_\varsigma})=\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.16b}$$

$$[R_{\alpha_\varsigma},R_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}R_{\gamma_\varsigma},[L_{\alpha_\varsigma},L_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}R_{\gamma_\varsigma},[R_{\alpha_\varsigma},L_{\beta_\varsigma}]=[L_{\alpha_\varsigma},R_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}L_{\gamma_\varsigma} \tag{1.16c}$$

$$R_{\alpha_\varsigma}\prec R_{\alpha_\varsigma}{}^{\rho_\varsigma}{}_{\eta_\varsigma},L_{\alpha_\varsigma}\prec L_{\alpha_\varsigma}{}^{\rho_\varsigma}{}_{\eta_\varsigma} \qquad \alpha_\varsigma,\beta_\varsigma,\gamma_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma},\rho_\varsigma,\eta_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot R} \tag{1.16d}$$

$$R^2=diag(2,2,2,1),L^2=diag(0,0,0,3) \tag{1.16e}$$

1.6.4 Group $SO(4)$ generating element matrices

$$\sigma_+=R+L=\{\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\} \tag{1.17a}$$

$$\sigma_+=\{-\sigma_y\otimes\sigma_x,-I\otimes\sigma_y,\sigma_y\otimes\sigma_z\},tr(\sigma_{+\alpha_\varsigma})=0,tr(\sigma_{+\alpha_\varsigma}\sigma_{+\beta_\varsigma})=4\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.17b}$$

$$\sigma_-=R-L=\{\begin{bmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix}\} \tag{1.17c}$$

$$\sigma_-=\{\sigma_x\otimes\sigma_y,-\sigma_z\otimes\sigma_y,\sigma_y\otimes I\},tr(\sigma_{-\alpha_\varsigma})=0,tr(\sigma_{-\alpha_\varsigma}\sigma_{-\beta_\varsigma})=4\delta_{\alpha_\varsigma\beta_\varsigma} \tag{1.17d}$$

$$[\sigma_{+\alpha_\varsigma},\sigma_{+\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{+\gamma_\varsigma},\{\sigma_{+\alpha_\varsigma},\sigma_{+\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_+^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.17e}$$

$$[\sigma_{-\alpha_\varsigma},\sigma_{-\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{-\gamma_\varsigma},\{\sigma_{-\alpha_\varsigma},\sigma_{-\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_-^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.17f}$$

$$[\sigma_{+\alpha_\varsigma},\sigma_{-\beta_\varsigma}]=[\sigma_{-\alpha_\varsigma},\sigma_{+\beta_\varsigma}]=0 \tag{1.17g}$$

$$\sigma_{+\alpha_\varsigma}\prec\sigma_{+\alpha_\varsigma}{}^{\rho_\varsigma}{}_{\eta_\varsigma},\sigma_{-\alpha_\varsigma}\prec\sigma_{-\alpha_\varsigma}{}^{\rho_\varsigma}{}_{\eta_\varsigma} \qquad \alpha_\varsigma,\beta_\varsigma,\gamma_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma},\rho_\varsigma,\eta_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot R} \tag{1.17h}$$

Unified representation of group $SO(4)$ generating element matrices

$$\sigma_\varsigma=\{\begin{bmatrix}0&0&0&i\varsigma\\0&0&-i&0\\0&i&0&0\\-i\varsigma&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&i\varsigma\\-i&0&0&0\\0&-i\varsigma&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\varsigma\\0&0&-i\varsigma&0\end{bmatrix}\} \tag{1.18a}$$

$$[\sigma_{\kappa\alpha_\varsigma},\sigma_{\tau\beta_\varsigma}]=i\delta_{\kappa\tau}\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\kappa\gamma_\varsigma},\{\sigma_{\kappa\alpha_\varsigma},\sigma_{\tau\beta_\varsigma}\}=2\delta_{\kappa\tau}\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_\varsigma^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.18b}$$

$$\sigma_{\kappa\alpha_\varsigma}\prec\sigma_{\kappa\alpha_\varsigma}{}^{\rho_\varsigma}{}_{\eta_\varsigma},\alpha_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma},\rho_\varsigma,\eta_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot R}\qquad\kappa,\tau,\varsigma=+,- \tag{1.18c}$$

Another understanding way of group $SO(4)$ generating element matrices

$$\sigma_{+\alpha_\varsigma}\prec\sigma_{+\alpha_\varsigma}{}^{a_\varsigma b_\varsigma},\sigma_{-\alpha'_\varsigma}\prec\sigma_{-\alpha'_\varsigma}{}^{a_\varsigma b_\varsigma}\quad\alpha_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma},\alpha'_\varsigma\sim e^{(i\omega-\varsigma\epsilon)\cdot\gamma},a_\varsigma,b_\varsigma\sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)} \tag{1.19}$$

1.6.5 Alternate representation transformation matrices

$$S_c(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\i&-i\end{bmatrix},S_c^+(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-i\\1&i\end{bmatrix},S_c(\frac{1}{2})S_c^+(\frac{1}{2})=S_c^+(\frac{1}{2})S_c(\frac{1}{2})=I \tag{1.20a}$$

$$S_{em}(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-1&1\\-i&-i\end{bmatrix},S_{em}^+(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-1&i\\1&i\end{bmatrix},S_{em}(\frac{1}{2})S_{em}^+(\frac{1}{2})=S_{em}^+(\frac{1}{2})S_{em}(\frac{1}{2})=I \tag{1.20b}$$

$$S_c(\frac{1}{2})(\sigma_x,\sigma_y,\sigma_z)S_c^+(\frac{1}{2})=(\sigma_z,\sigma_x,\sigma_y),S_{em}(\frac{1}{2})(\sigma_x,\sigma_y,\sigma_z)S_{em}^+(\frac{1}{2})=(-\sigma_z,-\sigma_x,\sigma_y) \tag{1.20c}$$

1.6.6 Electromagnetic pure imaginary representation transformation matrices and exchange matrices

$$S_{em}(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&-1\\i&0&0&i\\0&-1&-1&0\\0&i\varsigma&-i\varsigma&0\end{bmatrix},S_{em}^+(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-i&0&0\\0&0&-1&-i\varsigma\\0&0&-1&i\varsigma\\-1&-i&0&0\end{bmatrix},S_{ex}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \tag{1.21a}$$

$$\sigma_{-\varsigma}=S_{em}(\varsigma)(\sigma\otimes I)S_{em}^+(\varsigma),\sigma_\varsigma=S_{em}(\varsigma)(I\otimes\sigma)S_{em}^+(\varsigma),S_{em}(\varsigma)S_{em}^+(\varsigma)=S_{em}^+(\varsigma)S_{em}(\varsigma)=I \tag{1.21b}$$

$$(\sigma\otimes I)=S_{ex}(I\otimes\sigma)S_{ex},(I\otimes\sigma)=S_{ex}(\sigma\otimes I)S_{ex}\qquad S_{ex}^2=I \tag{1.21c}$$

$$S_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&-1\\i&0&i\\0&-2&0\end{bmatrix},S_m^-(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-i&0\\0&0&-1\\-1&-i&0\end{bmatrix} \tag{1.21d}$$

$$\gamma=S_m(1)\sigma(1)S_m^-(1)\qquad S_m(1)S_m^-(1)=S_m^-(1)S_m(1)=I,S_{em}\equiv-S_{em}(-1) \tag{1.21e}$$

1.6.7 Gravitational pure imaginary similarity transformation matrices

$$S_m(2)=\frac{1}{2}\begin{bmatrix}-1&0&2&0&-1\\-i&0&0&0&i\\0&2&0&-2&0\\1&0&2&0&1\\0&2i&0&2i&0\end{bmatrix},S_m^-(2)=\frac{1}{2}\begin{bmatrix}-1&2i&0&1&0\\0&0&1&0&-i\\1&0&0&1&0\\0&0&-1&0&-i\\-1&-2i&0&1&0\end{bmatrix} \tag{1.22a}$$

$$G_m=S_m(2)\sigma(2)S_m^-(2)\qquad S_m(2)S_m^-(2)=S_m^-(2)S_m(2)=I \tag{1.22b}$$

$$G_m=\{\begin{bmatrix}0&0&0&0&0\\0&0&-i&0&0\\0&i&0&0&0\\0&0&0&0&-2i\\i&0&0&2i&0\end{bmatrix},\begin{bmatrix}0&0&2i&0&0\\0&0&0&0&i\\-2i&0&0&-i&0\\0&0&0&0&0\\0&-i&0&0&0\end{bmatrix},\begin{bmatrix}0&-2i&0&0&0\\i&0&0&-i&0\\0&0&0&0&-i\\0&2i&0&0&0\\0&0&i&0&0\end{bmatrix}\} \tag{1.22c}$$

1.6.8 Full symmetric spinor condition matrix

$$T(s)=\begin{bmatrix}1&0&0&0&0\\0&\tau&0&0&0\\0&0&\cdots&0&0\\0&0&0&\tau&0\\0&0&0&0&1\end{bmatrix},\text{the number of }\tau\text{ is }2s-1,\tau=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix},\tau^n=\tau,T^n(s)=T(s) \tag{1.23a}$$

Full symmetric spinor condition : $\tilde{\psi}=T(s)\tilde{\psi}$ (1.23b)

Full symmetric spinor condition is similar to Majorana condition [7] and Weyl condition [8]

1.7 Vector index

$$a_\varsigma\sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\Leftrightarrow\begin{cases}a\sim e^{(i\omega\cdot R+\epsilon\cdot L)}\\a'\sim e^{(i\omega\cdot R-\epsilon\cdot L)}\end{cases},a'_\varsigma\sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\Leftrightarrow\begin{cases}a'\sim e^{(i\omega\cdot R-\epsilon\cdot L)}\\a\sim e^{(i\omega\cdot R+\epsilon\cdot L)}\end{cases} \tag{1.24}$$

$$\text{Index relationships:}\begin{cases}a_\varsigma\equiv a'_{-\varsigma}\\a_{-\varsigma}\equiv a'_\varsigma\end{cases},\begin{cases}a_+\equiv a'_-\equiv a\\a_-\equiv a'_+\equiv a'\end{cases},\text{Conjugate relationships:}\begin{cases}(a_\varsigma)^*\equiv a'_\varsigma\\(a'_\varsigma)^*\equiv a_\varsigma\end{cases},\begin{cases}(a)^*\equiv a'\\(a')^*\equiv a\end{cases} \tag{1.25}$$

The corresponding metric tensor of electromagnetic spinorial index: g

$$\begin{cases}g_{a_\varsigma b_\varsigma}=\delta_{a_\varsigma b_\varsigma}\succ I,g^{a_\varsigma b_\varsigma}=\delta^{a_\varsigma b_\varsigma}\succ I\\g_{a'_\varsigma b'_\varsigma}=\delta_{a'_\varsigma b'_\varsigma}\succ I,g^{a'_\varsigma b'_\varsigma}=\delta^{a'_\varsigma b'_\varsigma}\succ I\end{cases},\begin{cases}\psi_{a_\varsigma}=g_{a_\varsigma b_\varsigma}\psi^{b_\varsigma},\psi^{a_\varsigma}=g^{a_\varsigma b_\varsigma}\psi_{b_\varsigma}\\\psi^{a'_\varsigma}=g^{a'_\varsigma b'_\varsigma}\psi_{b'_\varsigma},\psi_{a'_\varsigma}=g_{a'_\varsigma b'_\varsigma}\psi^{b'_\varsigma}\end{cases} \tag{1.26}$$

The metric tensor is the unit matrix too. You don't need to distinguish covariant and contravariant tensors. The superscript and subscript can be exchanged at will.

1.8 Several mathematical formulas and properties

$$[A,B]=0\Rightarrow e^Ae^B=e^{A+B} \tag{1.27a}$$

$$\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1) \tag{1.27b}$$

$$\begin{bmatrix}I&\Delta\\0&I\end{bmatrix}\begin{bmatrix}I&-\Delta\\0&I\end{bmatrix}=\begin{bmatrix}I&-\Delta\\0&I\end{bmatrix}\begin{bmatrix}I&\Delta\\0&I\end{bmatrix}=I_4 \tag{1.27c}$$

2 Discoveries and proofs of constant tensors

2.1 Constant tensors $\delta_{a_\varsigma b_\varsigma},\delta^{a_\varsigma b_\varsigma},\delta_{a'_\varsigma b'_\varsigma},\delta^{a'_\varsigma b'_\varsigma}$

If $\psi_{a_\varsigma}\psi^{a_\varsigma}=\text{invariant}$, then $\psi^{a_\varsigma}\sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\Leftrightarrow\psi_{a_\varsigma}\sim e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)^T}=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}$ (1.28)

$I_4\prec\delta_{a_\varsigma b_\varsigma},I_4^-=I_4\prec\delta^{a_\varsigma b_\varsigma}$ (1.29)

Theorem 2.1.1. $I_4=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}I_4e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}$*, namely* $\delta_{a_\varsigma b_\varsigma},\delta^{a_\varsigma b_\varsigma}$ *are constant tensors,* $a_\varsigma,b_\varsigma\sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}$

Corollary 2.1.1. $I_4=e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}I_4e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}$*, namely* $\delta_{a'_\varsigma b'_\varsigma},\delta^{a'_\varsigma b'_\varsigma}$ *are constant tensors,* $a'_\varsigma,b'_\varsigma\sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}$

2.2 Constant tensors $\delta_{\alpha_\varsigma\beta_\varsigma},\delta^{\alpha_\varsigma\beta_\varsigma},\delta_{\alpha'_\varsigma\beta'_\varsigma},\delta^{\alpha'_\varsigma\beta'_\varsigma}$

If $\psi_{\alpha_\varsigma}\psi^{\alpha_\varsigma}=\text{invariant}$, then $\psi^{\alpha_\varsigma}\sim e^{(i\omega+\varsigma\epsilon)\cdot R}\Leftrightarrow\psi_{\alpha_\varsigma}\sim e^{-(i\omega+\varsigma\epsilon)\cdot R^T}=e^{(i\omega+\varsigma\epsilon)\cdot R}$ (1.30)

$I_4\prec\delta_{\alpha_\varsigma\beta_\varsigma},I_4^-=I_4\prec\delta^{\alpha_\varsigma\beta_\varsigma}$ (1.31)

Theorem 2.2.1. $I_4=e^{(i\omega+\varsigma\varepsilon)\cdot R}I_4e^{-(i\omega+\varsigma\epsilon)\cdot R}$*, namely* $\delta_{\alpha_\varsigma\beta_\varsigma},\delta^{\alpha_\varsigma\beta_\varsigma}$ *are constant tensors,* $\alpha_\varsigma,\beta_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot R}$

Corollary 2.2.1. $I_4=e^{(i\omega-\varsigma\varepsilon)\cdot R}I_4e^{-(i\omega-\varsigma\epsilon)\cdot R}$*, namely* $\delta_{\alpha'_\varsigma\beta'_\varsigma},\delta^{\alpha'_\varsigma\beta'_\varsigma}$ *are constant tensors,* $\alpha'_\varsigma,\beta'_\varsigma\sim e^{(i\omega-\varsigma\epsilon)\cdot R}$

2.3 Constant tensors [4] $\varepsilon_{A_\varsigma B_\varsigma}, \bar{\varepsilon}_{A_\varsigma B_\varsigma}, \varepsilon^{A_\varsigma B_\varsigma}, \bar{\varepsilon}^{A_\varsigma B_\varsigma}, \varepsilon_{A'_\varsigma B'_\varsigma}, \bar{\varepsilon}_{A'_\varsigma B'_\varsigma}, \varepsilon^{A'_\varsigma B'_\varsigma}, \bar{\varepsilon}^{A'_\varsigma B'_\varsigma}$

If $\psi_{A_\varsigma}\psi^{A_\varsigma} = \text{invariant}$, then $\psi^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \Leftrightarrow \psi_{A_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$ (1.32)

$\sigma_y \prec i\varepsilon_{A_\varsigma B_\varsigma} = -i\bar{\varepsilon}_{A_\varsigma B_\varsigma}, \sigma_y \prec -i\bar{\varepsilon}^{A_\varsigma B_\varsigma} = i\varepsilon^{A_\varsigma B_\varsigma}$ (1.33)

Theorem 2.3.1. $\sigma_y = e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma_y e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$, *namely* $\bar{\varepsilon}^{A_\varsigma B_\varsigma}, \varepsilon^{A_\varsigma B_\varsigma}$ *are constant tensors,* ${}^{A_\varsigma},{}^{B_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$

Corollary 2.3.1. $\sigma_y = e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma_y e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$, *namely* $\bar{\varepsilon}_{A'_\varsigma B'_\varsigma}, \varepsilon_{A'_\varsigma B'_\varsigma}$ *are constant tensors,* ${}_{A'_\varsigma},{}_{B'_\varsigma} \sim e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$

Theorem 2.3.2. $\sigma_y = e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}\sigma_y e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$, *namely* $\bar{\varepsilon}_{A_\varsigma B_\varsigma}, \varepsilon_{A_\varsigma B_\varsigma}$ *are constant tensors,* ${}_{A_\varsigma},{}_{B_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$

Corollary 2.3.2. $\sigma_y = e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}\sigma_y e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$, *namely* $\bar{\varepsilon}^{A'_\varsigma B'_\varsigma}, \varepsilon^{A'_\varsigma B'_\varsigma}$ *are constant tensors,* ${}^{A'_\varsigma},{}^{B'_\varsigma} \sim e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$

2.4 Constant tensors $\varepsilon^{A_\varsigma B_\varsigma}(s), \bar{\varepsilon}_{A_\varsigma B_\varsigma}(s)$

If $\psi_{A_\varsigma}\psi^{A_\varsigma} = \text{invariant}$, then $\psi^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}, \psi_{A_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}$ (1.34)

Lemma 2.4.1. $\sigma^T(s) = -\varepsilon(s)\sigma(s)\bar{\varepsilon}(s)$

Theorem 2.4.1. $\varepsilon(s) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\varepsilon(s)e^{(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}$, *namely* $\varepsilon^{A_\varsigma B_\varsigma}(s)$ *are constant tensors,* ${}^{A_\varsigma},{}^{B_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(s)$

Corollary 2.4.1. $\varepsilon(s) = e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)}\varepsilon(s)e^{(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)}$, *namely* $\varepsilon_{A'_\varsigma B'_\varsigma}(s)$ *are constant tensors,* ${}_{A'_\varsigma},{}_{B'_\varsigma} \sim e^{(i\omega-\varsigma\epsilon)\cdot\sigma}(s)$

Corollary 2.4.2. $\bar{\varepsilon}(s) = e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}\bar{\varepsilon}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$, *namely* $\bar{\varepsilon}_{A_\varsigma B_\varsigma}(s)$ *are constant tensors,* ${}_{A_\varsigma},{}_{B_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}$

Corollary 2.4.3. $\bar{\varepsilon}(s) = e^{-(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)}\bar{\varepsilon}(s)e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s)}$, *namely* $\bar{\varepsilon}^{A'_\varsigma B'_\varsigma}(s)$ *are constant tensors,* ${}^{A'_\varsigma},{}^{B'_\varsigma} \sim e^{-(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)}$

2.5 Antisymmetry constant tensors [27]

$\varepsilon_{a_\varsigma b_\varsigma c_\varsigma d_\varsigma}, \varepsilon^{a_\varsigma b_\varsigma c_\varsigma d_\varsigma}$ are constant tensors, $\varepsilon_{1234} = \varepsilon^{1234} = 1, a_\varsigma, b_\varsigma, c_\varsigma, d_\varsigma \sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}$ (1.35)

$\varepsilon_{a'_\varsigma b'_\varsigma c'_\varsigma d'_\varsigma}, \varepsilon^{a'_\varsigma b'_\varsigma c'_\varsigma d'_\varsigma}$ are constant tensors, $\varepsilon_{1'2'3'4'} = \varepsilon^{1'2'3'4'} = 1, a'_\varsigma, b'_\varsigma, c'_\varsigma, d'_\varsigma \sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}$ (1.36)

$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \varepsilon^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}$ are constant tensors, $\varepsilon_{123} = \varepsilon^{123} = 1, \alpha_\varsigma, \beta_\varsigma, \gamma_\varsigma \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma}$ (1.37)

$\varepsilon_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}, \varepsilon^{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}$ are constant tensors, $\varepsilon_{1'2'3'} = \varepsilon^{1'2'3'} = 1, \alpha'_\varsigma, \beta'_\varsigma, \gamma'_\varsigma \sim e^{(i\omega-\varsigma\epsilon)\cdot\gamma}$ (1.38)

$\varepsilon_{A_\varsigma B_\varsigma}, \varepsilon^{A_\varsigma B_\varsigma}$ are constant tensors, $\varepsilon_{12} = \varepsilon^{12} = 1, {}^{A_\varsigma}, {}^{B_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}, {}_{A_\varsigma}, {}_{B_\varsigma} \sim e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$ (1.39)

$\varepsilon_{A'_\varsigma B'_\varsigma}, \varepsilon^{A'_\varsigma B'_\varsigma}$ are constant tensors, $\varepsilon_{1'2'} = \varepsilon^{1'2'} = 1, {}^{A'_\varsigma}, {}^{B'_\varsigma} \sim e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}, {}_{A'_\varsigma}, {}_{B'_\varsigma} \sim e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma^T}$ (1.40)

2.6 Fundamental Theorem One and its relevant constant tensors

2.6.1 Preparation

Lemma 2.6.1. $\vartheta_a{}^b(\Gamma, i\varsigma)_b \equiv (-\omega\times\Gamma - \varsigma\epsilon, -i\epsilon\cdot\Gamma)_a \quad \vartheta_a{}^b \succ \vartheta \equiv (i\omega\cdot R + \epsilon\cdot L) = \begin{bmatrix} 0 & \omega_z & -\omega_y & i\epsilon_x \\ -\omega_z & 0 & \omega_x & i\epsilon_y \\ \omega_y & -\omega_x & 0 & i\epsilon_z \\ -i\epsilon_x & -i\epsilon_y & -i\epsilon_z & 0 \end{bmatrix}$

Lemma 2.6.2. $\frac{1}{2}i\omega\cdot[\Gamma, \Gamma_{\alpha_\varsigma}] = (\omega\times\Gamma)_{\alpha_\varsigma}, \forall\omega\to 0 \Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$

Lemma 2.6.3. $\frac{1}{2}\epsilon\cdot\{\Gamma, \Gamma_{\alpha_\varsigma}\} = \epsilon_{\alpha_\varsigma}, \forall\epsilon\to 0 \Leftrightarrow \{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}$

2.6.2 Fundamental Theorem One

Theorem 2.6.1. $(\Gamma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}(\Gamma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma} \Leftrightarrow \begin{cases} [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma} \\ \{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma} \end{cases}$

Proof: $(\Gamma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}(\Gamma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}, \forall\omega, \forall\epsilon$

$\Leftrightarrow (\Gamma, i\varsigma)_a = (\delta_a{}^b + \vartheta_a{}^b)(1 + (i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma)(\Gamma, i\varsigma)_b(1 - (i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma), \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow 0 = \vartheta_a{}^b(\Gamma, i\varsigma)_b + \frac{1}{2}i\omega\cdot[\Gamma, (\Gamma, i\varsigma)_a] + \frac{1}{2}\epsilon\cdot\{\Gamma, (\Gamma, i\varsigma)_a\}, \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow 0 = (-\omega\times\Gamma - \varsigma\epsilon, -i\epsilon\cdot\Gamma)_a + \frac{1}{2}i\omega\cdot[\Gamma, (\Gamma, i\varsigma)_a] + \frac{1}{2}\epsilon\cdot\{\Gamma, (\Gamma, i\varsigma)_a\}, \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow 0 = (-\omega\times\Gamma - \varsigma\epsilon)_{\alpha_\varsigma} + \frac{1}{2}i\omega\cdot[\Gamma, \Gamma_{\alpha_\varsigma}] + \frac{1}{2}\epsilon\cdot\{\Gamma, \Gamma_{\alpha_\varsigma}\}, \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow \frac{1}{2}i\omega\cdot[\Gamma, \Gamma_{\alpha_\varsigma}] = (\omega\times\Gamma)_{\alpha_\varsigma}, \frac{1}{2}\epsilon\cdot\{\Gamma, \Gamma_{\alpha_\varsigma}\} = \epsilon_{\alpha_\varsigma}, \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}, \{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}$ □

2.6.3 Constant tensors [4] $(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}, (\sigma, i\varsigma)_{a'}{}^{A'_\varsigma A_\varsigma}, (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}, (\sigma, -i\varsigma)^{a'}{}_{A_\varsigma A'_\varsigma}$

Corollary 2.6.1. $(\sigma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \rightarrow$ *Constant tensor:* $(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}$

Corollary 2.6.2. $(\sigma, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \rightarrow$ *Constant tensor:* $(\sigma, i\varsigma)_{a'}{}^{A'_\varsigma A_\varsigma}$

Corollary 2.6.3. $(\sigma, -i\varsigma)^a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]^a{}_b e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, -i\varsigma)^b e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \rightarrow$ *Constant tensor:* $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}$

Corollary 2.6.4. $(\sigma, -i\varsigma)^{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]^{a'}{}_{b'} e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, -i\varsigma)^{b'} e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \rightarrow$ *Constant tensor:* $(\sigma, -i\varsigma)^{a'}{}_{A_\varsigma A'_\varsigma}$

2.6.4 Transition

Corollary 2.6.5. $(\sigma\otimes I, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}(\sigma\otimes I, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}$

Corollary 2.6.6. $(\sigma\otimes I, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}(\sigma\otimes I, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}$

Corollary 2.6.7. $(I\otimes\sigma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}(\sigma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}$

Corollary 2.6.8. $(I\otimes\sigma, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}(\sigma, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}$

Corollary 2.6.9. $(\sigma_+, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}(\sigma_+, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$

Corollary 2.6.10. $(\sigma_+, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}(\sigma_+, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$

Corollary 2.6.11. $(\sigma_-, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}(\sigma_-, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$

Corollary 2.6.12. $(\sigma_-, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}(\sigma_-, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$

2.6.5 Constant tensors $(\sigma_+, i\varsigma)_a{}^{\alpha_\varsigma b'_\varsigma}, (\sigma_+, i\varsigma)_{a'}{}^{\alpha'_\varsigma b_\varsigma}, (\sigma_+, i\varsigma)_a{}^{b_\varsigma\alpha'_\varsigma}, (\sigma_+, i\varsigma)_{a'}{}^{b'_\varsigma\alpha_\varsigma}$

Corollary 2.6.13. $(\sigma_+, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_+, i\varsigma)_b e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_+, i\varsigma)_a{}^{\alpha_\varsigma b'_\varsigma}$

Corollary 2.6.14. $(\sigma_+, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_+, i\varsigma)_{b'} e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_+, i\varsigma)_{a'}{}^{\alpha'_\varsigma b_\varsigma}$

Corollary 2.6.15. $(\sigma_+, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_+, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_+, i\varsigma)_a{}^{b_\varsigma\alpha'_\varsigma}$

Corollary 2.6.16. $(\sigma_+, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_+, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_+, i\varsigma)_{a'}{}^{b'_\varsigma\alpha_\varsigma}$

2.6.6 Constant tensors $(\sigma_+, -i\varsigma)_a{}^{\alpha'_\varsigma b_\varsigma}, (\sigma_+, -i\varsigma)_{a'}{}^{\alpha_\varsigma b'_\varsigma}, (\sigma_+, -i\varsigma)_a{}^{b'_\varsigma\alpha_\varsigma}, (\sigma_+, -i\varsigma)_{a'}{}^{b_\varsigma\alpha'_\varsigma}$

Corollary 2.6.17. $(\sigma_+, -i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_+, -i\varsigma)_b e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_+, -i\varsigma)_a{}^{\alpha'_\varsigma b_\varsigma}$

Corollary 2.6.18. $(\sigma_+, -i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_+, -i\varsigma)_{b'} e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_+, -i\varsigma)_{a'}{}^{\alpha_\varsigma b'_\varsigma}$

Corollary 2.6.19. $(\sigma_+, -i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_+, -i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_+, -i\varsigma)_a{}^{b'_\varsigma\alpha_\varsigma}$

Corollary 2.6.20. $(\sigma_+, -i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_+, -i\varsigma)_{b'} e^{-(i\omega-\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_+, -i\varsigma)_{a'}{}^{b_\varsigma\alpha'_\varsigma}$

2.6.7 Constant tensors $(\sigma_-, i\varsigma)_a{}^{\alpha_\varsigma b_\varsigma}, (\sigma_-, i\varsigma)_{a'}{}^{\alpha'_\varsigma b'_\varsigma}, (\sigma_-, i\varsigma)_a{}^{b'_\varsigma\alpha'_\varsigma}, (\sigma_-, i\varsigma)_{a'}{}^{b_\varsigma\alpha_\varsigma}$

Corollary 2.6.21. $(\sigma_-, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_-, i\varsigma)_b e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_-, i\varsigma)_a{}^{\alpha_\varsigma b_\varsigma}$

Corollary 2.6.22. $(\sigma_-, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_-, i\varsigma)_{b'} e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)} \rightarrow$ *Constant tensor:* $(\sigma_-, i\varsigma)_{a'}{}^{\alpha'_\varsigma b'_\varsigma}$

Corollary 2.6.23. $(\sigma_-, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_-, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_-, i\varsigma)_a{}^{b'_\varsigma\alpha'_\varsigma}$

Corollary 2.6.24. $(\sigma_-, i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_-, i\varsigma)_{b'} e^{-(i\omega+\varsigma\epsilon)\cdot R} \rightarrow$ *Constant tensor:* $(\sigma_-, i\varsigma)_{a'}{}^{b_\varsigma\alpha_\varsigma}$

2.6.8 Constant tensors $(\sigma_-, -i\varsigma)_a{}^{\alpha'_\varsigma b'_\varsigma}, (\sigma_-, -i\varsigma)_{a'}{}^{\alpha_\varsigma b_\varsigma}, (\sigma_-, -i\varsigma)_a{}^{b_\varsigma \alpha_\varsigma}, (\sigma_-, -i\varsigma)_{a'}{}^{b'_\varsigma \alpha'_\varsigma}$

Corollary 2.6.25. $(\sigma_-, -i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_-, -i\varsigma)_b e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)} \to$ *Constant tensor:* $(\sigma_-, i\varsigma)_a{}^{\alpha'_\varsigma b'_\varsigma}$

Corollary 2.6.26. $(\sigma_-, -i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_-, -i\varsigma)_{b'} e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)} \to$ *Constant tensor:* $(\sigma_-, i\varsigma)_{a'}{}^{\alpha_\varsigma b_\varsigma}$

Corollary 2.6.27. $(\sigma_-, -i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_-, -i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot R} \to$ *Constant tensor:* $(\sigma_-, i\varsigma)_a{}^{b_\varsigma \alpha_\varsigma}$

Corollary 2.6.28. $(\sigma_-, -i\varsigma)_{a'} = [e^{(i\omega\cdot R-\epsilon\cdot L)}]_{a'}{}^{b'} e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_-, -i\varsigma)_{b'} e^{-(i\omega-\varsigma\epsilon)\cdot R} \to$ *Constant tensor:* $(\sigma_-, i\varsigma)_{a'}{}^{b'_\varsigma \alpha'_\varsigma}$

2.7 Fundamental Theorem Two and its relevant constant tensors

2.7.1 Fundamental Theorem Two

Theorem 2.7.1. $\Gamma_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}\Gamma_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma} \Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$

Proof: $\Gamma_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}\Gamma_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma}, \forall\omega, \forall\epsilon$

$\Leftrightarrow \Gamma_{\alpha_\varsigma} = [\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} + (i\omega+\varsigma\epsilon)\cdot\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}][1 + (i\omega+\varsigma\epsilon)\cdot\Gamma]\Gamma_{\beta_\varsigma}[1 - (i\omega+\varsigma\epsilon)\cdot\Gamma], \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow 0 = (i\omega+\varsigma\epsilon)\cdot\{\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}]\}, \forall\omega\to 0, \forall\epsilon\to 0$

$\Leftrightarrow \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}] = 0$

$\Leftrightarrow \gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma} + [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 0(\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \equiv i\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma})$

$\Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$ □

2.7.2 Constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s), \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)$

Corollary 2.7.1. $\sigma_{\alpha_\varsigma}(s) = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\sigma_{\beta_\varsigma}(s) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \to$ *Constant tensor:* $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s)$

Corollary 2.7.2. $\sigma^{\alpha'_\varsigma}(s) = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]^{\alpha'_\varsigma}{}_{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)}\sigma^{\beta'_\varsigma}(s) e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s)} \to$ *Constant tensor:* $\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)$

2.7.3 Constant tensors $(\sigma, ik)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s), (\sigma, ik)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)$

Corollary 2.7.3. $(\sigma, ik)_{\alpha_\varsigma}(s) = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}(\sigma, ik)_{\beta_\varsigma}(s) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \to$ *Constant tensor:* $(\sigma, ik)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s)$

Corollary 2.7.4. $(\sigma, ik)^{\alpha'_\varsigma}(s) = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]^{\alpha'_\varsigma}{}_{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)}(\sigma, ik)^{\beta'_\varsigma}(s) e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s)} \to$ *Constant tensor:* $(\sigma, ik)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)$

2.7.4 Constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}, \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

Corollary 2.7.5. $\sigma_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \to$ *Constant tensor:* $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}$

Corollary 2.7.6. $\sigma^{\alpha'_\varsigma} = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]^{\alpha'_\varsigma}{}_{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma^{\beta'_\varsigma} e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \to$ *Constant tensor:* $\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

2.7.5 Constant tensors $\gamma_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \gamma^{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}$

Corollary 2.7.7. $\gamma_{\alpha_\varsigma}(s) = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\gamma}\gamma_{\beta_\varsigma}(s) e^{-(i\omega+\varsigma\epsilon)\cdot\gamma} \to$ *Constant tensor:* $\gamma_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(\equiv -i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma})$

Corollary 2.7.8. $\gamma^{\alpha'_\varsigma}(s) = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]^{\alpha'_\varsigma}{}_{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\gamma}\gamma^{\beta'_\varsigma}(s) e^{-(i\omega-\varsigma\epsilon)\cdot\gamma} \to$ *Constant tensor:* $\gamma^{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}(\equiv -i\varepsilon^{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma})$

2.7.6 Constant tensors $(\sigma, ik)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}, (\sigma, ik)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

Corollary 2.7.9. $(\sigma, ik)_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, ik)_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \to$ *Constant tensor:* $(\sigma, ik)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}$

Corollary 2.7.10. $(\sigma, ik)^{\alpha'_\varsigma} = [e^{(i\omega-\varsigma\epsilon)\cdot R}]^{\alpha'_\varsigma}{}_{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, ik)^{\beta'_\varsigma} e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \to$ *Constant tensor:* $(\sigma, ik)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

2.7.7 Transition

Corollary 2.7.11. $\sigma_{\alpha_\varsigma} \otimes I = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}(\sigma_{\beta_\varsigma} \otimes I) e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma\otimes I}$

Corollary 2.7.12. $I\otimes\sigma_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}(I\otimes\sigma_{\beta_\varsigma})e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}I\otimes\sigma}$

Corollary 2.7.13. $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}\sigma_{+\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$

Corollary 2.7.14. $\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}\sigma_{-\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$

2.7.8 Constant tensors $\sigma_{+\alpha_\varsigma}{}^{a_\varsigma b_\varsigma},\sigma_{+\alpha'_\varsigma}{}^{a'_\varsigma b'_\varsigma},\sigma_{+\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},\sigma_{+\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.15. $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\sigma_{+\beta_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $\sigma_{+\alpha_\varsigma}{}^{a_\varsigma b_\varsigma}$

Corollary 2.7.16. $\sigma_{+\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\sigma_{+\beta'_\varsigma}e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $\sigma_{+\alpha'_\varsigma}{}^{a'_\varsigma b'_\varsigma}$

Corollary 2.7.17. $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}\sigma_{+\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $\sigma_{+\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.18. $\sigma_{+\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}\sigma_{+\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $\sigma_{+\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.9 Constant tensors $(\sigma_+,ik)_{\alpha_\varsigma}{}^{a_\varsigma b_\varsigma},(\sigma_+,ik)_{\alpha'_\varsigma}{}^{a'_\varsigma b'_\varsigma},(\sigma_+,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},(\sigma_+,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.19. $(\sigma_+,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_+,ik)_{\beta_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $(\sigma_+,ik)_{\alpha_\varsigma}{}^{a_\varsigma b_\varsigma}$

Corollary 2.7.20. $(\sigma_+,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_+,ik)_{\beta'_\varsigma}e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $(\sigma_+,ik)_{\alpha'_\varsigma}{}^{a'_\varsigma b'_\varsigma}$

Corollary 2.7.21. $(\sigma_+,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_+,ik)_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(\sigma_+,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.22. $(\sigma_+,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_+,ik)_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(\sigma_+,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.10 Constant tensors $\sigma_{-\alpha_\varsigma}{}^{a'_\varsigma b'_\varsigma},\sigma_{-\alpha'_\varsigma}{}^{a_\varsigma b_\varsigma},\sigma_{-\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},\sigma_{-\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.23. $\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\sigma_{-\beta_\varsigma}e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $\sigma_{-\alpha_\varsigma}{}^{a'_\varsigma b'_\varsigma}$

Corollary 2.7.24. $\sigma_{-\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\sigma_{-\beta'_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $\sigma_{-\alpha'_\varsigma}{}^{a_\varsigma b_\varsigma}$

Corollary 2.7.25. $\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}\sigma_{-\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $\sigma_{-\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.26. $\sigma_{-\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}\sigma_{-\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $\sigma_{-\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.11 Constant tensors $(\sigma_-,ik)_{\alpha_\varsigma}{}^{a'_\varsigma b'_\varsigma},(\sigma_-,ik)_{\alpha'_\varsigma}{}^{a_\varsigma b_\varsigma},(\sigma_-,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},(\sigma_-,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.27. $(\sigma_-,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}(\sigma_-,ik)_{\beta_\varsigma}e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $(\sigma_-,ik)_{\alpha_\varsigma}{}^{a'_\varsigma b'_\varsigma}$

Corollary 2.7.28. $(\sigma_-,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}(\sigma_-,ik)_{\beta'_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}\rightarrow$ *Constant tensor:* $(\sigma_-,ik)_{\alpha'_\varsigma}{}^{a_\varsigma b_\varsigma}$

Corollary 2.7.29. $(\sigma_-,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}(\sigma_-,ik)_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(\sigma_-,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.30. $(\sigma_-,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}(\sigma_-,ik)_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(\sigma_-,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.12 Constant tensors $R_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},R_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.31. $R_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}R_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $R_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.32. $R_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}R_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $R_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.13 Constant tensors $(R,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma},(R,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.33. $(R,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}(R,ik)_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(R,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.34. $(R,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}(R,ik)_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(R,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.14 Constant tensors $L_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}, L_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.35. $L_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}L_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $L_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.36. $L_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}L_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $L_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.7.15 Constant tensors $(L,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}, (L,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

Corollary 2.7.37. $(L,ik)_{\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot R}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot R}(L,ik)_{\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(L,ik)_{\alpha_\varsigma}{}^{\rho_\varsigma\eta_\varsigma}$

Corollary 2.7.38. $(L,ik)_{\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot R}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega-\varsigma\epsilon)\cdot R}(L,ik)_{\beta'_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot R}\rightarrow$ *Constant tensor:* $(L,ik)_{\alpha'_\varsigma}{}^{\rho'_\varsigma\eta'_\varsigma}$

2.8 Fundamental Theorem Three and its relevant constant tensors

2.8.1 Fundamental Theorem Three

In any N+1 dimensional spacetime, there is a following theorem.

Theorem 2.8.1. $\Gamma_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}\Gamma_{ef}}\Gamma_{cd}e^{-\frac{1}{2}\vartheta^{ef}\Gamma_{ef}}, \vartheta_{ab}=-\vartheta_{ba}$
$\Leftrightarrow [\Gamma_{ab},\Gamma_{cd}]=\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}$

Proof: $\Gamma_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}\Gamma_{ef}}\Gamma_{cd}e^{-\frac{1}{2}\vartheta^{ef}\Gamma_{ef}}, \forall\vartheta^{ef}$

$\Leftrightarrow \Gamma_{ab}=[\delta_a{}^c+\vartheta_a{}^c][\delta_b{}^d+\vartheta_b{}^d](1+\frac{1}{2}\vartheta^{ef}\Gamma_{ef})\Gamma_{cd}(1-\frac{1}{2}\vartheta^{ef}\Gamma_{ef}), \forall\vartheta^{ef}\rightarrow 0$

$\Leftrightarrow 0=\vartheta_a{}^c\Gamma_{cb}-\vartheta_b{}^d\Gamma_{da}-\frac{1}{2}\vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}], \forall\vartheta^{cd}\rightarrow 0$

$\Leftrightarrow \vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}]=2(\vartheta_a{}^c\Gamma_{cb}-\vartheta_b{}^d\Gamma_{da}), \forall\vartheta^{cd}\rightarrow 0$

$\Leftrightarrow \vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}]=\vartheta^{cd}(\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}), \forall\vartheta^{cd}\rightarrow 0$

$\Leftrightarrow [\Gamma_{ab},\Gamma_{cd}]=\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}$ □

2.8.2 Spin constant tensors [11] $S_{ab}{}^{\mathcal{A}}{}_{\mathcal{B}}, S_{a'b'}{}^{\mathcal{A}'}{}_{\mathcal{B}'}$

Corollary 2.8.1. $S_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}S_{ef}}S_{cd}e^{-\frac{1}{2}\vartheta^{ef}S_{ef}}, \vartheta=i\omega\cdot R+\epsilon\cdot L\rightarrow$ *Constant tensor:* $S_{ab}{}^{\mathcal{A}}{}_{\mathcal{B}}$

Corollary 2.8.2. $S_{a'b'}=[e^{\vartheta'}]_{a'}{}^{c'}[e^{\vartheta'}]_{b'}{}^{d'}e^{\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}}S_{c'd'}e^{-\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}}, \vartheta'=i\omega\cdot R-\epsilon\cdot L\rightarrow$ *Constant tensor:* $S_{a'b'}{}^{\mathcal{A}'}{}_{\mathcal{B}'}$

2.8.3 Spin constant tensors $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma), S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma)$

Corollary 2.8.3. $S_{ab}(s,\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}S_{ef}(s,\varsigma)}S_{cd}(s,\varsigma)e^{-\frac{1}{2}\vartheta^{ef}S_{ef}(s,\varsigma)}$

$\vartheta=i\omega\cdot R+\epsilon\cdot L, S_{ab}(s,\varsigma)\succ i\begin{bmatrix}0&\sigma_z(s)&-\sigma_y(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&\sigma_x(s)&-\varsigma\sigma_y(s)\\\sigma_y(s)&-\sigma_x(s)&0&-\varsigma\sigma_z(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&\varsigma\sigma_z(s)&0\end{bmatrix}\rightarrow$ *Constant tensor:* $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma)$

Corollary 2.8.4. $S_{ab}(s,-\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}S_{ef}(s,-\varsigma)}S_{cd}(s,-\varsigma)e^{-\frac{1}{2}\vartheta^{ef}S_{ef}(s,-\varsigma)}\rightarrow$ *Constant tensor:* $S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma)$

2.8.4 Spin constant tensors $S_{a'b'A'_\varsigma}{}^{B'_\varsigma}(s,\varsigma)(s,\varsigma), S_{a'b'}{}^{A_\varsigma}{}_{B_\varsigma}(s,-\varsigma)$

Corollary 2.8.5. $S_{a'b'}(s,\varsigma)=[e^{\vartheta'}]_{a'}{}^{c'}[e^{\vartheta'}]_{b'}{}^{d'}e^{\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(s,\varsigma)}S_{c'd'}(s,\varsigma)e^{-\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(s,\varsigma)}$

$\vartheta'=i\omega\cdot R-\epsilon\cdot L, S_{a'b'}(s,\varsigma)\succ i\begin{bmatrix}0&\sigma_z(s)&-\sigma_y(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&\sigma_x(s)&-\varsigma\sigma_y(s)\\\sigma_y(s)&-\sigma_x(s)&0&-\varsigma\sigma_z(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&\varsigma\sigma_z(s)&0\end{bmatrix}\rightarrow$ *Constant tensor:* $S_{a'b'A'_\varsigma}{}^{B'_\varsigma}(s,\varsigma)(s,\varsigma)$

Corollary 2.8.6. $S_{a'b'}(s,-\varsigma)=[e^{\vartheta'}]_{a'}{}^{c'}[e^{\vartheta'}]_{b'}{}^{d'}e^{\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(s,-\varsigma)}S_{c'd'}(s,-\varsigma)e^{-\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(s,-\varsigma)}$
$\rightarrow$ *Constant tensor:* $S_{a'b'}{}^{A_\varsigma}{}_{B_\varsigma}(s,-\varsigma)$

2.8.5 Spin constant tensors $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(\varsigma), S_{abA'_\varsigma}{}^{B'_\varsigma}(-\varsigma)$

Corollary 2.8.7. $S_{ab}(\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}S_{ef}(\varsigma)}S_{cd}(\varsigma)e^{-\frac{1}{2}\vartheta^{ef}S_{ef}(\varsigma)}$

$\vartheta=i\omega\cdot R+\epsilon\cdot L, S_{ab}(\varsigma)\succ\frac{i}{2}\begin{bmatrix}0&\sigma_z&-\sigma_y&-\varsigma\sigma_x\\-\sigma_z&0&\sigma_x&-\varsigma\sigma_y\\\sigma_y&-\sigma_x(s)&0&-\varsigma\sigma_z\\\varsigma\sigma_x&\varsigma\sigma_y&\varsigma\sigma_z&0\end{bmatrix}\rightarrow$ *Constant tensor:* $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(\varsigma)$

Corollary 2.8.8. $S_{ab}(-\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{1}{2}\vartheta^{ef}S_{ef}(-\varsigma)}S_{cd}(-\varsigma)e^{-\frac{1}{2}\vartheta^{ef}S_{ef}(-\varsigma)}\rightarrow$ *Constant tensor:* $S_{abA'_\varsigma}{}^{B'_\varsigma}(-\varsigma)$

2.8.6 Spin constant tensors $S_{a'b'A'_\varsigma}{}^{B'_\varsigma}(\varsigma), S_{a'b'}{}^{A_\varsigma}{}_{B_\varsigma}(-\varsigma)$

Corollary 2.8.9. $S_{a'b'}(\varsigma) = [e^{\vartheta'}]_{a'}{}^{c'}[e^{\vartheta'}]_{b'}{}^{d'} e^{\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(\varsigma)} S_{c'd'}(\varsigma) e^{-\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(\varsigma)}$

$\vartheta' = i\omega \cdot R - \epsilon \cdot L, S_{a'b'}(\varsigma) \succ \frac{i}{2}\begin{bmatrix} 0 & \sigma_z & -\sigma_y & -\varsigma\sigma_x \\ -\sigma_z & 0 & \sigma_x & -\varsigma\sigma_y \\ \sigma_y & -\sigma_x & 0 & -\varsigma\sigma_z \\ \varsigma\sigma_x & \varsigma\sigma_y & \varsigma\sigma_z & 0 \end{bmatrix} \to$ *Constant tensor:* $S_{a'b'A'_\varsigma}{}^{B'_\varsigma}(\varsigma)$

Corollary 2.8.10. $S_{a'b'}(-\varsigma) = [e^{\vartheta'}]_{a'}{}^{c'}[e^{\vartheta'}]_{b'}{}^{d'} e^{\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(-\varsigma)} S_{c'd'}(-\varsigma) e^{-\frac{1}{2}\vartheta'^{e'f'}S_{e'f'}(-\varsigma)} \to$ *Constant tensor:* $S_{a'b'}{}^{A_\varsigma}{}_{B_\varsigma}(-\varsigma)$

2.9 Fundamental Theorem Four and its relevant constant tensors

2.9.1 Fundamental Theorem Four

In any N+1 dimensional spacetime, there is a following theorem.

Theorem 2.9.1. $\Gamma_a = [e^{\vartheta}]_a{}^b e^{\frac{1}{2}\vartheta^{cd}S_{cd}} \Gamma_b e^{-\frac{1}{2}\vartheta^{cd}S_{cd}}, S_{cd} = \frac{1}{4}[\Gamma_c, \Gamma_d] \Leftrightarrow \frac{1}{4}[[\Gamma_c, \Gamma_d], \Gamma_a] = \Gamma_{[c}\delta_{d]a}$

Proof: $\Gamma_a = [e^{\vartheta}]_a{}^b e^{\frac{1}{2}\vartheta^{cd}S_{cd}} \Gamma_b e^{-\frac{1}{2}\vartheta^{cd}S_{cd}}$

$\Leftrightarrow \Gamma_a = (1+\vartheta)_a{}^b(1 + \frac{1}{2}\vartheta^{cd}S_{cd})\Gamma_b(1 - \frac{1}{2}\vartheta^{cd}S_{cd})$

$\Leftrightarrow 0 = \vartheta_a{}^b\Gamma_b + \frac{1}{2}\vartheta^{cd}[S_{cd}, \Gamma_a]$

$\Leftrightarrow 0 = -\frac{1}{2}\vartheta^{cd}\Gamma_{[c}\delta_{d]a} + \frac{1}{2}\vartheta^{cd}[S_{cd}, \Gamma_a]$

$\Leftrightarrow [S_{cd}, \Gamma_a] = \Gamma_{[c}\delta_{d]a}$

$\Leftrightarrow \frac{1}{4}[[\Gamma_c, \Gamma_d], \Gamma_a] = \Gamma_{[c}\delta_{d]a}$ □

Proposition 2.9.1. $[[\Gamma_c, \Gamma_d], \Gamma_a] = \frac{1}{2}(\{\Gamma_c, \{\Gamma_d, \Gamma_a\}\} - \{\{\Gamma_a, \Gamma_c\}, \Gamma_d\})$

2.9.2 Constant tensors [7] $\gamma_a{}^{\lambda_\varsigma}{}_{\mu_\varsigma}(\varsigma), \gamma_5{}^{\lambda_\varsigma}{}_{\mu_\varsigma}(\varsigma), \delta^{\lambda_\varsigma}{}_{\mu_\varsigma}$

Definiton 2.9.1. $\gamma_5(\varsigma) \equiv \gamma_x(\varsigma)\gamma_y(\varsigma)\gamma_z(\varsigma)\gamma_\pi(\varsigma), S_{ab}(e,\varsigma) \equiv \frac{1}{4}[\gamma_a(\varsigma), \gamma_b(\varsigma)]$

Definiton 2.9.2. ${}^{\lambda_\varsigma} \sim e^{\frac{1}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}, {}_{\mu_\varsigma} \sim e^{-\frac{1}{2}\vartheta^{ab}S^T_{ab}(e,\varsigma)}$

Definiton 2.9.3. *A special representation:* $[\gamma_a(\varsigma), \gamma_5(\varsigma)] \equiv [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$

Corollary 2.9.1. $\gamma_a(\varsigma) = [e^{\vartheta}]_a{}^b e^{\frac{1}{2}\vartheta^{cd}S_{cd}(e,\varsigma)}\gamma_b(\varsigma)e^{-\frac{1}{2}\vartheta^{cd}S_{cd}(e,\varsigma)} \to$ *Constant tensor:* $\gamma_a{}^{\lambda_\varsigma}{}_{\mu_\varsigma}(\varsigma)$

Corollary 2.9.2. $\gamma_5(\varsigma) = e^{\frac{1}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}\gamma_5(\varsigma)e^{-\frac{1}{2}\vartheta^{ab}S_{ab}(e,\varsigma)} \to$ *Constant tensor:* $\gamma_5{}^{\lambda_\varsigma}{}_{\mu_\varsigma}(\varsigma)$

Corollary 2.9.3. $I_4 = e^{\frac{1}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}I_4 e^{-\frac{1}{2}\vartheta^{ab}S_{ab}(e,\varsigma)} \to$ *Constant tensor:* $\delta^{\lambda_\varsigma}{}_{\mu_\varsigma}$

2.10 Fundamental Theorem Five and its relevant constant tensors

2.10.1 Fundamental Theorem Five

Theorem 2.10.1. $T_\alpha = [e^{\theta^\gamma f_\gamma}]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma} T_\beta e^{-i\theta^\gamma T_\gamma} \Leftrightarrow [T_\alpha, T_\beta] = if_{\alpha\beta}{}^\gamma T_\gamma$

Proof: $T_\alpha = [e^{\theta^\gamma f_\gamma}]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma} T_\beta e^{-\theta^{i\gamma} T_\gamma}, \forall\theta^\gamma$

$\Leftrightarrow T_\alpha = (\delta_\alpha{}^\beta + \theta^\gamma f_{\gamma\alpha}{}^\beta)(1 + i\theta^\gamma T_\gamma)T_\beta(1 - i\theta^\gamma T_\gamma), \forall\theta^\gamma \to 0$

$\Leftrightarrow 0 = \theta^\gamma(f_{\gamma\alpha}{}^\beta T_\beta + i[T_\gamma, T_\alpha]), \forall\theta^\gamma \to 0$

$\Leftrightarrow 0 = f_{\gamma\alpha}{}^\beta T_\beta + i[T_\gamma, T_\alpha]$

$\Leftrightarrow [T_\alpha, T_\beta] = if_{\alpha\beta}{}^\gamma T_\gamma$ □

Obviously, Fundamental Theorem Two is a special case of this theorem. $(f_{\alpha\beta}{}^\gamma = \varepsilon_{\alpha\beta}{}^\gamma, i\theta^\gamma = i\omega + \varsigma\epsilon)$

2.11 Fundamental Theorem Six and its relevant constant tensors

2.11.1 Fundamental Theorem Six

Theorem 2.11.1. $\Gamma = e^{(i\omega\cdot R + \varsigma\epsilon\cdot L)}\Gamma e^{-(i\omega\cdot R - \varsigma\epsilon\cdot L)} \Leftrightarrow [R, \Gamma] = 0, \{L, \Gamma\} = 0$

Proof: $\Gamma = e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\Gamma e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}, \forall\omega, \forall\epsilon$

$\Leftrightarrow \Gamma = [1 + (i\omega\cdot R + \varsigma\epsilon\cdot L)]\Gamma[1 - (i\omega\cdot R - \varsigma\epsilon\cdot L)], \forall\omega \to 0, \forall\epsilon \to 0$

$\Leftrightarrow 0 = (i\omega\cdot R + \varsigma\epsilon\cdot L)]\Gamma - \Gamma(i\omega\cdot R - \varsigma\epsilon\cdot L), \forall\omega \to 0, \forall\epsilon \to 0$

$\Leftrightarrow 0 = i\omega\cdot [R,\Gamma] + \varsigma\epsilon\cdot \{L,\Gamma\}, \forall\omega \to 0, \forall\epsilon \to 0$

$\Leftrightarrow [R,\Gamma] = 0, \{L,\Gamma\} = 0$ □

Corollary 2.11.1. $\eta = e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\eta e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}, \eta = diag(1,1,1,-1) \to$ *Constant tensor:* $\eta^{a_\varsigma}{}_{b'_\varsigma} \Leftrightarrow \eta^{a}{}_{b'}, \eta^{a'}{}_{b}, \eta^{ab'}, \eta^{a'b}$

2.12 Various acquiring methods of new constant tensors

First method: $\epsilon \leftrightarrow -\epsilon$

Second method: $\varsigma \leftrightarrow -\varsigma$

Third method: Matrices operations, such as conjugate, transpose, similarity transformation, representation transformation etc.

Fourth method: Operations, such as direct product, direct sum, shrinkage and addition, subtraction, multiplication, division etc.

In addition, the above methods have been applied to get all kinds of constant tensors in six fundamental theorems and their corollaries. The proofs of various corollaries are basically obvious. And in order to compact the contents, the proofs processes are omitted.

2.13 A review of six fundamental theorems

There are no special limits about transformation parameters $\omega, \epsilon, \vartheta^{ab}, \theta^{\alpha}$ in mathematical proofs of six basic theorems. They can take any complex numbers. So the obtained constant tensors have a lot of mathematical universality. For specific physical system, the parameters of internal gauge transformation can still take complex numbers. However, for external spacetime transformation, the transformation must satisfy Lorenz representation due to physical self consistent requirement [15]. So the transform matrices and transform parameters are limited. And ω, ϵ can only take the real numbers. In particular, it is pointed out that Fundamental Theorems Three and Fundamental Theorems Four are not only established in the four-dimensional spacetime, but also in any N+1 dimensional spacetime. This provides a mathematical analysis tool for physical research of high and low dimensional spacetime. The proofs from six fundamental theorems can also be obtained as follows: The commutation and anticommutation relationships between matrices mean the existence of the corresponding constant tensors. On the contrary, a constant tensor means the existence of corresponding commutation and anticommutation relationships. From this line of thought, we can find more meaningful constant tensors. From above it can also be easy to know that the commutation and anticommutation relationships of maxtrices mean themselves covariant. Namely, the commutation and anticommutation relationships of maxtrices themselves imply their establishment in any reference frame. This is a very interesting and wonderful mathematical characteristic. It leads us to endless aftertastes.

3 Properties of several important constant tensors

3.1 Relationships between constant tensors

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma) = -\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma) \succ\prec S_{a'b'A'_\varsigma}{}^{B'_\varsigma}(s,\varsigma) = -\sigma_\varsigma^{\alpha'_\varsigma}{}_{a'b'}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma}(s,\varsigma) \tag{1.41}$$

$$S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) = -\sigma_{-\varsigma ab}^{\alpha'_\varsigma}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) \succ\prec S_{a'b'}{}^{A_\varsigma}{}_{B_\varsigma}(s,-\varsigma) = -\sigma_{-\varsigma a'b'}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s,-\varsigma) \tag{1.42}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} = i\gamma_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \varepsilon_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma} = i\gamma_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma} \tag{1.43}$$

3.2 Properties of basic constant tensors [4, 5] $(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}, (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}$

3.2.1 Transpose properties

From the perspective of spinors:

$$(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma} = (\sigma^T, i\varsigma)_a{}^{A'_\varsigma A_\varsigma} \qquad (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} = (\sigma^T, -i\varsigma)^a{}_{A_\varsigma A'_\varsigma} \tag{1.44}$$

$$(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma} = -\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma, -i\varsigma)_{aB'_\varsigma B_\varsigma} \qquad (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} = -\varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}(\sigma, i\varsigma)^{aB_\varsigma B'_\varsigma} \tag{1.45}$$

$$\varepsilon_{A_\varsigma B_\varsigma}(\sigma, i\varsigma)_a{}^{B_\varsigma B'_\varsigma} = -(\sigma, -i\varsigma)_{aA'_\varsigma A_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma} \qquad (\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma} = -\varepsilon^{A_\varsigma B_\varsigma}(\sigma, -i\varsigma)_{aB'_\varsigma B_\varsigma} \tag{1.46}$$

$$\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma, -i\varsigma)_{aB'_\varsigma B_\varsigma} = -(\sigma, i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\varepsilon_{A_\varsigma B_\varsigma} \qquad (\sigma, -i\varsigma)_{aA'_\varsigma A_\varsigma}\varepsilon^{A_\varsigma B_\varsigma} = -\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma, i\varsigma)_a{}^{B_\varsigma B'_\varsigma} \tag{1.47}$$

From the perspective of matrices:

$$(\sigma, i\varsigma)_a = \varepsilon(\sigma^T, -i\varsigma)_a\varepsilon = [\varepsilon(\sigma, -i\varsigma)_a\varepsilon]^T \qquad (\sigma, -i\varsigma)_a = \varepsilon(\sigma^T, i\varsigma)_a\varepsilon = [\varepsilon(\sigma, i\varsigma)_a\varepsilon]^T \tag{1.48}$$

$$\varepsilon(\sigma, i\varsigma)_a = -(\sigma^T, -i\varsigma)_a\varepsilon \qquad (\sigma, i\varsigma)_a\varepsilon = -\varepsilon(\sigma^T, -i\varsigma)_a \tag{1.49}$$

$$\varepsilon(\sigma, -i\varsigma)_a = -(\sigma^T, i\varsigma)_a\varepsilon \qquad (\sigma, -i\varsigma)_a\varepsilon = -\varepsilon(\sigma^T, i\varsigma)_a \tag{1.50}$$

3.2.2 Orthogonality properties

From the perspective of spinors:

$$(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)^b{}_{A'_\varsigma A_\varsigma} = 2\delta_a{}^b \qquad (\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)^a{}_{B'_\varsigma B_\varsigma} = 2\delta^{A_\varsigma}{}_{B_\varsigma}\delta^{A'_\varsigma}{}_{B'_\varsigma} \tag{1.51}$$

$$(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^{aB_\varsigma B'_\varsigma} = -2\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma} \qquad (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}(\sigma, -i\varsigma)_{aB'_\varsigma B_\varsigma} = -2\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma} \tag{1.52}$$

$$(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)^a{}_{A'_\varsigma B_\varsigma} = 4\delta^{A_\varsigma}{}_{B_\varsigma} \qquad (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}(\sigma, i\varsigma)_a{}^{A_\varsigma B'_\varsigma} = 4\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.53}$$

From the perspective of matrices:

$$tr[(\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b] = 2\delta_{ab} \qquad tr[(\sigma, -i\varsigma)_a(\sigma, i\varsigma)_b] = 2\delta_{ab} \tag{1.54}$$

$$(\sigma, i\varsigma)_a(\sigma, -i\varsigma)^a = 4I \qquad (\sigma, -i\varsigma)^a(\sigma, i\varsigma)_a = 4I \tag{1.55}$$

3.3 Properties of basic constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s), \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)$

3.3.1 Preparation

Use the formula: $\sum_{k=1}^{2s} k^2 = \frac{8}{3}s(s+\frac{1}{2})(s+\frac{1}{4})$

$$tr[\sigma_x^2(s)] = tr[\sigma_y^2(s)] = \frac{1}{4}\sum_{k=1}^{2s} 2k(2s+1-k) = \frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.56}$$

$$tr[\sigma_z^2(s)] = \frac{1}{4}\sum_{k=1}^{2s}(2s-2k)^2 = \frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.57}$$

$$tr[\sigma_x^2(s)] = tr[\sigma_y^2(s)] = tr[\sigma_z^2(s)] = \frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.58}$$

3.3.2 Orthogonality properties

From the perspective of spinors:

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s)\sigma_{\beta_\varsigma}{}^{B_\varsigma}{}_{A_\varsigma}(s) = \frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha_\varsigma\beta_\varsigma} \qquad \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(s)\sigma^{\alpha_\varsigma C_\varsigma}{}_{B_\varsigma}(s) = s(s+1)\delta^{A_\varsigma}{}_{B_\varsigma} \tag{1.59}$$

$$\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)\sigma^{\beta'_\varsigma}{}_{B'_\varsigma}{}^{A'_\varsigma}(s) = \frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(s)\sigma_{\alpha'_\varsigma C'_\varsigma}{}^{B'_\varsigma}(s) = s(s+1)\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.60}$$

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s)\sigma^{\alpha_\varsigma B_\varsigma}{}_{A_\varsigma}(s) = 2s(s+\frac{1}{2})(s+1) \qquad \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(s)\sigma_{\alpha'_\varsigma B'_\varsigma}{}^{A'_\varsigma}(s) = 2s(s+\frac{1}{2})(s+1) \tag{1.61}$$

From the perspective of matrices:

$$tr[\sigma_{\alpha_\varsigma}(s)\sigma_{\beta_\varsigma}(s)] = \frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha_\varsigma\beta_\varsigma} \qquad \sigma^2(s) = s(s+1) \tag{1.62}$$

$$tr[\sigma^{\alpha'_\varsigma}(s)\sigma^{\beta'_\varsigma}(s)] = \frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma^2(s) = s(s+1) \tag{1.63}$$

3.3.3 Non trace properties

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{A_\varsigma}(s) = 0 \qquad tr[\sigma_{\alpha_\varsigma}(s)] = 0 \qquad \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{A'_\varsigma}(s) = 0 \qquad tr[\sigma^{\alpha'_\varsigma}(s)] = 0 \tag{1.64}$$

3.4 Properties of basic constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}, \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

3.4.1 Orthogonality properties

From the perspective of spinors:

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}\sigma_{\beta_\varsigma}{}^{B_\varsigma}{}_{A_\varsigma} = 2\delta_{\alpha_\varsigma\beta_\varsigma} \qquad \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}\sigma^{\alpha_\varsigma C_\varsigma}{}_{B_\varsigma} = 3\delta^{A_\varsigma}{}_{B_\varsigma} \tag{1.65}$$

$$\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}\sigma^{\beta'_\varsigma}{}_{B'_\varsigma}{}^{A'_\varsigma} = 2\delta^{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(s)\sigma_{\alpha'_\varsigma C'_\varsigma}{}^{B'_\varsigma} = 3\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.66}$$

From the perspective of spinors:

$$tr[\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}] = 2\delta_{\alpha_\varsigma\beta_\varsigma} \qquad \sigma^2 = 3 \tag{1.67}$$

$$tr[\sigma^{\alpha'^\varsigma}\sigma^{\beta'_\varsigma}] = 2\delta^{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma^2 = 3 \tag{1.68}$$

3.4.2 Non trace properties

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{A_\varsigma} = 0 \qquad tr[\sigma_{\alpha_\varsigma}] = 0 \qquad \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{A'_\varsigma} = 0 \qquad tr[\sigma^{\alpha'_\varsigma}] = 0 \tag{1.69}$$

3.5 Properties of extended constant tensors $(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}, (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}$

3.5.1 Transpose properties

From the perspective of spinors:

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma} = (\sigma^T, i\kappa)_{\alpha_\varsigma B_\varsigma}{}^{A_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma} = (\sigma^T, i\kappa)^{\alpha'_\varsigma B'_\varsigma}{}_{A'_\varsigma} \tag{1.70}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma} = -\varepsilon^{A_\varsigma D_\varsigma}\varepsilon_{B_\varsigma C_\varsigma}(\sigma, -i\kappa)_{\alpha_\varsigma}{}^{C_\varsigma}{}_{D_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma} = -\varepsilon_{A'_\varsigma D'_\varsigma}\varepsilon^{B'_\varsigma C'_\varsigma}(\sigma, -i\kappa)^{\alpha'_\varsigma}{}_{C'_\varsigma}{}^{D'_\varsigma} \tag{1.71}$$

$$\varepsilon_{A_\varsigma C_\varsigma}(\sigma, i\kappa)_{\alpha_\varsigma}{}^{C_\varsigma}{}_{B_\varsigma} = -(\sigma, -i\kappa)_{\alpha_\varsigma}{}^{C_\varsigma}{}_{A_\varsigma}\varepsilon_{C_\varsigma B_\varsigma} \qquad \varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\kappa)^{\alpha'_\varsigma}{}_{C'_\varsigma}{}^{B'_\varsigma} = -(\sigma, -i\kappa)^{\alpha'_\varsigma}{}_{C'_\varsigma}{}^{A'_\varsigma}\varepsilon^{C'_\varsigma B'_\varsigma} \tag{1.72}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}\varepsilon^{C_\varsigma B_\varsigma} = -\varepsilon^{A_\varsigma C_\varsigma}(\sigma, -i\kappa)_{\alpha_\varsigma}{}^{B_\varsigma}{}_{C_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}\varepsilon_{C'_\varsigma B'_\varsigma} = -\varepsilon_{A'_\varsigma C'_\varsigma}(\sigma, -i\kappa)^{\alpha'_\varsigma}{}_{B'_\varsigma}{}^{C'_\varsigma} \tag{1.73}$$

From the perspective of spinors:

$$(\sigma, i\kappa)_{\alpha_\varsigma} = \varepsilon(\sigma^T, -i\kappa)_{\alpha_\varsigma}\varepsilon = [\varepsilon(\sigma, -i\kappa)_{\alpha_\varsigma}\varepsilon]^T \qquad (\sigma, i\kappa)_{\alpha'_\varsigma} = \varepsilon(\sigma^T, -i\kappa)_{\alpha'_\varsigma}\varepsilon = [\varepsilon(\sigma, -i\kappa)_{\alpha'_\varsigma}\varepsilon]^T \tag{1.74}$$

$$\varepsilon(\sigma, i\kappa)_{\alpha_\varsigma} = -(\sigma^T, -i\kappa)_{\alpha_\varsigma}\varepsilon \qquad \varepsilon(\sigma, i\kappa)^{\alpha'_\varsigma} = -(\sigma^T, -i\kappa)^{\alpha'_\varsigma}\varepsilon \tag{1.75}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}\varepsilon = -\varepsilon(\sigma^T, -i\kappa)_{\alpha_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}\varepsilon = -\varepsilon(\sigma^T, -i\kappa)^{\alpha'_\varsigma} \tag{1.76}$$

3.5.2 Orthogonality properties

From the perspective of spinors:

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(\sigma, -i\kappa)_{\beta_\varsigma}{}^{B_\varsigma}{}_{A_\varsigma} = 2\delta_{\alpha_\varsigma\beta_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(\sigma, -i\kappa)^{\beta'_\varsigma}{}_{B'_\varsigma}{}^{A'_\varsigma} = 2\delta^{\alpha'_\varsigma\beta'_\varsigma} \tag{1.77}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(\sigma, -i\kappa)^{\alpha_\varsigma C_\varsigma}{}_{B_\varsigma} = 4\delta^{A_\varsigma}{}_{B_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(\sigma, -i\kappa)_{\alpha'_\varsigma C'_\varsigma}{}^{B'_\varsigma} = 4\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.78}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(\sigma, i\kappa)^{\alpha_\varsigma C_\varsigma}{}_{B_\varsigma} = 2\delta^{A_\varsigma}{}_{B_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(\sigma, i\kappa)_{\alpha'_\varsigma C'_\varsigma}{}^{B'_\varsigma} = 2\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.79}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(\sigma, -i\kappa)^{\alpha_\varsigma C_\varsigma}{}_{D_\varsigma} = 2\delta^{A_\varsigma}{}_{D_\varsigma}\delta_{B_\varsigma}{}^{C_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(\sigma, -i\kappa)_{\alpha'_\varsigma C'_\varsigma}{}^{D'_\varsigma} = 2\delta_{A'_\varsigma}{}^{D'_\varsigma}\delta^{B'_\varsigma}{}_{C'_\varsigma} \tag{1.80}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(\sigma, i\kappa)^{\alpha_\varsigma C_\varsigma}{}_{D_\varsigma} = -2\varepsilon^{A_\varsigma C_\varsigma}\varepsilon_{B_\varsigma D_\varsigma} \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{B'_\varsigma}(\sigma, i\kappa)_{\alpha'_\varsigma C'_\varsigma}{}^{D'_\varsigma} = -2\varepsilon_{A'_\varsigma C'_\varsigma}\varepsilon^{B'_\varsigma D'_\varsigma} \tag{1.81}$$

From the perspective of spinors:

$$tr[(\sigma, i\kappa)_{\alpha_\varsigma}(\sigma, -i\kappa)_{\beta_\varsigma}] = 2\delta_{\alpha_\varsigma\beta_\varsigma} \qquad tr[(\sigma, i\kappa)^{\alpha'_\varsigma}(\sigma, -i\kappa)^{\beta'_\varsigma}] = 2\delta^{\alpha'_\varsigma\beta'_\varsigma} \tag{1.82}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}(\sigma, -i\kappa)^{\alpha_\varsigma} = 4I \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}(\sigma, -i\kappa)_{\alpha'_\varsigma} = 4I \tag{1.83}$$

$$(\sigma, i\kappa)_{\alpha_\varsigma}(\sigma, i\kappa)^{\alpha_\varsigma} = 2I \qquad (\sigma, i\kappa)^{\alpha'_\varsigma}(\sigma, i\kappa)_{\alpha'_\varsigma} = 2I \tag{1.84}$$

3.6 Properties of basic constant tensors $\sigma^{\alpha}_{+ab}, \sigma^{\alpha'}_{-ab}, \sigma^{\alpha_\varsigma}_{\varsigma ab}, \sigma^{\alpha'_\varsigma}_{-\varsigma ab}$

3.6.1 Orthogonality properties

From the perspective of spinors:

$$\sigma^{\alpha}_{+ab}\sigma^{\beta ab}_{+} = -4\delta^{\alpha\beta}, \sigma^{\alpha'}_{-ab}\sigma^{\beta' ab}_{-} = -4\delta^{\alpha'\beta'} \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma ab}_{\varsigma} = -4\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.85}$$

$$\sigma^{\alpha}_{+ab}\sigma^{\beta' ab}_{-} = 0, \sigma^{\alpha'}_{-ab}\sigma^{\beta ab}_{+} = 0 \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta'_\varsigma ab}_{-\varsigma} = 0 \tag{1.86}$$

The above orthogonality relationships can be summarized as a following more compact relationship.

$$\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa ab}_{\kappa} = -4\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa} \tag{1.87}$$

From the perspective of spinors:

$$tr(\sigma^{\alpha}_{+}\sigma^{\beta}_{+}) = 4\delta^{\alpha\beta}, tr(\sigma^{\alpha'}_{-}\sigma^{\beta'}_{-}) = 4\delta^{\alpha'\beta'} \qquad tr(\sigma^{\alpha_\varsigma}_{\varsigma}\sigma^{\beta_\varsigma}_{\varsigma}) = 4\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.88}$$

$$tr(\sigma^{\alpha}_{+}\sigma^{\beta'}_{-}) = 0, tr(\sigma^{\alpha'}_{-}\sigma^{\beta}_{+}) = 0 \qquad tr(\sigma^{\alpha_\varsigma}_{\varsigma}\sigma^{\beta'_\varsigma}_{-\varsigma}) = 0 \tag{1.89}$$

The above orthogonality relationships can be summarized as a following more compact relationship.

$$tr(\sigma^{\alpha_\varsigma}_{\varsigma}\sigma^{\beta_\kappa}_{\kappa}) = 4\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa} \tag{1.90}$$

3.6.2 Duality properties

$$\sigma^{\alpha}_{+ab} = -*\sigma^{\alpha}_{+ab}, \sigma^{\alpha'}_{-ab} = *\sigma^{\alpha'}_{-ab} \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab} = -\varsigma * \sigma^{\alpha_\varsigma}_{\varsigma ab} \tag{1.91}$$

3.7 Properties of extended constant tensors $(\sigma_+, i\kappa)^{\alpha}{}_{ab}, (\sigma_-, i\kappa)^{\alpha'}{}_{ab}$

3.7.1 Orthogonality properties

From the perspective of spinors:

$$(\sigma_+, i\kappa)^{\alpha}{}_{ab}(\sigma_+, i\kappa)^{\beta ab} = -4\delta^{\alpha\beta}, (\sigma_-, i\kappa)^{\alpha'}{}_{ab}(\sigma_-, i\kappa)^{\beta' ab} = -4\delta^{\alpha'\beta'} \tag{1.92}$$

The above orthogonality relationships can be summarized as a following more compact relationship.

$$(\sigma_\varsigma, i\kappa)^{\alpha_\varsigma}{}_{ab}(\sigma_\varsigma, i\kappa)^{\beta_\varsigma ab} = -4\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.93}$$

From the perspective of spinors:

$$tr[(\sigma_+, i\kappa)^{\alpha}(\sigma_+, -i\kappa)^{\beta}] = 4\delta^{\alpha\beta}, tr[(\sigma_-, i\kappa)^{\alpha'}(\sigma_-, -i\kappa)^{\beta'}] = 4\delta^{\alpha'\beta'} \tag{1.94}$$

The above orthogonality relationships can be summarized as a following more compact relationship.

$$tr[(\sigma_\varsigma, i\kappa)^{\alpha_\varsigma}(\sigma_\varsigma, -i\kappa)^{\beta_\varsigma}] = 4\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.95}$$

3.7.2 Identity properties

$$(\sigma_+, -i)^{\alpha}{}_{ab} = (\sigma_-, -i)_{ab}{}^{\alpha} \qquad (\sigma_-, i)^{\alpha'}{}_{ab} = (\sigma_+, i)_{ab}{}^{\alpha'} \tag{1.96}$$

$$(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ab} = (\sigma_{-\varsigma}, -i\varsigma)_{ab}{}^{\alpha_\varsigma} \qquad (\sigma_{-\varsigma}, i\varsigma)^{\alpha'_\varsigma}{}_{ab} = (\sigma_\varsigma, i\varsigma)_{ab}{}^{\alpha'_\varsigma} \tag{1.97}$$

3.8 Properties of spin constant tensors $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s, \varsigma), S_{abA'_\varsigma}{}^{B'_\varsigma}(s, -\varsigma)$

3.8.1 Compound properties

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s, \varsigma) = -\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s) \qquad S_{abA'_\varsigma}{}^{B'_\varsigma}(s, -\varsigma) = -\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma}(s) \tag{1.98}$$

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s) = \frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s, \varsigma) \qquad \sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma}(s) = \frac{1}{4}\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}S_{abA'_\varsigma}{}^{B'_\varsigma}(s, -\varsigma) \tag{1.99}$$

3.8.2 Orthogonality properties

$$\begin{cases} S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma)S_{cd}{}^{B_\varsigma}{}_{A_\varsigma}(s,\varsigma) = \frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} \\ S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma)S_{cdB'_\varsigma}{}^{A'_\varsigma}(s,-\varsigma) = \frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} \end{cases} \tag{1.100}$$

$$\begin{cases} S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma)S^{ab B_\varsigma}{}_{A_\varsigma}(s,\varsigma) = -8s(s+\frac{1}{2})(s+1) \\ S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma)S^{ab}{}_{B'_\varsigma}{}^{A'_\varsigma}(s,-\varsigma) = -8s(s+\frac{1}{2})(s+1) \end{cases} \tag{1.101}$$

$$\begin{cases} S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}(s,\varsigma)S^{ab C_\varsigma}{}_{B_\varsigma}(s,\varsigma) = -4s(s+1)\delta^{A_\varsigma}{}_{B_\varsigma} \\ S_{abA'_\varsigma}{}^{C'_\varsigma}(s,-\varsigma)S^{ab}{}_{C'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) = -4s(s+1)\delta_{A'_\varsigma}{}^{B'_\varsigma} \end{cases} \tag{1.102}$$

$$\sigma^2(s) = -\frac{1}{4}S_{ab}(s,\varsigma)S^{ab}(s,\varsigma) = -\frac{1}{4}S_{ab}(s,-\varsigma)S^{ab}(s,-\varsigma) = s(s+1) \tag{1.103}$$

3.8.3 Duality properties

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma) = -\varsigma * S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(s,\varsigma) \qquad S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) = \varsigma * S_{abA'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) \tag{1.104}$$

3.9 Properties of spin constant tensors $S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}, S_{abA'_\varsigma}{}^{B'_\varsigma}$

3.9.1 Compound properties

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} \equiv S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(\varsigma) \equiv S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}(\frac{1}{2},\varsigma) \qquad S_{abA'_\varsigma}{}^{B'_\varsigma} \equiv S_{abA'_\varsigma}{}^{B'_\varsigma}(-\varsigma) \equiv S_{abA'_\varsigma}{}^{B'_\varsigma}(\frac{1}{2},-\varsigma) \tag{1.105}$$

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma} \qquad S_{abA'_\varsigma}{}^{B'_\varsigma} = -\frac{1}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma} \tag{1.106}$$

$$\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma} = \frac{1}{2}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} \qquad \sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma}(s) = \frac{1}{2}\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}S_{abA'_\varsigma}{}^{B'_\varsigma} \tag{1.107}$$

3.9.2 Orthogonality properties

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}S_{cd}{}^{B_\varsigma}{}_{A_\varsigma} = \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} \qquad S_{ab}{}^{A_\varsigma}{}_{B_\varsigma}S^{abB_\varsigma}{}_{A_\varsigma} = -6 \tag{1.108}$$

$$S_{abA'_\varsigma}{}^{B'_\varsigma}S_{cdB'_\varsigma}{}^{A'_\varsigma} = \frac{1}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} \qquad S_{abA'_\varsigma}{}^{B'_\varsigma}S^{ab}{}_{B'_\varsigma}{}^{A'_\varsigma} = -6 \tag{1.109}$$

$$S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}S^{abC_\varsigma}{}_{B_\varsigma} = -3\delta^{A_\varsigma}{}_{B_\varsigma} \qquad S_{abA'_\varsigma}{}^{C'_\varsigma}S^{ab}{}_{C'_\varsigma}{}^{B'_\varsigma} = -3\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.110}$$

$$S_{ab}(\varsigma)S^{ab}(\varsigma) = S_{ab}(-\varsigma)S^{ab}(-\varsigma) = -\sigma^2 = -3 \tag{1.111}$$

3.9.3 Duality properties

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} = -\varsigma * S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} \qquad S_{abA'_\varsigma}{}^{B'_\varsigma} = \varsigma * S_{abA'_\varsigma}{}^{B'_\varsigma} \tag{1.112}$$

3.9.4 Relationships between basic constant tensors

$$(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{bA'_\varsigma B_\varsigma} = \delta_{ab}\delta^{A_\varsigma}{}_{B_\varsigma} + 2S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} \tag{1.113}$$

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} = \frac{1}{4}(\sigma,i\varsigma)_{[a}{}^{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{b]A'_\varsigma B_\varsigma} \qquad \delta_{ab}\delta^{A_\varsigma}{}_{B_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{a}{}^{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{b\}A'_\varsigma B_\varsigma} \tag{1.114}$$

$$(\sigma,-i\varsigma)_{aA'_\varsigma A_\varsigma}(\sigma,i\varsigma)_b{}^{A_\varsigma B'_\varsigma} = \delta_{ab}\delta_{A'_\varsigma}{}^{B'_\varsigma} + 2S_{abA'_\varsigma}{}^{B'_\varsigma} \tag{1.115}$$

$$S_{abA'_\varsigma}{}^{B'_\varsigma} = \frac{1}{4}(\sigma,-i\varsigma)_{[aA'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{b]}{}^{A_\varsigma B'_\varsigma} \qquad \delta_{ab}\delta_{A'_\varsigma}{}^{B'_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)_{\{aA'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{b\}}{}^{A_\varsigma B'_\varsigma} \tag{1.116}$$

Relationships between basic constant tensors:

$$(\sigma,i\varsigma)_{[a}{}^{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{b]A'_\varsigma B_\varsigma} = -2\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma} \qquad (\sigma,-i\varsigma)_{[aA'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{b]}{}^{A_\varsigma B'_\varsigma} = -2\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma} \tag{1.117}$$

Better and more uniform writing:

$$S_{ab}{}^{A_\varsigma}{}_{B_\varsigma} = \frac{1}{4}(\sigma,i\varsigma)_{[a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A'_\varsigma}{}^{B'_\varsigma}(\sigma,-i\varsigma)_{b]B'_\varsigma B_\varsigma} \qquad \delta_{ab}\overline{\varepsilon}^{A_\varsigma}{}_{B_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A'_\varsigma}{}^{B'_\varsigma}(\sigma,-i\varsigma)_{b\}B'_\varsigma B_\varsigma} \tag{1.118}$$

$$S_{abA'_\varsigma}{}^{B'_\varsigma} = \frac{1}{4}(\sigma,-i\varsigma)_{[aA'_\varsigma A_\varsigma}\varepsilon^{A_\varsigma}{}_{B_\varsigma}(\sigma,i\varsigma)_{b]}{}^{B_\varsigma B'_\varsigma} \qquad \delta_{ab}\overline{\varepsilon}_{A'_\varsigma}{}^{B'_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)_{\{aA'_\varsigma A_\varsigma}\varepsilon^{A_\varsigma}{}_{B_\varsigma}(\sigma,i\varsigma)_{b\}}{}^{B_\varsigma B'_\varsigma} \tag{1.119}$$

3.10 Properties of compound constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s), \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s), \sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s), \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s)$

3.10.1 Definition

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) \equiv \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(s)\bar{\varepsilon}^{C_\varsigma B_\varsigma}(s) \\ \sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) \equiv \varepsilon_{A_\varsigma C_\varsigma}(s)\sigma^{\alpha_\varsigma C_\varsigma}{}_{B_\varsigma}(s)\end{cases} \qquad \begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) = [\sigma_{\alpha_\varsigma}(s)\bar{\varepsilon}(s)]^{A_\varsigma B_\varsigma} \\ \sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = [\varepsilon(s)\sigma^{\alpha_\varsigma}(s)]_{A_\varsigma B_\varsigma}\end{cases} \tag{1.120}$$

$$\begin{cases}\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) \equiv \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(s)\bar{\varepsilon}_{C'_\varsigma B'_\varsigma}(s) \\ \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) \equiv \varepsilon^{A'_\varsigma C'_\varsigma}(s)\sigma_{\alpha'_\varsigma C'_\varsigma}{}^{B'_\varsigma}(s)\end{cases} \qquad \begin{cases}\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) = [\sigma^{\alpha_\varsigma}(s)\bar{\varepsilon}(s)]_{A'_\varsigma B'_\varsigma} \\ \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = [\varepsilon(s)\sigma_{\alpha'_\varsigma}(s)]^{A'_\varsigma B'_\varsigma}\end{cases} \tag{1.121}$$

3.10.2 Symmetry and antisymmetry properties

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}\sigma_{\alpha_\varsigma}{}^{B_\varsigma A_\varsigma}(s) \\ \sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}\sigma^{\alpha_\varsigma}{}_{B_\varsigma A_\varsigma}(s)\end{cases} \qquad \begin{cases}\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}\sigma^{\alpha'_\varsigma}{}_{B'_\varsigma A'_\varsigma}(s) \\ \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}\sigma_{\alpha'_\varsigma}{}^{B'_\varsigma A'_\varsigma}(s)\end{cases} \tag{1.122}$$

3.10.3 Orthogonality properties

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s)\sigma^{\beta_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} \\ \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\sigma_{\beta'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha'_\varsigma}{}_{\beta'_\varsigma}\end{cases} \tag{1.123}$$

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}2s(s+\frac{1}{2})(s+1) \\ \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}2s(s+\frac{1}{2})(s+1)\end{cases} \tag{1.124}$$

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma C_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{C_\varsigma B_\varsigma}(s) = s(s+1)\delta^{A_\varsigma}{}_{B_\varsigma} \\ \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = s(s+1)\delta_{A'_\varsigma}{}^{B'_\varsigma}\end{cases} \tag{1.125}$$

$$\begin{cases}\sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(s)\sigma^{\alpha_\varsigma C_\varsigma B_\varsigma}(s) = s(s+1)\bar{\varepsilon}^{A_\varsigma B_\varsigma}(s) \qquad \sigma_{\alpha_\varsigma A_\varsigma C_\varsigma}(s)\sigma_{\alpha_\varsigma}{}^{C_\varsigma}{}_{B_\varsigma}(s) = s(s+1)\varepsilon_{A_\varsigma B_\varsigma}(s) \\ \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(s)\sigma_{\alpha'_\varsigma C'_\varsigma B'_\varsigma}(s) = s(s+1)\bar{\varepsilon}_{A'_\varsigma B'_\varsigma}(s) \qquad \sigma^{\alpha'_\varsigma A'_\varsigma C'_\varsigma}(s)\sigma^{\alpha'_\varsigma}{}_{C'_\varsigma}{}^{B'_\varsigma}(s) = s(s+1)\varepsilon^{A'_\varsigma B'_\varsigma}(s)\end{cases} \tag{1.126}$$

3.11 Properties of spin constant tensors $S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma), S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma), S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma), S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma)$

3.11.1 Definition

$$S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) \equiv S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}(s,\varsigma)\bar{\varepsilon}^{C_\varsigma B_\varsigma}(s) \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma) \equiv S^{ab}{}_{A'_\varsigma}{}^{C'_\varsigma}(s,-\varsigma)\bar{\varepsilon}_{C'_\varsigma B'_\varsigma}(s) \tag{1.127}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) \equiv \varepsilon_{A_\varsigma C_\varsigma}(s)S^{ab C_\varsigma}{}_{B_\varsigma}(s,\varsigma) \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) \equiv \varepsilon^{A'_\varsigma C'_\varsigma}(s)S_{ab C'_\varsigma}{}^{B'_\varsigma}(s,-\varsigma) \tag{1.128}$$

3.11.2 Corollary

$$S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma) = -\frac{1}{2}\sigma_{\varsigma\alpha'_\varsigma}{}^{ab}\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) \tag{1.129}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = -\frac{1}{2}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) = -\frac{1}{2}\sigma_\varsigma^{\alpha'_\varsigma}{}_{ab}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) \tag{1.130}$$

3.11.3 Symmetry and antisymmetry properties

$$S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) = (-1)^{2s+1}S_{ab}{}^{B_\varsigma A_\varsigma}(s,\varsigma) \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma) = (-1)^{2s+1}S^{ab}{}_{B'_\varsigma A'_\varsigma}(s,-\varsigma) \tag{1.131}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = (-1)^{2s+1}S^{ab}{}_{B_\varsigma A_\varsigma}(s,\varsigma) \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) = (-1)^{2s+1}S_{ab}{}^{B'_\varsigma A'_\varsigma}(s,-\varsigma) \tag{1.132}$$

3.11.4 Orthogonality properties

$$\begin{cases} S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma)S^{cd}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = (-1)^{2s+1}\frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma}{}^{cd} \\ S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma)S_{cd}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) = (-1)^{2s+1}\frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}\sigma^{\alpha'_\varsigma}_{-\varsigma cd} \end{cases} \tag{1.133}$$

$$\begin{cases} S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma)S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = (-1)^{2s}8s(s+\frac{1}{2})(s+1) \\ S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma)S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) = (-1)^{2s}8s(s+\frac{1}{2})(s+1) \end{cases} \tag{1.134}$$

$$\begin{cases} S_{ab}{}^{A_\varsigma C_\varsigma}(s,\varsigma)S^{ab}{}_{C_\varsigma B_\varsigma}(s,\varsigma) = -4s(s+1)\delta^{A_\varsigma}{}_{B_\varsigma} \\ S^{ab}{}_{A'_\varsigma C'_\varsigma}(s,-\varsigma)S_{ab}{}^{C'_\varsigma B'_\varsigma}(s,-\varsigma) = -4s(s+1)\delta_{A'_\varsigma}{}^{B'_\varsigma} \end{cases} \tag{1.135}$$

3.11.5 Duality properties

$$S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) = -\varsigma * S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma) = \varsigma * S^{ab}{}_{A'_\varsigma B'_\varsigma}(s,-\varsigma) \tag{1.136}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = -\varsigma * S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) = \varsigma * S_{ab}{}^{A'_\varsigma B'_\varsigma}(s,-\varsigma) \tag{1.137}$$

3.12 Properties of spin constant tensors $S_{ab}{}^{A_\varsigma B_\varsigma}, S^{ab}{}_{A'_\varsigma B'_\varsigma}, S^{ab}{}_{A_\varsigma B_\varsigma}, S_{ab}{}^{A'_\varsigma B'_\varsigma}$

3.12.1 Definition

$$S_{ab}{}^{A_\varsigma B_\varsigma} \equiv S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}\bar{\varepsilon}^{C_\varsigma B_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} \equiv S^{ab}{}_{A'_\varsigma}{}^{C'_\varsigma}\bar{\varepsilon}_{C'_\varsigma B'_\varsigma} \tag{1.138}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} \equiv \varepsilon_{A_\varsigma C_\varsigma}S^{abC_\varsigma}{}_{B_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} \equiv \varepsilon^{A'_\varsigma C'_\varsigma}S_{abC'_\varsigma}{}^{B'_\varsigma} \tag{1.139}$$

3.12.2 Corollary

$$S_{ab}{}^{A_\varsigma B_\varsigma} = -\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = -\frac{1}{4}\sigma_{\varsigma\alpha'_\varsigma}{}^{ab}\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma} \tag{1.140}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = -\frac{1}{4}\sigma^{\alpha'_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma} \tag{1.141}$$

3.12.3 Symmetry and antisymmetry properties

$$S_{ab}{}^{A_\varsigma B_\varsigma} = S_{ab}{}^{B_\varsigma A_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = S^{ab}{}_{B'_\varsigma A'_\varsigma} \tag{1.142}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = S^{ab}{}_{B_\varsigma A_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = S_{ab}{}^{B'_\varsigma A'_\varsigma} \tag{1.143}$$

3.12.4 Orthogonality properties

$$S_{ab}{}^{A_\varsigma B_\varsigma}S^{cd}{}_{A_\varsigma B_\varsigma} = \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma}{}^{cd} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}S_{cd}{}^{A'_\varsigma B'_\varsigma} = \frac{1}{2}\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}\sigma^{\alpha'_\varsigma}_{-\varsigma cd} \tag{1.144}$$

$$S_{ab}{}^{A_\varsigma B_\varsigma}S^{ab}{}_{A_\varsigma B_\varsigma} = -6 \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma}S_{ab}{}^{A'_\varsigma B'_\varsigma} = -6 \tag{1.145}$$

$$S_{ab}{}^{A_\varsigma C_\varsigma}S^{ab}{}_{C_\varsigma B_\varsigma} = -3\delta^{A_\varsigma}{}_{B_\varsigma} \qquad S^{ab}{}_{A'_\varsigma C'_\varsigma}S_{ab}{}^{C'_\varsigma B'_\varsigma} = -3\delta_{A'_\varsigma}{}^{B'_\varsigma} \tag{1.146}$$

3.12.5 Duality properties

$$S_{ab}{}^{A_\varsigma B_\varsigma} = -\varsigma * S_{ab}{}^{A_\varsigma B_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = \varsigma * S^{ab}{}_{A'_\varsigma B'_\varsigma} \tag{1.147}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = -\varsigma * S^{ab}{}_{A_\varsigma B_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = \varsigma * S_{ab}{}^{A'_\varsigma B'_\varsigma} \tag{1.148}$$

3.12.6 Relationships between constant tensors

$$S_{ab}{}^{A_\varsigma B_\varsigma} = \frac{1}{4}(\sigma,i\varsigma)_{[a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma,i\varsigma)_{b]}{}^{B_\varsigma B'_\varsigma} \qquad \delta_{ab}\bar{\varepsilon}^{A_\varsigma B_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma,i\varsigma)_{b\}}{}^{B_\varsigma B'_\varsigma} \tag{1.149}$$

$$S^{ab}{}_{A'_\varsigma B'_\varsigma} = \frac{1}{4}(\sigma,-i\varsigma)^{[a}{}_{A'_\varsigma A_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}(\sigma,-i\varsigma)^{b]}{}_{B'_\varsigma B_\varsigma} \qquad \delta^{ab}\bar{\varepsilon}_{A'_\varsigma B'_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)^{\{a}{}_{A'_\varsigma A_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}(\sigma,-i\varsigma)^{b\}}{}_{B'_\varsigma B_\varsigma} \tag{1.150}$$

$$S_{ab}{}^{A'_\varsigma B'_\varsigma} = \frac{1}{4}(\sigma,i\varsigma)_{[a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}(\sigma,i\varsigma)_{b]}{}^{B_\varsigma B'_\varsigma} \qquad \delta_{ab}\bar{\varepsilon}^{A'_\varsigma B'_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{a}{}^{A_\varsigma A'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}(\sigma,i\varsigma)_{b\}}{}^{B_\varsigma B'_\varsigma} \tag{1.151}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = \frac{1}{4}(\sigma,-i\varsigma)^{[a}{}_{A'_\varsigma A_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma,-i\varsigma)^{b]}{}_{B'_\varsigma B_\varsigma} \qquad \delta^{ab}\bar{\varepsilon}_{A_\varsigma B_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)^{\{a}{}_{A'_\varsigma A_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma,-i\varsigma)^{b\}}{}_{B'_\varsigma B_\varsigma} \tag{1.152}$$

3.13 Normalization of compound constant tensors $\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s), \sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s), \sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s), \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s)$

3.13.1 Definition

$$k(s) = [(-1)^{2s}\frac{8}{3}s(s+\frac{1}{2})(s+1)]^{-\frac{1}{2}} \tag{1.153}$$

$$\begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) \equiv k(s)\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) \\ \widehat{\sigma}^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) \equiv k(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s)\end{cases} \quad \begin{cases}\widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) \equiv k(s)\sigma^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) \\ \widehat{\sigma}_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) \equiv k(s)\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s)\end{cases} \tag{1.154}$$

3.13.2 Symmetry and antisymmetry properties

$$\begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}\widehat{\sigma}_{\alpha_\varsigma}{}^{B_\varsigma A_\varsigma}(s) \\ \widehat{\sigma}^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = (-1)^{2s+1}\widehat{\sigma}^{\alpha_\varsigma}{}_{B_\varsigma A_\varsigma}(s)\end{cases} \quad \begin{cases}\widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}\widehat{\sigma}^{\alpha'_\varsigma}{}_{B'_\varsigma A'_\varsigma}(s) \\ \widehat{\sigma}_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = (-1)^{2s+1}\widehat{\sigma}_{\alpha'_\varsigma}{}^{B'_\varsigma A'_\varsigma}(s)\end{cases} \tag{1.155}$$

3.13.3 Orthogonality properties

$$\begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s)\widehat{\sigma}^{\beta_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = -\frac{1}{4}\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} \\ \widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\widehat{\sigma}_{\beta'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = -\frac{1}{4}\delta^{\alpha'_\varsigma}{}_{\beta'_\varsigma}\end{cases} \quad \begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s)\widehat{\sigma}^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s) = -\frac{3}{4} \\ \widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\widehat{\sigma}_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = -\frac{3}{4}\end{cases} \tag{1.156}$$

$$\begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma C_\varsigma}(s)\widehat{\sigma}^{\alpha_\varsigma}{}_{C_\varsigma B_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\delta^{A_\varsigma}{}_{B_\varsigma} \\ \widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma B'_\varsigma}(s)\widehat{\sigma}_{\alpha'_\varsigma}{}^{A'_\varsigma B'_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\delta_{A'_\varsigma}{}^{B'_\varsigma}\end{cases} \tag{1.157}$$

$$\begin{cases}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}(s)\widehat{\sigma}^{\alpha_\varsigma C_\varsigma B_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\overline{\varepsilon}^{A_\varsigma B_\varsigma}(s) \\ \widehat{\sigma}^{\alpha'_\varsigma}{}_{A'_\varsigma}{}^{C'_\varsigma}(s)\widehat{\sigma}_{\alpha'_\varsigma C'_\varsigma B'_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\overline{\varepsilon}_{A'_\varsigma B'_\varsigma}(s)\end{cases} \quad \begin{matrix}\widehat{\sigma}_{\alpha_\varsigma A_\varsigma C_\varsigma}(s)\widehat{\sigma}_{\alpha_\varsigma}{}^{C_\varsigma}{}_{B_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\varepsilon_{A_\varsigma B_\varsigma}(s) \\ \widehat{\sigma}^{\alpha'_\varsigma A'_\varsigma C'_\varsigma}(s)\widehat{\sigma}^{\alpha'_\varsigma}{}_{C'_\varsigma}{}^{B'_\varsigma}(s) = \frac{3(-1)^{2s}}{4(2s+1)}\varepsilon^{A'_\varsigma B'_\varsigma}(s)\end{matrix} \tag{1.158}$$

3.14 Properties of compound constant tensor $\Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s')$

3.14.1 Definition

$$\Sigma_{ab}^{A_{\varsigma_s} B_{\varsigma_s}}(s) \equiv \begin{cases} k(s)S_{ab}{}^{A_\varsigma B_\varsigma}(s,\varsigma) = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\widehat{\sigma}_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}(s), s>0 \\ k(-s)S_{ab A'_\varsigma B'_\varsigma}(-s,-\varsigma) = -\frac{1}{2}\sigma_\varsigma^{\alpha'_\varsigma}{}_{ab}\widehat{\sigma}_{\alpha'_\varsigma A'_\varsigma B'_\varsigma}(-s), s<0 \end{cases} \tag{1.159}$$

$$\Sigma^{ab}_{A_{\varsigma_s} B_{\varsigma_s}}(s) \equiv \begin{cases} k(s)S^{ab}{}_{A_\varsigma B_\varsigma}(s,\varsigma) = -\frac{1}{2}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\widehat{\sigma}^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}(s), s>0 \\ k(-s)S^{ab A'_\varsigma B'_\varsigma}(-s,-\varsigma) = -\frac{1}{2}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\widehat{\sigma}^{\alpha_\varsigma A_\varsigma B_\varsigma}(-s), s<0 \end{cases} \tag{1.160}$$

$$k(s) = [(-1)^{2s}\frac{8}{3}s(s+\frac{1}{2})(s+1)]^{-\frac{1}{2}} \tag{1.161}$$

Definition: $\Sigma^{A_{\varsigma_s} B_{\varsigma_s}}_{A_{\varsigma_{s'}} B_{\varsigma_{s'}}}(s,s') \equiv \Sigma_{ab}^{A_{\varsigma_s} B_{\varsigma_s}}(s)\Sigma^{ab}_{A_{\varsigma_{s'}} B_{\varsigma_{s'}}}(s')$ (1.162)

For convenience let's omit the symbol ς, then get $\Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s') \equiv \Sigma_{ab}^{A_s B_s}(s)\Sigma^{ab}_{A_{s'} B_{s'}}(s')$ (1.163)

3.14.2 Transfer properties

$$\Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s')\Sigma^{A_{s'} B_{s'}}_{A_{s''} B_{s''}}(s',s'') = \Sigma^{A_s B_s}_{A_{s''} B_{s''}}(s,s'') \tag{1.164}$$

3.14.3 Symmetry and antisymmetry properties

$$\Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s') = (-1)^{2s+1}\Sigma^{B_s A_s}_{A_{s'} B_{s'}}(s,s') \qquad \Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s') = (-1)^{2s'+1}\Sigma^{A_s B_s}_{B_{s'} A_{s'}}(s,s') \tag{1.165}$$

$$\Sigma^{A_s B_s}_{A_{s'} B_{s'}}(s,s') = (-1)^{2(s+s')}\Sigma^{B_s A_s}_{B_{s'} A_{s'}}(s,s') \tag{1.166}$$

3.15 Properties of vector spin tensors S_{abcd} and antisymmetric tensors ε_{abcd}

Proposition 3.15.1. $S_{abcd} = -\frac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} + \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} = \delta_{a[c}\delta_{d]b} = \delta_{c[a}\delta_{b]d}$

Proposition 3.15.2. $\varepsilon_{abcd} = -\frac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} - \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd})$

The above two propositions are unfolded by different situations. Then they can be proved.

$$\begin{cases} S_{(*ab)(*cd)} = S_{abcd} \\ \varepsilon_{(*ab)(*cd)} = \varepsilon_{abcd} \end{cases} \qquad \begin{cases} S_{(*ab)cd} = S_{ab(*cd)} = \varepsilon_{abcd} \\ \varepsilon_{(*ab)cd} = \varepsilon_{ab(*cd)} = S_{abcd} \end{cases} \tag{1.167}$$

$$S_{abcd} = S_{cdab}, S_{abcd} = -S_{bacd}, S_{abcd} = S_{abdc}, S_{abcd} = \frac{1}{2} S_{abef} S^{ef}{}_{cd}, \vartheta_{ab} = \frac{1}{2} S_{abcd} \vartheta^{cd} \tag{1.168}$$

$$\begin{cases} \sigma^{\alpha'}_{-ab} \sigma_{-\alpha' cd} = -(S_{abcd} + \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \varepsilon_{abcd}) \\ \sigma^{\alpha}_{+ab} \sigma_{+\alpha cd} = -(S_{abcd} - \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varepsilon_{abcd}) \end{cases} \tag{1.169}$$

$$\sigma^{\alpha_\varsigma}_{\varsigma ab} \sigma_{\varsigma\alpha cd} = -(S_{abcd} - \varsigma\varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varsigma\varepsilon_{abcd}) \tag{1.170}$$

3.16 Properties of spin tensors $S_{ab}(\varsigma)$

$$S_{ab}(\varsigma) = -\frac{1}{2} \sigma_\varsigma^{\alpha_\varsigma}{}_{ab} \sigma_{\alpha_\varsigma} = \frac{1}{4} (\sigma, i\varsigma)_{[a} (\sigma, -i\varsigma)_{b]} \quad \delta_{ab} = \frac{1}{2} (\sigma, i\varsigma)_{\{a} (\sigma, -i\varsigma)_{b\}} \tag{1.171}$$

$$\begin{cases} [S_{ab}(\varsigma), S_{cd}(\varsigma)] = \delta_{a[c} S_{d]b}(\varsigma) + S_{a[c}(\varsigma)\delta_{d]b} = -\delta_{c[a} S_{b]d}(\varsigma) - S_{c[a}(\varsigma)\delta_{b]d} \\ \{S_{ab}(\varsigma), S_{cd}(\varsigma)\} = \frac{1}{2} \sigma^{\alpha_\varsigma}_{\varsigma ab} \sigma_{\varsigma\alpha cd} = \frac{1}{2} (\varsigma\varepsilon_{abcd} - S_{abcd}) \end{cases} \tag{1.172}$$

$$\varepsilon_{abcd} = \varsigma 2tr[S_{ab}(\varsigma) S_{cd}(\varsigma) - S_{ab}(-\varsigma) S_{cd}(-\varsigma)] \qquad S_{ab}(\varsigma) = -\varsigma * S_{ab}(\varsigma) \tag{1.173}$$

$$\begin{cases} 2S_{ab}(\varsigma)(\sigma, i\varsigma)_c = (\sigma, i\varsigma)_{[a}\delta_{b]c} + \varsigma\varepsilon_{abcd}(\sigma, i\varsigma)^d \\ 2(\sigma, -i\varsigma)_c S_{ab}(\varsigma) = \delta_{c[a}(\sigma, -i\varsigma)_{b]} - \varsigma\varepsilon_{abcd}(\sigma, -i\varsigma)^d \end{cases} \tag{1.174}$$

3.17 Properties of Dirac spin tensors $S_{ab}(e, \varsigma)$ [24]

$$[\gamma_a(\varsigma), \gamma_5(\varsigma)] = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z] \tag{1.175}$$

$$S_{ab}(e, \varsigma) = \frac{1}{4} [\gamma_a(\varsigma), \gamma_b(\varsigma)] = S_{ab}(\varsigma) \oplus S_{ab}(-\varsigma) \quad \delta_{ab} = \frac{1}{2} \{\gamma_a(\varsigma), \gamma_b(\varsigma)\} \tag{1.176}$$

$$\begin{cases} [S_{ab}(e,\varsigma), S_{cd}(e,\varsigma)] = \delta_{a[c} S_{d]b}(e,\varsigma) + S_{a[c}(e,\varsigma)\delta_{d]b} = -\delta_{c[a} S_{b]d}(e,\varsigma) - S_{c[a}(e,\varsigma)\delta_{b]d} \\ \{S_{ab}(e,\varsigma), S_{cd}(e,\varsigma)\} = \frac{1}{2} \sigma^{\alpha_\varsigma}_{\varsigma ab} \sigma_{\varsigma\alpha'_\varsigma cd} \oplus \frac{1}{2} \sigma^{\alpha_\varsigma}_{-\varsigma ab} \sigma_{-\varsigma\alpha'_\varsigma cd} = \frac{1}{2} [\gamma_5(\varsigma)\varepsilon_{abcd} - S_{abcd}] \end{cases} \tag{1.177}$$

$$[S_{ab}(e,\varsigma), \gamma_c(\varsigma)] = \gamma_{[a}\delta_{b]c} \qquad \{S_{ab}(e,\varsigma), \gamma_c(\varsigma)\} = \varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma) \tag{1.178}$$

$$S_{ab}(e,\varsigma) = -\gamma_5(\varsigma) * S_{ab}(e,\varsigma) \tag{1.179}$$

3.18 Relationships between constant tensors $\varepsilon_{abcd}, \gamma_a(\varsigma)$ [24]

$$\varepsilon_{abcd}\gamma^a(\varsigma)\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = 24\gamma_5(\varsigma) \tag{1.180}$$

$$\varepsilon_{abcd}\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = -6\gamma_5(\varsigma)\gamma_a(\varsigma) \tag{1.181}$$

$$\varepsilon_{abcd}\gamma^c(\varsigma)\gamma^d(\varsigma) = -4\gamma_5(\varsigma) S_{ab}(e,\varsigma) \tag{1.182}$$

$$\varepsilon_{abcd}\gamma^d(\varsigma) = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma) - [\delta_{ab}\gamma_c(\varsigma) + \gamma_{[a}(\varsigma)\delta_{b]c}]\} \tag{1.183}$$

$$\varepsilon_{abcd} = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma) \tag{1.184}$$

$$- [\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2\delta_{ab} S_{cd}(e,\varsigma) + 2S_{ab}(e,\varsigma)\delta_{cd} + 2\delta_{a[c} S_{d]b}(e,\varsigma) + 2S_{a[c}(e,\varsigma)\delta_{d]b}]\} \tag{1.185}$$

3.19 Relationships between constant tensors $\varepsilon_{abcd}, (\sigma, i\varsigma)_a$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^a(\sigma, -i\varsigma)^b(\sigma, i\varsigma)^c(\sigma, -i\varsigma)^d = 24\varsigma \tag{1.186}$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^b(\sigma, -i\varsigma)^c(\sigma, i\varsigma)^d = -6\varsigma(\sigma, i\varsigma)^a \tag{1.187}$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^c(\sigma, -i\varsigma)^d = -4\varsigma S_{ab}(\varsigma) \tag{1.188}$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^d = \varsigma\{(\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b(\sigma, i\varsigma)_c - [\delta_{ab}(\sigma, i\varsigma)_c + (\sigma, i\varsigma)_{[a}\delta_{b]c}]\} \tag{1.189}$$

$$\varepsilon_{abcd} = \varsigma\{(\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b(\sigma, i\varsigma)_c(\sigma, -i\varsigma)_d \tag{1.190}$$

$$- [\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2\delta_{ab} S_{cd}(\varsigma) + 2S_{ab}(\varsigma)\delta_{cd} + 2\delta_{a[c} S_{d]b}(\varsigma) + 2S_{a[c}(\varsigma)\delta_{d]b}]\} \tag{1.191}$$

3.20 Relationships between constant tensors $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \sigma_{\alpha_\varsigma}$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} \tag{1.192}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} = -i(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}\sigma_{\gamma_\varsigma} - \delta_{\beta_\varsigma\gamma_\varsigma}\sigma_{\alpha_\varsigma} + \delta_{\gamma_\varsigma\alpha_\varsigma}\sigma_{\beta_\varsigma} - \delta_{\alpha_\varsigma\beta_\varsigma}\sigma_{\gamma_\varsigma}) \tag{1.193}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\gamma_\varsigma} = -i(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma} - \delta_{\alpha_\varsigma\beta_\varsigma}) = -\frac{1}{2}i[\sigma_{\alpha_\varsigma}, \sigma_{\beta_\varsigma}] \tag{1.194}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\beta_\varsigma}\sigma^{\gamma_\varsigma} = 2i\sigma_{\alpha_\varsigma} \qquad \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\alpha_\varsigma}\sigma^{\beta_\varsigma}\sigma^{\gamma_\varsigma} = 6i \tag{1.195}$$

$$2S_{\alpha_\varsigma\beta_\varsigma}\sigma_{\gamma_\varsigma} = \sigma_{[\alpha_\varsigma}\delta_{\beta_\varsigma]\gamma_\varsigma} + i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \qquad 2\sigma_{\gamma_\varsigma}S_{\alpha_\varsigma\beta_\varsigma} = \delta_{\gamma_\varsigma[\alpha_\varsigma}\sigma_{\beta_\varsigma]} + i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \tag{1.196}$$

$$[S_{\alpha_\varsigma\beta_\varsigma}, \sigma_{\gamma_\varsigma}] = \sigma_{[\alpha_\varsigma}\delta_{\beta_\varsigma]\gamma_\varsigma} \qquad \{S_{\alpha_\varsigma\beta_\varsigma}, \sigma_{\gamma_\varsigma}\} = i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \tag{1.197}$$

3.21 Relationships between constant tensors $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \varepsilon_{abcd}$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} \tag{1.198}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma d}A^d \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}A^4 \tag{1.199}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma cd}F^{cd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(F^{\gamma_\varsigma 4} - F^{4\gamma_\varsigma}) \tag{1.200}$$

$$\varepsilon_{\alpha_\varsigma bcd}H^{bcd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(H^{\beta_\varsigma\gamma_\varsigma 4} - H^{\beta_\varsigma 4\gamma_\varsigma} + H^{4\beta_\varsigma\gamma_\varsigma}) \tag{1.201}$$

$$\varepsilon_{abcd}R^{abcd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(R^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} - R^{\alpha_\varsigma\beta_\varsigma 4\gamma_\varsigma} + R^{\alpha_\varsigma 4\beta_\varsigma\gamma_\varsigma} - R^{4\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}) \tag{1.202}$$

3.22 Relationships between constant tensors $\varepsilon_{A_\varsigma B_\varsigma}, \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}$

$$\varepsilon_{A_\varsigma B_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma 3} \tag{1.203}$$

$$\varepsilon_{A_\varsigma B_\varsigma\gamma_\varsigma}A^{\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}A^3 \tag{1.204}$$

$$\varepsilon_{A_\varsigma\beta_\varsigma\gamma_\varsigma}F^{\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}(F^{B_\varsigma 3} - F^{3B_\varsigma}) \tag{1.205}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}H^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}(H^{A_\varsigma B_\varsigma 3} - H^{A_\varsigma 3B_\varsigma} + H^{3A_\varsigma B_\varsigma}) \tag{1.206}$$

3.23 Special constant tensors

3.23.1 Introducing constant matrices $\bar{\mathbb{N}}(s), \mathbb{N}(s)$

$$\mathbb{N}(s) \equiv 1 \oplus (\begin{bmatrix}1\\1\end{bmatrix} \otimes I_{2s-1}) \oplus 1, \bar{\mathbb{N}}^T(s) \equiv \varepsilon(\frac{1}{2}) \otimes \varepsilon(s-\frac{1}{2})\mathbb{N}(s)\bar{\varepsilon}(s) \tag{1.207}$$

$$\mathbb{N}(s) = \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}, \cdots, \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&0&\vdots&0&0\\0&0&\vdots&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \tag{1.208}$$

$$\bar{\mathbb{N}}(s) = \begin{bmatrix}1&0\\0&1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}2&0&0&0\\0&1&1&0\\0&0&0&2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}3&0&0&0&0&0\\0&1&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&0&3\end{bmatrix}, \frac{1}{4}\begin{bmatrix}4&0&0&0&0&0&0&0\\0&1&3&0&0&0&0&0\\0&0&0&2&2&0&0&0\\0&0&0&0&0&3&1&0\\0&0&0&0&0&0&0&4\end{bmatrix}, \cdots \tag{1.209}$$

$$\mathcal{N}^T(s) = \frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}\frac{1}{2}&0&0&0\\0&1&1&0\\0&0&0&\frac{1}{2}\end{bmatrix}, \frac{1}{2}\begin{bmatrix}\frac{1}{3}&0&0&0&0&0\\0&1&\frac{1}{2}&0&0&0\\0&0&0&\frac{1}{2}&1&0\\0&0&0&0&0&\frac{1}{3}\end{bmatrix}, \frac{1}{2}\begin{bmatrix}\frac{1}{4}&0&0&0&0&0&0&0\\0&1&\frac{1}{3}&0&0&0&0&0\\0&0&0&\frac{1}{2}&\frac{1}{2}&0&0&0\\0&0&0&0&0&\frac{1}{3}&1&0\\0&0&0&0&0&0&0&\frac{1}{4}\end{bmatrix} \tag{1.210}$$

$$tr[\mathbb{N}(s)\bar{\mathbb{N}}(s)] = tr[\bar{\mathbb{N}}(s)\mathbb{N}(s)] = 2s+1, tr[\mathcal{N}(s)\bar{\mathbb{N}}(s)] = tr[\bar{\mathbb{N}}(s)\mathcal{N}(s)] = 1 \tag{1.211}$$

3.23.2 Constant tensors properties of matrices $(\sigma \otimes I_{2s}, i\varsigma)_a, (\sigma \otimes I_{2s}, -i\varsigma)_a$

Corollary 3.23.1. $(\sigma \otimes I_{2s}, -i\varsigma)_a = e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma \otimes I_{2s}, -i\varsigma)_a e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Corollary 3.23.2. $(\sigma \otimes I_{2s}, -i\varsigma)_a = e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma \otimes I_{2s}, -i\varsigma)_a e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Corollary 3.23.3. $(\sigma \otimes I_{2s}, i\varsigma)_a = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma \otimes I_{2s}, i\varsigma)_a e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Corollary 3.23.4. $(\sigma \otimes I_{2s}, i\varsigma)_a = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma \otimes I_{2s}, i\varsigma)_a e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

3.23.3 Constant matrices $\bar{\mathbb{N}}(s), \mathbb{N}(s)$

Lemma 3.23.1. $\bar{\mathbb{N}}(s)\mathbb{N}(s) = I_{2s+1}, \mathbb{N}(s)\bar{\mathbb{N}}(s) \neq kI_{4s}, \tilde{\psi}(s,\varsigma) \equiv \mathbb{N}(s)\bar{\mathbb{N}}(s)\tilde{\psi}(s,\varsigma)$

Lemma 3.23.2. $[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})]\mathbb{N}(s) = \mathbb{N}(s)\sigma(s), \bar{\mathbb{N}}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})] = \sigma(s)\bar{\mathbb{N}}(s)$

Corollary 3.23.5. $[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2}), \mathbb{N}(s)\bar{\mathbb{N}}(s)] = 0$

Lemma 3.23.3. $\bar{\mathbb{N}}(s)\sigma(\frac{1}{2}) \otimes I_{2s}\mathbb{N}(s) = \frac{1}{2s}\sigma(s)$

Corollary 3.23.6. $\bar{\mathbb{N}}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})]\mathbb{N}(s) = \sigma(s)$

Corollary 3.23.7. $\bar{\mathbb{N}}(s)I \otimes \sigma(s-\frac{1}{2})\mathbb{N}(s) = (1-\frac{1}{2s})\sigma(s)$

Corollary 3.23.8. $(\sigma \otimes I_{2s}, -i\varsigma)^a\mathbb{N}(s)s\bar{\mathbb{N}}(s)(\sigma \otimes I_{2s}, i\varsigma)_a = (2s+1)I_{4s}$

3.23.4 Constant tensors properties of matrices $\bar{\mathbb{N}}(s), \mathbb{N}(s)$

Theorem 3.23.1. $\mathbb{N}(s) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}\mathbb{N}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

Proof: $[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})]N(s) = N(s)\sigma(s)$

$\Leftrightarrow 0 = [(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2}) \otimes I_{2s-1} + (i\omega+\varsigma\epsilon)\cdot I \otimes \sigma(s-\frac{1}{2})]N(s) - (i\omega+\varsigma\epsilon)\cdot N(s)\sigma(s)$

$\Leftrightarrow N(s) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}N(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$ □

Theorem 3.23.2. $\bar{\mathbb{N}}(s) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{\mathbb{N}}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Proof: $\bar{\mathbb{N}}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})] = \sigma(s)\bar{\mathbb{N}}(s)$

$\Leftrightarrow 0 = (i\omega+\varsigma\epsilon)\cdot\sigma(s)\bar{\mathbb{N}}(s) - \bar{\mathbb{N}}(s)[(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2}) \otimes I_{2s-1} + (i\omega+\varsigma\epsilon)\cdot I \otimes \sigma(s-\frac{1}{2})]$

$\Leftrightarrow \bar{\mathbb{N}}(s) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{\mathbb{N}}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

3.23.5 Introducing constant tensors $\mathbb{Z}_a(s,\varsigma), \bar{\mathbb{Z}}_a(s,\varsigma)$

Definiton 3.23.1. $\mathbb{Z}_a(s,\varsigma) \equiv s\bar{\mathbb{N}}(s)(\sigma \otimes I_{2s}, i\varsigma)_a, \bar{\mathbb{Z}}_a(s,\varsigma) \equiv (\sigma \otimes I_{2s}, -i\varsigma)_a\mathbb{N}(s), \mathcal{Z}_a^-(s,\varsigma) \equiv (\sigma \otimes I_{2s}, -i\varsigma)_a\mathcal{N}(s)$

Corollary 3.23.9. $\mathbb{Z}_{\{a}(s,\varsigma)\bar{\mathbb{Z}}_{b\}}(s,\varsigma) = 2sI_{2s+1}\delta_{ab}, \bar{\mathbb{Z}}_a(s,\varsigma)\mathbb{Z}_a(s,\varsigma) \neq kI_{4s}$

Corollary 3.23.10. $\bar{\mathbb{Z}}_a(s,\varsigma)\mathbb{Z}^a(s,\varsigma) = (2s+1)I_{4s}$

3.23.6 Constant tensors $\mathbb{N}_m(1), \bar{\mathbb{N}}_m(1), \mathbb{Z}_a(1,\varsigma)_m, \bar{\mathbb{Z}}_a(1,\varsigma)_m$ in electromagnetic representation

Definiton 3.23.2. $\bar{\mathbb{N}}_m(1) \equiv S_m(1)\bar{\mathbb{N}}(1)S_{em}^+(\varsigma), \mathbb{N}_m(1) \equiv S_{em}(\varsigma)\mathbb{N}(1)S_m^-(1)$

Corollary 3.23.11. $\bar{\mathbb{N}}_m(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbb{N}_m(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbb{N}_m(1)\bar{\mathbb{N}}_m(1) = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \bar{\mathbb{N}}_m(1)\mathbb{N}_m(1) = I_3$

Definiton 3.23.3. $\mathbb{Z}_a(1,\varsigma)_m \equiv S_m(1)\mathbb{Z}_a(1,\varsigma)S_{em}^+(\varsigma) = \bar{\mathbb{N}}_m(1)(\sigma_{-\varsigma}, i\varsigma)_a$

$\bar{\mathbb{Z}}_a(1,\varsigma)_m \equiv S_{em}(\varsigma)\bar{\mathbb{Z}}_a(1,\varsigma)S_m^-(1) = (\sigma_{-\varsigma}, -i\varsigma)_a\mathbb{N}_m(1)$

3.23.7 Constant tensors $\mathbb{N}_m(2), \bar{\mathbb{N}}_m(2), \mathbb{Z}_a(2,\varsigma)_m, \bar{\mathbb{Z}}_a(2,\varsigma)_m$ in electromagnetic representation

Definiton 3.23.4. $\bar{\mathbb{N}}_m(2) \equiv S_m(2)\bar{\mathbb{N}}(2)S_{em}^+(\varsigma) \otimes S_{em}^+(\varsigma)(\frac{1}{2}), \mathbb{N}_m(2) \equiv S_{em}(\varsigma) \otimes S_{em}(\varsigma)(\frac{1}{2})\mathbb{N}(2)S_m^-(2)$

Corollary 3.23.12. $\bar{\mathbb{N}}_m(2) = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -\varsigma \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \varsigma & 0 & 0 & 2 & 0 \end{bmatrix}, \mathbb{N}_m(2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\mathbb{N}_m(2)\bar{\mathbb{N}}_m(2) = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -\varsigma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \varsigma & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\mathbb{N}}_m(2)\mathbb{N}_m(2) = I_5$$

Corollary 3.23.13. $\mathbb{Z}_a(2,\varsigma)_m \equiv S_m(2)\mathbb{Z}_a(2,\varsigma)S_{em}^+(\varsigma) \otimes S_{em}^+(\varsigma)(\frac{1}{2}) = \bar{\mathbb{N}}_m(2)(\sigma_{-\varsigma} \otimes I, i\varsigma)_a$

$\bar{\mathbb{Z}}_a(2,\varsigma)_m \equiv S_{em}(\varsigma) \otimes S_{em}(\varsigma)(\frac{1}{2})\bar{\mathbb{Z}}_a(2,\varsigma)S_m^-(2) = (\sigma_{-\varsigma} \otimes I, -i\varsigma)_a\mathbb{N}_m(2)$

4 General theory of constant tensors

4.1 General definition of constant tensors

Definition: Lorentz transformation:$\Lambda[L_i] \equiv e^{\frac{1}{2}\vartheta^{ab}S_{ab}[L_i]}$, Yang-Mills guage transformation:$\Lambda[Y_j] \equiv e^{i\theta^\alpha T_\alpha[Y_j]}$ (1.212)

General definition of constant tensors:

In any reference frame $C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n}$ is constant and equal, and satisfies the following transformation:

$$C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \prod_{i=1}^{n}\Lambda^{L'_i}_{L_i}[L_i]\prod_{j=1}^{m}\Lambda^{Y_j}_{Y'_j}[Y_j]C^{Y'_1Y'_2\cdots Y'_m}_{L'_1L'_2\cdots L'_n} \tag{1.213}$$

then $C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n}$ are constant tensors, $L_i \sim e^{\frac{1}{2}\vartheta^{ab}S_{ab}[L_i]}, Y_j \sim e^{i\theta^\alpha T_\alpha[Y_j]}$

Infinitesimal transformation:

$$0 = \delta C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \frac{1}{2}\vartheta^{ab}\sum_{i=1}^{n}S_{ab}{}^{L'_i}_{L_i}[L_i]C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L'_i\cdots L_n} + i\theta^\alpha\sum_{j=1}^{m}T_\alpha{}^{Y_j}_{Y'_j}[L_i]C^{Y_1Y_2\cdots Y'_j\cdots Y_m}_{L_1L_2\cdots L_n}, \forall\vartheta^{ab}, \forall\theta^\alpha \tag{1.214}$$

$$\Leftrightarrow \sum_{i=1}^{n}S_{ab}{}^{L'_i}_{L_i}[L_i]C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L'_i\cdots L_n} = 0, \sum_{j=1}^{m}T_\alpha{}^{Y_j}_{Y'_j}[L_i]C^{Y_1Y_2\cdots Y'_j\cdots Y_m}_{L_1L_2\cdots L_n} = 0 \tag{1.215}$$

4.2 Covariant derivative of all constant tensors is zero

$$D_uC^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \partial_uC^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} + \frac{1}{2}\omega_u{}^{ab}\sum_{i=1}^{n}S_{ab}{}^{L'_i}_{L_i}[L_i]C^{Y_1Y_2\cdots Y_m}_{L_1\cdots L'_i\cdots L_n} + iA_u{}^\alpha\sum_{j=1}^{m}T_\alpha{}^{Y_j}_{Y'_j}[L_i]C^{Y_1\cdots Y'_j\cdots Y_m}_{L_1L_2\cdots L_n} \tag{1.216}$$

$$D_uC^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = 0 + 0 + 0 = 0 \tag{1.217}$$

Therefore the covariant derivative of all constant tensors is zero. This is a very good and convenient property.

Chapter 2

Physical application of constant tensors

1 Applying constant tensors to define spinors of electromagnetic field [10]

1.1 Integer spinorial description of electromagnetic field

Electromagnetic tensor: $$F^{ab} = \begin{bmatrix} 0 & B^z & -B^y & -iE^x \\ -B^z & 0 & B^x & -iE_y \\ B^y & -B^x & 0 & -iE^z \\ iE^x & iE^y & iE^z & 0 \end{bmatrix} \tag{2.1}$$

Dual tensor: $$*F^{ab} = \begin{bmatrix} 0 & -iE^z & iE^y & B^x \\ -iE^z & 0 & -iE^x & B_y \\ -iE^y & iE^x & 0 & B^z \\ -B^x & -B^y & -B^z & 0 \end{bmatrix} \tag{2.2}$$

Definiton 1.1.1. *Electromagnetic integer spinor:* $\psi^{\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab} = (E - i\varsigma B)^{\alpha_\varsigma}$

Corollary 1.1.1. $F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$

Proof: $F^{ab} = -F^{ba}$

$\Leftrightarrow F_{ab} = \frac{1}{2}S_{adcd}F^{cd}, *F_{ab} \equiv \frac{1}{2}\varepsilon_{adcd}F^{cd}$

$\Leftrightarrow F_{ab} - \varsigma * F_{ab} = \frac{1}{2}(S_{adcd} - \varsigma\varepsilon_{adcd})F^{cd}$

$\Leftrightarrow F_{ab} - \varsigma * F_{ab} = -\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}F^{cd}$

$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$ ☐

Corollary 1.1.2. $\psi^{\alpha_\varsigma} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$

Proof: $F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$

$\Rightarrow \psi^{\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab} = \frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab} - \frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_{\varsigma\beta_\varsigma}{}^{ab}\psi^{\beta_\varsigma}$

$\Rightarrow \psi^{\alpha_\varsigma} = \frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab} + 2\delta^\alpha{}_\beta\psi^{\beta_\varsigma}$

$\Rightarrow \psi^{\alpha_\varsigma} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$ ☐

Corollary 1.1.3. $\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$

Corollary 1.1.4. $\psi^{\alpha_\varsigma} = \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}(F^{ab} - \varsigma * F^{ab})$

Proof: $\psi^{\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab}, \psi^{\alpha_\varsigma} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$

$\Rightarrow \psi^{\alpha_\varsigma} = \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}(F_{ab} - \varsigma * F_{ab})$ ☐

Corollary 1.1.5. $F^{ab} - \varsigma * F^{ab} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}(F^{cd} - \varsigma * F^{cd})$

Proof: $F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}, \psi^{\alpha_\varsigma} = \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}(F^{ab} - \varsigma * F^{ab})$

$\Rightarrow F^{ab} - \varsigma * F^{ab} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}(F^{cd} - \varsigma * F^{cd})$ ☐

Corollary 1.1.6. $F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} - \sigma_{+\alpha}{}^{ab}\psi^\alpha), *F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} + \sigma_{+\alpha}{}^{ab}\psi^\alpha)$

Proof: $F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$
$\Leftrightarrow F^{ab} - *F^{ab} = -\sigma_{+\alpha}{}^{ab}\psi^{\alpha}, F^{ab} + *F^{ab} = \sigma_{-\alpha'}{}^{ab}\psi^{\alpha'}$
$\Leftrightarrow F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha}), *F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} + \sigma_{+\alpha}{}^{ab}\psi^{\alpha})$
$\Leftrightarrow F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha})$
$\Leftrightarrow *F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} + \sigma_{+\alpha}{}^{ab}\psi^{\alpha})$ □

Corollary 1.1.7. $F^{ab} = -F^{ba} \Leftrightarrow F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha}),$

Proof: $F^{ab} = -F^{ba}$
$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$
$\Leftrightarrow F^{ab} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha})$
$\Leftrightarrow F^{ab} = \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}\psi^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma})$ □

1.2 Half integer spinorial description of electromagnetic field [4,5]

Definiton 1.2.1. *Electromagnetic spinor:* $\psi^{A_\varsigma B_\varsigma} \equiv -\frac{1}{\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}F^{ab} = -\frac{1}{\sqrt{2}}\varsigma\varepsilon^{B_\varsigma C_\varsigma}S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}F^{ab}$

Corollary 1.2.1. $\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma}$

Proof: $\psi^{A_\varsigma B_\varsigma} \equiv -\frac{1}{\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}F^{ab}$
$\Leftrightarrow \psi^{A_\varsigma B_\varsigma} = \frac{1}{2\sqrt{2}}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}F^{ab}$
$\Leftrightarrow \psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma}$ □

Corollary 1.2.2. $\psi^{A_\varsigma B_\varsigma} = \psi^{B_\varsigma A_\varsigma}$

Proof: $\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma} = \sigma_{\alpha_\varsigma}{}^{B_\varsigma A_\varsigma}$
$\Rightarrow \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{B_\varsigma A_\varsigma}\psi^{\alpha_\varsigma}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma} = \psi^{B_\varsigma A_\varsigma}$ □

Corollary 1.2.3. $\psi^{\alpha_\varsigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$

Proof: $\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma}$
$\Leftrightarrow \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\frac{1}{\sqrt{2}}\sigma_{\beta_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\beta_\varsigma}$
$\Leftrightarrow \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma} = \delta^{\alpha_\varsigma}{}_{\beta_\varsigma}\psi^{\beta_\varsigma}$
$\Leftrightarrow \psi^{\alpha_\varsigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$ □

Corollary 1.2.4. $F^{ab} - \varsigma * F^{ab} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$

Proof: $F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$
$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$
$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$ □

Corollary 1.2.5. $\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma} * F^{ab}$

Proof: $\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma}, \psi^{\alpha_\varsigma} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma} = -\frac{1}{2\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma} * F^{ab}$ □

Corollary 1.2.6. $\psi^{A_\varsigma B_\varsigma} = -\frac{1}{2\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}(F^{ab} - \varsigma * F^{ab})$

Proof: $\psi^{A_\varsigma B_\varsigma} \equiv -\frac{1}{\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}F^{ab}, \psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma} * F^{ab}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma} = -\frac{1}{2\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}(F^{ab} - \varsigma * F^{ab})$ □

Corollary 1.2.7. $F^{ab} - \varsigma * F^{ab} = -\frac{1}{2}S^{ab}{}_{A_\varsigma B_\varsigma}S_{cd}{}^{A_\varsigma B_\varsigma}(F^{cd} - \varsigma * F^{cd})$

Proof: $F^{ab} - \varsigma * F^{ab} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}, \psi^{A_\varsigma B_\varsigma} = -\frac{1}{2\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}(F^{ab} - \varsigma * F^{ab})$
$\Rightarrow F^{ab} - \varsigma * F^{ab} = -\frac{1}{2}S^{ab}{}_{A_\varsigma B_\varsigma}S_{cd}{}^{A_\varsigma B_\varsigma}(F^{cd} - \varsigma * F^{cd})$ □

Corollary 1.2.8. $F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} - S^{ab}{}_{AB}\psi^{AB}), *F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} + S^{ab}{}_{AB}\psi^{AB})$

Proof: $F^{ab} - \varsigma * F^{ab} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$

$\Leftrightarrow F^{ab} - *F^{ab} = \sqrt{2}S^{ab}{}_{AB}\psi^{AB}, F^{ab} + *F^{ab} = -\sqrt{2}S^{abA'B'}\psi_{A'B'}$

$\Leftrightarrow F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} - S^{ab}{}_{AB}\psi^{AB}), *F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} + S^{ab}{}_{AB}\psi^{AB})$

$\Leftrightarrow F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} - S^{ab}{}_{AB}\psi^{AB})$

$\Leftrightarrow *F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} + S^{ab}{}_{AB}\psi^{AB})$ □

Corollary 1.2.9. $F^{ab} = -F^{ba} \Leftrightarrow F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} - S^{ab}{}_{AB}\psi^{AB})$

Proof: $F^{ab} = -F^{ba}$

$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma}$

$\Leftrightarrow F^{ab} - \varsigma * F^{ab} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}$

$\Leftrightarrow F^{ab} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'} - S^{ab}{}_{AB}\psi^{AB})$ □

1.3 Half integer spinorial description of electromagnetic field source [4, 5]

Definiton 1.3.1. *Spinor of electromagnetic field source:* $J_{A'_\varsigma}{}^{B_\varsigma} \equiv \frac{1}{\sqrt{2}}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}J_a$

Corollary 1.3.1. $J_a = \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma}$

Proof: $J_{A'_\varsigma}{}^{B_\varsigma} \equiv \frac{1}{\sqrt{2}}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}J_a$

$\Leftrightarrow \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma} = \frac{1}{2}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}(\sigma, -i\varsigma)^a{}_{A'_\varsigma C_\varsigma}\bar{\varepsilon}^{C_\varsigma B_\varsigma}J_a$

$\Leftrightarrow \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma} = \frac{1}{2}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\delta_{A_\varsigma}{}^{C_\varsigma}(\sigma, -i\varsigma)^b{}_{A'_\varsigma C_\varsigma}J_b$

$\Leftrightarrow \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma} = \frac{1}{2}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)^b{}_{A'_\varsigma A_\varsigma}J_b$

$\Leftrightarrow \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma} = \delta_a{}^b J_b$

$\Leftrightarrow J_a = \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma}$ □

2 Applying constant tensors to define spinors of gravitational field [14–17]

2.1 Preparation

Lemma 2.1.1. $X^*_{a'_\varsigma} = \eta_{a'_\varsigma}{}^{a_\varsigma}X_{a_\varsigma}, X^*_{a'_\varsigma b'_\varsigma} = \eta_{a'_\varsigma}{}^{a_\varsigma}\eta_{b'_\varsigma}{}^{b_\varsigma}X_{a_\varsigma b_\varsigma}, X^*_{a'_\varsigma b'_\varsigma c'_\varsigma} = \eta_{a'_\varsigma}{}^{a_\varsigma}\eta_{b'_\varsigma}{}^{b_\varsigma}\eta_{c'_\varsigma}{}^{c_\varsigma}X_{a_\varsigma b_\varsigma c_\varsigma}\cdots$

Lemma 2.1.2. $X^*_{a'_\varsigma}B^{*a'_\varsigma} = X_{a_\varsigma}B^{a_\varsigma}, X^*_{a'_\varsigma b'_\varsigma}Y^{*a'_\varsigma b'_\varsigma} = X_{a_\varsigma b_\varsigma}Y^{a_\varsigma b_\varsigma}, X^*_{a'_\varsigma b'_\varsigma c'_\varsigma}Y^{*a'_\varsigma b'_\varsigma c'_\varsigma} = X_{a_\varsigma b_\varsigma c_\varsigma}Y^{a_\varsigma b_\varsigma c_\varsigma}, \cdots$

2.2 Curvature tensor of gravitational field

Antisymmetry: $R^{abcd} = -R^{bacd}, R^{abcd} = -R^{abdc}$ (2.3)

Symmetry: $R^{abcd} = R^{cdab}$ (2.4)

Alternate symmetry: $R^{abcd} + R^{adbc} + R^{acdb} = 0$ (2.5)

Definiton 2.2.1. *Ricci tensor:* $R^{ab} \equiv g_{cd}R^{cadb}$, *Curvature scalar:* $R \equiv g_{ab}R^{ab} = R_{ab}{}^{ab}$

Definiton 2.2.2. $C^{abcd} \equiv R^{abcd} + \frac{1}{2}g^{a[d}R^{c]b} + \frac{1}{2}g^{b[c}R^{d]a} + \frac{1}{6}g^{a[c}g^{d]b}R$

Corollary 2.2.1. $C^{ab} = g_{cd}C^{cadb} = 0$

Corollary 2.2.2. $R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R$

Corollary 2.2.3. $R^{abcd} + R^{adbc} + R^{acdb} \equiv 0 \Rightarrow g_{cd}R^{ca(*db)} \equiv 0$

Proof: $R^{abcd} + R^{adbc} + R^{acdb} \equiv 0$

$\Rightarrow \varepsilon_{ebcd}(R^{abcd} + R^{adbc} + R^{acdb}) \equiv 0$

$\Rightarrow \varepsilon_{ebcd}R^{abcd} + \varepsilon_{edbc}R^{adbc} + \varepsilon_{ecdb}R^{acdb} \equiv 0$

$\Rightarrow 3\varepsilon_{ebcd}R^{abcd} \equiv 0$

$\Rightarrow g_{cd}R^{ca(*db)} \equiv 0$ □

2.3 Yang-Mills [9] description of gravitational field

Definiton 2.3.1. *Yang-Mills tensor of gravitational field:* $F^{ab\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}R^{abcd}$,
Yang-Mills tensor of Weyl tensor: $C^{ab\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}C^{abcd}$

Corollary 2.3.1. $R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R$
$\Rightarrow F^{ab\alpha_\varsigma} = C^{ab\alpha_\varsigma} - \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}(g^{a[d}R^{c]b} + g^{b[c}R^{d]a} + \frac{1}{3}Rg^{a[c}g^{d]b})$
$\Rightarrow F^{ab\alpha_\varsigma} = C^{ab\alpha_\varsigma} + \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma a}{}_cR^{cb} - \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma b}{}_cR^{ca} - \frac{1}{6}\varsigma\sigma_\varsigma^{\alpha_\varsigma ab}R$

Corollary 2.3.2. $R^{abcd} - \varsigma R^{ab(*cd)} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{cd}F^{ab\alpha_\varsigma}$

Corollary 2.3.3. $F^{ab\alpha_\varsigma} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}R^{ab(*cd)}$

Corollary 2.3.4. $\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}R^{abcd} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}R^{ab(*cd)}$

Corollary 2.3.5. $F^{ab\alpha_\varsigma} = \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}(R^{abcd} - \varsigma R^{ab(*cd)})$

Corollary 2.3.6. $R^{abcd} - \varsigma R^{ab(*cd)} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{cd}\sigma_\varsigma^{\alpha_\varsigma}{}_{ef}(R^{abef} - \varsigma R^{ab(*ef)})$

Corollary 2.3.7. $R^{abcd} = \frac{1}{2}(\sigma_{-\alpha'}{}^{cd}F^{ab\alpha'} - \sigma_{+\alpha}{}^{cd}F^{ab\alpha}), R^{ab(*cd)} = \frac{1}{2}(\sigma_{-\alpha'}{}^{cd}F^{ab\alpha'} + \sigma_{+\alpha}{}^{cd}F^{ab\alpha})$

Corollary 2.3.8. $R^{ab} = \frac{1}{2}(-F^{\alpha'}\sigma_{-\alpha'} + F^{\alpha}\sigma_{+\alpha})^{ab}, 0 = \frac{1}{2}(F^{\alpha}\sigma_{+\alpha} + F^{\alpha'}\sigma_{-\alpha'})^{ab}$

Corollary 2.3.9. $R^{ab} = \varsigma(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, (F^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma})^{ab} = -(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}$

Corollary 2.3.10. $R = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}F_{ab}{}^{\alpha_\varsigma}$

2.4 Integer spinorial description of gravitational field

2.4.1 Integer spinor of gravitational field

Definiton 2.4.1. *Gravitational integer spinor:* $\psi^{\alpha_\varsigma\beta_\kappa} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab\beta_\kappa} = \frac{1}{4}\varsigma\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{abcd}$
Weyl gravitational integer spinor: $C^{\alpha_\varsigma\beta_\kappa} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}C^{ab\beta_\kappa} = \frac{1}{4}\varsigma\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}C^{abcd}$

Corollary 2.4.1. $\psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} - \frac{1}{2}\varsigma\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ac}\sigma_\kappa^{\beta_\kappa c}{}_bR^{ab} + \frac{1}{3}\varsigma\kappa\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$

Proof: $\psi^{\alpha_\varsigma\beta_\kappa} \equiv \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab\beta_\kappa}$
$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}(C^{ab\beta_\kappa} + \frac{1}{2}\kappa\sigma_\kappa^{\beta_\kappa a}{}_cR^{cb} - \frac{1}{2}\kappa\sigma_\kappa^{\beta_\kappa b}{}_cR^{ca} - \frac{1}{6}\kappa\sigma_\kappa^{\beta_\kappa ab}R)$
$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} - \frac{1}{2}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\kappa\sigma_\kappa^{\beta_\kappa b}{}_cR^{ca} + \frac{1}{3}\varsigma\kappa\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$
$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} - \frac{1}{2}\varsigma\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ac}\sigma_\kappa^{\beta_\kappa c}{}_bR^{ab} + \frac{1}{3}\varsigma\kappa\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$ □

Corollary 2.4.2. $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ac}\sigma_\varsigma^{\beta_\varsigma c}{}_bR^{ab} + \frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$

Corollary 2.4.3. $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$

Proof: $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ac}\sigma_\varsigma^{\beta_\varsigma c}{}_bR^{ab} + \frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow \psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{2}(\delta^{\alpha_\varsigma\beta_\varsigma}\delta_{ab} + i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma_\varsigma^{\gamma_\varsigma}{}_{ab})R^{ab} + \frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow \psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$ □

Corollary 2.4.4. $F^{ab\beta_\kappa} - \varsigma * F^{ab\beta_\kappa} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\beta_\kappa}$

Corollary 2.4.5. $\psi^{\alpha_\varsigma\beta_\kappa} = -\frac{1}{2}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab\beta_\kappa}$

Corollary 2.4.6. $\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}F^{ab\beta_\kappa} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab} * F^{ab\beta_\kappa}$

Corollary 2.4.7. $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}(F^{ab\beta_\kappa} - \varsigma * F^{ab\beta_\kappa})$

Corollary 2.4.8. $F^{ab\beta_\kappa} - \varsigma * F^{ab\beta_\kappa} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_\varsigma^{\alpha_\varsigma}{}_{cd}(F^{cd\beta_\kappa} - \varsigma * F^{cd\beta_\kappa})$

Corollary 2.4.9. $F^{ab\beta_\kappa} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\beta_\kappa} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha\beta_\kappa}), *F^{ab\beta_\kappa} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\beta_\kappa} + \sigma_{+\alpha}{}^{ab}\psi^{\alpha\beta_\kappa})$

Corollary 2.4.10. $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{1}{4}\varsigma\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{abcd} = -\frac{1}{4}\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{ab(*cd)}$
$= \frac{1}{4}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{(*ab)(*cd)} = -\frac{1}{4}\kappa\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{(*ab)cd}$

2.4.2 Properties of gravitational integer spinor $\psi^{\alpha_\varsigma\beta_\kappa}$

Corollary 2.4.11. $\psi^{\alpha_\varsigma\beta_\kappa} = \psi^{\beta_\kappa\alpha_\varsigma}$

Proof: $R^{abcd} = R^{cdab}$
$\Rightarrow \frac{1}{4}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{abcd} = \frac{1}{4}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{cdab}$
$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = \psi^{\beta_\kappa\alpha_\varsigma}$ □

Corollary 2.4.12. $\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = -\frac{1}{2}R$

Proof: $\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} = -(S_{abcd} - \varsigma\varepsilon_{abcd})$
$\Rightarrow \frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}R^{abcd} = -\frac{1}{4}(S_{abcd} - \varsigma\varepsilon_{abcd})R^{abcd}$
$\Rightarrow \frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}R^{abcd} = -\frac{1}{2}(R_{ab}{}^{ab} - \varsigma R_{*ab}{}^{ab})$
$\Rightarrow \psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = -\frac{1}{2}(R_{ab}{}^{ab} - \varsigma R_{*ab}{}^{ab})$
$\Rightarrow \psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = -\frac{1}{2}R$ □

Corollary 2.4.13. $\psi^{\alpha'_\varsigma\beta'_\kappa} = (\psi^{\alpha_\varsigma\beta_\kappa})^*$

Proof: $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{1}{4}\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{abcd}$
$\Leftrightarrow (\psi^{\alpha_\varsigma\beta_\kappa})^* = \frac{1}{4}(\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\sigma_\kappa^{\beta_\kappa}{}_{cd}R^{abcd})^* = \frac{1}{4}\sigma_\varsigma^{\alpha'_\varsigma}{}_{a'b'}\sigma_\kappa^{\beta'_\kappa}{}_{c'd'}\eta^{a'}{}_a\eta^{b'}{}_b\eta^{c'}{}_c\eta^{d'}{}_dR^{abcd}$
$\Leftrightarrow (\psi^{\alpha_\varsigma\beta_\kappa})^* = \frac{1}{4}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\kappa}_{-\kappa cd}R^{abcd}$
$\Leftrightarrow \psi^{\alpha'_\varsigma\beta'_\kappa} = (\psi^{\alpha_\varsigma\beta_\kappa})^*$ □

2.5 Spreading of gravitational field curvature tensor

Corollary 2.5.1. $R^{abcd} = \frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'} - \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'} - \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta} + \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta})$

Proof: $R^{abcd} = \frac{1}{2}(\sigma_{-\beta'}{}^{cd}F^{ab\beta'} - \sigma_{+\beta}{}^{cd}F^{ab\beta})$
$\Leftrightarrow R^{abcd} = \frac{1}{4}[\sigma_{-\beta'}{}^{cd}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\beta'} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha\beta'}) - \sigma_{+\beta}{}^{cd}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\beta} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha\beta})]$
$\Leftrightarrow R^{abcd} = \frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'} - \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'} - \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta} + \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta})$ □

Corollary 2.5.2. $R^{ab(*cd)} = \frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'} - \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'} + \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta} - \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta})$

Corollary 2.5.3. $R^{(*ab)cd} = \frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'} + \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'} - \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta} - \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta})$

Corollary 2.5.4. $R^{(*ab)(*cd)} = \frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'} + \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'} + \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta} + \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta})$

Corollary 2.5.5. $R^{abcd} + R^{ab(*cd)} + R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha'\beta'}$

Corollary 2.5.6. $-R^{abcd} - R^{ab(*cd)} + R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}\psi^{\alpha\beta'}$

Corollary 2.5.7. $-R^{abcd} + R^{ab(*cd)} - R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha'\beta}$

Corollary 2.5.8. $R^{abcd} - R^{ab(*cd)} - R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}\psi^{\alpha\beta}$

Corollary 2.5.9. $-R^{abcd} - \varsigma R^{ab(*cd)} + \varsigma R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{-\varsigma\beta'_\varsigma}{}^{cd}\psi^{\alpha\beta'_\varsigma}$

Corollary 2.5.10. $R^{abcd} - \varsigma R^{ab(*cd)} - \varsigma R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma\beta_\varsigma}{}^{cd}\psi^{\alpha_\varsigma\beta_\varsigma}$

Corollary 2.5.11. $\varsigma\kappa R^{abcd} - \varsigma R^{ab(*cd)} - \kappa R^{(*ab)cd} + R^{(*ab)(*cd)} = \sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\kappa\beta_\kappa}{}^{cd}\psi^{\alpha_\varsigma\beta_\kappa}$

Definiton 2.5.1. $R^{ab} \equiv \delta_{cd}R^{acdb} = -\frac{1}{4}\delta_{cd}(\sigma_{-\alpha'}{}^{ca}\sigma_{-\beta'}{}^{db}\psi^{\alpha'\beta'} - \sigma_{+\alpha}{}^{ac}\sigma_{-\beta'}{}^{db}\psi^{\alpha\beta'} - \sigma_{-\alpha'}{}^{ac}\sigma_{+\beta}{}^{db}\psi^{\alpha'\beta} + \sigma_{+\alpha}{}^{ac}\sigma_{+\beta}{}^{db}\psi^{\alpha\beta})$

Corollary 2.5.12. $R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}\delta^{ab}R + \delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'}$

Proof: $R^{ab} = -\frac{1}{4}(\sigma_{-\alpha'}\sigma_{-\beta'}\psi^{\alpha'\beta'} - \sigma_{+\alpha}\sigma_{-\beta'}\psi^{\alpha\beta'} - \sigma_{-\alpha'}\sigma_{+\beta}\psi^{\alpha'\beta} + \sigma_{+\alpha}\sigma_{+\beta}\psi^{\alpha\beta})^{ab}$
$\Leftrightarrow R^{ab} = -\frac{1}{8}(\{\sigma_{-\alpha'}, \sigma_{-\beta'}\}\psi^{\alpha'\beta'} - 2\{\sigma_{+\alpha}, \sigma_{-\beta'}\}\psi^{\alpha\beta'} + \{\sigma_{+\alpha}, \sigma_{+\beta}\}\psi^{\alpha\beta})^{ab}$
$\Leftrightarrow R^{ab} = -\frac{1}{4}(-R - 2\{\sigma_{+\alpha}, \sigma_{-\beta'}\}\psi^{\alpha\beta'})^{ab}$
$\Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}\delta^{ab}R + \delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'}$ □

Corollary 2.5.13. $C^{\alpha\beta'}=0$

Proof: $C^{abcd}=\frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}C^{\alpha'\beta'}-\sigma_{+\alpha}{}^{ab}\sigma_{-\beta'}{}^{cd}C^{\alpha\beta'}-\sigma_{-\alpha'}{}^{ab}\sigma_{+\beta}{}^{cd}C^{\alpha'\beta}+\sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}C^{\alpha\beta})$

$\Rightarrow 0=g_{bc}C^{abcd}=\frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'b}{}^{d}C^{\alpha'\beta'}-\sigma_{+\alpha}{}^{ab}\sigma_{-\beta'b}{}^{d}C^{\alpha\beta'}-\sigma_{-\alpha'}{}^{ab}\sigma_{+\beta b}{}^{d}C^{\alpha'\beta}+\sigma_{+\alpha}{}^{ab}\sigma_{+\beta b}{}^{d}C^{\alpha\beta})$

$\Rightarrow \sigma_{-\alpha'}\sigma_{-\beta'}C^{\alpha'\beta'}-\sigma_{+\alpha}\sigma_{-\beta'}C^{\alpha\beta'}-\sigma_{-\alpha'}\sigma_{+\beta}C^{\alpha'\beta}+\sigma_{+\alpha}\sigma_{+\beta}C^{\alpha\beta}=0$

$\Rightarrow (\delta^{\alpha'\beta'}I+i\varepsilon_{\alpha'\beta'\gamma'}\sigma_{-}^{\gamma'})C^{\alpha'\beta'}-2\sigma_{+\alpha}\sigma_{-\beta'}C^{\alpha\beta'}+(\delta^{\alpha\beta}I+i\varepsilon_{\alpha\beta\gamma}\sigma_{+}^{\gamma})C^{\alpha\beta}=0$

$\Rightarrow \delta_{\alpha'\beta'}IC^{\alpha'\beta'}-2\sigma_{+\alpha}\sigma_{-\beta'}C^{\alpha\beta'}+\delta_{\alpha\beta}IC^{\alpha\beta}=0$

$\Rightarrow \sigma_{+\alpha}\sigma_{-\beta'}C^{\alpha\beta'}=0$

$\Rightarrow C^{\alpha\beta'}=0$ □

Corollary 2.5.14. $C^{abcd}=\frac{1}{4}(\sigma_{-\alpha'}{}^{ab}\sigma_{-\beta'}{}^{cd}C^{\alpha'\beta'}+\sigma_{+\alpha}{}^{ab}\sigma_{+\beta}{}^{cd}C^{\alpha\beta}), C^{\alpha'\beta'}=(C^{\alpha\beta})^*$

Corollary 2.5.15. $C^{(*ab)cd}=C^{ab(*cd)}, C^{abcd}=C^{(*ab)(*cd)}$

Corollary 2.5.16. $\psi^{\alpha_\varsigma\beta'_\varsigma}=\frac{1}{2}\sigma_{\varsigma}^{\alpha_\varsigma}{}_{ac}\sigma_{-\varsigma}^{\beta'_\varsigma c}{}_{b}R^{ab}=\frac{1}{2}(\sigma_{\varsigma}^{\alpha_\varsigma}\sigma_{-\varsigma}^{\beta'_\varsigma})_{ab}R^{ab}$

Corollary 2.5.17. $\frac{1}{4}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}(\sigma_{+\rho}\sigma_{-\sigma'})^{ab}=\delta^{\alpha}{}_{\rho}\delta^{\beta'}{}_{\sigma'}$

A more general proof that it does not depend on the definition of various amounts.

Corollary 2.5.18. $\psi^{\alpha_\varsigma\beta'}=\frac{1}{2}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}R^{ab}\Leftrightarrow R^{ab}=\frac{1}{4}\delta^{ab}R+\frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}$

Proof: $\psi^{\alpha_\varsigma\beta'}=\frac{1}{2}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}R^{ab}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{cd}R^{cd}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(\sigma_{+\alpha}{}^{ae}\sigma_{-\beta'e}{}^{b})(\sigma_{+cf}^{\alpha}\sigma_{-}^{\beta'f}{}_{d})R^{cd}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(\sigma_{+\alpha}{}^{ae}\sigma_{+}^{\alpha cf})(\sigma_{-\beta'eb}\sigma_{-}^{\beta'}{}_{fd})R_{c}{}^{d}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(S^{aecf}-\varepsilon^{aecf})(S_{ebfd}+\varepsilon_{ebfd})R_{c}{}^{d}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(2\delta^{ac}\delta_{bd}+2\delta^{a}{}_{d}\delta_{b}{}^{c}-\delta^{a}{}_{b}\delta^{c}{}_{d})R_{c}{}^{d}$

$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha_\varsigma\beta'}=\frac{1}{4}(4R^{a}{}_{b}-\delta^{a}{}_{b}R)$

$\Leftrightarrow R^{ab}=\frac{1}{4}\delta^{ab}R+\frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}$ □

2.6 Ashtekar guage description of gravitational field curvature tensor [28]

Gravitational field curvature tensor: $R_{ab}{}^{cd}=\partial_a\omega_b{}^{cd}-\partial_b\omega_a{}^{cd}+\omega_{[a}{}^{ce}\omega_{b]e}{}^{d}$ (2.6)

Definiton 2.6.1. *Introducing Ashtekar variable [28]:* $A_a{}^{\alpha_\varsigma}\equiv\frac{1}{2}\varsigma\sigma_{\varsigma cd}^{\alpha_\varsigma}\omega_a{}^{cd}$

Corollary 2.6.1. $[A_a{}^{\alpha_\varsigma}]^*=A^*_{a'}{}^{\alpha'_\varsigma}=\eta_{a'}{}^{a}A_a{}^{\alpha'_\varsigma}$

Proof: $[A_a{}^{\alpha_\varsigma}]^*=A^*_{a'}{}^{\alpha'_\varsigma}=-\frac{1}{2}\varsigma\sigma_{\varsigma c'd'}^{\alpha'_\varsigma}\omega^*_{a'}{}^{c'd'}=-\frac{1}{2}\varsigma\eta_{a'}{}^{a}\sigma_{-\varsigma cd}^{\alpha'_\varsigma}\omega_a{}^{cd}=\eta_{a'}{}^{a}A_a{}^{\alpha'_\varsigma}$ □

Corollary 2.6.2. $[F_{ab}{}^{\alpha_\varsigma}]^*=F^*_{a'b'}{}^{\alpha'_\varsigma}=\eta_{a'}{}^{a}\eta_{b'}{}^{b}F_{ab}{}^{\alpha'_\varsigma}$

Proof: $[F_{ab}{}^{\alpha_\varsigma}]^*=F^*_{a'b'}{}^{\alpha'_\varsigma}=-\frac{1}{2}\varsigma\sigma_{\varsigma c'd'}^{\alpha'_\varsigma}R^*_{a'b'}{}^{c'd'}=-\frac{1}{2}\varsigma\eta_{a'}{}^{a}\eta_{b'}{}^{b}\sigma_{-\varsigma cd}^{\alpha'_\varsigma}R_{ab}{}^{cd}=\eta_{a'}{}^{a}\eta_{b'}{}^{b}F_{ab}{}^{\alpha'_\varsigma}$ □

Corollary 2.6.3. $\omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}+\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma_{\varsigma\alpha_\varsigma}{}^{cd}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma})$

Proof: $\omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\omega_a{}^{ce}\omega_{be}{}^{d}-\omega_b{}^{ce}\omega_{ae}{}^{d}$

$\Leftrightarrow \omega_a{}^{ce}\omega_{be}{}^{d}=\delta_{ef}[\frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{ce}A_a{}^{\alpha'_\varsigma}-\sigma_{\varsigma\alpha_\varsigma}{}^{ce}A_a{}^{\alpha_\varsigma})][\frac{1}{2}\varsigma(\sigma_{-\varsigma\beta'_\varsigma}{}^{fd}A_b{}^{\beta'_\varsigma}-\sigma_{\varsigma\beta_\varsigma}{}^{fd}A_b{}^{\beta_\varsigma})]$

$\Leftrightarrow \omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\frac{1}{4}\delta_{ef}(\sigma_{-\varsigma[\alpha'_\varsigma}{}^{ce}\sigma_{-\varsigma\beta'_\varsigma]}{}^{fd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta'_\varsigma}-\sigma_{\varsigma[\alpha_\varsigma}{}^{ce}\sigma_{-\varsigma\beta'_\varsigma]}{}^{fd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta'_\varsigma}$

$-\sigma_{-\varsigma[\alpha'_\varsigma}{}^{ce}\sigma_{\varsigma\beta_\varsigma]}{}^{fd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta_\varsigma}+\sigma_{\varsigma[\alpha_\varsigma}{}^{ce}\sigma_{\varsigma\beta_\varsigma]}{}^{fd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\frac{1}{4}(2i\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{-\varsigma\gamma'_\varsigma}{}^{cd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta'_\varsigma}+0+0+2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{cd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\frac{i}{2}(\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{-\varsigma\gamma'_\varsigma}{}^{cd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta'_\varsigma}+\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{cd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[a}{}^{ce}\omega_{b]e}{}^{d}=\frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}+\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma_{\varsigma\alpha_\varsigma}{}^{cd}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma})$ □

Corollary 2.6.4. $\begin{cases}F_{ab}{}^{\alpha_\varsigma}=\partial_aA_b{}^{\alpha_\varsigma}-\partial_bA_a{}^{\alpha_\varsigma}-i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}\\F_{ab}{}^{\alpha'_\varsigma}=\partial_aA_b{}^{\alpha'_\varsigma}-\partial_bA_a{}^{\alpha'_\varsigma}+i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}\end{cases}$

Proof: $R_{ab}{}^{cd} = \partial_a\omega_b{}^{cd} - \partial_b\omega_a{}^{cd} + \omega_{[a}{}^{ce}\omega_{b]e}{}^{d}$

$\Leftrightarrow R_{ab}{}^{cd} = \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}\partial_a A_b{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd}\partial_a A_b{}^{\alpha_\varsigma}) - \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}\partial_b A_a{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd}\partial_b A_a{}^{\alpha_\varsigma})$
$\quad + \frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma} + \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma_{\varsigma\alpha_\varsigma}{}^{cd}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma})$

$\Leftrightarrow R_{ab}{}^{cd} = \frac{1}{2}\varsigma\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}(\partial_{[a}A_{b]}{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}) - \frac{1}{2}\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{cd}(\partial_{[a}A_{b]}{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma})$

$\Leftrightarrow \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}F_{ab}{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd}F_{ab}{}^{\alpha_\varsigma}) = \frac{1}{2}\varsigma\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd}(\partial_{[a}A_{b]}{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma})$
$\quad - \frac{1}{2}\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{cd}(\partial_{[a}A_{b]}{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma})$

$$\Leftrightarrow \begin{cases} F_{ab}{}^{\alpha_\varsigma} = \partial_a A_b{}^{\alpha_\varsigma} - \partial_b A_a{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma} \\ F_{ab}{}^{\alpha'_\varsigma} = \partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma} \end{cases}$$ □

Corollary 2.6.5. $\frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_{[a}{}^{ce}\omega_{b]e}{}^{d} = -i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}$

Proof: $\omega_{[a}{}^{ce}\omega_{b]e}{}^{d} = \frac{i}{2}(\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{-\varsigma\gamma'_\varsigma}{}^{cd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta'_\varsigma} + \varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{cd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma})$

$\Rightarrow \frac{1}{2}\varsigma\sigma^{\rho_\varsigma}_{\varsigma cd}\omega_{[a}{}^{ce}\omega_{b]e}{}^{d} = \frac{i}{4}\varsigma\sigma^{\rho_\varsigma}_{\varsigma cd}(\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{-\varsigma\gamma'_\varsigma}{}^{cd}A_a{}^{\alpha'_\varsigma}A_b{}^{\beta'_\varsigma} + \varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{cd}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma})$

$\Leftrightarrow \frac{1}{2}\varsigma\sigma^{\rho_\varsigma}_{\varsigma cd}\omega_{[a}{}^{ce}\omega_{b]e}{}^{d} = 0 - i\varsigma\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\delta^{\rho_\varsigma}{}_{\gamma_\varsigma}A_a{}^{\alpha_\varsigma}A_b{}^{\beta_\varsigma}$

$\Leftrightarrow \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_{[a}{}^{ce}\omega_{b]e}{}^{d} = -i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}$ □

Corollary 2.6.6. $F_{ab}{}^{\alpha_\varsigma} = \partial_a A_b{}^{\alpha_\varsigma} - \partial_b A_a{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma} \Leftrightarrow F_{ab}{}^{\alpha'_\varsigma} = \partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}$

Proof: $F_{ab}{}^{\alpha_\varsigma} = \partial_a A_b{}^{\alpha_\varsigma} - \partial_b A_a{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}$

$\Leftrightarrow [F_{ab}{}^{\alpha_\varsigma}]^* = F^*_{a'b'}{}^{\alpha'_\varsigma} = \partial_{a'}A^*_{b'}{}^{\alpha'_\varsigma} - \partial_{b'}A^*_{a'}{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A^*_{a'}{}^{\beta'_\varsigma}A^*_{b'}{}^{\gamma'_\varsigma}$

$\Leftrightarrow F^*_{a'b'}{}^{\alpha'_\varsigma} = \eta_{a'}{}^{a}\eta_{b'}{}^{b}(\partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma})$

$\Leftrightarrow F^*_{a'b'}{}^{\alpha'_\varsigma} = \eta_{a'}{}^{a}\eta_{b'}{}^{b}F_{ab}{}^{\alpha'_\varsigma} = \eta_{a'}{}^{a}\eta_{b'}{}^{b}(\partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma})$

$\Leftrightarrow F_{ab}{}^{\alpha'_\varsigma} = \partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta'_\varsigma}A_b{}^{\gamma'_\varsigma}$ □

Corollary 2.6.7. $R_{ab}{}^{cd} = \partial_a\omega_b{}^{cd} - \partial_b\omega_a{}^{cd} + \omega_{[a}{}^{ce}\omega_{b]e}{}^{d} \Leftrightarrow F_{ab}{}^{\alpha_\varsigma} = \partial_a A_b{}^{\alpha_\varsigma} - \partial_b A_a{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}$

Corollary 2.6.8. $R_{ab}{}^{cd} = \partial_a\omega_b{}^{cd} - \partial_b\omega_a{}^{cd} + \omega_{[a}{}^{ce}\omega_{b]e}{}^{d} \Leftrightarrow F_{ab}{}^{\alpha'_\varsigma} = \partial_a A_b{}^{\alpha'_\varsigma} - \partial_b A_a{}^{\alpha'_\varsigma} + i\varsigma\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma'_\varsigma}$

2.7 Half integer spinorial description of gravitational field [4, 5]

Definiton 2.7.1. *Gravitational spinor:* $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \frac{1}{2}\varsigma\kappa S_{ab}{}^{A_\varsigma B_\varsigma}S_{cd}{}^{C_\kappa D_\kappa}R^{abcd}$
Weyl spinor: $C^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \frac{1}{2}\varsigma\kappa S_{ab}{}^{A_\varsigma B_\varsigma}S_{cd}{}^{C_\kappa D_\kappa}C^{abcd}$

Corollary 2.7.1. $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \frac{1}{2}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\kappa}{}^{C_\kappa D_\kappa}\psi^{\alpha_\varsigma\beta_\kappa}$

Proof: $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \frac{1}{2}\varsigma\kappa S_{ab}{}^{A_\varsigma B_\varsigma}S_{cd}{}^{C_\kappa D_\kappa}R^{abcd}$

$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \frac{1}{8}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ab}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\cdot\kappa\sigma^{\beta_\kappa}_{\kappa}{}_{cd}\sigma_{\beta_\kappa}{}^{C_\varsigma D_\varsigma}R^{abcd}$

$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \frac{1}{2}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\kappa}{}^{C_\kappa D_\kappa}\psi^{\alpha_\varsigma\beta_\kappa}$ □

Corollary 2.7.2. $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{1}{2}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\sigma^{\beta_\kappa}{}_{C_\kappa D_\kappa}\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa}$

Corollary 2.7.3. $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{12}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma C_\varsigma D_\varsigma}R$

Proof: $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} - \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$

$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{12}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\delta^{\alpha_\varsigma\beta_\varsigma}R$

$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{12}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma C_\varsigma D_\varsigma}R$ □

Corollary 2.7.4. $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{B_\varsigma A_\varsigma C_\kappa D_\kappa}, \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{A_\varsigma B_\varsigma D_\kappa C_\kappa}, \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{C_\kappa D_\kappa A_\varsigma B_\varsigma}$

Corollary 2.7.5. $$\psi^{\alpha_\varsigma\beta_\varsigma} = \psi^{\beta_\varsigma\alpha_\varsigma} \Leftrightarrow \begin{cases} \psi^{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma} \\ \psi^{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = \psi^{1_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma} = \psi^{1_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma} = \psi^{2_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma} \\ \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma}, \psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = \psi^{1_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma} = \psi^{2_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = \psi^{2_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma} \\ \psi^{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma} \\ \psi^{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma} \end{cases}$$

Corollary 2.7.6. $\psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \varepsilon_{A_\varsigma C_\varsigma}\varepsilon_{B_\varsigma D_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = 2(\psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma})$

Corollary 2.7.7. $\psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$

Proof: $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \frac{1}{2}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\kappa}{}^{C_\kappa D_\kappa}\psi^{\alpha_\varsigma\beta_\kappa}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \frac{1}{2}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$ □

Corollary 2.7.8. $\psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = -\frac{1}{4}R$

Corollary 2.7.9. $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$ *is a full symmetric spinor.* $\Leftrightarrow$ $\psi^{\alpha_\varsigma\beta_\varsigma}$ *is a non trace symmetry tensor.*
Namely, $\psi^{\alpha_\varsigma\beta_\varsigma} = \psi^{\beta_\varsigma\alpha_\varsigma}, \psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = 0$

2.8 Integer spinorial description of gravitational field source

Definiton 2.8.1. *Integer spinor of gravitational field source:* $J_a{}^{\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd}J_a{}^{cd}$

Corollary 2.8.1. $[J_a{}^{\alpha_\varsigma}]^* = J^*_{a'}{}^{\alpha'_\varsigma} = \eta_{a'}{}^a J_a{}^{\alpha'_\varsigma}$

Corollary 2.8.2. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{cd}J_a{}^{\alpha_\varsigma}$

Corollary 2.8.3. $J_a{}^{\alpha_\varsigma} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma}{}_{cd} * J_a{}^{cd}$

Corollary 2.8.4. $\sigma^{\alpha_\varsigma}_{\varsigma}{}_{cd}J_a{}^{cd} = -\varsigma\sigma^{\alpha_\varsigma}_{\varsigma}{}_{cd} * J_a{}^{cd}$

Corollary 2.8.5. $J_a{}^{\alpha_\varsigma} = \frac{1}{4}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma}{}_{cd}(J_a{}^{cd} - \varsigma * J_a{}^{cd})$

Corollary 2.8.6. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{cd}\sigma^{\alpha_\varsigma}_{\varsigma}{}_{ef}(J_a{}^{ef} - \varsigma * J_a{}^{ef})$

Corollary 2.8.7. $J_a{}^{cd} = \frac{1}{2}(\sigma_{-\alpha'}{}^{cd}J_a{}^{\alpha'} - \sigma_{+\alpha}{}^{cd}J_a{}^{\alpha}), *J_a{}^{cd} = \frac{1}{2}(\sigma_{-\alpha'}{}^{cd}J_a{}^{\alpha'} + \sigma_{+\alpha}{}^{cd}J_a{}^{\alpha})$

Corollary 2.8.8. $J_a{}^{cd} = -J_a{}^{dc} \Leftrightarrow J_a{}^{cd} = \frac{1}{2}(\sigma_{-\alpha'}{}^{cd}J_a{}^{\alpha'} - \sigma_{+\alpha}{}^{cd}J_a{}^{\alpha_\varsigma}),$

2.9 Vector-spinor description of gravitational field source [4,5]

Definiton 2.9.1. *Vector-spinor of gravitational field source:* $J_a{}^{A_\varsigma B_\varsigma} \equiv -\frac{1}{\sqrt{2}}\varsigma S_{cd}{}^{A_\varsigma B_\varsigma}J_a{}^{cd}$

Corollary 2.9.1. $J_a{}^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}J_a{}^{\alpha_\varsigma}$

Corollary 2.9.2. $J_a{}^{A_\varsigma B_\varsigma} = J_a{}^{B_\varsigma A_\varsigma}$

Corollary 2.9.3. $J_a{}^{\alpha_\varsigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}J_a{}^{A_\varsigma B_\varsigma}$

Corollary 2.9.4. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = \sqrt{2}\varsigma S^{cd}{}_{A_\varsigma B_\varsigma}J_a{}^{A_\varsigma B_\varsigma}$

Corollary 2.9.5. $J_a{}^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}S_{cd}{}^{A_\varsigma B_\varsigma} * J_a{}^{cd}$

Corollary 2.9.6. $J_a{}^{A_\varsigma B_\varsigma} = -\frac{1}{2\sqrt{2}}\varsigma S_{cd}{}^{A_\varsigma B_\varsigma}(J_a{}^{cd} - \varsigma * J_a{}^{cd})$

Corollary 2.9.7. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = -\frac{1}{2}S^{cd}{}_{A_\varsigma B_\varsigma}S_{ef}{}^{A_\varsigma B_\varsigma}(J_a{}^{ef} - \varsigma * J_a{}^{ef})$

Corollary 2.9.8. $J_a{}^{cd} = -\frac{1}{\sqrt{2}}(S^{cdA'B'}J_{aA'B'} - S^{cd}{}_{AB}J_a{}^{AB}), *J_a{}^{cd} = -\frac{1}{\sqrt{2}}(S^{cdA'B'}J_{aA'B'} + S^{cd}{}_{AB}J_a{}^{AB})$

Corollary 2.9.9. $J_a{}^{cd} = -J_a{}^{dc} \Leftrightarrow J_a{}^{cd} = -\frac{1}{\sqrt{2}}(S^{cdA'B'}J_{aA'B'} - S^{cd}{}_{AB}J_a{}^{AB})$

2.10 Half integer spinorial description of gravitational field source [4, 5]

Definiton 2.10.1. *Spinor of gravitational field source:* $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} \equiv -\frac{1}{2}\kappa(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma} S_{cd}{}^{C_\kappa D_\kappa} J_a{}^{cd}$

Corollary 2.10.1. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = \frac{1}{\sqrt{2}}\varsigma\kappa(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma} J_a{}^{C_\kappa D_\kappa}$

Corollary 2.10.2. $J_a{}^{C_\kappa D_\kappa} = \frac{1}{\sqrt{2}}\varsigma\kappa(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Corollary 2.10.3. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = \frac{1}{2}\varsigma\kappa(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\sigma_{\alpha_\kappa}{}^{C_\kappa D_\kappa} J_a{}^{\alpha_\kappa}$

Corollary 2.10.4. $J_a{}^{\alpha_\kappa} = \frac{1}{2}\varsigma\kappa(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}\sigma^{\alpha_\kappa}{}_{C_\kappa D_\kappa} J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Corollary 2.10.5. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = \frac{1}{2}\varsigma\kappa(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma} S_{cd}{}^{C_\kappa D_\kappa} * J_a{}^{cd}$

Corollary 2.10.6. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = J_{A'_\varsigma}{}^{B_\varsigma D_\kappa C_\kappa}$

Corollary 2.10.7. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = \kappa(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma} S^{cd}{}_{C_\kappa D_\kappa} J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Corollary 2.10.8. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = \frac{1}{2}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma}$

Corollary 2.10.9. $J_a{}^{\alpha_\varsigma} = \frac{1}{2}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}{}_{C_\varsigma D_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$

Corollary 2.10.10. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{B_\varsigma D_\varsigma C_\varsigma} \Leftrightarrow \begin{cases} J_{1'_\varsigma}{}^{1_\varsigma 1_\varsigma 1_\varsigma} \\ J_{1'_\varsigma}{}^{1_\varsigma 1_\varsigma 2_\varsigma} = J_{1'_\varsigma}{}^{1_\varsigma 2_\varsigma 1_\varsigma}, J_{1'_\varsigma}{}^{2_\varsigma 1_\varsigma 2_\varsigma} = J_{1'_\varsigma}{}^{2_\varsigma 2_\varsigma 1_\varsigma} \\ J_{2'_\varsigma}{}^{1_\varsigma 1_\varsigma 2_\varsigma} = J_{2'_\varsigma}{}^{1_\varsigma 2_\varsigma 1_\varsigma}, J_{2'_\varsigma}{}^{2_\varsigma 1_\varsigma 2_\varsigma} = J_{2'_\varsigma}{}^{2_\varsigma 2_\varsigma 1_\varsigma} \\ J_{2'_\varsigma}{}^{2_\varsigma 2_\varsigma 2_\varsigma} \end{cases}$

Corollary 2.10.11. $J_{A'_\varsigma}{}^{1_\varsigma 2_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{2_\varsigma 1_\varsigma D_\varsigma} \Leftrightarrow \begin{cases} \varsigma J_\pi{}^{x_\varsigma} = J_y{}^{z_\varsigma} - J_z{}^{y_\varsigma}, \varsigma J_\pi{}^{y_\varsigma} = J_z{}^{x_\varsigma} - J_x{}^{z_\varsigma}, \varsigma J_\pi{}^{z_\varsigma} = J_x{}^{y_\varsigma} - J_y{}^{x_\varsigma} \\ J_x{}^{x_\varsigma} + J_y{}^{y_\varsigma} + J_z{}^{z_\varsigma} = 0 \end{cases}$

Proof: $J_{A'_\varsigma}{}^{1_\varsigma 2_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{2_\varsigma 1_\varsigma D_\varsigma}$

$$\Leftrightarrow \tfrac{1}{2}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma 1_\varsigma}\sigma_{\alpha_\varsigma}{}^{2_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma} = \tfrac{1}{2}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma 2_\varsigma}\sigma_{\alpha_\varsigma}{}^{1_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma}$$
$$\Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma 2_\varsigma}\bar{\varepsilon}^{2_\varsigma 1_\varsigma}\sigma_{\alpha_\varsigma}{}^{2_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma} = (\sigma, -i\varsigma)^a{}_{A'_\varsigma 1_\varsigma}\bar{\varepsilon}^{1_\varsigma 2_\varsigma}\sigma_{\alpha_\varsigma}{}^{1_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma}$$
$$\Leftrightarrow -(\sigma, -i\varsigma)^a{}_{A'_\varsigma 2_\varsigma}\sigma_{\alpha_\varsigma}{}^{2_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma} = (\sigma, -i\varsigma)^a{}_{A'_\varsigma 1_\varsigma}\sigma_{\alpha_\varsigma}{}^{1_\varsigma D_\varsigma} J_a{}^{\alpha_\varsigma}$$
$$\Leftrightarrow [(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}\bar{\varepsilon}]_{A'_\varsigma}{}^{D_\varsigma} J_a{}^{\alpha_\varsigma} = 0$$
$$\Leftrightarrow [(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}\bar{\varepsilon}] J_a{}^{\alpha_\varsigma} = 0$$
$$\Leftrightarrow (-iJ_x{}^{y_\varsigma} + iJ_y{}^{x_\varsigma} + i\varsigma J_\pi{}^{z_\varsigma})\cdot\sigma_x + (-iJ_x{}^{x_\varsigma} - iJ_y{}^{y_\varsigma} - iJ_z{}^{z_\varsigma})\cdot\sigma_y$$
$$+ (iJ_y{}^{z_\varsigma} - iJ_z{}^{y_\varsigma} - i\varsigma J_\pi{}^{x_\varsigma})\cdot\sigma_z + (-J_x{}^{z_\varsigma} + J_z{}^{x_\varsigma} - \varsigma J_\pi{}^{y_\varsigma})\cdot I = 0$$
$$\Leftrightarrow \varsigma J_\pi{}^{x_\varsigma} = J_y{}^{z_\varsigma} - J_z{}^{y_\varsigma}, \varsigma J_\pi{}^{y_\varsigma} = J_z{}^{x_\varsigma} - J_x{}^{z_\varsigma}, \varsigma J_\pi{}^{z_\varsigma} = J_x{}^{y_\varsigma} - J_y{}^{x_\varsigma}, J_x{}^{x_\varsigma} + J_y{}^{y_\varsigma} + J_z{}^{z_\varsigma} = 0$$

□

Corollary 2.10.12. $[(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}] J_a{}^{\alpha_\varsigma} = 0 \Leftrightarrow \begin{cases} \varsigma J_\pi{}^{x_\varsigma} = J_y{}^{z_\varsigma} - J_z{}^{y_\varsigma}, \varsigma J_\pi{}^{y_\varsigma} = J_z{}^{x_\varsigma} - J_x{}^{z_\varsigma}, \varsigma J_\pi{}^{z_\varsigma} = J_x{}^{y_\varsigma} - J_y{}^{x_\varsigma} \\ J_x{}^{x_\varsigma} + J_y{}^{y_\varsigma} + J_z{}^{z_\varsigma} = 0 \end{cases}$

Corollary 2.10.13. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$ *is full symmetry for superscripts* $B_\varsigma C_\varsigma D_\varsigma$. $\Leftrightarrow [(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}] J_a{}^{\alpha_\varsigma} = 0$

Corollary 2.10.14. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$ *is full symmetry for superscripts* $B_\varsigma C_\varsigma D_\varsigma$. $\Leftrightarrow [(\sigma, -i\varsigma)^a S_{cd}(\varsigma)] J_a{}^{cd} = 0$

3 Applying constant tensors to define spinors of Yang-Mills field [9]

3.1 Integer spinorial description of Yang-Mills field

Definiton 3.1.1. *Yang-Mills integer spinor:* $\psi^{\alpha_\varsigma\sigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_\varsigma{}_{ab} F^{ab\sigma} = (E - i\varsigma B)^{\alpha_\varsigma\sigma}$

Corollary 3.1.1. $F^{ab\sigma} - \varsigma * F^{ab\sigma} = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}$

Corollary 3.1.2. $\psi^{\alpha_\varsigma\sigma} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_\varsigma{}_{ab} * F^{ab\sigma}$

Corollary 3.1.3. $\sigma_{\varsigma}^{\alpha_\varsigma}{}_{ab}F^{ab\sigma} = -\varsigma\sigma_{\varsigma}^{\alpha_\varsigma}{}_{ab} * F^{ab\sigma}$

Corollary 3.1.4. $\psi^{\alpha_\varsigma\sigma} = \frac{1}{4}\varsigma\sigma_{\varsigma}^{\alpha_\varsigma}{}_{ab}(F^{ab} - \varsigma * F^{ab\sigma})$

Corollary 3.1.5. $F^{ab\sigma} - \varsigma * F^{ab\sigma} = -\frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma}^{\alpha_\varsigma}{}_{cd}(F^{cd\sigma} - \varsigma * F^{cd\sigma})$

Corollary 3.1.6. $F^{ab\sigma} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\sigma} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha\sigma}), *F^{ab\sigma} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\sigma} + \sigma_{+\alpha}{}^{ab}\psi^{\alpha\sigma})$

Corollary 3.1.7. $F^{ab\sigma} = -F^{ba\sigma} \Leftrightarrow F^{ab\sigma} = \frac{1}{2}(\sigma_{-\alpha'}{}^{ab}\psi^{\alpha'\sigma} - \sigma_{+\alpha}{}^{ab}\psi^{\alpha\sigma}),$

3.2 Half integer spinorial description of Yang-Mills field [4, 5]

Definiton 3.2.1. *Yang-Mills spinor:* $\psi^{A_\varsigma B_\varsigma\sigma} \equiv -\frac{1}{\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}F^{ab\sigma} = -\frac{1}{\sqrt{2}}\varsigma\varepsilon^{B_\varsigma C_\varsigma}S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}F^{ab\sigma}$

Corollary 3.2.1. $\psi^{A_\varsigma B_\varsigma\sigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma\sigma}$

Corollary 3.2.2. $\psi^{A_\varsigma B_\varsigma\sigma} = \psi^{B_\varsigma A_\varsigma\sigma}$

Corollary 3.2.3. $\psi^{\alpha_\varsigma\sigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma\sigma}$

Corollary 3.2.4. $F^{ab\sigma} - \varsigma * F^{ab\sigma} = \sqrt{2}\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma\sigma}$

Corollary 3.2.5. $\psi^{A_\varsigma B_\varsigma\sigma} = \frac{1}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma} * F^{ab\sigma}$

Corollary 3.2.6. $\psi^{A_\varsigma B_\varsigma\sigma} = -\frac{1}{2\sqrt{2}}\varsigma S_{ab}{}^{A_\varsigma B_\varsigma}(F^{ab\sigma} - \varsigma * F^{ab\sigma})$

Corollary 3.2.7. $F^{ab\sigma} - \varsigma * F^{ab\sigma} = -\frac{1}{2}S^{ab}{}_{A_\varsigma B_\varsigma}S_{cd}{}^{A_\varsigma B_\varsigma}(F^{cd\sigma} - \varsigma * F^{cd\sigma})$

Corollary 3.2.8. $F^{ab\sigma} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'}{}^{\sigma} - S^{ab}{}_{AB}\psi^{AB\sigma}), *F^{ab\sigma} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'}{}^{\sigma} + S^{ab}{}_{AB}\psi^{AB\sigma})$

Corollary 3.2.9. $F^{ab\sigma} = -F^{ba\sigma} \Leftrightarrow F^{ab\sigma} = -\frac{1}{\sqrt{2}}(S^{abA'B'}\psi_{A'B'}{}^{\sigma} - S^{ab}{}_{AB}\psi^{AB\sigma})$

3.3 Half integer spinorial description of Yang-Mills field source [4, 5]

Definiton 3.3.1. *Spinor of Yang-Mills field source:* $J_{A'_\varsigma}{}^{B_\varsigma\sigma} \equiv \frac{1}{\sqrt{2}}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}J_a{}^{\sigma}$

Corollary 3.3.1. $J_a{}^{\sigma} = \frac{1}{\sqrt{2}}(\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma\sigma}$

3.4 Self review

In fact, both electromagnetic and gravitational fields can be attributed to Yang-Mills field cases. When σ is empty, it is an electromagnetic field. When $\sigma = \beta_\kappa$, it is a gravitational field. When σ is more than one letter, it can describe more general cases. σ can be both internal and external indicators, and even a mixture of the two ones. So the mathematical form of Yang-Mills field is a very general case.

4 Constant tensor and representation transformation

4.1 Full symmetry tensor and representation transformation of electromagnetic field

Definiton 4.1.1. $\Psi(1,\varsigma) \equiv [\psi^{x_\varsigma}, \psi^{y_\varsigma}, \psi^{z_\varsigma}]^T, \psi(1,\varsigma) \equiv [\psi^{1_\varsigma 1_\varsigma}, \psi^{1_\varsigma 2_\varsigma}, \psi^{2_\varsigma 2_\varsigma}]^T$

Definiton 4.1.2. $\hat{\Psi}(1,\varsigma) = \tilde{\Psi}(1,\varsigma) \equiv [\psi^{x_\varsigma}, \psi^{y_\varsigma}, \psi^{z_\varsigma}, 0]^T, \hat{\psi}(1,\varsigma) = \tilde{\psi}(1,\varsigma) \equiv [\psi^{1_\varsigma 1_\varsigma}, \psi^{1_\varsigma 2_\varsigma}, \psi^{1_\varsigma 2_\varsigma}, \psi^{2_\varsigma 2_\varsigma}]^T$

Proposition 4.1.1. $\psi^{\alpha_\varsigma} = \frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma} \Leftrightarrow \tilde{\Psi}(1,\varsigma) = S_{em}(\varsigma)\tilde{\psi}(1,\varsigma) \Leftrightarrow \Psi(1,\varsigma) = S_m(1,\varsigma)\psi(1,\varsigma)$

Proposition 4.1.2. $\psi^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma} \Leftrightarrow \tilde{\psi}(1) = S^+_{em}(\varsigma)\tilde{\Psi}(1) \Leftrightarrow \psi(1) = S^+_m(1,\varsigma)\Psi(1)$

Proposition 4.1.3. $\psi^{A_\varsigma B_\varsigma} = \psi^{B_\varsigma A_\varsigma} \Leftrightarrow \tilde{\psi}(1,\varsigma) = S_{ex}\tilde{\psi}(1,\varsigma)$

Proposition 4.1.4. $\tilde{\psi}(1,\varsigma) = S^+_{em}(\varsigma)\tilde{\Psi}(1) \Leftrightarrow \tilde{\Psi}(1) = S_{em}(\varsigma)\tilde{\psi}(1) \Leftrightarrow \Psi(1) = S_m(1,\varsigma)\psi(1) \Leftrightarrow \psi(1) = S^+_m(1,\varsigma)\Psi(1)$

Proposition 4.1.5. $\Psi(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma} \Leftrightarrow \tilde{\Psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot R}$
$\Leftrightarrow \tilde{\psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \Leftrightarrow \psi(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(1)}$

4.2 Quasi full symmetric tensor and representation transformation of electromagnetic field source

Definiton 4.2.1. $\hat{\mathcal{J}}(1,\varsigma)=\tilde{\mathcal{J}}(1,\varsigma)\equiv[J_x,J_y,J_z,J_\pi]^T$, $\hat{J}(1,\varsigma)=\tilde{J}(1,\varsigma)\equiv[J_{1'_\varsigma}{}^{1_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma}]^T$

Corollary 4.2.1. $J_a=\frac{1}{\sqrt{2}}(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma}\Leftrightarrow\tilde{\mathcal{J}}(1,\varsigma)=S_{em}(\varsigma)\tilde{J}(1,\varsigma)$

Corollary 4.2.2. $J_{A'_\varsigma}{}^{B_\varsigma}=\frac{1}{\sqrt{2}}(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}J_a\Leftrightarrow\tilde{J}(1,\varsigma)=S^+_{em}(\varsigma)\tilde{\mathcal{J}}(1,\varsigma)$

4.3 Full symmetric tensor and representation transformation of Yang-Mills field

Definiton 4.3.1. $\Psi^\sigma(1,\varsigma)\equiv[\psi^{x_\varsigma\sigma},\psi^{y_\varsigma\sigma},\psi^{z_\varsigma\sigma}]^T$, $\psi^\sigma(1,\varsigma)\equiv[\psi^{1_\varsigma1_\varsigma\sigma},\psi^{1_\varsigma2_\varsigma\sigma},\psi^{2_\varsigma2_\varsigma\sigma}]^T$

Definiton 4.3.2. $\hat{\Psi}^\sigma(1,\varsigma)=\tilde{\Psi}^\sigma(1,\varsigma)\equiv[\psi^{x_\varsigma\sigma},\psi^{y_\varsigma\sigma},\psi^{z_\varsigma\sigma},0]^T$, $\hat{\psi}^\sigma(1,\varsigma)=\tilde{\psi}^\sigma(1,\varsigma)\equiv[\psi^{1_\varsigma1_\varsigma\sigma},\psi^{1_\varsigma2_\varsigma\sigma},\psi^{1_\varsigma2_\varsigma\sigma},\psi^{2_\varsigma2_\varsigma\sigma}]^T$

Proposition 4.3.1. $\psi^{\alpha_\varsigma\sigma}=\frac{1}{\sqrt{2}}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma\sigma}\Leftrightarrow\tilde{\Psi}^\sigma(1,\varsigma)=S_{em}(\varsigma)\tilde{\psi}^\sigma(1,\varsigma)\Leftrightarrow\Psi^\sigma(1,\varsigma)=S_m(1,\varsigma)\psi^\sigma(1,\varsigma)$

Proposition 4.3.2. $\psi^{A_\varsigma B_\varsigma\sigma}=\frac{1}{\sqrt{2}}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma\sigma}\Leftrightarrow\tilde{\psi}^\sigma(1)=S^+_{em}(\varsigma)\tilde{\Psi}^\sigma(1)\Leftrightarrow\psi^\sigma(1)=S^+_m(1,\varsigma)\Psi^\sigma(1)$

Proposition 4.3.3. $\psi^{A_\varsigma B_\varsigma\sigma}=\psi^{B_\varsigma A_\varsigma\sigma}\Leftrightarrow\tilde{\psi}^\sigma(1,\varsigma)=S_{ex}\tilde{\psi}^\sigma(1,\varsigma)$

Proposition 4.3.4. $\tilde{\psi}^\sigma(1,\varsigma)=S^+_{em}(\varsigma)\tilde{\Psi}^\sigma(1)\Leftrightarrow\tilde{\Psi}^\sigma(1)=S_{em}(\varsigma)\tilde{\psi}^\sigma(1)$
$\Leftrightarrow\Psi^\sigma(1)=S_m(1,\varsigma)\psi^\sigma(1)\Leftrightarrow\psi^\sigma(1)=S^+_m(1,\varsigma)\Psi^\sigma(1)$

4.4 Quasi full symmetric tensor and representation transformation of Yang-Mills field source

Definiton 4.4.1. $\hat{\mathcal{J}}^\sigma(1,\varsigma)=\tilde{\mathcal{J}}^\sigma(1,\varsigma)\equiv[J_x{}^\sigma,J_y{}^\sigma,J_z{}^\sigma,J_\pi{}^\sigma]^T$, $\hat{J}^\sigma(1,\varsigma)=\tilde{J}^\sigma(1,\varsigma)\equiv[J_{1'_\varsigma}{}^{1_\varsigma\sigma},J_{2'_\varsigma}{}^{1_\varsigma\sigma},J_{1'_\varsigma}{}^{2_\varsigma\sigma},J_{2'_\varsigma}{}^{2_\varsigma\sigma}]^T$

Corollary 4.4.1. $J_a{}^\sigma=\frac{1}{\sqrt{2}}(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma\sigma}\Leftrightarrow\tilde{\mathcal{J}}^\sigma(1,\varsigma)=S_{em}(\varsigma)\tilde{J}^\sigma(1,\varsigma)$

Corollary 4.4.2. $J_{A'_\varsigma}{}^{B_\varsigma\sigma}=\frac{1}{\sqrt{2}}(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}J_a{}^\sigma\Leftrightarrow\tilde{J}^\sigma(1,\varsigma)=S^+_{em}(\varsigma)\tilde{\mathcal{J}}^\sigma(1,\varsigma)$

4.5 Full symmetry tensor and representation transformation of gravitational field

Definiton 4.5.1. $\psi^{\alpha_\varsigma\beta_\varsigma}=C^{\alpha_\varsigma\beta_\varsigma}$, $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}=C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$

Definiton 4.5.2. $\Psi(2,\varsigma)\equiv[\psi^{x_\varsigma x_\varsigma},\psi^{y_\varsigma x_\varsigma},\psi^{z_\varsigma x_\varsigma},\psi^{y_\varsigma y_\varsigma},\psi^{z_\varsigma y_\varsigma}]^T$, $\psi(2,\varsigma)\equiv[\psi^{1_\varsigma1_\varsigma1_\varsigma1_\varsigma},\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},\psi^{2_\varsigma2_\varsigma2_\varsigma2_\varsigma}]^T$

Definiton 4.5.3. $\tilde{\Psi}(2,\varsigma)\equiv[\psi^{x_\varsigma x_\varsigma},\psi^{y_\varsigma x_\varsigma},\psi^{z_\varsigma x_\varsigma},0,|\psi^{x_\varsigma y_\varsigma},\psi^{y_\varsigma y_\varsigma},\psi^{z_\varsigma y_\varsigma},0]^T$
$\tilde{\psi}(2,\varsigma)\equiv[\psi^{1_\varsigma1_\varsigma1_\varsigma1_\varsigma},\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},\psi^{2_\varsigma2_\varsigma2_\varsigma2_\varsigma}]^T$

Definiton 4.5.4. $\hat{\Psi}(2,\varsigma)\equiv[\psi^{x_\varsigma x_\varsigma},\psi^{y_\varsigma x_\varsigma},\psi^{z_\varsigma x_\varsigma},0,|\psi^{x_\varsigma y_\varsigma},\psi^{y_\varsigma y_\varsigma},\psi^{z_\varsigma y_\varsigma},0,|\psi^{x_\varsigma z_\varsigma},\psi^{y_\varsigma z_\varsigma},\psi^{z_\varsigma z_\varsigma},0,|0,0,0,0]^T$
$\hat{\psi}(2,\varsigma)\equiv[\psi^{1_\varsigma1_\varsigma1_\varsigma1_\varsigma},\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},|\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},|$
$\psi^{1_\varsigma1_\varsigma1_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},|\psi^{1_\varsigma1_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},\psi^{1_\varsigma2_\varsigma2_\varsigma2_\varsigma},\psi^{2_\varsigma2_\varsigma2_\varsigma2_\varsigma}]^T$

Proposition 4.5.1. $\psi^{\alpha_\varsigma\beta_\varsigma}=\frac{1}{2}\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}{}_{C_\varsigma D_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$
$\Leftrightarrow\hat{\Psi}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\varsigma)\hat{\psi}(2,\varsigma)\Leftrightarrow\Psi(2,\varsigma)=S_m(2)\psi(2,\varsigma)\Leftrightarrow\tilde{\Psi}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})\tilde{\psi}(2,\varsigma)$

Proposition 4.5.2. $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}=\frac{1}{2}\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$
$\Leftrightarrow\hat{\psi}(2,\varsigma)=S^+_{em}(\varsigma)\otimes S^+_{em}(\varsigma)\hat{\Psi}(2,\varsigma)\Leftrightarrow\psi(2,\varsigma)=S^+_m(2)\Psi(2,\varsigma)\Leftrightarrow\tilde{\psi}(2,\varsigma)=S^+_{em}(\varsigma)\otimes S^+_{em}(\frac{1}{2})\tilde{\Psi}(2,\varsigma)$

4.6 Quasi full symmetry tensor and representation transformation of gravitational field source

Definiton 4.6.1. $\hat{\mathcal{J}}(2,\varsigma)\equiv[J_x{}^{x_\varsigma},J_y{}^{x_\varsigma},J_z{}^{x_\varsigma},J_\pi{}^{x_\varsigma},|J_x{}^{y_\varsigma},J_y{}^{y_\varsigma},J_z{}^{y_\varsigma},J_\pi{}^{y_\varsigma},|J_x{}^{z_\varsigma},J_y{}^{z_\varsigma},J_z{}^{z_\varsigma},J_\pi{}^{z_\varsigma},|0,0,0,0]^T$

Definiton 4.6.2. $\hat{J}(2,\varsigma)\equiv[J_{1'_\varsigma}{}^{1_\varsigma1_\varsigma1_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma1_\varsigma1_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma1_\varsigma1_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma1_\varsigma1_\varsigma},|J_{1'_\varsigma}{}^{1_\varsigma2_\varsigma1_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma2_\varsigma1_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma2_\varsigma1_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma2_\varsigma1_\varsigma},|$
$J_{1'_\varsigma}{}^{1_\varsigma1_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma1_\varsigma2_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma1_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma1_\varsigma2_\varsigma},|J_{1'_\varsigma}{}^{1_\varsigma2_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma2_\varsigma2_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma2_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma2_\varsigma2_\varsigma},]^T$

Definiton 4.6.3. $\tilde{\mathcal{J}}(2,\varsigma)\equiv[J_x{}^{x_\varsigma},J_y{}^{x_\varsigma},J_z{}^{x_\varsigma},J_\pi{}^{x_\varsigma},|J_x{}^{y_\varsigma},J_y{}^{y_\varsigma},J_z{}^{y_\varsigma},J_\pi{}^{y_\varsigma}]^T$

Definiton 4.6.4. $\tilde{J}(2,\varsigma)\equiv[J_{1'_\varsigma}{}^{1_\varsigma1_\varsigma1_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma1_\varsigma1_\varsigma},J_{1'_\varsigma}{}^{1_\varsigma1_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma1_\varsigma2_\varsigma},|J_{1'_\varsigma}{}^{1_\varsigma2_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{1_\varsigma2_\varsigma2_\varsigma},J_{1'_\varsigma}{}^{2_\varsigma2_\varsigma2_\varsigma},J_{2'_\varsigma}{}^{2_\varsigma2_\varsigma2_\varsigma}]^T$

Proposition 4.6.1. $J_a{}^{\alpha_\varsigma}=\frac{1}{2}(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}{}_{C_\varsigma D_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$
$\Leftrightarrow\hat{\mathcal{J}}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\varsigma)\hat{J}(2,\varsigma)\Leftrightarrow\tilde{\mathcal{J}}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})\tilde{J}(2,\varsigma)$

Proposition 4.6.2. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}=\frac{1}{2}(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}J_a{}^{\alpha_\varsigma}$
$\Leftrightarrow\hat{J}(2,\varsigma)=S^+_{em}(\varsigma)\otimes S^+_{em}(\varsigma)\hat{\mathcal{J}}(2,\varsigma)\Leftrightarrow\tilde{J}(2,\varsigma)=S^+_{em}(\varsigma)\otimes S^+_{em}(\frac{1}{2})\tilde{\mathcal{J}}(2,\varsigma)$

4.7 Full symmetric tensor and representation transformation of spin-s field

Definiton 4.7.1. $\hat{\psi}(s,\varsigma) \equiv \psi^{\overbrace{A_\varsigma \otimes B_\varsigma \otimes C_\varsigma \otimes D_\varsigma \otimes \cdots}^{2s}}, \hat{\Psi}(n,\varsigma) \equiv \psi^{\overbrace{\alpha_\varsigma \otimes \beta_\varsigma \otimes \gamma_\varsigma \otimes \cdots}^{n}}$

Definiton 4.7.2. $\psi(s,\varsigma) \equiv [\overbrace{\psi^{1_\varsigma}, \psi^{2_\varsigma}, \cdots, \psi^{(2s)_\varsigma}, \psi^{(2s+1)_\varsigma}}^{2s+1}]^T$

$$\equiv [\psi^{\overbrace{1_\varsigma 1_\varsigma \cdots 1_\varsigma 1_\varsigma}^{2s}}, \psi^{\overbrace{1_\varsigma 1_\varsigma \cdots 1_\varsigma 2_\varsigma}^{2s}}, \psi^{\overbrace{1_\varsigma 1_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s}}, \cdots, \psi^{\overbrace{1_\varsigma 2_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s}}, \psi^{\overbrace{2_\varsigma 2_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s}}]^T$$

Definiton 4.7.3. $\tilde{\psi}(s,\varsigma) \equiv [\overbrace{\psi^{1_\varsigma}, \psi^{2_\varsigma}, \psi^{2_\varsigma}, \psi^{3_\varsigma}, \psi^{3_\varsigma}, \cdots, \psi^{(2s)_\varsigma}, \psi^{(2s)_\varsigma}, \psi^{(2s+1)_\varsigma}}^{4s}]^T$

Proposition 4.7.1. $\psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = (\frac{1}{\sqrt{2}})^n (\overbrace{\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{C_\varsigma D_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma} \cdots}^{n}) \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}}$

$$\Leftrightarrow \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}} = (\frac{1}{\sqrt{2}})^n (\overbrace{\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma} \cdots}^{n}) \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}}$$

Proposition 4.7.2. $\psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = (\frac{1}{\sqrt{2}})^n (\overbrace{\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{C_\varsigma D_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma} \cdots}^{n}) \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}}$

$$\Leftrightarrow \hat{\Psi}(n,\varsigma) = (\overbrace{S_{em}(\varsigma) \otimes S_{em}(\varsigma) \otimes S_{em}(\varsigma) \cdots}^{n}) \hat{\psi}(n,\varsigma)$$

Proposition 4.7.3. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}} = (\frac{1}{\sqrt{2}})^n (\overbrace{\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma} \cdots}^{n}) \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}}$

$$\Leftrightarrow \hat{\psi}(n,\varsigma) = (\overbrace{S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \cdots}^{n}) \hat{\Psi}(n,\varsigma)$$

Proposition 4.7.4. $\hat{\Psi}(n,\varsigma) = (\overbrace{S_{em}(\varsigma) \otimes S_{em}(\varsigma) \otimes S_{em}(\varsigma) \cdots}^{n}) \hat{\psi}(n,\varsigma)$

$$\Leftrightarrow \hat{\psi}(n,\varsigma) = (\overbrace{S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \cdots}^{n}) \hat{\Psi}(n,\varsigma)$$

4.8 Quasi full symmetric tensor and representation transformation of spin-s field source

Definiton 4.8.1. $\hat{J}(s,\varsigma) \equiv J_{A_\varsigma \otimes}{}^{\overbrace{B_\varsigma \otimes C_\varsigma \otimes D_\varsigma \otimes \cdots}^{2s-1}}, \hat{\mathcal{J}}(n,\varsigma) = J_{a\otimes}{}^{\overbrace{\alpha_\varsigma \otimes \beta_\varsigma \otimes \cdots}^{n-1}}$

Definiton 4.8.2. $\tilde{J}(s,\varsigma) \equiv [\overbrace{\tilde{J}^{1_\varsigma}, \tilde{J}^{2_\varsigma}, \tilde{J}^{3_\varsigma}, \cdots, \tilde{J}^{(4s-1)_\varsigma}, \tilde{J}^{(4s)_\varsigma}}^{4s}]^T$

$$\equiv [J_{1'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 1_\varsigma 1_\varsigma}^{2s-1}}, J_{2'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 1_\varsigma 1_\varsigma}^{2s-1}}, J_{1'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 1_\varsigma 2_\varsigma}^{2s-1}}, J_{2'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 1_\varsigma 2_\varsigma}^{2s-1}}, \cdots, J_{1'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s-1}}, J_{2'_\varsigma}{}^{\overbrace{1_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s-1}}, J_{1'_\varsigma}{}^{\overbrace{2_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s-1}}, J_{2'_\varsigma}{}^{\overbrace{2_\varsigma \cdots 2_\varsigma 2_\varsigma}^{2s-1}}]^T$$

Proposition 4.8.1. $J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}} = (\frac{1}{\sqrt{2}})^n (\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma} \bar{\varepsilon}_{A_\varsigma B_\varsigma} \cdot (\overbrace{\sigma^{\alpha_\varsigma}{}_{C_\varsigma D_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma} \cdots}^{n-1}) J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}}$

$$\Leftrightarrow J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}} = (\frac{1}{\sqrt{2}})^n (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \bar{\varepsilon}^{A_\varsigma B_\varsigma} \cdot (\overbrace{\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma} \cdots}^{n-1}) J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}}$$

Proposition 4.8.2. $J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}} = (\frac{1}{\sqrt{2}})^n (\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma} \bar{\varepsilon}_{A_\varsigma B_\varsigma} \cdot (\overbrace{\sigma^{\alpha_\varsigma}{}_{C_\varsigma D_\varsigma} \cdot \sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma} \cdots}^{n-1}) J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}}$

$$\Leftrightarrow \hat{\mathcal{J}}(n,\varsigma) = (\overbrace{S_{em}(\varsigma) \otimes S_{em}(\varsigma) \otimes S_{em}(\varsigma) \cdots}^{n}) \hat{J}(n,\varsigma)$$

Proposition 4.8.3. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}} = (\frac{1}{\sqrt{2}})^n (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \bar{\varepsilon}^{A_\varsigma B_\varsigma} \cdot (\overbrace{\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma} \cdot \sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma} \cdots}^{n-1}) J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}}$

$$\Leftrightarrow \hat{J}(n,\varsigma) = (\overbrace{S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \cdots}^{n}) \hat{\mathcal{J}}(n,\varsigma)$$

Proposition 4.8.4. $\hat{\mathcal{J}}(n,\varsigma) = (\overbrace{S_{em}(\varsigma) \otimes S_{em}(\varsigma) \otimes S_{em}(\varsigma) \cdots}^{n}) \hat{J}(n,\varsigma)$

$$\Leftrightarrow \hat{J}(n,\varsigma) = (\overbrace{S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma) \cdots}^{n}) \hat{\mathcal{J}}(n,\varsigma)$$

4.9 Transformation relationships between spin-s particle field quantities

Theorem 4.9.1. $\psi(s,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \Leftrightarrow \tilde{\psi}(s,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(s-\frac{1}{2})]}$

Proof: $\psi(s,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

$\Leftrightarrow \psi'(s,\varsigma) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\psi(s,\varsigma)$

$\Leftrightarrow \mathbb{N}(s)\psi'(s,\varsigma) = \mathbb{N}(s)e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\psi(s,\varsigma)$

$\Leftrightarrow \tilde{\psi}'(s,\varsigma) = \mathbb{N}(s)e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{\mathbb{N}}(s)\tilde{\psi}(s,\varsigma)$

$\Leftrightarrow \tilde{\psi}'(s,\varsigma) = e^{(i\omega+\varsigma\epsilon)\cdot\mathbb{N}(s)\sigma(s)\bar{\mathbb{N}}(s)}\tilde{\psi}(s,\varsigma)$

$\Leftrightarrow \tilde{\psi}'(s,\varsigma) = e^{(i\omega+\varsigma\epsilon)\cdot\mathbb{N}(s)\bar{\mathbb{N}}(s)[\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(s-\frac{1}{2})]\mathbb{N}(s)\bar{\mathbb{N}}(s)}\tilde{\psi}(1,\varsigma)$

$\Leftrightarrow \tilde{\psi}'(s,\varsigma) = e^{(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(s-\frac{1}{2})]}\mathbb{N}(s)\bar{\mathbb{N}}(s)\tilde{\psi}(s,\varsigma)$

$\Leftrightarrow \tilde{\psi}'(s,\varsigma) = e^{(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(s-\frac{1}{2})]}\tilde{\psi}(s,\varsigma)$

$\Leftrightarrow \tilde{\psi}(s,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(s-\frac{1}{2})]}$

$\Leftrightarrow \tilde{\psi}(s,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

Proposition 4.9.1. $\tilde{\Psi}(s,\varsigma) = S_{em}(s,\varsigma)\tilde{\psi}(s,\varsigma), \Psi(s,\varsigma) = S_m(s)\psi(s,\varsigma)$

$\Rightarrow \tilde{\Psi}(s,\varsigma) = \mathbb{N}_m(s)\Psi(s,\varsigma), \mathbb{N}_m(s) = S_{em}(s,\varsigma)\mathbb{N}(s)S_m^-(s)$

$\Rightarrow \Psi(s,\varsigma) = \bar{\mathbb{N}}_m(s)\tilde{\Psi}(s,\varsigma), \bar{\mathbb{N}}_m(s) = S_{em}^-(s,\varsigma)\bar{\mathbb{N}}(s)S_m(s)$

Proof: $\tilde{\Psi}(s,\varsigma) = S_{em}(s,\varsigma)\tilde{\psi}(s,\varsigma), \Psi(s,\varsigma) = S_m(s)\psi(s,\varsigma)$

$\Leftrightarrow \tilde{\Psi}(s,\varsigma) = S_{em}(s,\varsigma)\mathbb{N}(s)\psi(s,\varsigma), \Psi(s,\varsigma) = S_m(s)\psi(s,\varsigma)$

$\Rightarrow \tilde{\Psi}(s,\varsigma) = S_{em}(s,\varsigma)\mathbb{N}(s)S_m^-(s)\Psi(s,\varsigma)$

$\Rightarrow \tilde{\Psi}(s,\varsigma) = \mathbb{N}_m(s)\Psi(s,\varsigma), \mathbb{N}_m(s) = S_{em}(s,\varsigma)\mathbb{N}(s)S_m^-(s)$

$\Rightarrow \Psi(s,\varsigma) = \bar{\mathbb{N}}_m(s)\tilde{\Psi}(s,\varsigma), \bar{\mathbb{N}}_m(s) = S_{em}^-(s,\varsigma)\bar{\mathbb{N}}(s)S_m(s)$ □

Proposition 4.9.2. $\tilde{\Psi}(s,\varsigma) = S_{em}(s)\tilde{\psi}(s,\varsigma), \tilde{\Psi}(s,\varsigma) = \mathbb{N}_m(s)\Psi(s,\varsigma)$

$\Rightarrow \Psi(s,\varsigma) = S_m(s)\psi(s,\varsigma), S_m(s) = \bar{\mathbb{N}}_m(s)S_{em}(s)\mathbb{N}_m(s)$

Proof: $\tilde{\Psi}(s,\varsigma) = S_{em}(s)\tilde{\psi}(s,\varsigma), \tilde{\Psi}(s,\varsigma) = \mathbb{N}_m(s)\Psi(s,\varsigma)$

$\Rightarrow \mathbb{N}_m(s)\Psi(s,\varsigma) = S_{em}(s)\tilde{\psi}(s,\varsigma)$

$\Rightarrow \Psi(s,\varsigma) = \bar{\mathbb{N}}_m(s)S_{em}(s)\mathbb{N}(s)\psi(s,\varsigma)$

$\Rightarrow \Psi(s,\varsigma) = S_m(s)\psi(s,\varsigma), S_m(s) = \bar{\mathbb{N}}_m(s)S_{em}(s)\mathbb{N}(s)$ □

5 Symmetry condition analysis

5.1 Symmetry condition analysis of field quantities

Definiton 5.1.1. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma} = (\frac{1}{\sqrt{2}})^n(\overbrace{\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots}^{n})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n} Z_\varsigma}$

Corollary 5.1.1. $\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n} Z_\varsigma} = (\frac{1}{\sqrt{2}})^n(\overbrace{\sigma^{\alpha_\varsigma}{}_{A_\varsigma B_\varsigma}\cdot\sigma^{\beta_\varsigma}{}_{C_\varsigma D_\varsigma}\cdot\sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma}\cdots}^{n})\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma}$

Proposition 5.1.1. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma}$ *is full symmetry in () and between () for superscripts* $(A_\varsigma B_\varsigma), (C_\varsigma D_\varsigma), \cdots$

$\Leftrightarrow \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n} Z_\varsigma}$ *is full symmetry for superscripts* $\alpha_\varsigma\beta_\varsigma\cdots$

Proposition 5.1.2. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma} = \psi^{\overbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma} \Leftrightarrow \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n} Z_\varsigma} = 0$

Proof: $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma} = \psi^{\overbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma}$

$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n} Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma}(\frac{1}{\sqrt{2}})^n(\overbrace{\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots}^{n})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n} Z_\varsigma} = 0$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot(\tfrac{1}{\sqrt{2}})^n(\overbrace{\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots}^{n-2})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{C_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}\bar{\varepsilon}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$ □

Proposition 5.1.3. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}=\psi^{\overbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}B_\varsigma}\Leftrightarrow\sigma_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}[Z_\varsigma]}=0$

Proof: $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}=\psi^{\overbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}B_\varsigma}$

$$\Leftrightarrow \varepsilon_{B_\varsigma Z_\varsigma}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}=0$$

$$\Leftrightarrow \varepsilon_{B_\varsigma Z_\varsigma}(\tfrac{1}{\sqrt{2}})^n(\overbrace{\sigma_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots}^{n})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{Z_\varsigma}\cdot(\tfrac{1}{\sqrt{2}})^n(\overbrace{\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots}^{n-2})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}{}^{A_\varsigma}{}_{Z_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$

$$\Leftrightarrow \sigma_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}[Z_\varsigma]}=0$$ □

Proposition 5.1.4. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}$ *is full symmetry for superscripts* $A_\varsigma B_\varsigma\cdots$

$\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}$ *is full symmetry for superscripts* $\alpha_\varsigma\beta_\varsigma\cdots,\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$

Proof: $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}$ is full symmetry for superscripts $A_\varsigma B_\varsigma\cdots$

$$\Leftrightarrow\begin{cases}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}\text{ is full symmetry in () and between () for superscripts }(A_\varsigma B_\varsigma),(C_\varsigma D_\varsigma),\cdots\\ \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}=\psi^{\overbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}\end{cases}$$

$$\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}\text{ is full symmetry for superscripts }\alpha_\varsigma\beta_\varsigma\cdots,\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0$$ □

Two important theorems about field quantities are obtained:

Theorem 5.1.1. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}}$ *is full symmetry for superscripts.*

$\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}}$ *is full symmetry for superscripts,* $\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}}=0$

Theorem 5.1.2. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}$ *is full symmetry for superscripts.*

$\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}$ *is full symmetry for superscripts* $\alpha_\varsigma\beta_\varsigma\cdots,\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0,\sigma_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}[Z_\varsigma]}=0$

Proof: $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}$ is full symmetry for superscripts.

$$\Leftrightarrow\begin{cases}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}\text{ is full symmetry for superscripts }A_\varsigma B_\varsigma\cdots\\ \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}Z_\varsigma}=\psi^{\overbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n}B_\varsigma}\end{cases}$$

$$\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}\text{ is full symmetry for superscripts }\alpha_\varsigma\beta_\varsigma\cdots,\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}Z_\varsigma}=0,\sigma_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\cdots}^{n}[Z_\varsigma]}=0$$ □

5.2 Source symmetry condition analysis

Definiton 5.2.1. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\cdot\overbrace{(\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}$

Corollary 5.2.1. $J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=(\frac{1}{\sqrt{2}})^n(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\bar{\varepsilon}_{A_\varsigma B_\varsigma}\cdot\overbrace{(\sigma^{\alpha_\varsigma}{}_{C_\varsigma D_\varsigma}\cdot\sigma^{\beta_\varsigma}{}_{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$

Proposition 5.2.1. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ *is full symmetry in () and between () for superscripts* $(C_\varsigma D_\varsigma),(E_\varsigma F_\varsigma),\cdots$

$\Leftrightarrow J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}$ *is full symmetry for superscripts* $\alpha_\varsigma\beta_\varsigma\cdots$

Proposition 5.2.2. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

Proof: $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$

$\Leftrightarrow\varepsilon_{B_\varsigma C_\varsigma}J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=0$

$\Leftrightarrow\varepsilon_{B_\varsigma C_\varsigma}(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\cdot\overbrace{(\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma C_\varsigma}\cdot\overbrace{(\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma C_\varsigma}\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\overbrace{(\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{G_\varsigma H_\varsigma}\cdots)}^{n-2}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma C_\varsigma}\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}\bar{\varepsilon}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$ □

Proposition 5.2.3. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}B_\varsigma}$

Proof: $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}B_\varsigma}$

$\Leftrightarrow\varepsilon_{B_\varsigma Z_\varsigma}J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=0$

$\Leftrightarrow\varepsilon_{B_\varsigma Z_\varsigma}(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\cdot\overbrace{(\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma Z_\varsigma}\cdot\overbrace{(\sigma_{\alpha_\varsigma}{}^{C_\varsigma D_\varsigma}\cdot\sigma_{\beta_\varsigma}{}^{E_\varsigma F_\varsigma}\cdots)}^{n-1}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\frac{1}{\sqrt{2}})^n(\sigma,-i\varsigma)^a{}_{A'_\varsigma Z_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

$\Leftrightarrow(\sigma,-i\varsigma)^aJ_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}[Z_\varsigma]}=0$ □

Proposition 5.2.4. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ *is full symmetry for superscripts* $B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots$

$\Leftrightarrow J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}$ *is full symmetry for superscripts* $\alpha_\varsigma\beta_\varsigma\cdots,(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$

Proof: $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ is full symmetry for superscripts $B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots$

$\Leftrightarrow\begin{cases}J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}\text{ is full symmetry in () and between () for superscripts }(C_\varsigma D_\varsigma),(E_\varsigma F_\varsigma),\cdots\\J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{C_\varsigma B_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}\end{cases}$

$\Leftrightarrow J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}$ is full symmetry for superscripts $\alpha_\varsigma\beta_\varsigma\cdots$, $(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}\cdot J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$ □

Two important theorems about source are obtained:

Theorem 5.2.1. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}}$ *is full symmetry for superscripts.*

$$\Leftrightarrow J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}} \textit{ is full symmetry for superscripts}, (\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}}=0$$

Theorem 5.2.2. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ *is full symmetry for superscripts.*

$$\Leftrightarrow \begin{cases} J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma} \textit{ is full symmetry for Greek alphabet.} \\ (\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0, (\sigma,-i\varsigma)^a J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}[Z_\varsigma]}=0 \end{cases}$$

Proof: $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ is full symmetry for superscripts.

$\Leftrightarrow J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}$ is full symmetry for superscripts $B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots$, $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{Z_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma\cdots}^{2n-1}B_\varsigma}$

$\Leftrightarrow J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}$ is full symmetry for Greek alphabet, $(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}Z_\varsigma}=0$, $(\sigma,-i\varsigma)^a J_a{}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n-1}[Z_\varsigma]}=0$

□

Chapter 3

Spinorial equations of various particles

1 Yang-Mills theory [9, 11]

1.1 Matrix description of Yang-Mills theory

Definiton 1.1.1. $\begin{cases} F_{uv}{}^{\sigma}T_{\sigma} = \partial_u A_v{}^{\sigma}T_{\sigma} - \partial_v A_u{}^{\sigma}T_{\sigma} + ig[A_u{}^{\rho}T_{\rho}, A_v{}^{\tau}T_{\tau}] \\ [T_{\rho}, T_{\tau}] = if_{\rho\tau}{}^{\sigma}T_{\sigma}, C^{\sigma}T_{\sigma} = 0 \Leftrightarrow C^{\sigma} = 0 \end{cases}$

Definiton 1.1.2. *Gauge transformation:*

$$\begin{cases} \psi \to U\psi \\ A_u{}^{\sigma}T_{\sigma} \to UA_u{}^{\sigma}T_{\sigma}U^{-1} + \frac{i}{g}(\partial_u U)U^{-1} \end{cases}$$

Corollary 1.1.1. $D_u\psi \to UD_u\psi, D_u = \partial_u + igA_u{}^{\sigma}T_{\sigma}$

Proof: $D_u\psi = (\partial_u + igA_u{}^{\sigma}T_{\sigma})\psi \to [\partial_u + UigA_u{}^{\sigma}T_{\sigma}U^{-1} - (\partial_u U)U^{-1}](U\psi)$

$\Leftrightarrow D_u\psi \to [\partial_u(U\psi) + UigA_u{}^{\sigma}T_{\sigma}\psi - (\partial_u U)\psi]$

$\Leftrightarrow D_u\psi \to U(\partial_u + igA_u{}^{\sigma}T_{\sigma})\psi$

$\Leftrightarrow D_u\psi \to UD_u\psi, D_u = \partial_u + igA_u{}^{\sigma}T_{\sigma}$ □

Lemma 1.1.1. $\partial_u(U^{-1}) = -U^{-1}\partial_u(U)U^{-1}$

Proof: $\partial_u(UU^{-1}) = \partial_u(I)$

$\Leftrightarrow \partial_u(U)U^{-1} + U\partial_u(U^{-1}) = 0$

$\Leftrightarrow U\partial_u(U^{-1}) = -\partial_u(U)U^{-1}$

$\Leftrightarrow \partial_u(U^{-1}) = -U^{-1}\partial_u(U)U^{-1}$ □

Corollary 1.1.2. $F_{uv}{}^{\sigma}T_{\sigma} \to UF_{uv}{}^{\sigma}T_{\sigma}U^{-1}$

Proof: $F_{uv}{}^{\sigma}T_{\sigma} = \partial_u A_v{}^{\sigma}T_{\sigma} - \partial_v A_u{}^{\sigma}T_{\sigma} + ig[A_u{}^{\rho}T_{\rho}, A_v{}^{\tau}T_{\tau}]$

$\to \partial_u[UA_v{}^{\sigma}T_{\sigma}U^{-1} + \frac{i}{g}(\partial_v U)U^{-1}] - \partial_v[UA_u{}^{\sigma}T_{\sigma}U^{-1} + \frac{i}{g}(\partial_u U)U^{-1}]$

$+ ig[UA_u{}^{\rho}T_{\rho}U^{-1} + \frac{i}{g}(\partial_u U)U^{-1}, UA_v{}^{\tau}T_{\tau}U^{-1} + \frac{i}{g}(\partial_v U)U^{-1}]$

$\Leftrightarrow F_{uv}{}^{\sigma}T_{\sigma} \to U(\partial_u A_v{}^{\sigma}T_{\sigma} - \partial_v A_u{}^{\sigma}T_{\sigma} + ig[A_u{}^{\rho}T_{\rho}, A_v{}^{\tau}T_{\tau}])U^{-1}$

$\Leftrightarrow F_{uv}{}^{\sigma}T_{\sigma} \to UF_{uv}{}^{\sigma}T_{\sigma}U^{-1}$ □

Corollary 1.1.3. $D_w F_{uv}{}^{\sigma}T_{\sigma} \to UD_w F_{uv}{}^{\sigma}T_{\sigma}U^{-1}, D_w = \nabla_w + ig[A_w{}^{\sigma}T_{\sigma}, \quad]$

Proof: $D_w F_{uv}{}^{\sigma}T_{\sigma} = \nabla_w F_{uv}{}^{\sigma}T_{\sigma} + ig[A_w{}^{\rho}T_{\rho}, F_{uv}{}^{\tau}T_{\tau}]$

$\to \nabla_w(UF_{uv}{}^{\sigma}T_{\sigma}U^{-1}) + ig[UA_w{}^{\rho}T_{\rho}U^{-1} + \frac{i}{g}(\partial_w U)U^{-1}, UF_{uv}{}^{\tau}T_{\tau}U^{-1}]$

$\Leftrightarrow D_w F_{uv}{}^{\sigma}T_{\sigma} \to U(\nabla_w F_{uv}{}^{\sigma}T_{\sigma} + ig[A_w{}^{\rho}T_{\rho}, F_{uv}{}^{\tau}T_{\tau}])U^{-1}$

$\Leftrightarrow D_w F_{uv}{}^{\sigma}T_{\sigma} \to UD_w F_{uv}{}^{\sigma}T_{\sigma}U^{-1}$ □

1.2 Components description of Yang-Mills theory

Proposition 1.2.1. $F_{uv}{}^{\sigma}T_{\sigma} = \partial_u A_v{}^{\sigma}T_{\sigma} - \partial_v A_u{}^{\sigma}T_{\sigma} + ig[A_u{}^{\rho}T_{\rho}, A_v{}^{\tau}T_{\tau}] \Leftrightarrow F_{uv}{}^{\sigma} = \partial_u A_v{}^{\sigma} - \partial_v A_u{}^{\sigma} - gf_{\rho\tau}{}^{\sigma}A_u{}^{\rho}A_v{}^{\tau}$

Definiton 1.2.1. $U(\theta) = e^{ig\theta^{\sigma}T_{\sigma}}$

Corollary 1.2.1. *Gauge transformation:* $\begin{cases} \delta\psi = ig\theta^\sigma T_\sigma \psi \\ \delta A_u{}^\sigma = ig\theta^\rho (if_{\rho\tau}{}^\sigma) A_u{}^\tau - \partial_u \theta^\sigma \end{cases}$

Corollary 1.2.2. $\delta F_{uv}{}^\sigma = ig\theta^\rho (if_{\rho\tau}{}^\sigma) F_{uv}{}^\tau$

1.3 Introduction of Yang-Mills equation

Introducing Yang-Mills equation:

$$D^u F_{uv}{}^\sigma = -J_v{}^\sigma, D^u * F_{uv}{}^\sigma \equiv 0 \tag{3.1}$$

1.4 Frame description of Yang-Mills equation

Definiton 1.4.1. $F_{ab}{}^\sigma \equiv e_a{}^u e_b{}^v F_{uv}{}^\sigma, A_a{}^\sigma \equiv e_a{}^u A_u{}^\sigma, \partial_a \equiv e_a{}^u \partial_u$

Corollary 1.4.1. $F_{ab}{}^\sigma = \partial_a A_b{}^\sigma - \partial_b A_a{}^\sigma - g f_{\rho\tau}{}^\sigma A_a{}^\rho A_b{}^\tau$

Frame description of Yang-Mills equation:

$$D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0 \tag{3.2}$$

2 Spinorial form of Yang-Mills equation

2.1 Integer spinorial form of Yang-Mills equation

Lemma 2.1.1. $D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}) = \varsigma J^{b\sigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma$

Proof: $D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}) = \varsigma J^{b\sigma}$

$\Leftrightarrow D_a[(\sigma_\varsigma, -i\varsigma)_{\alpha_\varsigma}{}^{ab}\tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma)] = \varsigma J^{b\sigma}$

$\Leftrightarrow D_a[(\sigma_{-\varsigma}, -i\varsigma)^{ab}{}_{\alpha_\varsigma}\tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma)] = \varsigma J^{b\sigma}$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^{ab}{}_{\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J^{b\sigma}$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma$ □

Theorem 2.1.1. $D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0$

Proof: $D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0$

$\Leftrightarrow D_a F^{ab\sigma} = -J^{b\sigma}, D_a * F^{ab\sigma} \equiv 0, D^a * F_{ab}{}^\sigma \equiv 0$

$\Leftrightarrow D_a(F^{ab\sigma} - \varsigma * F^{ab\sigma}) = -J^{b\sigma}, D^a * F_{ab}{}^\sigma \equiv 0$

$\Leftrightarrow D_a(-\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}) = -J^{b\sigma}, D^a * F_{ab}{}^\sigma \equiv 0$

$\Leftrightarrow D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}) = \varsigma J^{b\sigma}, D^a * F_{ab}{}^\sigma \equiv 0$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0$ □

Proposition 2.1.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^\sigma(1,\varsigma)$

Corollary 2.1.1. $D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^\sigma(1,\varsigma), D^a * F_{ab}{}^\sigma \equiv 0$

Corollary 2.1.2. $(-\sigma_-, i)^a D_a \tilde{\Psi}^\sigma(1,+) = -\tilde{\mathcal{J}}^\sigma(1,+) \Leftrightarrow (\sigma_+, i)^a D_a \tilde{\Psi}^\sigma(1,-) = -\tilde{\mathcal{J}}^\sigma(1,-)$

2.2 Relationships between half integer and integer spinorial form of Yang-Mills equation

Theorem 2.2.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \sigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma\sigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma$

Proof: $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \sigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma\sigma}$

$\Leftrightarrow [\frac{1}{\sqrt{2}}(\sigma, i\varsigma)_b{}^{C_\varsigma A'_\varsigma}\bar{\varepsilon}_{C_\varsigma B_\varsigma}](\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma\sigma} = \varsigma[\frac{1}{\sqrt{2}}(\sigma, i\varsigma)_b{}^{C_\varsigma A'_\varsigma}\bar{\varepsilon}_{C_\varsigma B_\varsigma}] J_{A'_\varsigma}{}^{B_\varsigma\sigma}$

$\Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon_{B_\varsigma C_\varsigma}[(\sigma, i\varsigma)_b{}^{C_\varsigma A'_\varsigma}(\sigma, -i\varsigma)_{aA'_\varsigma A_\varsigma}] D^a \psi^{A_\varsigma B_\varsigma\sigma} = \varsigma J_b{}^\sigma$

$\Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon_{B_\varsigma C_\varsigma}[\delta_{ba}\delta^{C_\varsigma}{}_{A_\varsigma} + 2S_{ba}{}^{C_\varsigma}{}_{A_\varsigma}] D^a \psi^{A_\varsigma B_\varsigma\sigma} = \varsigma J_b{}^\sigma$

$\Leftrightarrow [\frac{1}{\sqrt{2}}\delta_{ba}\varepsilon_{B_\varsigma A_\varsigma} + \sqrt{2}\varepsilon_{B_\varsigma C_\varsigma} S_{ba}{}^{C_\varsigma}{}_{A_\varsigma}] D^a \psi^{A_\varsigma B_\varsigma\sigma} = \varsigma J_b{}^\sigma$

$\Leftrightarrow D^a(\sqrt{2} S_{abC_\varsigma A_\varsigma}\psi^{A_\varsigma B_\varsigma\sigma}) = -\varsigma J_b{}^\sigma$

$\Leftrightarrow D^a(-\frac{1}{\sqrt{2}}\sigma_{\varsigma\alpha_\varsigma ab}\sigma^{\alpha_\varsigma}{}_{C_\varsigma A_\varsigma}\psi^{A_\varsigma B_\varsigma\sigma}) = -\varsigma J_b{}^\sigma$

$\Leftrightarrow D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\sigma}) = \varsigma J^{b\sigma}$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma}(1,\varsigma) = \varsigma J_b{}^\sigma$ □

Proposition 2.2.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \sigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma \sigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^\sigma(1,\varsigma) = \varsigma \tilde{J}^\sigma(1,\varsigma)$

Corollary 2.2.1. $(\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^\sigma(1,\varsigma) = \varsigma \tilde{J}^\sigma(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^\sigma(1,\varsigma)$

2.3 Half integer spinorial form of Yang-Mills equation [4, 5]

Corollary 2.3.1. $D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0 \Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \sigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma \sigma}, D^a * F_{ab}{}^\sigma \equiv 0$

Corollary 2.3.2. $D^a F_{ab}{}^\sigma = -J_b{}^\sigma, D^a * F_{ab}{}^\sigma \equiv 0 \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^\sigma(1,\varsigma) = \varsigma \tilde{J}^\sigma(1,\varsigma), D^a * F_{ab}{}^\sigma \equiv 0$

2.4 Conjecture

Theorem 2.4.1. $D^u * F_{uv}{}^\sigma = 0 \Leftrightarrow F_{uv}{}^\sigma = \partial_u A_v{}^\sigma - \partial_v A_u{}^\sigma - g f_{\rho\tau}{}^\sigma A_u{}^\rho A_v{}^\tau \Leftrightarrow D^u * F_{uv}{}^\sigma \equiv 0$

Theorem 2.4.2. $D^u F_{uv}{}^\sigma = -J_v{}^\sigma, D^u * F_{uv}{}^\sigma = 0 \Leftrightarrow D^u F_{uv}{}^\sigma = -J_v{}^\sigma, F_{uv}{}^\sigma = \partial_u A_v{}^\sigma - \partial_v A_u{}^\sigma - g f_{\rho\tau}{}^\sigma A_u{}^\rho A_v{}^\tau$

3 Spinorial form of electromagnetic field equation

The electromagnetic field equation can be regarded as a special case of Yang-Mills equation. Namely, σ is empty case. So there are the follwing quite similar conclusions.

3.1 Electromagnetic field equation

Electromagnetic field equation: $D^u F_{uv} = -J_v, D^u * F_{uv} \equiv 0$ (3.3)

Its frame description: $D^a F_{ab} = -J_b, D^a * F_{ab} \equiv 0$ (3.4)

3.2 Integer spinorial form of electromagnetic field equation

Theorem 3.2.1. $D^a F_{ab} = -J_b, D^a * F_{ab} \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma}(1,\varsigma) = \varsigma J_b, D^a * F_{ab} \equiv 0$

Corollary 3.2.1. $D^a F_{ab} = -J_b, D^a * F_{ab} \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}(1,\varsigma) = \varsigma \tilde{\mathcal{J}}(1,\varsigma), D^a * F_{ab} \equiv 0$

Corollary 3.2.2. $(-\sigma_-, i)^a D_a \tilde{\Psi}(1,+) = -\tilde{\mathcal{J}}(1,+) \Leftrightarrow (\sigma_+, i)^a D_a \tilde{\Psi}(1,-) = -\tilde{\mathcal{J}}(1,-)$

3.3 Relationships between half integer and integer spinorial form of electromagnetic field

Theorem 3.3.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma}(1,\varsigma) = \varsigma J_b$

Proposition 3.3.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}(1,\varsigma) = \varsigma \tilde{J}(1,\varsigma)$

Corollary 3.3.1. $(\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}(1,\varsigma) = \varsigma \tilde{J}(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}(1,\varsigma) = \varsigma \tilde{\mathcal{J}}(1,\varsigma)$

3.4 Half integer spinorial form of electromagnetic equation [4, 5]

Corollary 3.4.1. $D^a F_{ab} = -J_b, D^a * F_{ab} \equiv 0 \Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma}, D^a * F_{ab} \equiv 0$

Corollary 3.4.2. $D^a F_{ab} = -J_b, D^a * F_{ab} \equiv 0 \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}(1,\varsigma) = \varsigma \tilde{J}(1,\varsigma), D^a * F_{ab} \equiv 0$

3.5 Conjecture

Theorem 3.5.1. $D^a * F_{ab} = 0 \Leftrightarrow F_{ab} = \partial_a A_b - \partial_b A_a \Leftrightarrow D^a * F_{ab}{}^\sigma \equiv 0$

Theorem 3.5.2. $D^a F_{ab} = -J_b, D^a * F_{ab} = 0 \Leftrightarrow D^a F_{ab} = -J_b, F_{ab} = \partial_a A_b - \partial_b A_a$

4 Yang-Mills gauge theory explanation of gravitation [29–32]

4.1 Mathematical preparation

Definiton 4.1.1. $\theta^{\alpha_\varsigma}(\varsigma) \equiv \frac{1}{2}\varsigma \sigma^{\alpha_\varsigma}_{\varsigma ab} \vartheta^{ab} = -\varsigma(i\omega + \varsigma\epsilon)^{\alpha_\varsigma}$

Corollary 4.1.1. $\frac{1}{2}\vartheta^{ab} S_{ab}(s,\varsigma) = -\varsigma \theta^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) = (i\omega + \varsigma\epsilon) \cdot \sigma(s)$

Corollary 4.1.2. $\frac{1}{2}\omega_u{}^{ab} S_{ab}(s,\varsigma) = -\varsigma A_u{}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s)$

Lemma 4.1.1. $[\omega_u{}^{cd}(\frac{1}{2}S_{cd}), \omega_v{}^{ef}(\frac{1}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^d(\frac{1}{2}S_{cd})$

Proof: $[\omega_u{}^{cd}S_{cd}, \omega_v{}^{ef}S_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[S_{cd}, S_{ef}]$
$\Leftrightarrow [\omega_u{}^{cd}S_{cd}, \omega_v{}^{ef}S_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[\delta_{cf}S_{de} - \delta_{ce}S_{df} + \delta_{de}S_{cf} - \delta_{df}S_{ce}]$
$\Leftrightarrow [\omega_u{}^{cd}S_{cd}, \omega_v{}^{ef}S_{ef}] = 4\omega_u{}^{ce}\omega_{ve}{}^{d}S_{cd}$
$\Leftrightarrow [\omega_u{}^{cd}S_{cd}, \omega_v{}^{ef}S_{ef}] = 2\omega_{[u}{}^{ce}\omega_{v]e}{}^{d}S_{cd}$
$\Leftrightarrow [\omega_u{}^{cd}(\frac{1}{2}S_{cd}), \omega_v{}^{ef}(\frac{1}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}(\frac{1}{2}S_{cd})$ □

Corollary 4.1.3. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{cd}(\frac{1}{2}S_{cd}) = \partial_u\omega_v{}^{cd}(\frac{1}{2}S_{cd}) - \partial_v\omega_u{}^{cd}(\frac{1}{2}S_{cd}) + [\omega_u{}^{cd}(\frac{1}{2}S_{cd}), \omega_v{}^{ef}(\frac{1}{2}S_{ef})]$

Corollary 4.1.4. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{<cd>} = \partial_u\omega_v{}^{<cd>} - \partial_v\omega_u{}^{<cd>} + [\omega_u{}^{<cd>}, \omega_v{}^{<ef>}]$

Corollary 4.1.5. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) = \partial_u\omega_v{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) - \partial_v\omega_u{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) + [\omega_u{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma), \omega_v{}^{ef}\frac{1}{2}S_{ef}(s,\varsigma)]$

Corollary 4.1.6. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u A_v{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \partial_v A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \varsigma[A_u{}^{\beta_\varsigma}\sigma_{\beta_\varsigma}(s), A_v{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)]$

Corollary 4.1.7. $\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma) \to U(\theta)\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta)$

Corollary 4.1.8. $[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}(s)$

Proof: $[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)$
$\Leftrightarrow \frac{1}{16}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma\beta_\varsigma}{}^{cd}[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \frac{1}{16}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma\beta_\varsigma}{}^{cd}[\delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)]$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{16}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma\beta_\varsigma}{}^{cd}[\delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)]$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{4}\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\sigma_{\varsigma\beta_\varsigma}{}^{cd}\delta_{ad}S_{bc}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{4}[\delta_{\alpha_\varsigma\beta_\varsigma}\delta^{bc} + i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{bc}(s)]S_{bc}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}(s)$ □

4.2 Matrix description of gravitational field's Yang-Mills theory

Matrix description of gravitational field's Yang-Mills theory:

$$\begin{cases} F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u A_v{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \partial_v A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \varsigma[A_u{}^{\beta_\varsigma}\sigma_{\beta_\varsigma}(s), A_v{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)] \\ [\sigma_{\beta_\varsigma}(s), \sigma_{\gamma_\varsigma}(s)] = i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s), C^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = 0 \Leftrightarrow C^{\alpha_\varsigma} = 0 \end{cases} \tag{3.5}$$

Gauge transformation:

$$\begin{cases} \psi(s,\varsigma) \to U(\theta)\psi(s,\varsigma), U(\theta) = e^{-\varsigma\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)}\{= e^{\frac{1}{2}\vartheta^{ab}S_{ab}(s,\varsigma)} = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\} \\ A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta) \end{cases} \tag{3.6}$$

The above is the Yang-Mills theory of $g = i\varsigma$ and $T = \sigma(s)$ case. So there are the following similar conclusions.

Corollary 4.2.1. $D_u\psi(s,\varsigma) \to U(\theta)D_u\psi(s,\varsigma), D_u = \partial_u - \varsigma A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u - \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_u{}^{cd}\sigma_{\alpha_\varsigma}(s) = \partial_u + \frac{1}{2}\omega_u{}^{cd}S_{cd}(s)$

Corollary 4.2.2. $F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta)$

Corollary 4.2.3. $D_w F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U D_w F_{uv}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}, D_w = \nabla_w - \varsigma[A_w{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s), \quad]$

4.3 Matrix description of gravitational field's Yang-Mills-like theory

Matrix description of gravitational field's Yang-Mills-like theory:

$$\begin{cases} R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) = \partial_u\omega_v{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) - \partial_v\omega_u{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) + [\omega_u{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma), \omega_v{}^{ef}\frac{1}{2}S_{ef}(s,\varsigma)] \\ [S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma) \\ c^{ab}S_{ab}(s,\varsigma) = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0 \end{cases} \tag{3.7}$$

Gauge transformation:

$$\begin{cases} \psi(s,\varsigma) \to U(\theta)\psi(s,\varsigma), U(\theta) = e^{\frac{1}{2}\vartheta^{ab}S_{ab}(s,\varsigma)}\{= e^{-\varsigma\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)} = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\} \\ \frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma) \to U(\theta)\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta) \end{cases} \tag{3.8}$$

Corollary 4.3.1. $D_u\psi(s,\varsigma) \to U(\theta)D_u\psi(s,\varsigma), D_u = \partial_u + \frac{1}{2}\omega_u{}^{cd}S_{cd}(s) = \partial_u - \varsigma A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u - \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_u{}^{cd}\sigma_{\alpha_\varsigma}(s)$

Corollary 4.3.2. $R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) \to U(\theta)R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma)U^{-1}(\theta)$

Corollary 4.3.3. $D_w R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma) \to UD_w R_{uv}{}^{cd}\frac{1}{2}S_{cd}(s,\varsigma)U^{-1}, D_w = \nabla_w + [\frac{1}{2}\omega_w{}^{cd}S_{cd}(s), \quad]$

4.4 Matrix description of gravitational field's general Yang-Mills-like theory

Matrix description of gravitational field's general Yang-Mills-like theory:

$$\begin{cases} R_{uv}{}^{cd}\frac{1}{2}S_{cd} = \partial_u\omega_v{}^{cd}\frac{1}{2}S_{cd} - \partial_v\omega_u{}^{cd}\frac{1}{2}S_{cd} + [\omega_u{}^{cd}\frac{1}{2}S_{cd}, \omega_v{}^{ef}\frac{1}{2}S_{ef}] \\ [S_{ab}, S_{cd}] = \delta_{ad}S_{bc} - \delta_{ac}S_{bd} + \delta_{bc}S_{ad} - \delta_{bd}S_{ac} \\ c^{ab}S_{ab} = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0 \end{cases} \tag{3.9}$$

Gauge transformation:

$$\begin{cases} \psi \to U(\theta)\psi, U(\theta) = e^{-\varsigma\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)}\{= e^{\frac{1}{2}\vartheta^{ab}S_{ab}} = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\} \\ \frac{1}{2}\omega_u{}^{ab}S_{ab} \to U(\theta)\frac{1}{2}\omega_u{}^{ab}S_{ab}U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta) \end{cases} \tag{3.10}$$

Corollary 4.4.1. $D_u\psi \to U(\theta)D_u\psi, D_u = \partial_u + \frac{1}{2}\omega_u{}^{cd}S_{cd}$

Corollary 4.4.2. $R_{uv}{}^{cd}\frac{1}{2}S_{cd} \to U(\theta)R_{uv}{}^{cd}\frac{1}{2}S_{cd}U^{-1}(\theta)$

Corollary 4.4.3. $D_w R_{uv}{}^{cd}\frac{1}{2}S_{cd} \to UD_w R_{uv}{}^{cd}\frac{1}{2}S_{cd}U^{-1}, D_w = \nabla_w + [\frac{1}{2}\omega_w{}^{cd}S_{cd}, \quad]$

4.5 Components description of gravitational field's Yang-Mills theory

Corollary 4.5.1. $A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow A_u{}^{\alpha'_\varsigma}\sigma_{\alpha'_\varsigma}(s) \to U(-\theta^*)A_u{}^{\alpha'_\varsigma}\sigma_{\alpha'_\varsigma}(s)U^{-1}(-\theta^*) + \varsigma[\partial_u U(-\theta^*)]U^{-1}(-\theta^*)$

Theorem 4.5.1. $A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow \delta A_u{}^{\alpha_\varsigma} = -\varsigma\theta^{\beta_\varsigma}(i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma})A_u{}^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma}$

Theorem 4.5.2. $\delta A_u{}^{\alpha_\varsigma} = -\varsigma\theta^{\beta_\varsigma}(i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma})A_u{}^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma} \Leftrightarrow \delta\omega_u{}^{ab} = \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} + \partial_u\vartheta^{ab}$

Proof: $\delta A_u{}^{\alpha_\varsigma} = -\varsigma\theta^{\beta_\varsigma}(i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma})A_u{}^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma}$
$\Leftrightarrow A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [-\frac{1}{2}\varsigma\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to U(\theta)[-\frac{1}{2}\varsigma\omega_u{}^{ab}S_{ab}(s,\varsigma)]U^{-1}(\theta) - \varsigma[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to U(\theta)[\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)]U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to \frac{1}{2}(\omega_u{}^{ab} + \partial_u\vartheta^{ab})S_{ab}(s,\varsigma) + \frac{1}{4}\vartheta^{ab}\omega_u{}^{cd}[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)]$
$\Leftrightarrow [\frac{1}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to \frac{1}{2}(\omega_u{}^{ab} + \partial_u\vartheta^{ab})S_{ab}(s,\varsigma) + \frac{1}{2}(\vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb})S_{ab}(s,\varsigma)$
$\Leftrightarrow \omega_u{}^{ab} \to \omega_u{}^{ab} + \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} + \partial_u\vartheta^{ab}$
$\Leftrightarrow \delta\omega_u{}^{ab} = \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} + \partial_u\vartheta^{ab}$ □

Corollary 4.5.2. *Gauge transformation:* $\begin{cases} \delta\psi(s,\varsigma) = -\varsigma\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)\psi(s,\varsigma) \\ \delta A_u{}^{\alpha_\varsigma} = -\varsigma\theta^{\beta_\varsigma}(i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma})A_u{}^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma} \Leftrightarrow \delta A_u{}^{\alpha'_\varsigma} = \varsigma\theta^{*\beta'_\varsigma}(i\varepsilon_{\beta'_\varsigma\gamma'_\varsigma}{}^{\alpha'_\varsigma})A_u{}^{\gamma'_\varsigma} - \partial_u\theta^{*\alpha'_\varsigma} \end{cases}$

Corollary 4.5.3. $\delta F_{uv}{}^{\alpha_\varsigma} = -\varsigma\theta^{\beta_\varsigma}(i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma})F_{uv}{}^{\gamma_\varsigma}, \delta F_{uv}{}^{[\alpha_\varsigma]} = -\varsigma\theta^{\beta_\varsigma}\gamma_{\beta_\varsigma}F_{uv}{}^{[\alpha_\varsigma]}, \delta F_{uv}{}^{[\alpha_\varsigma]} = (i\omega + \varsigma\epsilon)\cdot\gamma F_{uv}{}^{[\alpha_\varsigma]}$

Corollary 4.5.4. $\delta\omega_u{}^{ab} = \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{ab}\delta A_u{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{ab}\delta A_u{}^{\alpha_\varsigma})$

4.6 Bianchi identity of gravitational field [14–17]

Bianchi identity: $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0$ (3.11)

Corollary 4.6.1. $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0\Rightarrow R^{(*ab)cd}{}_{;a}\equiv 0$

Proof: $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0$
$\Rightarrow \varepsilon_{fcde}(R^{abcd;e}+R^{abde;c}+R^{abec;d})\equiv 0$
$\Rightarrow 3\varepsilon_{fcde}R^{abcd;e}\equiv 0$
$\Rightarrow R^{ab(*cd)}{}_{;d}\equiv 0$
$\Rightarrow R^{(*ab)cd}{}_{;a}\equiv 0$ □

Corollary 4.6.2. $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0\Rightarrow R^{abcd}{}_{;a}\equiv -R^{b[c;d]}$

Proof: $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0$
$\Rightarrow R^{abcd}{}_{;a}-R^{bd;c}+R^{bc;d}\equiv 0$
$\Rightarrow R^{abcd}{}_{;a}=R^{bd;c}-R^{bc;d}$
$\Rightarrow R^{cdba}{}_{;a}\equiv R^{b[c;d]}$
$\Rightarrow R^{abcd}{}_{;a}\equiv -R^{b[c;d]}$ □

Corollary 4.6.3. $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0\Rightarrow (R^{ab}-\frac{1}{2}g^{ab}R)_{;b}\equiv 0$

Proof: $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0$
$\Rightarrow R^{abcd}{}_{;a}\equiv -R^{b[c;d]}$
$\Rightarrow R^{ac}{}_{;a}\equiv R^{;c}-R^{ac}{}_{;a}$
$\Rightarrow R^{ac}{}_{;a}\equiv \frac{1}{2}R^{;c}$
$\Rightarrow (R^{ab}-\frac{1}{2}g^{ab}R)_{;b}\equiv 0$ □

4.7 The identity of gravitational field Weyl tensor [17]

Definiton 4.7.1. $C^{abcd}\equiv R^{abcd}+\frac{1}{2}g^{a[d}R^{c]b}+\frac{1}{2}g^{b[c}R^{d]a}+\frac{1}{6}g^{a[c}g^{d]b}R$

Corollary 4.7.1. $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0\Rightarrow C^{abcd}{}_{;a}\equiv -\frac{1}{2}R^{b[c;d]}+\frac{1}{12}g^{b[c}R^{;d]}$

Proof: $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0$
$\Rightarrow R^{abcd}{}_{;a}\equiv -R^{b[c;d]}, R^{ba}{}_{;a}\equiv \frac{1}{2}R^{;b}$
$\Rightarrow C^{abcd}{}_{;a}\equiv R^{abcd}{}_{;a}+\frac{1}{2}g^{a[d}R^{c]b}{}_{;a}+\frac{1}{2}g^{b[c}R^{d]a}{}_{;a}+\frac{1}{6}g^{a[c}g^{d]b}R_{;a}$
$\Rightarrow C^{abcd}{}_{;a}\equiv -R^{b[c;d]}+\frac{1}{2}R^{b[c;d]}+\frac{1}{4}g^{b[c}R^{;d]}-\frac{1}{6}g^{b[c}R^{;d]}$
$\Rightarrow C^{abcd}{}_{;a}\equiv -\frac{1}{2}R^{b[c;d]}+\frac{1}{12}g^{b[c}R^{;d]}$ □

Corollary 4.7.2. $C^{(*ab)cd}\equiv R^{(*ab)cd}+\frac{1}{2}\varepsilon^{abe[c}R^{d]}{}_{e}+\frac{1}{6}\varepsilon^{abcd}R$

4.8 Gravitational field's Yang-Mills gauge identity

Corollary 4.8.1. $R^{abcd;e}+R^{abde;c}+R^{abec;d}\equiv 0\Rightarrow \begin{cases} R^{abcd}{}_{;a}\equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a}\equiv 0 \end{cases}$

Lemma 4.8.1. $D^aF_{ab}{}^{\alpha_\varsigma}=-J_b{}^{\alpha_\varsigma}\Leftrightarrow D^aF_{ab}{}^{\alpha'_\varsigma}=-J_b{}^{\alpha'_\varsigma}$

Proof: $D^aF_{ab}{}^{\alpha_\varsigma}=-J_b{}^{\alpha_\varsigma}$
$\Leftrightarrow (D^aF_{ab}{}^{\alpha_\varsigma})^*=-(J_b{}^{\alpha_\varsigma})^*$
$\Leftrightarrow \eta^{a'}{}_cD^c(\eta_{a'}{}^a\eta_{b'}{}^bF_{ab}{}^{\alpha'_\varsigma})=-\eta_{b'}{}^bJ_b{}^{\alpha'_\varsigma}$
$\Leftrightarrow \eta_{b'}{}^bD^a(F_{ab}{}^{\alpha'_\varsigma})=-\eta_{b'}{}^bJ_b{}^{\alpha'_\varsigma}$
$\Leftrightarrow D^a(F_{ab}{}^{\alpha'_\varsigma})=-J_b{}^{\alpha'_\varsigma}$ □

Lemma 4.8.2. $D^a*F_{ab}{}^{\alpha_\varsigma}\equiv 0\Leftrightarrow D^a*F_{ab}{}^{\alpha'_\varsigma}\equiv 0$

Proof: $D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0$

$\Leftrightarrow (D^a * F_{ab}{}^{\alpha_\varsigma})^* \equiv 0$

$\Leftrightarrow \eta^{a'}{}_c D^c(\eta_{a'}{}^a \eta_{b'}{}^b * F_{ab}{}^{\alpha'_\varsigma}) \equiv 0$

$\Leftrightarrow \eta_{b'}{}^b D^a * F_{ab}{}^{\alpha'_\varsigma} \equiv 0$

$\Leftrightarrow D^a * F_{ab}{}^{\alpha'_\varsigma} \equiv 0$ □

Theorem 4.8.1. $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} \equiv -J^{b\alpha_\varsigma}, J^{b\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \end{cases}$

Proof: $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R_{ab}{}^{cd;a} \equiv -R_b{}^{[c;d]} \\ R_{*ab}{}^{cd;a} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} F_{ab}{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd} F_{ab}{}^{\alpha_\varsigma})^{;a} \equiv -\frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} J_b{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd} J_b{}^{\alpha_\varsigma}) \\ \frac{1}{2}\varsigma(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} * F_{ab}{}^{\alpha'_\varsigma} - \sigma_{\varsigma\alpha_\varsigma}{}^{cd} * F_{ab}{}^{\alpha_\varsigma})^{;a} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} \equiv -J_b{}^{\alpha_\varsigma}, D^a F_{ab}{}^{\alpha'_\varsigma} \equiv -J_b{}^{\alpha'_\varsigma} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0, D^a * F_{ab}{}^{\alpha'_\varsigma} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} \equiv -J^{b\alpha_\varsigma}, J^{b\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \end{cases}$ □

5 Spinorial form of gravitational field's Yang-Mills gauge identity

5.1 Gravitational field's gauge identity

gravitational field's Yang-Mills gauge identity:

$$\begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} \equiv -J^{b\alpha_\varsigma}, J^{b\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \end{cases} \quad (3.12)$$

The above gravitational Yang-Mills gauge identity can also be regarded as a special case of Yang-Mills equation. Namely $\sigma = \alpha_\varsigma$ case. And all equal signs are replaced with identity sign. So there are the following completely similar conclusions.

5.2 Integer spinorial form of gravitational field's Yang-Mills gauge identity

Corollary 5.2.1. $D^a F_{ab}{}^{\beta_\kappa} \equiv -J_b{}^{\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\beta_\kappa}(1,\varsigma) \equiv \varsigma J_b{}^{\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0$

Corollary 5.2.2. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\beta_\kappa}(1,\varsigma) \equiv \varsigma J_b{}^{\beta_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta_\kappa}(1,\varsigma) \equiv \varsigma \tilde{\mathcal{J}}^{\beta_\kappa}(1,\varsigma)$

Corollary 5.2.3. $D^a F_{ab}{}^{\beta_\kappa} \equiv -J_b{}^{\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0 \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta_\kappa}(1,\varsigma) \equiv \varsigma \tilde{\mathcal{J}}^{\beta_\kappa}(1,\varsigma), D^a * F_{ab}{}^{\beta_\kappa} \equiv 0$

5.3 Relationships between half integer and integer spinorial form of gravitational field's gauge identity

Theorem 5.3.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \beta_\kappa} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma\beta_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\beta_\kappa}(1,\varsigma) = \varsigma J_b{}^{\beta_\kappa}$

Proposition 5.3.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \beta_\kappa} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma\beta_\kappa} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^{\beta_\kappa}(1,\varsigma) = \varsigma \tilde{J}^{\beta_\kappa}(1,\varsigma)$

Corollary 5.3.1. $(\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^{\beta_\kappa}(1,\varsigma) = \varsigma \tilde{J}^{\beta_\kappa}(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta_\kappa}(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^{\beta_\kappa}(1,\varsigma)$

5.4 Half integer and integer spinorial form of gravitational field's gauge identity [4, 5]

Corollary 5.4.1. $D^a F_{ab}{}^{\beta_\kappa} = -J_b{}^{\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0 \Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \beta_\kappa} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0$

Corollary 5.4.2. $D^a F_{ab}{}^{\beta_\kappa} = -J_b{}^{\beta_\kappa}, D^a * F_{ab}{}^{\beta_\kappa} \equiv 0 \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^{\beta_\kappa}(1,\varsigma) = \varsigma \tilde{J}^{\beta_\kappa}(1,\varsigma), D^a * F_{ab}{}^{\beta_\kappa} \equiv 0$

5.5 Half integer spinorial form of gravitational field's gauge identity [4, 5]

Theorem 5.5.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma \beta_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma\beta_\kappa} \Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Proof: $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma \beta_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma \beta_\kappa}$
$\Leftrightarrow \frac{1}{\sqrt{2}}\sigma_{\beta_\kappa}{}^{C_\kappa D_\kappa}(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma \beta_\kappa} \equiv \frac{1}{\sqrt{2}}\sigma_{\beta_\kappa}{}^{C_\kappa D_\kappa}\varsigma J_{A'_\varsigma}{}^{B_\varsigma \beta_\kappa}$
$\Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$ □

Corollary 5.5.1. $D^aF_{ab}{}^{\beta_\kappa} \equiv -J_b{}^{\beta_\kappa}, D^a*F_{ab}{}^{\beta_\kappa} \equiv 0 \Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}, D^a*F_{ab}{}^{\beta_\kappa} \equiv 0$

Corollary 5.5.2. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa}(1,\varsigma) \equiv \varsigma J_b{}^{\beta_\kappa} \Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Corollary 5.5.3. $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma \beta_\varsigma} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma \beta_\varsigma} \Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$

Corollary 5.5.4. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv \varsigma J_b{}^{\beta_\varsigma} \Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$

6 Weyl spinorial form of gravitational field's gauge identity

6.1 Weyl spinorial form of gravitational field's gauge identity

Definiton 6.1.1. $\tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv [C^{\alpha_\varsigma\beta_\varsigma}, 0^{\beta_\varsigma}]$

Theorem 6.1.1. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv \frac{1}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}(R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{;d]})$

Proof: $D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\beta_\varsigma}) \equiv \varsigma J^{b\beta_\varsigma}, J^{b\beta_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\beta_\varsigma}_{\varsigma cd}R^{b[c;d]}$
$\Leftrightarrow D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}C^{\alpha_\varsigma\beta_\varsigma}) \equiv \frac{1}{6}D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\delta^{\alpha_\varsigma\beta_\varsigma}R) + \varsigma J^{b\beta_\varsigma}$
$\Leftrightarrow D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}C^{\alpha_\varsigma\beta_\varsigma}) \equiv \frac{1}{6}\sigma_\varsigma{}^{\beta_\varsigma ab}R_{;a} + \frac{1}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}R^{b[c;d]}$
$\Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv \frac{1}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}(R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{;d]})$ □

Definiton 6.1.2. $\bar{J}^{bcd} \equiv R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{;d]}, \bar{J}^{b\beta_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\beta_\varsigma}_{\varsigma cd}\bar{J}^{bcd}, \bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma} \equiv \frac{1}{2}(\sigma,-i\varsigma)^b{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\bar{J}_b{}^{\beta_\varsigma}$

Corollary 6.1.1. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv \varsigma\bar{J}_b{}^{\beta_\varsigma} \Leftrightarrow (\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_aC^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \varsigma\bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}$

Corollary 6.1.2. $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_aC^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \varsigma\bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma} \Rightarrow \bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}B_\varsigma C_\varsigma D_\varsigma$

Corollary 6.1.3. $\bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}B_\varsigma C_\varsigma D_\varsigma \Leftrightarrow [(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}]\bar{J}_a{}^{\alpha_\varsigma} \equiv 0 \Leftrightarrow [(\sigma,-i\varsigma)^aS_{cd}(\varsigma)]\bar{J}_a{}^{cd} \equiv 0$

Theorem 6.1.2. $R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a} \Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a}$

Proof: $R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a}$
$\Leftrightarrow [\frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}]_{;b} \equiv \frac{1}{2}R^{;a}$
$\Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a}$ □

Corollary 6.1.4. $D_a(\sigma_{\varsigma\alpha_\varsigma}{}^{ab}\psi^{\alpha_\varsigma\beta'_\varsigma}) \equiv -\frac{1}{2}\sigma^{\beta'_\varsigma}_{-\varsigma cd}R^{b[c;d]}, \psi^{\alpha_\varsigma\beta'_\varsigma} \equiv \frac{1}{2}(\sigma^{\alpha_\varsigma}_\varsigma\sigma^{\beta'_\varsigma}_{-\varsigma})_{ab}R^{ab} \Rightarrow R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a}$

6.2 Similar electromagnetic field form I of gravitational field's gauge identity

Corollary 6.2.1. $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_aC^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \varsigma\bar{J}_{A'_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma}\otimes I,-i\varsigma)^aD_a\tilde{\Psi}(2,\varsigma) \equiv \varsigma\tilde{\mathcal{J}}(2,\varsigma), \tilde{\Psi}(2,\varsigma) \equiv \tilde{C}(2,\varsigma)$

The above second equation is formally equivalent to two electromagnetic field equations with both electric charge and magnetic charge. And it satisfies Lorenz covariant. It characterizes non torsion gravitational fields. It has no connection with establishing or not of Einstein equation. Therefore some analytical techniques of electromagnetic field can be used here, so that some new gravitational properties can be obtained.

Definiton 6.2.1. $\Omega(\varsigma) = (\begin{bmatrix} 0 & 0 \\ -\sigma_{\varsigma y} & \sigma_{\varsigma x} \end{bmatrix}, \begin{bmatrix} \sigma_{\varsigma y} & -\sigma_{\varsigma x} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix})$

Corollary 6.2.2. $\tilde{\Psi}(2,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot R\otimes I_4 + (i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$

Proof: $\Lambda[\tilde{\Psi}(2,\varsigma)] = S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{3}{2})}S^+_{em}(\varsigma)\otimes S^+_{em}(\frac{1}{2})$
$= e^{(i\omega+\varsigma\epsilon)\cdot[R\otimes I_4+\Omega(\varsigma)]} = e^{(i\omega+\varsigma\epsilon)\cdot R\otimes I_4 + (i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$ □

Corollary 6.2.3. $\tilde{\mathcal{J}}(2,\varsigma) \sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)\otimes I_4 + (i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$

Proof: $\Lambda[\tilde{\mathcal{J}}(2,\varsigma)] = S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{3}{2})}S^+_{em}(\varsigma)\otimes S^+_{em}(\frac{1}{2})$
$= e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)\otimes I_4 + (i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$ □

6.3 Similar electromagnetic field form II of gravitational field's gauge identity

Corollary 6.3.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv \varsigma \bar{J}_b{}^{\beta_\varsigma} \Leftrightarrow (\gamma, -i\varsigma)^a{}_{l\alpha_\varsigma} D_a C^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma \bar{J}_l{}^{\beta_\varsigma}$

After spreading, the corollary can be proved. The above covariant equation shows that $(\gamma, -i\varsigma)_a$ acts with a covariation under some special condition. The general covariant equations are constructed with Pauli matrix. But here a complete covariant equation is constructed with photon spin matrix. I've seen the special case first time.

7 Gravitational field's Yang-Mills gauge physical equation

7.1 Einstein equation [14] and gravitational field's Yang-Mills gauge equation

Einstein equation: $R^{ab} - \dfrac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab}$ (3.13)

Corollary 7.1.1. $R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow T^{ab}{}_{;b} = 0$

Proof: $R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab}$
$\Rightarrow (R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab}); b = -8\pi G T^{ab}; b$
$\Rightarrow 0 = -8\pi G T^{ab}; b$
$\Rightarrow T^{ab}{}_{;b} = 0$ □

Corollary 7.1.2. $R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2} g^{ab} T) + \Lambda g^{ab}$

Proof: $R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \to R = 8\pi G T + 4\Lambda$
$\Leftrightarrow R^{ab} - \frac{1}{2} g^{ab}(8\pi G T + 4\Lambda) + \Lambda g^{ab} = -8\pi G T^{ab}$
$\Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2} g^{ab} T) + \Lambda g^{ab} \to R = 8\pi G T + 4\Lambda$ □

Corollary 7.1.3. $\begin{cases} R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd}{}_{;a} = 8\pi G(T^{b[c;d]} - \frac{1}{2} g^{b[c} T^{;d]}), R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases}$

Corollary 7.1.4. $\begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \\ J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2} g^{b[c} T^{;d]}) \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} = -J^{b\alpha_\varsigma} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \\ J^{b\alpha_\varsigma} \equiv \frac{1}{2}\varsigma \sigma^{\alpha_\varsigma}_{\varsigma cd} J^{bcd}, J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2} g^{b[c} T^{;d]}) \end{cases}$

Corollary 7.1.5. $\begin{cases} R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \end{cases} \Rightarrow \begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} = -J^{b\alpha_\varsigma} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \end{cases}$

Corollary 7.1.6. $\begin{cases} D^a F_{ab}{}^{\alpha_\varsigma} = -J^{b\alpha_\varsigma} \\ D^a * F_{ab}{}^{\alpha_\varsigma} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}{}^{[\alpha_\varsigma]} = -J^{b[\alpha_\varsigma]} \\ D^a * F_{ab}{}^{[\alpha_\varsigma]} \equiv 0 \end{cases}$

7.2 Spinorial form of gravitational field's Yang-Mills gauge equation

As long as Einstein equation $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2} g^{ab} T) + \Lambda g^{ab}$ is put into the spinorial form of gravitational field's gauge identity and corresponding identity signs are replaced with equal sign, then we can get the spinorial form of gravitational field's Yang-Mills gauge equation. In form, it is completely consistent with the spinorial form of gravitational field's gauge identity. No longer repeat. In essence, the gravitational field's Yang-Mills gauge equation is just the identity of gravitational field. It has nothing to do with whether or not the establishment of Einstein equation. But the real description of physics is Einstein equation. Only when Einstein equation is applied to the source term of gravitational field's gauge identity, the gravitational field's Yang-Mills gauge equation really has a physical gravitational source term. Then the gravitational field's Yang-Mills gauge equation becomes a real physical equation. So this is quite different from electromagnetic field and Yang-Mills field cases. The gauge equations of electromagnetic field and Yang-Mills field are not only identities, but also describe the real physics directly.

8 Equivalent matrix form of general relativistic Einstein equation [14–17]

8.1 Preparation

Corollary 8.1.1. $R^{ab} = \varsigma(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, (F^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma})^{ab} = -(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, F^{\alpha_\varsigma}{}_{ab} = F_{ab}{}^{\alpha_\varsigma}, R = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}F_{ab}{}^{\alpha_\varsigma}$

Corollary 8.1.2. $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$

Definiton 8.1.1. $\bar{T}^{ab} = 8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}, \bar{\mathcal{T}}^b \equiv [\bar{T}_x{}^b, \bar{T}_y{}^b, \bar{T}_z{}^b, \bar{T}_\pi{}^b]^T$

Definiton 8.1.2. $\mathcal{F}_{ab} \equiv [F_{ab}{}^{x_\varsigma}, F_{ab}{}^{y_\varsigma}, F_{ab}{}^{z_\varsigma}, 0_{ab}]^T = F_{ab}{}^{[\alpha_\varsigma]}, \mathcal{R} = [R, 0]$

Definiton 8.1.3. $\mathcal{A}_a(\varsigma) \equiv [A_a{}^{x_\varsigma}, A_a{}^{y_\varsigma}, A_a{}^{z_\varsigma}, 0_a]^T = A_a{}^{[\alpha_\varsigma]}(\varsigma), \mathcal{J}_a(\varsigma) \equiv [J_a{}^{x_\varsigma}, J_a{}^{y_\varsigma}, J_a{}^{z_\varsigma}, 0_a]^T = J_a{}^{[\alpha_\varsigma]}$

Definiton 8.1.4. $\mathbb{F}_{ab} \equiv [F_{ab}{}^{x_\varsigma}, F_{ab}{}^{y_\varsigma}, F_{ab}{}^{z_\varsigma}, -i\bar{T}_{ab}]^T$

Corollary 8.1.3. $F_{ab}{}^{\alpha_\varsigma} = \partial_a A_b{}^{\alpha_\varsigma} - \partial_b A_a{}^{\alpha_\varsigma} - i\varsigma\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_a{}^{\beta_\varsigma}A_b{}^{\gamma_\varsigma}$
$\Leftrightarrow \mathcal{F}_{ab}(\varsigma) = \partial_a\mathcal{A}_b(\varsigma) - \partial_b\mathcal{A}_a(\varsigma) + \varsigma\mathcal{A}_a^T(\varsigma)\mathcal{R}\mathcal{A}_b(\varsigma) = [\partial_a + \frac{1}{2}\varsigma\mathcal{A}_a^T(\varsigma)\mathcal{R}]\mathcal{A}_b(\varsigma) - [\partial_b + \frac{1}{2}\varsigma\mathcal{A}_b^T(\varsigma)\mathcal{R}]\mathcal{A}_a(\varsigma)$

8.2 Equivalent matrix form of Einstein equation

Corollary 8.2.1. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab} = \varsigma\bar{\mathcal{T}}^b$

Proof: $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$
$\Leftrightarrow (F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab} = -\varsigma\bar{T}^{ab}$
$\Leftrightarrow (\sigma_{\varsigma\alpha_\varsigma}F^{\alpha_\varsigma})^{ab} = -\varsigma\bar{T}^{ab}$
$\Leftrightarrow [(\sigma_\varsigma, -i\varsigma)_{\alpha_\varsigma}F^{\alpha_\varsigma}]^{ab} = -\varsigma\bar{T}^{ab}$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ac}F^{cb}{}_{\alpha_\varsigma} = -\varsigma\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ca}F^{cb}{}_{\alpha_\varsigma} = \varsigma\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{ca}{}^{\alpha_\varsigma}F^{cb}{}_{\alpha_\varsigma} = \varsigma\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(\varsigma) = \varsigma\bar{\mathcal{T}}^b$ □

Self evaluation: Here gets a beautiful and concise spinorial equation. It's equivalent to Einstein equation and very interesting.

Corollary 8.2.2. $(\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab} = \varsigma\bar{\mathcal{T}}^b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a\mathbb{F}^{ab}(\varsigma) = 0$

Proof: $(\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(\varsigma) = \varsigma\bar{\mathcal{T}}^b$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ca}F^{cb}{}_{\alpha_\varsigma} = \varsigma\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ca}F^{cb}{}_{\alpha_\varsigma} - i\varsigma\delta_{ca}(-i\bar{T}^{cb}) = 0$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a\mathbb{F}^{ab}(\varsigma) = 0$ □

Corollary 8.2.3. $(\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(\varsigma) = \varsigma\bar{\mathcal{T}}^b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0(\textit{gauge condition})$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a[\partial_a + \varsigma\mathcal{A}_a^T(\varsigma)\mathcal{R}]\mathcal{A}_b(\varsigma) = \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0$

Approximate processing:

Corollary 8.2.4. $(\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0, \partial^a A_a(\varsigma) \approx \frac{1}{2}\varsigma(\sigma_{-\varsigma}, i\varsigma)^b\bar{\mathcal{T}}_b$

Proof: $(\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0, (\sigma_{-\varsigma}, i\varsigma)^b(\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma(\sigma_{-\varsigma}, i\varsigma)^b\bar{\mathcal{T}}_b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0, [2\delta^{ab} - (\sigma_{-\varsigma}, i\varsigma)^a(\sigma_{-\varsigma}, -i\varsigma)^b]\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma(\sigma_{-\varsigma}, i\varsigma)^b\bar{\mathcal{T}}_b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx \varsigma\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0, \partial^a A_a(\varsigma) \approx \frac{1}{2}\varsigma(\sigma_{-\varsigma}, i\varsigma)^b\bar{\mathcal{T}}_b$ □

Corollary 8.2.5. $(\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx 0, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\mathcal{A}_b(\varsigma) \approx 0, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0, \partial^a A_a(\varsigma) \approx 0$

9 Analysis of Rarita-Schwinger equation [21]

9.1 Preparation

Rarita-Schwinger Lagrangian: $\mathcal{L}_{RS} = -\bar{\psi}^a \varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma)$

Lemma 9.1.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) + \frac{1}{2}m\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma)\psi^b(e,\varsigma) = 0$
Use formula: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma) = 2S_{ab}(e,\varsigma)\gamma_c(\varsigma) - \gamma_{[a}(\varsigma)\delta_{b]c}, \varepsilon_{abcd}S^{cd}(e,\varsigma) = -2\gamma_5(\varsigma)S_{ab}(e,\varsigma)$
$\Leftrightarrow [2S_{ab}(e,\varsigma)\gamma_c(\varsigma) - \gamma_{[a}(\varsigma)\delta_{b]c}]D^b\psi^c(e,\varsigma) - m\gamma_5(\varsigma)S_{ab}(e,\varsigma)\psi^b(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$ □

Lemma 9.1.2. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$

$$\Rightarrow \begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$

$$\Rightarrow \begin{cases} [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m][D^a\psi_a(e,\varsigma)] - [\gamma_a(\varsigma)D^a]D_c\psi^c(e,\varsigma) - D^aD_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0 \\ 4[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m][\gamma^a(\varsigma)\psi_a(e,\varsigma)] - 4D_c\psi^c(e,\varsigma) - [\gamma^a(\varsigma)D_a][\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$$

$\Leftrightarrow \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0$ □

Lemma 9.1.3. $\begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0 \\ D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] = 0, m = 0 \end{cases}$$

9.2 Equivalent form of Rarita-Schwinger equation with mass

Corollary 9.2.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0, m \neq 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0$ □

Corollary 9.2.2. $[\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, m \neq 0$

Corollary 9.2.3. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0, m \neq 0 \Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0$

9.3 Equivalent form of Rarita-Schwinger equation without mass

Corollary 9.3.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0 \Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0, D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b]\psi_a(\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$
$\Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0, D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$ □

Corollary 9.3.2. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0 \Rightarrow D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$,

Corollary 9.3.3. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0$
$\Rightarrow [\gamma_b(\varsigma)D^b]\psi^a(e,\varsigma) = D^a[\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow \gamma_a(\varsigma)\gamma_b(\varsigma)D^b\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow [2\delta_{ab} - \gamma_b(\varsigma)\gamma_a(\varsigma)]D^b\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$,

Corollary 9.3.4. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0, D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0$

Corollary 9.3.5. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0 \Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0$

Corollary 9.3.6. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0 \Leftrightarrow \gamma_a(\varsigma)F^{ab}(e,\varsigma) = 0, F^{ab}(e,\varsigma) \equiv D^a\psi^b(e,\varsigma) - D^b\psi^a(e,\varsigma)$

Corollary 9.3.7. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0()$
$\Leftrightarrow \gamma_b(\varsigma)D^b\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0$

9.4 Equivalent form of Rarita-Schwinger equation of Weyl type

Corollary 9.4.1. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0, D_a\psi^a(\varsigma) = [(\sigma,i\varsigma)_aD^a][(\sigma,-i\varsigma)_b\psi^b(\varsigma)]$

Corollary 9.4.2. $(\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0 \Rightarrow D_a\psi^a(\varsigma) = [(\sigma,i\varsigma)_aD^a][(\sigma,-i\varsigma)_b\psi^b(\varsigma)]$

Corollary 9.4.3. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0$

Corollary 9.4.4. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma,-i\varsigma)_aF^{ab}(\varsigma) = 0, F^{ab}(\varsigma) \equiv D^a\psi^b(\varsigma) - D^b\psi^a(\varsigma)$

Corollary 9.4.5. $F_{ab}(\varsigma) \equiv D_a\psi_b(\varsigma) - D_b\psi_a(\varsigma) \Leftrightarrow F_{ab}(\varsigma) \equiv \partial_a\psi_b(\varsigma) - \partial_b\psi_a(\varsigma) - \frac{1}{2}\varsigma\mathcal{A}_a{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}\psi_b(\varsigma) + \frac{1}{2}\varsigma\mathcal{A}_b{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}\psi_a(\varsigma)$

Corollary 9.4.6. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma) = 0, (\sigma,-i\varsigma)_a\psi^a(\varsigma) = 0 \Leftrightarrow (\sigma,-i\varsigma)_bD^b\psi^a(\varsigma) = 0, (\sigma,-i\varsigma)_a\psi^a(\varsigma) = 0$

10 Analysis of full symmetry Penrose equation with arbitrary spin [4, 5]

10.1 Equivalent matrix form of full symmetric Penrose equation with arbitrary spin

Convention : $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s}}$ is full symmetry for superscripts, $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s-1}}$ is full symmetry for superscripts

Theorem 10.1.1. $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s}} = \varsigma J_{A_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s-1}} \Leftrightarrow (\sigma\otimes I_{2^{2s-1}},-i\varsigma)^aD_a\hat{\psi}(s,\varsigma) = \varsigma\hat{J}(s,\varsigma)$

The above theorem can be obtained by rewriting the components into a matrix.

Theorem 10.1.2. $(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^aD_a\hat{\psi}(s,\varsigma) = \varsigma\hat{J}(s,\varsigma) \Leftrightarrow (\sigma\otimes I_{2s},-i\varsigma)^aD_a\tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma)$

The above theorem can be obtained by removing redundant equations.

Corollary 10.1.1. $(\sigma,-i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s}} = \varsigma J_{A_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s-1}} \Leftrightarrow (\sigma\otimes I_{2s},-i\varsigma)^aD_a\tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma)$

Theorem 10.1.3. $(\sigma\otimes I_{2^{2n-1}},-i\varsigma)^aD_a\hat{\psi}(n,\varsigma) = \varsigma\hat{J}(n,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}\otimes I_{4^{n-1}},-i\varsigma)^aD_a\hat{\Psi}(n,\varsigma) = \varsigma\hat{\mathcal{J}}(n,\varsigma)$

The above theorem can be obtained by making a representation transformation.

Theorem 10.1.4. $(\sigma\otimes I_{2n},-i\varsigma)^aD_a\tilde{\psi}(n,\varsigma) = \varsigma\tilde{J}(n,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}\otimes I_n,-i\varsigma)^aD_a\tilde{\Psi}(n,\varsigma) = \varsigma\tilde{\mathcal{J}}(n,\varsigma)$

For $n = 1, 2$, the above theorem can be obtained by making a representation transformation.
For $n > 2$, it will be proved later.

10.2 Equivalent integer spinorial form of full symmetric Penrose equation with arbitrary spin

Convention :

$\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n} Z_\varsigma}$ is full symmetry for superscripts, $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1} Z_\varsigma}$ is full symmetry for superscripts $\psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} Z_\varsigma}$ is full symmetry for superscripts $\alpha_\varsigma \beta_\varsigma \cdots$, $J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1} Z_\varsigma}$ is full symmetry for superscripts $\alpha_\varsigma \beta_\varsigma \cdots$

Theorem 10.2.1. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n} Z_\varsigma} = \varsigma J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1} Z_\varsigma}$

$$\Leftrightarrow \sigma_{\varsigma\alpha_\varsigma}{}^{ab} D_a \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} Z_\varsigma} = \varsigma J_b{}^{\overbrace{\beta_\varsigma \gamma_\varsigma \cdots}^{n-1} Z_\varsigma}, \begin{cases} \delta_{\alpha_\varsigma \beta_\varsigma} \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} Z_\varsigma} = 0, (\sigma, -i\varsigma)^a \sigma_{\alpha_\varsigma} J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1} Z_\varsigma} = 0 \\ \sigma_{\alpha_\varsigma} \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} [Z_\varsigma]} = 0, (\sigma, -i\varsigma)^a J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1} [Z_\varsigma]} = 0 \end{cases}$$

Theorem 10.2.2. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n} Z_\varsigma} = \varsigma J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1} Z_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \Psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} Z_\varsigma} = \varsigma J_b{}^{\overbrace{\beta_\varsigma \gamma_\varsigma \cdots}^{n-1} Z_\varsigma} \\ \delta_{\alpha_\varsigma \beta_\varsigma} \Psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} Z_\varsigma} = 0, (\sigma, -i\varsigma)^a (\sigma, i\varsigma)_{\alpha_\varsigma} J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1} Z_\varsigma} = 0 \\ (\sigma, -i\varsigma)_{\alpha_\varsigma} \Psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n} [Z_\varsigma]} = 0, (\sigma, -i\varsigma)^a J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1} [Z_\varsigma]} = 0 \end{cases}$$

Theorem 10.2.3. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}} = \varsigma J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}}$

$$\Leftrightarrow \sigma_{\varsigma\alpha_\varsigma}{}^{ab} D_a \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = \varsigma J_b{}^{\overbrace{\beta_\varsigma \gamma_\varsigma \cdots}^{n-1}}, \delta_{\alpha_\varsigma \beta_\varsigma} \psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = 0, (\sigma, -i\varsigma)^a \sigma_{\alpha_\varsigma} J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}} = 0$$

Theorem 10.2.4. $(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} D_a \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n}} = \varsigma J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma E_\varsigma F_\varsigma \cdots}^{2n-1}}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \Psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = \varsigma J_b{}^{\overbrace{\beta_\varsigma \gamma_\varsigma \cdots}^{n-1}} \\ \delta_{\alpha_\varsigma \beta_\varsigma} \Psi^{\overbrace{\alpha_\varsigma \beta_\varsigma \gamma_\varsigma \cdots}^{n}} = 0, (\sigma, -i\varsigma)^a (\sigma, i\varsigma)_{\alpha_\varsigma} J_a{}^{\overbrace{\alpha_\varsigma \beta_\varsigma \cdots}^{n-1}} = 0 \end{cases}$$

10.3 Equivalent form of Penrose type gravitino equation

Definiton 10.3.1. $F^{ab}(\varsigma) \equiv F^{ab[C_\varsigma]}, J_a(\varsigma) \equiv J_a{}^{[C_\varsigma]}$

Corollary 10.3.1. $J_a{}^{C_\varsigma} = \frac{1}{\sqrt{2}} (\sigma, i\varsigma)_a{}^{A_\varsigma A'_\varsigma} \bar{\varepsilon}_{A_\varsigma B_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} \Leftrightarrow J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} \equiv \frac{1}{\sqrt{2}} (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \bar{\varepsilon}^{A_\varsigma B_\varsigma} J_a{}^{C_\varsigma}$

Corollary 10.3.2. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} = J_{A'_\varsigma}{}^{C_\varsigma B_\varsigma} \Leftrightarrow (\sigma, -i\varsigma)^a J_a(\varsigma) = 0$

Proof: $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} = J_{A'_\varsigma}{}^{C_\varsigma B_\varsigma}$
$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} = 0$
$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma} \frac{1}{\sqrt{2}} (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \bar{\varepsilon}^{A_\varsigma B_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \delta^{A_\varsigma}{}_{C_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma C_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a J_a{}^{[C_\varsigma]} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a J_a(\varsigma) = 0$ □

Corollary 10.3.3. $\psi^{A_\varsigma B_\varsigma C_\varsigma} \equiv -\frac{1}{\sqrt{2}} \varsigma S_{ab}{}^{A_\varsigma B_\varsigma} F^{abC_\varsigma}$

Corollary 10.3.4. $\psi^{A_\varsigma B_\varsigma C_\varsigma} = \psi^{A_\varsigma C_\varsigma B_\varsigma} \Leftrightarrow (\sigma, i\varsigma)_a (\sigma, -i\varsigma)_b F^{ab}(\varsigma) = 0$

Proof: $\psi^{A_\varsigma B_\varsigma C_\varsigma} = \psi^{A_\varsigma C_\varsigma B_\varsigma}$

$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma} = 0$

$\Leftrightarrow -\frac{1}{\sqrt{2}}\varsigma\varepsilon_{B_\varsigma C_\varsigma}S_{ab}{}^{A_\varsigma B_\varsigma}F^{abC_\varsigma} = 0$

$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma}S_{ab}{}^{A_\varsigma}{}_{D_\varsigma}\bar{\varepsilon}^{D_\varsigma B_\varsigma}F^{abC_\varsigma} = 0$

$\Leftrightarrow S_{ab}{}^{A_\varsigma}{}_{D_\varsigma}\delta^{D_\varsigma}{}_{C_\varsigma}F^{abC_\varsigma} = 0$

$\Leftrightarrow S_{ab}{}^{A_\varsigma}{}_{C_\varsigma}F^{abC_\varsigma} = 0$

$\Leftrightarrow \frac{1}{4}(\sigma, i\varsigma)_{[a}(\sigma, -i\varsigma)_{b]}F^{ab[C_\varsigma]} = 0$

$\Leftrightarrow (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab[C_\varsigma]} = 0$

$\Leftrightarrow (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab}(\varsigma) = 0$ □

Corollary 10.3.5. $\begin{cases}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\varsigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} \\ \psi^{A_\varsigma B_\varsigma C_\varsigma}, J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} B_\varsigma C_\varsigma\end{cases} \Leftrightarrow \begin{cases}D^aF_{ab}{}^{[C_\varsigma]} = -J_b{}^{[C_\varsigma]}, D^a * F_{ab}{}^{[C_\varsigma]} \equiv 0 \\ (\sigma, -i\varsigma)_aJ^{a[C_\varsigma]} = 0, (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab[C_\varsigma]} = 0\end{cases}$

Corollary 10.3.6. $\begin{cases}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma}D_a\psi^{A_\varsigma B_\varsigma C_\varsigma} = \varsigma J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} \\ \psi^{A_\varsigma B_\varsigma C_\varsigma} \textit{ is full symmetry}, J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} \textit{ is symmetry for } B_\varsigma C_\varsigma\end{cases}$

$$\Leftrightarrow \begin{cases}D^aF_{ab}(\varsigma) = -J_b(\varsigma), F^{ab}(\varsigma) = D^a\psi^b(\varsigma) - D^b\psi^a(\varsigma) \\ (\sigma, -i\varsigma)_aJ^a(\varsigma) = 0, (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab}(\varsigma) = 0\end{cases}$$

11 Comparative research of various field equations

11.1 Comparison between Einstein equation and guage equation of gravitational field

$$\begin{cases}\text{Einstein equation of gravitational field: } (\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(\varsigma) = \varsigma\bar{\mathcal{T}}^b \\ \text{Guage equation of gravitational field: } D_a\mathcal{F}^{ab}(\varsigma) = -\mathcal{J}^b(\varsigma) \\ \mathcal{F}_{ab}(\varsigma) = (\partial_a - \frac{1}{2}\varsigma A_a{}^{\alpha_\varsigma}\mathcal{R}_{\alpha_\varsigma})\mathcal{A}_b(\varsigma) - (\partial_b - \frac{1}{2}\varsigma A_b{}^{\alpha_\varsigma}\mathcal{R}_{\alpha_\varsigma})\mathcal{A}_a(\varsigma)\end{cases} \tag{3.14}$$

In form it's equal to $(\sigma_{-\varsigma}, -i\varsigma)_a \leftrightarrow D_a$

11.2 Comparison between Weyl and Penrose type gravitino equation

$$\begin{cases}\text{Weyl type gravitino equation: } (\sigma, -i\varsigma)_aF^{ab}(\varsigma) = 0 \\ \text{Penrose type gravitino equation: } D_aF^{ab}(\varsigma) = -J^b(\varsigma) \\ F_{ab}(\varsigma) = (\partial_a - \frac{1}{2}\varsigma A_a{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma})\psi_b(\varsigma) - (\partial_b - \frac{1}{2}\varsigma A_b{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma})\psi_a(\varsigma)\end{cases} \tag{3.15}$$

In form it's equal to $(\sigma, -i\varsigma)_a \leftrightarrow D_a$, Two forms are also very similar in gravitational field and gravitino case.

Gravitino equation: $\gamma_a(\varsigma)F^{ab}(e, \varsigma) = 0, F^{ab}(e, \varsigma) \equiv D^a\psi^b(e, \varsigma) - D^b\psi^a(e, \varsigma)$ (3.16)

11.3 Comparison between massless field equations after introducing gauge condition

$$\begin{cases} \text{Einstein equation equation of gravitational field: } (\sigma_{-\varsigma}, -i\varsigma)^a[\partial_a + \varsigma\mathcal{A}_a^T(\varsigma)\mathcal{R}]\mathcal{A}_b(\varsigma) = \varsigma\bar{\mathcal{T}}_b \\ \text{Gauge condition: } (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0 \end{cases} \tag{3.17}$$

$$\begin{cases} \text{Guage equation of gravitational field: } D_a\mathcal{F}^{ab}(\varsigma) = -J^b(\varsigma) \\ \text{Gauge condition: } D_a\mathcal{A}^a(\varsigma) = 0 \end{cases} \tag{3.18}$$

$$\begin{cases} \text{Weyl type gravitino equation: } (\sigma, -i\varsigma)_b D^b\psi^a(\varsigma) = 0 \\ \text{Gauge condition: } (\sigma, -i\varsigma)_a\psi^a(\varsigma) = 0 \end{cases} \tag{3.19}$$

$$\begin{cases} \text{Penrose type gravitino equation: } D^aF_{ab}(\varsigma) = -J_b(\varsigma), F^{ab}(\varsigma) = D^a\psi^b(\varsigma) - D^b\psi^a(\varsigma) \\ \text{Gauge condition: } (\sigma, -i\varsigma)_a\psi^a(\varsigma) = 0, D_a\psi^a(\varsigma) = 0, (\sigma, -i\varsigma)_aJ^a(\varsigma) = 0 \end{cases} \tag{3.20}$$

$$\begin{cases} \text{Gravitino equation: } \gamma_a(\varsigma)F^{ab}(e,\varsigma) = 0, F^{ab}(e,\varsigma) \equiv D^a\psi^b(e,\varsigma) - D^b\psi^a(e,\varsigma) \\ \text{Gauge condition: } \gamma_a(\varsigma)\psi^a(\varsigma) = 0 \end{cases} \tag{3.21}$$

Chapter 4

Spin equations of all kinds of particles

1 Construction of spin equation

1.1 A new particle equation constructed directly from the spin amounts

The following particle equation is directly constructed by the spin amounts.

$$[(s+\phi)D_a + S_{ab}D^b]\psi = \mathbb{J}_a \tag{4.1}$$

ψ is a particle state spinor. s is the particle spin. S_{ab} is the particle spin tensor. ϕ is a scalar field. $\mathbb{J}_a$ is a spinorial source. D_a is the covariant derivative.

1.2 Properties of the new particle equation

Theorem 1.2.1. *$[(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi = 0 \Rightarrow \phi = 0$ or $\phi = -(2s+1)$ or $\sigma(s)\cdot\nabla\psi = 0, \partial_\pi\psi = 0$*

Proof: $[(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi = 0$

$$\Leftrightarrow \begin{cases} [(s+\phi)\partial_x + i\sigma_z(s)\partial_y - i\sigma_y(s)\partial_z - i\varsigma\sigma_x(s)\partial_\pi]\psi = 0 \\ [(s+\phi)\partial_y + i\sigma_x(s)\partial_z - i\sigma_z(s)\partial_x - i\varsigma\sigma_y(s)\partial_\pi]\psi = 0 \\ [(s+\phi)\partial_z + i\sigma_y(s)\partial_x - i\sigma_x(s)\partial_y - i\varsigma\sigma_z(s)\partial_\pi]\psi = 0 \\ [(s+\phi)\partial_\pi + i\varsigma\sigma_x(s)\partial_x + i\varsigma\sigma_y(s)\partial_y + i\varsigma\sigma_z(s)\partial_z]\psi = 0 \end{cases}$$

$$\Rightarrow \begin{cases} [(s+1+\phi)\sigma(s)\cdot\nabla - i\varsigma\sigma^2(s)\partial_\pi]\psi = 0 \\ [(s+\phi)\partial_\pi + i\varsigma\sigma\cdot\nabla]\psi = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma(s)\cdot\nabla\psi = i\varsigma(s+\phi)\partial_\pi\psi \\ [(s+\phi)(s+1+\phi) - s(s+1)]\partial_\pi\psi = 0 \end{cases}$$

$\Leftrightarrow \phi = 0$ or $\phi = -(2s+1)$ or $\sigma(s)\cdot\nabla\psi = 0, \partial_\pi\psi = 0$ ☐

Theorem 1.2.2. *$[s\partial_a + S_{ab}\partial^b]\psi = 0 \Rightarrow \partial_a\partial^a\psi = 0$*

Proof: $[s\partial_a + S_{ab}\partial^b]\psi = 0$

$\Rightarrow \partial^a[s\partial_a + S_{ab}\partial^b]\psi = 0$

$\Leftrightarrow [s\partial_a\partial^a + S_{ab}\partial^a\partial^b]\psi = 0$

$\Leftrightarrow [s\partial_a\partial^a + 0]\psi = 0$

$\Leftrightarrow \partial_a\partial^a\psi = 0$ ☐

Namely, this equation describes a massless particle.

Theorem 1.2.3. $\begin{cases} \textit{When } \phi \neq 0, [(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \textit{ has no plane wave solution.} \\ \textit{When } \phi = 0, [(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \textit{ has plane wave solutions.} \end{cases}$

Proof: Because this equation describes massless particles. The particle movement direction can always be selected as Z.

Get $p_a = (0,0,p,ip)$, then

$[(s+\phi)p_a + S_{ab}(s,\varsigma)p^b]\psi(s,\varsigma) = 0$

$$\Leftrightarrow \begin{cases} [(s+\phi)p_x + i\sigma_z(s)p_y - i\sigma_y(s)p_z - i\varsigma\sigma_x(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_y + i\sigma_x(s)p_z - i\sigma_z(s)p_x - i\varsigma\sigma_y(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_z + i\sigma_y(s)p_x - i\sigma_x(s)p_y - i\varsigma\sigma_z(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_\pi + i\varsigma\sigma_x(s)p_x + i\varsigma\sigma_y(s)p_y + i\varsigma\sigma_z(s)p_z]\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [-i\sigma_y(s)p_z - i\varsigma\sigma_x(s)p_\pi]\psi(s,\varsigma) = 0 \\ [i\sigma_x(s)p_z - i\varsigma\sigma_y(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_z - i\varsigma\sigma_z(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_\pi + i\varsigma\sigma_z(s)p_z]\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\sigma_x(s) - i\varsigma\sigma_y(s)]p\psi(s,\varsigma) = 0 \Leftrightarrow \psi_m(s,\varsigma) = 0, m = s-1, \cdots, -(s-1), \varsigma s \\ [(s+\phi) + \varsigma\sigma_z(s)]p\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_m(s,\varsigma) = 0, m = s-1, \cdots, -(s-1), \varsigma s \\ \phi\psi_{-\varsigma s}(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{When } \phi \neq 0, \psi(s,\varsigma) = 0, \text{namely, all spin components are 0.} \\ \text{When } \phi = 0, \psi(s,\varsigma) = [\frac{1}{2}(\varsigma-1)\psi_s, 0, \cdots, 0, \frac{1}{2}(\varsigma+1)\psi_{-s}]^T e^{ip\cdot x} \\ \text{Namely, } (-\varsigma s) \text{ component may not be 0, the rest components are 0.} \end{cases}$$

$$\Leftrightarrow \begin{cases} \phi \neq 0, [(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \text{ has no plane wave solution.} \\ \phi = 0, [(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \text{ has plane wave solutions.} \end{cases}$$ □

In this equation ϕ is like a switch.

Corollary 1.2.1. *$[s\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0$ has a plane wave solution: $\psi(s,\varsigma) = [\frac{1}{2}(\varsigma-1)\psi_s, 0, \cdots, 0, \frac{1}{2}(\varsigma+1)\psi_{-s}]^T e^{ip\cdot x}$*

1.3 Definition of spin equation

Definiton 1.3.1. *$[sD_a + S_{ab}D^b]\psi = \mathbb{J}_a$ is called spin equation.*

1.4 Definition of switch spin equation

Definiton 1.4.1. *$[(s+\phi)D_a + S_{ab}D^b]\psi = \mathbb{J}_a$ is called switch spin equation. ϕ is called switch scalar field.*

2 Various particle spin equations

2.1 Neutrino [8] spin equation

Theorem 2.1.1. $[\frac{1}{2}D_a + S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = 0$

Proof: $[\frac{1}{2}D_a + S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0$

$$\Leftrightarrow \begin{cases} (D_x + i\sigma_z D_y - i\sigma_y D_z - i\varsigma\sigma_x D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (D_y + i\sigma_x D_z - i\sigma_z D_x - i\varsigma\sigma_y D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (D_z + i\sigma_y D_x - i\sigma_x D_y - i\varsigma\sigma_z D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (D_\pi + i\varsigma\sigma_x D_x + i\varsigma\sigma_y D_y + i\varsigma\sigma_z D_z)\psi(\frac{1}{2},\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma_x(D_x + i\sigma_z D_y - i\sigma_y D_z - i\varsigma\sigma_x D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ \sigma_y(D_y + i\sigma_x D_z - i\sigma_z D_x - i\varsigma\sigma_y D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ \sigma_z(D_z + i\sigma_y D_x - i\sigma_x D_y - i\varsigma\sigma_z D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ -i\varsigma(D_\pi + i\varsigma\sigma_x D_x + i\varsigma\sigma_y D_y + i\varsigma\sigma_z D_z)\psi(\frac{1}{2},\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\sigma_x D_x + \sigma_y D_y + \sigma_z D_z - i\varsigma D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (\sigma_x D_x + \sigma_y D_y + \sigma_z D_z - i\varsigma D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (\sigma_x D_x + \sigma_y D_y + \sigma_z D_z - i\varsigma D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \\ (\sigma_x D_x + \sigma_y D_y + \sigma_z D_z - i\varsigma D_\pi)\psi(\frac{1}{2},\varsigma) = 0 \end{cases}$$

$\Leftrightarrow (\sigma, -i\varsigma)^a D_a \psi(\frac{1}{2},\varsigma) = 0$ □

A more concise, more analytical, but more abstract proof of the law is as follows:

Proof: $[\frac{1}{2}D_a + S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow [\frac{1}{2}\delta_{ab}D^b + S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow \frac{1}{2}[(\sigma, i\varsigma)_{\{a}(\sigma, -i\varsigma)_{b\}} + (\sigma, i\varsigma)_{[a}(\sigma, -i\varsigma)_{b]}]D^b\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b D^b\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow (\sigma, -i\varsigma)_b D^b\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow (\sigma, -i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = 0$ □

This spin equation is completely equivalent to the neutrino equation. So It's a spin description form of the neutrino equation.

2.2 Electron [7] spin equation in arbitrary N+1 dimensional spacetime

Electron spin equation in arbitrary n=N+1 dimensional spacetime:

Theorem 2.2.1. $[\frac{1}{2}(D_a + m\gamma_a) + S_{ab}D^b]\psi = 0, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \Leftrightarrow (\gamma^a D_a + m)\psi = 0$

Proof: $[\frac{1}{2}(D_a + m\gamma_a) + S_{ab}D^b]\psi = 0, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$

$\Leftrightarrow [(2S_{ab} + \delta_{ab})D^b + \gamma_a m]\psi = 0, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$

$\Leftrightarrow [\frac{1}{2}([\gamma_a, \gamma_b] + \{\gamma_a, \gamma_b\})D_b + \gamma_a m]\psi = 0$

$\Leftrightarrow \gamma_a(\gamma_b D^b + m)\psi = 0$

$\Leftrightarrow (\gamma_a D^a + m)\psi = 0$

$\Leftrightarrow (\gamma^a D_a + m)\psi = 0$ □

Electron spin equations in 4 dimensional spacetime:

Corollary 2.2.1. $\{\frac{1}{2}[D_a + m\gamma_a(\varsigma)] + S_{ab}(e,\varsigma)D^b\}\psi(e,\varsigma) = 0 \Leftrightarrow [\gamma^a(\varsigma)D_a + m]\psi(e,\varsigma) = 0$

2.3 Spin equation of electromagnetic field [10]

Theorem 2.3.1. $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b, S_{ab} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = \varsigma\tilde{\mathcal{J}}(1,\varsigma)$

Proof: $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b, S_{ab} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{xb}J^b \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{yb}J^b \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{zb}J^b \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{\pi b}J^b \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma}(1,\varsigma) \\ \Psi^{y_\varsigma}(1,\varsigma) \\ \Psi^{z_\varsigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} -J^\pi \\ -\varsigma J^z \\ \varsigma J^y \end{bmatrix} \\ \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma}(1,\varsigma) \\ \Psi^{y_\varsigma}(1,\varsigma) \\ \Psi^{z_\varsigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} \varsigma J^z \\ -J^\pi \\ -\varsigma J^x \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma}(1,\varsigma) \\ \Psi^{y_\varsigma}(1,\varsigma) \\ \Psi^{z_\varsigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} -\varsigma J^y \\ \varsigma J^x \\ -J^\pi \end{bmatrix} \\ iD_\pi \Psi(1,\varsigma) = \varsigma\gamma \cdot \nabla_d \Psi(1,\varsigma) - \vec{J} \end{cases}$$

$$\Leftrightarrow \begin{cases} iD_\pi \Psi(1,\varsigma) = i\varsigma \nabla_d \times \Psi(1,\varsigma) - \vec{J} \\ \nabla_d \cdot \Psi(1,\varsigma) = -iJ^\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} iD_\pi \Psi(1,\varsigma) = \varsigma\gamma \cdot \nabla_d \Psi(1,\varsigma) - \vec{J} \\ \nabla_d \cdot \Psi(1,\varsigma) = -iJ^\pi \end{cases}$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}(1,\varsigma) = \varsigma \tilde{\mathcal{J}}(1,\varsigma)$$ □

This spin equation is completely equivalent to electromagnetic field equation. So It's a spin description form of electromagnetic field equation.

Lemma 2.3.1. $\mathbb{J}_a{}^{\beta_\varsigma} = -\varsigma \sigma_\varsigma^{\beta_\varsigma}{}_{ab} J^b \Leftrightarrow \begin{cases} \mathbb{J}_y{}^{z_\varsigma} = -\mathbb{J}_z{}^{y_\varsigma} = -\varsigma \mathbb{J}_\pi{}^{x_\varsigma} \equiv -i\varsigma J^x \\ \mathbb{J}_z{}^{x_\varsigma} = -\mathbb{J}_x{}^{z_\varsigma} = -\varsigma \mathbb{J}_\pi{}^{y_\varsigma} \equiv -i\varsigma J^y \\ \mathbb{J}_x{}^{y_\varsigma} = -\mathbb{J}_y{}^{x_\varsigma} = -\varsigma \mathbb{J}_\pi{}^{z_\varsigma} \equiv -i\varsigma J^z \\ \mathbb{J}_x{}^{x_\varsigma} = \mathbb{J}_y{}^{y_\varsigma} = \mathbb{J}_z{}^{z_\varsigma} \equiv -iJ^\pi \end{cases}$

To expand, it can be proved. The above spin equation is about a special source term. What about a general source term? See the following theorem.

Theorem 2.3.2. $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_a{}^{\beta_\varsigma}, S_{ab} = -\varsigma \sigma_\varsigma^{\alpha_\varsigma}{}_{ab} \gamma_{\alpha_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}(1,\varsigma) = \varsigma \tilde{\mathcal{J}}(1,\varsigma), \mathbb{J}_a{}^{\beta_\varsigma} = -\varsigma \sigma_\varsigma^{\beta_\varsigma}{}_{ab} J^b$

Proof: $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_a{}^{\beta_\varsigma}, S_{ab} = -\varsigma \sigma_\varsigma^{\alpha_\varsigma}{}_{ab} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_x{}^{\beta_\varsigma} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_y{}^{\beta_\varsigma} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_z{}^{\beta_\varsigma} \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma}(1,\varsigma) = \mathbb{J}_\pi{}^{\beta_\varsigma} \end{cases}$$

$$\Leftrightarrow\begin{cases}\begin{bmatrix}D_x & D_y & D_z\\ -D_y & D_x & -\varsigma D_\pi\\ -D_z & \varsigma D_\pi & D_x\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma}(1,\varsigma)\\ \Psi^{y_\varsigma}(1,\varsigma)\\ \Psi^{z_\varsigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_x{}^{x_\varsigma}\\ \mathbb{J}_x{}^{y_\varsigma}\\ \mathbb{J}_x{}^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}\nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_x{}^{x_\varsigma}\\ [\nabla_d\times\Psi(1,\varsigma)]^{z_\varsigma}-\varsigma D_\pi\Psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_x{}^{y_\varsigma}\\ -[\nabla_d\times\Psi(1,\varsigma)]^{y_\varsigma}+\varsigma D_\pi\Psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_x{}^{z_\varsigma}\end{cases}\\ \begin{bmatrix}D_y & -D_x & \varsigma D_\pi\\ D_x & D_y & D_z\\ -\varsigma D_\pi & -D_z & D_y\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma}(1,\varsigma)\\ \Psi^{y_\varsigma}(1,\varsigma)\\ \Psi^{z_\varsigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_y{}^{x_\varsigma}\\ \mathbb{J}_y{}^{y_\varsigma}\\ \mathbb{J}_y{}^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}-[\nabla_d\times\Psi(1,\varsigma)]^{z_\varsigma}+\varsigma D_\pi\Psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_y{}^{x_\varsigma}\\ \nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_y{}^{y_\varsigma}\\ [\nabla_d\times\Psi(1,\varsigma)]^{x_\varsigma}-\varsigma D_\pi\Psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_y{}^{z_\varsigma}\end{cases}\\ \begin{bmatrix}D_z & -\varsigma D_\pi & -D_x\\ \varsigma D_\pi & D_z & -D_y\\ D_x & D_y & D_z\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma}(1,\varsigma)\\ \Psi^{y_\varsigma}(1,\varsigma)\\ \Psi^{z_\varsigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_z{}^{x_\varsigma}\\ \mathbb{J}_z{}^{y_\varsigma}\\ \mathbb{J}_z{}^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}[\nabla_d\times\Psi(1,\varsigma)]^{y_\varsigma}-\varsigma D_\pi\Psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_z{}^{x_\varsigma}\\ -[\nabla_d\times\Psi(1,\varsigma)]^{x_\varsigma}+\varsigma D_\pi\Psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_z{}^{y_\varsigma}\\ \nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_z{}^{z_\varsigma}\end{cases}\\ D_\pi\Psi(1,\varsigma)+i\varsigma\gamma\cdot\nabla_d\Psi=\mathbb{J}_\pi\Leftrightarrow D_\pi\Psi(1,\varsigma)-\varsigma\nabla_d\times\Psi(1,\varsigma)=\mathbb{J}_\pi\end{cases}$$

$$\Leftrightarrow\begin{cases}\mathbb{J}_y{}^{z_\varsigma}=-\mathbb{J}_z{}^{y_\varsigma}=-\varsigma\mathbb{J}_\pi{}^{x_\varsigma}\equiv -i\varsigma J^x\\ \mathbb{J}_z{}^{x_\varsigma}=-\mathbb{J}_x{}^{z_\varsigma}=-\varsigma\mathbb{J}_\pi{}^{y_\varsigma}\equiv -i\varsigma J^y\\ \mathbb{J}_x{}^{y_\varsigma}=-\mathbb{J}_y{}^{x_\varsigma}=-\varsigma\mathbb{J}_\pi{}^{z_\varsigma}\equiv -i\varsigma J^z\\ \mathbb{J}_x{}^{x_\varsigma}=\mathbb{J}_y{}^{y_\varsigma}=\mathbb{J}_z{}^{z_\varsigma}\equiv -iJ^\pi\\ D_\pi\Psi(1,\varsigma)-\varsigma\nabla_d\times\Psi(1,\varsigma)=i\vec{J}\\ \nabla_d\cdot\Psi(1,\varsigma)=-iJ^\pi\end{cases}$$

$$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma)=\varsigma\tilde{\mathcal{J}}(1,\varsigma),\mathbb{J}_a{}^{\beta_\varsigma}=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b$$ □

This theorem shows that the source term of the spin equation is limited, and it is not arbitrary. The spin equation has solutions only for the source term of the previous theorem case, and other cases have no solution.

Corollary 2.3.1. $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma},S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma},S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma},\mathbb{J}_a{}^{\beta_\varsigma}=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b$

Corollary 2.3.2. $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma}$ *has solutions,* $S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}\Leftrightarrow\mathbb{J}_a{}^{\beta_\varsigma}=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b$

2.4 Spin equation of Yang-Mills field [9]

Theorem 2.4.1. $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^{b\sigma},S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\Psi}^\sigma(1,\varsigma)=\varsigma\tilde{\mathcal{J}}^\sigma(1,\varsigma)$

Proof: $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^{b\sigma},S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow\begin{cases}(D_x+i\gamma_zD_y-i\gamma_yD_z-i\varsigma\gamma_xD_\pi)\Psi^\sigma(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{xb}J^{b\sigma}\\ (D_y+i\gamma_xD_z-i\gamma_zD_x-i\varsigma\gamma_yD_\pi)\Psi^\sigma(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{yb}J^{b\sigma}\\ (D_z+i\gamma_yD_x-i\gamma_xD_y-i\varsigma\gamma_zD_\pi)\Psi^\sigma(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{zb}J^{b\sigma}\\ (D_\pi+i\varsigma\gamma_xD_x+i\varsigma\gamma_yD_y+i\varsigma\gamma_zD_z)\Psi^\sigma(1,\varsigma)=-\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{\pi b}J^{b\sigma}\end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma \sigma}(1,\varsigma) \\ \Psi^{y_\varsigma \sigma}(1,\varsigma) \\ \Psi^{z_\varsigma \sigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} -J^{\pi\sigma} \\ -\varsigma J^{z\sigma} \\ \varsigma J^{y\sigma} \end{bmatrix} \\ \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma \sigma}(1,\varsigma) \\ \Psi^{y_\varsigma \sigma}(1,\varsigma) \\ \Psi^{z_\varsigma \sigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} \varsigma J^{z\sigma} \\ -J^{\pi\sigma} \\ -\varsigma J^{x\sigma} \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \Psi^{x_\varsigma \sigma}(1,\varsigma) \\ \Psi^{y_\varsigma \sigma}(1,\varsigma) \\ \Psi^{z_\varsigma \sigma}(1,\varsigma) \end{bmatrix} = i \begin{bmatrix} -\varsigma J^{y\sigma} \\ \varsigma J^{x\sigma} \\ -J^{\pi\sigma} \end{bmatrix} \\ i\partial_\pi \Psi^\sigma(1,\varsigma) = \varsigma\gamma \cdot \nabla_d \Psi^\sigma(1,\varsigma) - \vec{J}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi^\sigma(1,\varsigma) = i\varsigma \nabla_d \times \Psi^\sigma(1,\varsigma) - \vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1,\varsigma) = -iJ^{\pi\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi^\sigma(1,\varsigma) = \varsigma\gamma \cdot \nabla_d \Psi^\sigma(1,\varsigma) - \vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1,\varsigma) = -iJ^{\pi\sigma} \end{cases}$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^\sigma(1,\varsigma)$$ □

This spin equation is completely equivalent to Yang-Mills equation. So It's a spin description form of Yang-Mills equation.

Lemma 2.4.1. $\mathbb{J}_a{}^{\beta_\varsigma \sigma} = -\varsigma \sigma_\varsigma^{\beta_\varsigma}{}_{ab} J^{b\sigma} \Leftrightarrow \begin{cases} \mathbb{J}_y{}^{z_\varsigma \sigma} = -\mathbb{J}_z{}^{y_\varsigma \sigma} = -\varsigma \mathbb{J}_\pi{}^{x_\varsigma \sigma} \equiv -i\varsigma J^{x\sigma} \\ \mathbb{J}_z{}^{x_\varsigma \sigma} = -\mathbb{J}_x{}^{z_\varsigma \sigma} = -\varsigma \mathbb{J}_\pi{}^{y_\varsigma \sigma} \equiv -i\varsigma J^{y\sigma} \\ \mathbb{J}_x{}^{y_\varsigma \sigma} = -\mathbb{J}_y{}^{x_\varsigma \sigma} = -\varsigma \mathbb{J}_\pi{}^{z_\varsigma \sigma} \equiv -i\varsigma J^{z\sigma} \\ \mathbb{J}_x{}^{x_\varsigma \sigma} = \mathbb{J}_y{}^{y_\varsigma \sigma} = \mathbb{J}_z{}^{z_\varsigma \sigma} \equiv -iJ^{\pi\sigma} \end{cases}$

To expand, it can be proved. The above spin equation is about a special source term. What about a general source term? See the following theorem.

Theorem 2.4.2. $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_a{}^{\beta_\varsigma \sigma}, S_{ab} = -\varsigma \sigma_\varsigma^{\alpha_\varsigma}{}_{ab} \gamma_{\alpha_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = \varsigma \tilde{\mathcal{J}}^\sigma(1,\varsigma), \mathbb{J}_a{}^{\beta_\varsigma \sigma} = -\varsigma \sigma_\varsigma^{\beta_\varsigma}{}_{ab} J^{b\sigma}$

Proof: $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_a{}^{\beta_\varsigma \sigma}, S_{ab} = -\varsigma \sigma_\varsigma^{\alpha_\varsigma}{}_{ab} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_x{}^{\beta_\varsigma \sigma} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_y{}^{\beta_\varsigma \sigma} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_z{}^{\beta_\varsigma \sigma} \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma} \Psi^{\gamma_\varsigma \sigma}(1,\varsigma) = \mathbb{J}_\pi{}^{\beta_\varsigma \sigma} \end{cases}$$

$$\Leftrightarrow\begin{cases}\begin{bmatrix}D_x & D_y & D_z\\ -D_y & D_x & -\varsigma D_\pi\\ -D_z & \varsigma D_\pi & D_x\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma\sigma}(1,\varsigma)\\ \Psi^{y_\varsigma\sigma}(1,\varsigma)\\ \Psi^{z_\varsigma\sigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_x{}^{x_\varsigma\sigma}\\ \mathbb{J}_x{}^{y_\varsigma\sigma}\\ \mathbb{J}_x{}^{z_\varsigma\sigma}\end{bmatrix}\Leftrightarrow\begin{cases}\nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_x{}^{x_\varsigma\sigma}\\ [\nabla_d\times\Psi(1,\varsigma)]^{z_\varsigma}-\varsigma D_\pi\Psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_x{}^{y_\varsigma\sigma}\\ -[\nabla_d\times\Psi(1,\varsigma)]^{y_\varsigma}+\varsigma D_\pi\Psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_x{}^{z_\varsigma\sigma}\end{cases}\\ \begin{bmatrix}D_y & -D_x & \varsigma D_\pi\\ D_x & D_y & D_z\\ -\varsigma D_\pi & -D_z & D_y\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma\sigma}(1,\varsigma)\\ \Psi^{y_\varsigma\sigma}(1,\varsigma)\\ \Psi^{z_\varsigma\sigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_y{}^{x_\varsigma\sigma}\\ \mathbb{J}_y{}^{y_\varsigma\sigma}\\ \mathbb{J}_y{}^{z_\varsigma\sigma}\end{bmatrix}\Leftrightarrow\begin{cases}-[\nabla_d\times\Psi(1,\varsigma)]^{z_\varsigma}+\varsigma D_\pi\Psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_y{}^{x_\varsigma\sigma}\\ \nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_y{}^{y_\varsigma\sigma}\\ [\nabla_d\times\Psi(1,\varsigma)]^{x_\varsigma}-\varsigma D_\pi\Psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_y{}^{z_\varsigma\sigma}\end{cases}\\ \begin{bmatrix}D_z & -\varsigma D_\pi & -D_x\\ \varsigma D_\pi & D_z & -D_y\\ D_x & D_y & D_z\end{bmatrix}\begin{bmatrix}\Psi^{x_\varsigma\sigma}(1,\varsigma)\\ \Psi^{y_\varsigma\sigma}(1,\varsigma)\\ \Psi^{z_\varsigma\sigma}(1,\varsigma)\end{bmatrix}=\begin{bmatrix}\mathbb{J}_z{}^{x_\varsigma\sigma}\\ \mathbb{J}_z{}^{y_\varsigma\sigma}\\ \mathbb{J}_z{}^{z_\varsigma\sigma}\end{bmatrix}\Leftrightarrow\begin{cases}[\nabla_d\times\Psi(1,\varsigma)]^{y_\varsigma}-\varsigma D_\pi\Psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_z{}^{x_\varsigma\sigma}\\ -[\nabla_d\times\Psi(1,\varsigma)]^{x_\varsigma}+\varsigma D_\pi\Psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_z{}^{y_\varsigma\sigma}\\ \nabla_d\cdot\Psi(1,\varsigma)=\mathbb{J}_z{}^{z_\varsigma\sigma}\end{cases}\\ D_\pi\Psi^\sigma(1,\varsigma)+i\varsigma\gamma\cdot\nabla_d\Psi^\sigma(1,\varsigma)=\mathbb{J}_\pi{}^\sigma\Leftrightarrow D_\pi\Psi^\sigma(1,\varsigma)-\varsigma\nabla_d\times\Psi^\sigma(1,\varsigma)=\mathbb{J}_\pi{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}\mathbb{J}_y{}^{z_\varsigma\sigma}=-\mathbb{J}_z{}^{y_\varsigma\sigma}=-\varsigma\mathbb{J}_\pi{}^{x_\varsigma\sigma}\equiv -i\varsigma J^{x\sigma}\\ \mathbb{J}_z{}^{x_\varsigma\sigma}=-\mathbb{J}_x{}^{z_\varsigma\sigma}=-\varsigma\mathbb{J}_\pi{}^{y_\varsigma\sigma}\equiv -i\varsigma J^{y\sigma}\\ \mathbb{J}_x{}^{y_\varsigma\sigma}=-\mathbb{J}_y{}^{x_\varsigma\sigma}=-\varsigma\mathbb{J}_\pi{}^{z_\varsigma\sigma}\equiv -i\varsigma J^{z\sigma}\\ \mathbb{J}_x{}^{x_\varsigma\sigma}=\mathbb{J}_y{}^{y_\varsigma\sigma}=\mathbb{J}_z{}^{z_\varsigma\sigma}\equiv -iJ^{\pi\sigma}\\ D_\pi\Psi^\sigma(1,\varsigma)-\varsigma\nabla_d\times\Psi^\sigma(1,\varsigma)=i\vec{J}^\sigma\\ \nabla_d\cdot\Psi^\sigma(1,\varsigma)=-iJ^{\pi\sigma}\end{cases}$$

$$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\Psi}^\sigma(1,\varsigma)=\varsigma\tilde{\mathcal{J}}^\sigma(1,\varsigma),\mathbb{J}_a{}^{\beta_\varsigma\sigma}=-\varsigma\sigma^{\beta_\varsigma}_{\varsigma\ ab}J^{b\sigma}$$ □

This theorem shows that the source term of the spin equation is limited, and it is not arbitrary. The spin equation has solutions only for the source term of the previous theorem case, and other cases have no solution. From the above we can see that the spin equation of electromagnetic field is a special case of Yang-Mills spin equation. Namely, σ is empty.

Corollary 2.4.1. $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma\sigma},S_{ab}=-\varsigma\sigma^{\alpha_\varsigma}_{\varsigma\ ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma\sigma},S_{ab}=-\varsigma\sigma^{\alpha_\varsigma}_{\varsigma\ ab}\gamma_{\alpha_\varsigma},\mathbb{J}_a{}^{\beta_\varsigma\sigma}=-\varsigma\sigma^{\beta_\varsigma}_{\varsigma\ ab}J^{b\sigma}$

Corollary 2.4.2. $(D_a+S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma)=\mathbb{J}_a{}^{\beta_\varsigma\sigma}$ *has solutions,* $S_{ab}=-\varsigma\sigma^{\alpha_\varsigma}_{\varsigma\ ab}\gamma_{\alpha_\varsigma}\Leftrightarrow\mathbb{J}_a{}^{\beta_\varsigma\sigma}=-\varsigma\sigma^{\beta_\varsigma}_{\varsigma\ ab}J^{b\sigma}$

2.5 Spin-s particle spin equation

2.5.1 Several important theorems about spin-s equations

Lemma 2.5.1.
$$\begin{cases}s-\sigma_z(s)=\begin{bmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&\cdots&0&0\\0&0&0&2s-1&0\\0&0&0&0&2s\end{bmatrix},\sigma_x(s)-i\sigma_y(s)=\begin{bmatrix}0&0&0&0&0\\1&0&0&0&0\\0&2&0&\cdots&0\\0&0&\cdots&0&0\\0&0&0&2s&0\end{bmatrix}\\ s+\sigma_z(s)=\begin{bmatrix}2s&0&0&0&0\\0&2s-1&0&0&0\\0&0&\cdots&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix},\sigma_x(s)+i\sigma_y(s)=\begin{bmatrix}0&2s&0&0&0\\0&0&(2s-1)&0&0\\0&0&0&\cdots&0\\0&0&\cdots&0&1\\0&0&0&0&0\end{bmatrix}\end{cases}$$

Theorem 2.5.1. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{Z}_a\varsigma\tilde{J}(s,\varsigma)\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)^a D_a\tilde{\psi}(s,\varsigma)=\varsigma\tilde{J}(s,\varsigma)$

Proof: $[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{Z}_a\varsigma\tilde{J}(s,\varsigma)$

$$\Leftrightarrow \begin{cases} [sD_x + i\sigma_z(s)D_y - i\sigma_y(s)D_z - i\varsigma\sigma_x(s)D_\pi]\psi(s,\varsigma) = \mathbb{Z}_x\varsigma\tilde{J}(s,\varsigma) \\ [sD_y + i\sigma_x(s)D_z - i\sigma_z(s)D_x - i\varsigma\sigma_y(s)D_\pi]\psi(s,\varsigma) = \mathbb{Z}_y\varsigma\tilde{J}(s,\varsigma) \\ [sD_z + i\sigma_y(s)D_x - i\sigma_x(s)D_y - i\varsigma\sigma_z(s)D_\pi]\psi(s,\varsigma) = \mathbb{Z}_z\varsigma\tilde{J}(s,\varsigma) \\ [sD_\pi + i\varsigma\sigma_x(s)D_x + i\varsigma\sigma_y(s)D_y + i\varsigma\sigma_z(s)D_z]\psi(s,\varsigma) = \mathbb{Z}_\pi\varsigma\tilde{J}(s,\varsigma) \end{cases}$$

$$\Leftrightarrow \begin{cases} \{[s-\sigma_z(s)](D_x - iD_y) + [\sigma_x(s) - i\sigma_y(s)](D_z - i\varsigma D_\pi)\}\psi(s,\varsigma) = (\mathbb{Z}_x - i\mathbb{Z}_y)\varsigma\tilde{J}(s,\varsigma) \\ \{[\sigma_x(s) + i\sigma_y(s)](D_x - iD_y) + [s+\sigma_z(s)](D_z - i\varsigma D_\pi)\}\psi(s,\varsigma) = (\mathbb{Z}_z - i\varsigma\mathbb{Z}_\pi)\varsigma\tilde{J}(s,\varsigma) \\ \{[s+\sigma_z(s)](D_x + iD_y) - [\sigma_x(s) + i\sigma_y(s)](D_z + i\varsigma D_\pi)\}\psi(s,\varsigma) = (\mathbb{Z}_x + i\mathbb{Z}_y)\varsigma\tilde{J}(s,\varsigma) \\ \{-[\sigma_x(s) - i\sigma_y(s)](D_x + iD_y) + [s-\sigma_z(s)](D_z + i\varsigma D_\pi)\}\psi(s,\varsigma) = (\mathbb{Z}_z + i\varsigma\mathbb{Z}_\pi)\varsigma\tilde{J}(s,\varsigma) \end{cases}$$

$$\Leftrightarrow \begin{cases} (\sigma, -i\varsigma)^a D_a \begin{bmatrix} \psi^1(s,\varsigma) \\ \psi^2(s,\varsigma) \end{bmatrix} = \begin{bmatrix} \varsigma\tilde{J}^1(s,\varsigma) \\ \varsigma\tilde{J}^2(s,\varsigma) \end{bmatrix} \\ (\sigma, -i\varsigma)^a D_a \begin{bmatrix} \psi^2(s,\varsigma) \\ \psi^3(s,\varsigma) \end{bmatrix} = \begin{bmatrix} \varsigma\tilde{J}^3(s,\varsigma) \\ \varsigma\tilde{J}^4(s,\varsigma) \end{bmatrix} \\ \quad \ldots\ldots \\ (\sigma, -i\varsigma)^a D_a \begin{bmatrix} \psi^{2s-1}(s,\varsigma) \\ \psi^{2s}(s,\varsigma) \end{bmatrix} = \begin{bmatrix} \varsigma\tilde{J}^{4s-3}(s,\varsigma) \\ \varsigma\tilde{J}^{4s-2}(s,\varsigma) \end{bmatrix} \\ (\sigma, -i\varsigma)^a D_a \begin{bmatrix} \psi^{2s}(s,\varsigma) \\ \psi^{2s+1}(s,\varsigma) \end{bmatrix} = \begin{bmatrix} \varsigma\tilde{J}^{4s-1}(s,\varsigma) \\ \varsigma\tilde{J}^{4s}(s,\varsigma) \end{bmatrix} \end{cases}$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)^a D_a \tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma)$$ □

Lemma 2.5.2. $\mathbb{J}_a(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)$

$$\Leftrightarrow \begin{cases} {\mathbb{J}_x}^1(s,\varsigma) - i{\mathbb{J}_y}^1(s,\varsigma) = 0, {\mathbb{J}_z}^1(s,\varsigma) + i\varsigma{\mathbb{J}_\pi}^1(s,\varsigma) = 0 \\ \begin{bmatrix} [{\mathbb{J}_x}^2(s,\varsigma) - i{\mathbb{J}_y}^2(s,\varsigma)] \\ \frac{1}{2s}[{\mathbb{J}_x}^1(s,\varsigma) + i{\mathbb{J}_y}^1(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2s}[{\mathbb{J}_z}^1(s,\varsigma) - i\varsigma{\mathbb{J}_\pi}^1(s,\varsigma)] \\ -[{\mathbb{J}_z}^2(s,\varsigma) + i\varsigma{\mathbb{J}_\pi}^2(s,\varsigma)] \end{bmatrix} \equiv \begin{bmatrix} \varsigma\tilde{J}^1(s,\varsigma) \\ \varsigma\tilde{J}^2(s,\varsigma) \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2}[{\mathbb{J}_x}^3(s,\varsigma) - i{\mathbb{J}_y}^3(s,\varsigma)] \\ \frac{1}{2s-1}[{\mathbb{J}_x}^2(s,\varsigma) + i{\mathbb{J}_y}^2(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2s-1}[{\mathbb{J}_z}^2(s,\varsigma) - i\varsigma{\mathbb{J}_\pi}^2(s,\varsigma)] \\ -\frac{1}{2}[{\mathbb{J}_z}^3(s,\varsigma) + i\varsigma{\mathbb{J}_\pi}^3(s,\varsigma)] \end{bmatrix} \equiv \begin{bmatrix} \varsigma\tilde{J}^3(s,\varsigma) \\ \varsigma\tilde{J}^4(s,\varsigma) \end{bmatrix} \\ \quad \ldots\ldots \\ \begin{bmatrix} \frac{1}{2s-1}[{\mathbb{J}_x}^{2s}(s,\varsigma) - i{\mathbb{J}_y}^{2s-1}(s,\varsigma)] \\ \frac{1}{2}[{\mathbb{J}_x}^{2s-1}(s,\varsigma) + i{\mathbb{J}_y}^2(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[{\mathbb{J}_z}^{2s-1}(s,\varsigma) - i\varsigma{\mathbb{J}_\pi}^2(s,\varsigma)] \\ -\frac{1}{2s-1}[{\mathbb{J}_z}^{2s}(s,\varsigma) + i\varsigma{\mathbb{J}_\pi}^{2s-1}(s,\varsigma)] \end{bmatrix} \equiv \begin{bmatrix} \varsigma\tilde{J}^{4s-3}(s,\varsigma) \\ \varsigma\tilde{J}^{4s-2}(s,\varsigma) \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2s}[{\mathbb{J}_x}^{2s+1}(s,\varsigma) - i{\mathbb{J}_y}^{2s+1}(s,\varsigma)] \\ [{\mathbb{J}_x}^{2s}(s,\varsigma) + i{\mathbb{J}_y}^{2s}(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} [{\mathbb{J}_z}^{2s}(s,\varsigma) - i\varsigma{\mathbb{J}_\pi}^{2s}(s,\varsigma)] \\ -\frac{1}{2s}[{\mathbb{J}_z}^{2s+1}(s,\varsigma) + i\varsigma{\mathbb{J}_\pi}^{2s+1}(s,\varsigma)] \end{bmatrix} \equiv \begin{bmatrix} \varsigma\tilde{J}^{4s-1}(s,\varsigma) \\ \varsigma\tilde{J}^{4s}(s,\varsigma) \end{bmatrix} \\ {\mathbb{J}_x}^{2s+1}(s,\varsigma) + i{\mathbb{J}_y}^{2s+1}(s,\varsigma) = 0, {\mathbb{J}_z}^{2s+1}(s,\varsigma) - i\varsigma{\mathbb{J}_\pi}^{2s+1}(s,\varsigma) = 0 \end{cases}$$

To expand, it can be proved.

Theorem 2.5.2. $[sD_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma) \Leftrightarrow \begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)^a D_a \tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma) \\ \mathbb{J}_a(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma) \end{cases}$

Proof: $[sD_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma)$

$$\Leftrightarrow \begin{cases} [sD_x + i\sigma_z(s)D_y - i\sigma_y(s)D_z - i\varsigma\sigma_x(s)D_\pi]\psi(s,\varsigma) = \mathbb{J}_x(s,\varsigma) \\ [sD_y + i\sigma_x(s)D_z - i\sigma_z(s)D_x - i\varsigma\sigma_y(s)D_\pi]\psi(s,\varsigma) = \mathbb{J}_y(s,\varsigma) \\ [sD_z + i\sigma_y(s)D_x - i\sigma_x(s)D_y - i\varsigma\sigma_z(s)D_\pi]\psi(s,\varsigma) = \mathbb{J}_z(s,\varsigma) \\ [sD_\pi + i\varsigma\sigma_x(s)D_x + i\varsigma\sigma_y(s)D_y + i\varsigma\sigma_z(s)D_z]\psi(s,\varsigma) = \mathbb{J}_\pi(s,\varsigma) \end{cases}$$

$$\Leftrightarrow \begin{cases} \{[s-\sigma_z(s)](D_x-iD_y)+[\sigma_x(s)-i\sigma_y(s)](D_z-i\varsigma D_\pi)\}\psi(s,\varsigma)=\mathbb{J}_x(s,\varsigma)-i\mathbb{J}_y(s,\varsigma) \\ \{[\sigma_x(s)+i\sigma_y(s)](D_x-iD_y)+[s+\sigma_z(s)](D_z-i\varsigma D_\pi)\}\psi(s,\varsigma)=\mathbb{J}_z(s,\varsigma)-i\varsigma\mathbb{J}_\pi(s,\varsigma) \\ \{[s+\sigma_z(s)](D_x+iD_y)-[\sigma_x(s)+i\sigma_y(s)](D_z+i\varsigma D_\pi)\}\psi(s,\varsigma)=\mathbb{J}_x(s,\varsigma)+i\mathbb{J}_y(s,\varsigma) \\ \{-[\sigma_x(s)-i\sigma_y(s)](D_x+iD_y)+[s-\sigma_z(s)](D_z+i\varsigma D_\pi)\}\psi(s,\varsigma)=\mathbb{J}_z(s,\varsigma)+i\varsigma\mathbb{J}_\pi(s,\varsigma) \end{cases}$$

$$\Leftrightarrow \begin{cases} (\sigma,-i\varsigma)^a D_a \begin{bmatrix}\psi^1(s,\varsigma)\\ \psi^2(s,\varsigma)\end{bmatrix} = \begin{bmatrix}[\mathbb{J}_x{}^2(s,\varsigma)-i\mathbb{J}_y{}^2(s,\varsigma)]\\ \frac{1}{2s}[\mathbb{J}_x{}^1(s,\varsigma)+i\mathbb{J}_y{}^1(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2s}[\mathbb{J}_z{}^1(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^1(s,\varsigma)]\\ -[\mathbb{J}_z{}^2(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\end{bmatrix} \\ (\sigma,-i\varsigma)^a D_a \begin{bmatrix}\psi^2(s,\varsigma)\\ \psi^3(s,\varsigma)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}[\mathbb{J}_x{}^3(s,\varsigma)-i\mathbb{J}_y{}^3(s,\varsigma)]\\ \frac{1}{2s-1}[\mathbb{J}_x{}^2(s,\varsigma)+i\mathbb{J}_y{}^2(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2s-1}[\mathbb{J}_z{}^2(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\\ -\frac{1}{2}[\mathbb{J}_z{}^3(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^3(s,\varsigma)]\end{bmatrix} \\ \quad \cdots\cdots \\ (\sigma,-i\varsigma)^a D_a \begin{bmatrix}\psi^{2s-1}(s,\varsigma)\\ \psi^{2s}(s,\varsigma)\end{bmatrix} = \begin{bmatrix}\frac{1}{2s-1}[\mathbb{J}_x{}^{2s}(s,\varsigma)-i\mathbb{J}_y{}^{2s-1}(s,\varsigma)]\\ \frac{1}{2}[\mathbb{J}_x{}^{2s-1}(s,\varsigma)+i\mathbb{J}_y{}^2(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2}[\mathbb{J}_z{}^{2s-1}(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\\ -\frac{1}{2s-1}[\mathbb{J}_z{}^{2s}(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^{2s-1}(s,\varsigma)]\end{bmatrix} \\ (\sigma,-i\varsigma)^a D_a \begin{bmatrix}\psi^{2s}(s,\varsigma)\\ \psi^{2s+1}(s,\varsigma)\end{bmatrix} = \begin{bmatrix}\frac{1}{2s}[\mathbb{J}_x{}^{2s+1}(s,\varsigma)-i\mathbb{J}_y{}^{2s+1}(s,\varsigma)]\\ [\mathbb{J}_x{}^{2s}(s,\varsigma)+i\mathbb{J}_y{}^{2s}(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}[\mathbb{J}_z{}^{2s}(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^{2s}(s,\varsigma)]\\ -\frac{1}{2s}[\mathbb{J}_z{}^{2s+1}(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^{2s+1}(s,\varsigma)]\end{bmatrix} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix}[\mathbb{J}_x{}^2(s,\varsigma)-i\mathbb{J}_y{}^2(s,\varsigma)]\\ \frac{1}{2s}[\mathbb{J}_x{}^1(s,\varsigma)+i\mathbb{J}_y{}^1(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2s}[\mathbb{J}_z{}^1(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^1(s,\varsigma)]\\ -[\mathbb{J}_z{}^2(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\end{bmatrix} \equiv \begin{bmatrix}\varsigma\tilde{J}^{1_\varsigma}(s,\varsigma)\\ \varsigma\tilde{J}^{2_\varsigma}(s,\varsigma)\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2}[\mathbb{J}_x{}^3(s,\varsigma)-i\mathbb{J}_y{}^3(s,\varsigma)]\\ \frac{1}{2s-1}[\mathbb{J}_x{}^2(s,\varsigma)+i\mathbb{J}_y{}^2(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2s-1}[\mathbb{J}_z{}^2(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\\ -\frac{1}{2}[\mathbb{J}_z{}^3(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^3(s,\varsigma)]\end{bmatrix} \equiv \begin{bmatrix}\varsigma\tilde{J}^{3_\varsigma}(s,\varsigma)\\ \varsigma\tilde{J}^{4_\varsigma}(s,\varsigma)\end{bmatrix} \\ \quad \cdots\cdots \\ \begin{bmatrix}\frac{1}{2s-1}[\mathbb{J}_x{}^{2s}(s,\varsigma)-i\mathbb{J}_y{}^{2s-1}(s,\varsigma)]\\ \frac{1}{2}[\mathbb{J}_x{}^{2s-1}(s,\varsigma)+i\mathbb{J}_y{}^2(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}\frac{1}{2}[\mathbb{J}_z{}^{2s-1}(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^2(s,\varsigma)]\\ -\frac{1}{2s-1}[\mathbb{J}_z{}^{2s}(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^{2s-1}(s,\varsigma)]\end{bmatrix} \equiv \begin{bmatrix}\varsigma\tilde{J}^{(4s-3)_\varsigma}(s,\varsigma)\\ \varsigma\tilde{J}^{(4s-2)_\varsigma}(s,\varsigma)\end{bmatrix} \\ \begin{bmatrix}\frac{1}{2s}[\mathbb{J}_x{}^{2s+1}(s,\varsigma)-i\mathbb{J}_y{}^{2s+1}(s,\varsigma)]\\ [\mathbb{J}_x{}^{2s}(s,\varsigma)+i\mathbb{J}_y{}^{2s}(s,\varsigma)]\end{bmatrix} = \begin{bmatrix}[\mathbb{J}_z{}^{2s}(s,\varsigma)-i\varsigma\mathbb{J}_\pi{}^{2s}(s,\varsigma)]\\ -\frac{1}{2s}[\mathbb{J}_z{}^{2s+1}(s,\varsigma)+i\varsigma\mathbb{J}_\pi{}^{2s+1}(s,\varsigma)]\end{bmatrix} \equiv \begin{bmatrix}\varsigma\tilde{J}^{(4s-1)_\varsigma}(s,\varsigma)\\ \varsigma\tilde{J}^{(4s)_\varsigma}(s,\varsigma)\end{bmatrix} \\ (\sigma\otimes I_{2s},-i\varsigma)^a D_a\tilde{\psi}(s,\varsigma)=\varsigma\tilde{J}(s,\varsigma),\ \tilde{J}(s,\varsigma)\equiv[\tilde{J}^{1_\varsigma}(s,\varsigma),\tilde{J}^{2_\varsigma}(s,\varsigma)\cdots\tilde{J}^{(4s-1)_\varsigma}(s,\varsigma),\tilde{J}^{(4s)_\varsigma}(s,\varsigma)]^T \end{cases}$$

$$\Leftrightarrow (\sigma\otimes I_{2s},-i\varsigma)^a D_a\tilde{\psi}(s,\varsigma)=\varsigma\tilde{J}(s,\varsigma),\mathbb{J}_a(s,\varsigma)=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma) \qquad \square$$

This theorem shows that the source term of the spin equation is limited, and it is not arbitrary. The spin equation has solutions only for the source term of the previous theorem case, and other cases have no solution.

Corollary 2.5.1. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)\Leftrightarrow\begin{cases}[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)\\ \mathbb{J}_a(s,\varsigma)=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)\end{cases}$

Corollary 2.5.2. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)\Leftrightarrow\mathbb{J}_a(s,\varsigma)=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma),\varsigma\tilde{J}(s,\varsigma)=\mathcal{Z}_a^-(s,\varsigma)\mathbb{J}^a(s,\varsigma)$

Corollary 2.5.3. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi=\mathbb{J}_a\Rightarrow\sigma(s)\cdot\mathbb{J}+i\varsigma(s+1)\mathbb{J}_\pi=0$

Proof: $[sD_a+S_{ab}(s,\varsigma)D^b]\psi=\mathbb{J}_a$

$$\Leftrightarrow\begin{cases}[sD_x+i\sigma_z(s)D_y-i\sigma_y(s)D_z-i\varsigma\sigma_x(s)D_\pi]\psi=\mathbb{J}_x\\ [sD_y+i\sigma_x(s)D_z-i\sigma_z(s)D_x-i\varsigma\sigma_y(s)D_\pi]\psi=\mathbb{J}_y\\ [sD_z+i\sigma_y(s)D_x-i\sigma_x(s)D_y-i\varsigma\sigma_z(s)D_\pi]\psi=\mathbb{J}_z\\ [sD_\pi+i\varsigma\sigma_x(s)D_x+i\varsigma\sigma_y(s)D_y+i\varsigma\sigma_z(s)D_z]\psi=\mathbb{J}_\pi\end{cases}$$

$$\Rightarrow\begin{cases}[(s+1)\sigma(s)\cdot\nabla_d-i\varsigma\sigma^2(s)D_\pi]\psi=\sigma\cdot\mathbb{J}\\ [sD_\pi+i\varsigma\sigma\cdot\nabla_d]\psi=\mathbb{J}_\pi\end{cases}$$

$$\Leftrightarrow\begin{cases}[(s+1)\sigma(s)\cdot\nabla_d-i\varsigma s(s+1)D_\pi]\psi=\sigma(s)\cdot\mathbb{J}\\ [(s+1)\sigma(s)\cdot\nabla_d-i\varsigma s(s+1)D_\pi]\psi=-i\varsigma(s+1)\mathbb{J}_\pi\end{cases}$$

$\Rightarrow \sigma(s)\cdot\mathbb{J}+i\varsigma(s+1)\mathbb{J}_\pi=0$ □

Corollary 2.5.4. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)\Rightarrow[\sigma(s),i\varsigma(s+1)]^a\bar{\mathbb{N}}(s)(\sigma\otimes I_{2s},i\varsigma)_a\tilde{J}(s,\varsigma)=0$

Corollary 2.5.5. $[\sigma(s),i\varsigma(s+1)]^a\mathbb{Z}_a(s,\varsigma)=0$

Proof: $[\sigma(s),i\varsigma(s+1)]^a\mathbb{Z}_a(s,\varsigma)$

$=[\sigma(s),i\varsigma(s+1)]^a\bar{\mathbb{N}}(s)(\sigma\otimes I_{2s},i\varsigma)_a$

$=\bar{\mathbb{N}}(s)[s\sigma\otimes I_{2s},i\varsigma(s+1)]^a\mathbb{N}(s)\bar{\mathbb{N}}(s)(\sigma\otimes I_{2s},i\varsigma)_a$

$=[\bar{\mathbb{N}}(s)(\sigma\otimes I_{2s},-i\varsigma)^a\mathbb{N}(s)s\bar{\mathbb{N}}(s)(\sigma\otimes I_{2s},i\varsigma)_a\varsigma\tilde{J}(s,\varsigma)-(2s+1)\bar{\mathbb{N}}(s)]$

$=[\bar{\mathbb{N}}(s)(2s+1)I_{4s}-(2s+1)\bar{\mathbb{N}}(s)]$

$=0$ □

Theorem 2.5.3. $[sD_a+S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)$

$\Leftrightarrow[(s-l)D_a+S_{ab}(s-l,\varsigma)D^b]\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma)=\mathbb{Z}_a(s-l,\varsigma)\varsigma\tilde{J}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma),l=0,\frac{1}{2},1,\cdots,s$

Theorem 2.5.4. $[sD_a+S_{ab}(s,\varsigma)_mD^b]\Psi(s,\varsigma)=\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{\mathcal{J}}(s,\varsigma)$

$\Leftrightarrow[(s-l)D_a+S_{ab}(s-l,\varsigma)D^b]\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma)=\mathbb{Z}_a(s-l,\varsigma)\varsigma\tilde{J}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma),l=0,\frac{1}{2},1,\cdots,s$

3 Switch spin equation

3.1 Switch neutrino spin equation without sources

Theorem 3.1.1. $[(\frac{1}{2}+\phi)D_a+S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma)=0$

$$\Leftrightarrow\begin{cases}(\sigma,-i\varsigma)^aD_a\psi(\frac{1}{2},\varsigma)=0,\phi=0\\ \sigma_xD_x\psi(\frac{1}{2},\varsigma)=\sigma_yD_y\psi(\frac{1}{2},\varsigma)=\sigma_zD_z\psi(\frac{1}{2},\varsigma)=-i\varsigma D_\pi\psi(\frac{1}{2},\varsigma),\phi=-2\\ \psi(\frac{1}{2},\varsigma)=constant,\phi\neq0,-2\end{cases}$$

Proof: $[(\frac{1}{2}+\phi)D_a+S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma)=0$

$\Leftrightarrow[\frac{1}{2}D_a+S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma)=-\phi D_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow\sigma_a[\frac{1}{2}D_a+S_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma)=-(\sigma,-i\varsigma)_a\phi D_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow(\sigma,-i\varsigma)^bD_b\psi(\frac{1}{2},\varsigma)=-2\phi(\sigma,-i\varsigma)_aD_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow(\sigma,-i\varsigma)^aD_a\psi(\frac{1}{2},\varsigma)=-2\phi\sigma_xD_x\psi(\frac{1}{2},\varsigma)=-2\phi\sigma_yD_y\psi(\frac{1}{2},\varsigma)=-2\phi\sigma_zD_z\psi(\frac{1}{2},\varsigma)=-2\phi(-i\varsigma)D_\pi\psi(\frac{1}{2},\varsigma)$

$$\Leftrightarrow\begin{cases}(\sigma,-i\varsigma)^aD_a\psi(\frac{1}{2},\varsigma)=0,\phi=0\\ \sigma_xD_x\psi(\frac{1}{2},\varsigma)=\sigma_yD_y\psi(\frac{1}{2},\varsigma)=\sigma_zD_z\psi(\frac{1}{2},\varsigma)=-i\varsigma D_\pi\psi(\frac{1}{2},\varsigma),\phi=-2\\ D_a\psi(\frac{1}{2},\varsigma)=0,\phi\neq0,-2\end{cases}$$ □

Corollary 3.1.1. $[(\frac{1}{2}+\phi)\partial_a+S_{ab}(\varsigma)\partial^b]\psi(\frac{1}{2},\varsigma)=0$

$$\Leftrightarrow\begin{cases}(\sigma,-i\varsigma)^a\partial_a\psi(\frac{1}{2},\varsigma)=0,\phi=0\\ \sigma_x\partial_x\psi(\frac{1}{2},\varsigma)=\sigma_y\partial_y\psi(\frac{1}{2},\varsigma)=\sigma_z\partial_z\psi(\frac{1}{2},\varsigma)=-i\varsigma\partial_\pi\psi(\frac{1}{2},\varsigma),\phi=-2\\ \psi(\frac{1}{2},\varsigma)=constant,\phi\neq0,-2\end{cases}$$

Corollary 3.1.2. $\sigma_x\partial_x\psi(\frac{1}{2},\varsigma)=\sigma_y\partial_y\psi(\frac{1}{2},\varsigma)=\sigma_z\partial_z\psi(\frac{1}{2},\varsigma)=-i\varsigma\partial_\pi\psi(\frac{1}{2},\varsigma)$

$\Rightarrow\psi(\frac{1}{2},\varsigma)=(x\sigma_x+y\sigma_y+z\sigma_z+i\varsigma\pi)\pi_0\Leftrightarrow\psi^{A_\varsigma}(\frac{1}{2},\varsigma)=x^a(\sigma,i\varsigma)_a{}^{A_\varsigma A'_\varsigma}\pi_{A'_\varsigma}$

Above conclusion is just the Penrose twistor [5,6] projection relationship.

3.2 Switch electromagnetic field spin equation without sources

Theorem 3.2.1. $[(1+\phi)D_a+S_{ab}D^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma)=0,S_{ab}=-\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow\begin{cases}(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}(1,\varsigma)=0,\phi=0\\ \begin{cases}-D_y\Psi_{z_\varsigma}=D_z\Psi_{y_\varsigma}=\varsigma D_\pi\Psi_{x_\varsigma},-D_z\Psi_{x_\varsigma}=D_x\Psi_{z_\varsigma}=\varsigma D_\pi\Psi_{y_\varsigma}\\ -D_x\Psi_{y_\varsigma}=D_y\Psi_{x_\varsigma}=\varsigma D_\pi\Psi_{z_\varsigma},D_x\Psi_{x_\varsigma}=D_y\Psi_{y_\varsigma}=D_z\Psi_{z_\varsigma}\end{cases},\phi=3\\ D_a\Psi_{b_\varsigma}=0,\phi\neq0,3\end{cases}$$

Proof: $[(1+\phi)D_a + S_{ab}D^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = 0, S_{ab} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow (D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\phi D_a\Psi^{\beta_\varsigma}(1,\varsigma)$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = \varsigma\tilde{\mathcal{J}}(1,\varsigma), -\phi D_a\Psi^{\beta_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = 0, \phi = 0 \\ \begin{cases} -D_y\Psi_{z_\varsigma} = D_z\Psi_{y_\varsigma} = \varsigma D_\pi\Psi_{x_\varsigma}, -D_z\Psi_{x_\varsigma} = D_x\Psi_{z_\varsigma} = \varsigma D_\pi\Psi_{y_\varsigma} \\ -D_x\Psi_{y_\varsigma} = D_y\Psi_{x_\varsigma} = \varsigma D_\pi\Psi_{z_\varsigma}, D_x\Psi_{x_\varsigma} = D_y\Psi_{y_\varsigma} = D_z\Psi_{z_\varsigma} \end{cases}, \phi = 3 \\ D_a\Psi_{b_\varsigma} = 0, \phi \neq 0, 3 \end{cases}$$ □

Corollary 3.2.1. $[(1+\phi)\partial_a + S_{ab}\partial^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = 0, S_{ab} = -\varsigma\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a\tilde{\Psi}(1,\varsigma) = 0, \phi = 0 \\ \begin{cases} -\partial_y\Psi_{z_\varsigma} = \partial_z\Psi_{y_\varsigma} = \varsigma\partial_\pi\Psi_{x_\varsigma}, -\partial_z\Psi_{x_\varsigma} = \partial_x\Psi_{z_\varsigma} = \varsigma\partial_\pi\Psi_{y_\varsigma} \\ -\partial_x\Psi_{y_\varsigma} = \partial_y\Psi_{x_\varsigma} = \varsigma\partial_\pi\Psi_{z_\varsigma}, \partial_x\Psi_{x_\varsigma} = \partial_y\Psi_{y_\varsigma} = \partial_z\Psi_{z_\varsigma} \end{cases}, \phi = 3 \\ \Psi_{\alpha_\varsigma} = constant, \phi \neq 0, 3 \end{cases}$$

Corollary 3.2.2. $$\begin{cases} -\partial_y\Psi_{z_\varsigma} = \partial_z\Psi_{y_\varsigma} = \varsigma\partial_\pi\Psi_{x_\varsigma}, -\partial_z\Psi_{x_\varsigma} = \partial_x\Psi_{z_\varsigma} = \varsigma\partial_\pi\Psi_{y_\varsigma} \\ -\partial_x\Psi_{y_\varsigma} = \partial_y\Psi_{x_\varsigma} = \varsigma\partial_\pi\Psi_{z_\varsigma}, \partial_x\Psi_{x_\varsigma} = \partial_y\Psi_{y_\varsigma} = \partial_z\Psi_{z_\varsigma} \end{cases} \Rightarrow \Psi^{\alpha_\varsigma}(1,\varsigma) = x^a\sigma_\varsigma^{\alpha_\varsigma}{}_{ab}C^b$$

3.3 Vector field spin equation and its switch spin equation without sources in arbitrary N+1 dimensional spacetime

Vector field spin equation in arbitrary N+1 dimensional spacetime

Theorem 3.3.1. $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac} \Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$

Proof: $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac}$

$\Leftrightarrow [D_a\delta_{cd} + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})D^b]A^d = X_{ac}$

$\Leftrightarrow D_aA_c + \delta_{ac}D_bA^b - D_cA_a = X_{ac}$

$\Leftrightarrow D_aA_b - D_bA_a + \delta_{ab}D_cA^c = X_{ab}$

$\Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$ □

Corollary 3.3.1. $(D_a\delta_{cd} + S_{abcd}D^b)A^d = 0 \Leftrightarrow D_aA_b - D_bA_a = 0, D_aA^a = 0$

The origin of the scalar field:

Corollary 3.3.2. $(\partial_a\delta_{cd} + S_{abcd}\partial^b)A^d = 0 \Leftrightarrow \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0 \Leftrightarrow \partial^a\partial_a\phi = 0, A_a = \partial_a\phi$

Vector field spin equation without source in arbitrary N+1 dimensional spacetime

Corollary 3.3.3. $$[(1+\phi)D_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b\neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$$

Proof: $[(1+\phi)D_a\delta_{cd} + S_{abcd}D^b]A^d = 0$

$\Leftrightarrow (D_a\delta_{cd} + S_{abcd}D^b)A^d = -\phi D_aA_c$

$\Leftrightarrow -\phi D_aA_b = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$

$\Leftrightarrow -\phi D_aA_a = D_cA^c, -\phi(D_aA_{b\neq a} + D_bA_{a\neq b}) = 0, (2+\phi)(D_aA_b - D_bA_a) = 0$

$$\Leftrightarrow \begin{cases} -\phi D_aA_a = D_cA^c, (4+\phi)D_aA^a = 0 \\ -\phi(D_aA_{b\neq a} + D_bA_{a\neq b}) = 0, (2+\phi)(D_aA_b - D_bA_a) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b\neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$$ □

Corollary 3.3.4. $[(1+\phi)\partial_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0, \phi = 0 \\ \partial_aA_b + \partial_bA_a = 0, \phi = -2 \\ \partial_aA_{b\neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi, \phi = -4 \\ A_a = constant, \phi \neq 0, -2, -4 \end{cases}$

Corollary 3.3.5. $\partial_aA_{b\neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi \Rightarrow A_a = kx_a$

3.4 Switch electron spin equation without sources in arbitrary N+1 dimensional spacetime

Switch electron spin equation in arbitrary N+1 dimensional spacetime:

Theorem 3.4.1. $[(\frac{1}{2}+\phi)(D_a + m\gamma_a) + S_{ab}D^b]\psi = 0, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \Leftrightarrow (\gamma^aD_a + m)\psi = -2\phi\gamma_bD_b\psi$

Proof: $[(\frac{1}{2}+\phi)(D_a + m\gamma_a) + S_{ab}D^b]\psi = 0, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$
$\Leftrightarrow [\frac{1}{2}(D_a + m\gamma_a) + S_{ab}D^b]\psi = -\phi D_a\psi, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$
$\Leftrightarrow [(2S_{ab} + \delta_{ab})D_b + \gamma_a m]\psi = -2\phi D_a\psi, S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$
$\Leftrightarrow [\frac{1}{2}([\gamma_a, \gamma_b] + \{\gamma_a, \gamma_b\})D_b + \gamma_a m]\psi = -2\phi D_a\psi$
$\Leftrightarrow \gamma_a(\gamma_bD^b + m)\psi = -2\phi D_a\psi$
$\Leftrightarrow (\gamma_bD^b + m)\psi = -2\phi\gamma_aD_a\psi$
$\Leftrightarrow (\gamma^aD_a + m)\psi = -2\phi\gamma_bD_b\psi$ □

Corollary 3.4.1. $(\gamma^aD_a + m)\psi = -2\phi\gamma_bD_b\psi, \phi \neq 0$

$$\Leftrightarrow \begin{cases} \psi = 0, \phi = -\frac{n}{2}, m \neq 0 \\ \gamma_1D_{x_1}\psi = \gamma_2D_{x_2}\psi = \cdots = \gamma_nD_{x_n}\psi = -(n+2\phi)^{-1}m\psi, \phi \neq -\frac{n}{2}, m \neq 0 \\ \gamma_1D_{x_1}\psi = \gamma_2D_{x_2}\psi = \cdots = \gamma_nD_{x_n}\psi, \phi = -\frac{n}{2}, m = 0 \\ \gamma_1D_{x_1}\psi = \gamma_2D_{x_2}\psi = \cdots = \gamma_nD_{x_n}\psi = 0, \phi \neq -\frac{n}{2}, m = 0 \end{cases}$$

Corollary 3.4.2. $(\gamma^aD_a + m)\psi = -2\phi\gamma_bD_b\psi, \phi \neq 0 \Rightarrow \begin{cases} \psi = 0, \phi = -\frac{n}{2}, m \neq 0 \\ \psi = 0, \phi \neq -\frac{n}{2}, m \neq 0 \\ \psi = x^a\gamma_a\lambda, \phi = -\frac{n}{2}, m = 0 \\ \psi = constant, \phi \neq -\frac{n}{2}, m = 0 \end{cases}$

3.5 Spin-s particle switch spin equation without sources

Corollary 3.5.1. $[(s+\phi)D_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = 0$
$\Leftrightarrow \exists\tilde{J}(s,\varsigma), (\sigma\otimes I_{2s}, -i\varsigma)^aD_a\tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma), -\phi D_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)$

Proposition 3.5.1. $(\sigma\otimes I_{2s}, -i\varsigma)^aD_a\tilde{\psi}(s,\varsigma) = \varsigma\tilde{J}(s,\varsigma), -\phi D_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma), \phi \neq 0$
$\Leftrightarrow (\phi + 2s + 1)\tilde{J}(s,\varsigma) = 0, -\phi D_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma), \phi \neq 0$

Corollary 3.5.2. $[(s+\phi)D_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = 0 \Leftrightarrow \begin{cases} (\sigma\otimes I_{2s}, -i\varsigma)^aD_a\tilde{\psi}(s,\varsigma) = 0, \phi = 0 \\ D_a\psi(s,\varsigma) = \frac{1}{2s+1}\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma), \phi = -(2s+1) \\ D_a\psi(s,\varsigma) = 0, \phi \neq 0, -(2s+1) \end{cases}$

Corollary 3.5.3. $[-(s+1)D_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = 0 \Leftrightarrow D_a\psi(s,\varsigma) = \frac{1}{2s+1}\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma), \forall\tilde{J}(s,\varsigma)$

Corollary 3.5.4. $[-(s+1)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \Rightarrow$ *It has a solution:* $\psi(s,\varsigma) = \frac{1}{2s+1}x^a\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)$

Corollary 3.5.5. $[(s+\phi)\partial_a + S_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0$

$$\Rightarrow \begin{cases} \textit{1. When } \phi = 0, (\sigma \otimes I_{2s}, -i\varsigma)^a \partial_a \tilde{\psi}(s,\varsigma) = 0 \\ \textit{There are plane wave solutions, it has solutions of characterizing particles.} \\ \textit{2. When } \phi = -(2s+1), \psi(s,\varsigma) = \frac{1}{2s+1} x^a \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma) \\ \textit{No plane wave solutions, it degenerates to solutions of characterizing spacetime.} \\ \textit{3. When } \phi \neq 0, -(2s+1), \psi(s,\varsigma) = \textit{constant} \\ \textit{Only constant solutions, it degenerates to solutions of characterizing the void.} \end{cases}$$

Corollary 3.5.6. $-\phi\partial_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma) \Leftrightarrow$

$$\begin{cases} \partial_x\psi^1(s,\varsigma) - i\partial_y\psi^1(s,\varsigma) = 0, \partial_z\psi^1(s,\varsigma) + i\varsigma\partial_\pi\psi^1(s,\varsigma) = 0 \\ \begin{bmatrix} [\partial_x\psi^2(s,\varsigma) - i\partial_y\psi^2(s,\varsigma)] \\ \frac{1}{2s}[\partial_x\psi^1(s,\varsigma) + i\partial_y\psi^1(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2s}[\partial_z\psi^1(s,\varsigma) - i\varsigma\partial_\pi\psi^1(s,\varsigma)] \\ -[\partial_z\psi^2(s,\varsigma) + i\varsigma\partial_\pi\psi^2(s,\varsigma)] \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2}[\partial_x\psi^3(s,\varsigma) - i\partial_y\psi^3(s,\varsigma)] \\ \frac{1}{2s-1}[\partial_x\psi^2(s,\varsigma) + i\partial_y\psi^2(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2s-1}[\partial_z\psi^2(s,\varsigma) - i\varsigma\partial_\pi\psi^2(s,\varsigma)] \\ -\frac{1}{2}[\partial_z\psi^3(s,\varsigma) + i\varsigma\partial_\pi\psi^3(s,\varsigma)] \end{bmatrix} \\ \cdots\cdots \\ \begin{bmatrix} \frac{1}{2s-1}[\partial_x\psi^{2s}(s,\varsigma) - i\partial_y\psi^{2s-1}(s,\varsigma)] \\ \frac{1}{2}[\partial_x\psi^{2s-1}(s,\varsigma) + i\partial_y\psi^2(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[\partial_z\psi^{2s-1}(s,\varsigma) - i\varsigma\partial_\pi\psi^2(s,\varsigma)] \\ -\frac{1}{2s-1}[\partial_z\psi^{2s}(s,\varsigma) + i\varsigma\partial_\pi\psi^{2s-1}(s,\varsigma)] \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2s}[\partial_x\psi^{2s+1}(s,\varsigma) - i\partial_y\psi^{2s+1}(s,\varsigma)] \\ [\partial_x\psi^{2s}(s,\varsigma) + i\partial_y\psi^{2s}(s,\varsigma)] \end{bmatrix} = \begin{bmatrix} [\partial_z\psi^{2s}(s,\varsigma) - i\varsigma\partial_\pi\psi^{2s}(s,\varsigma)] \\ -\frac{1}{2s}[\partial_z\psi^{2s+1}(s,\varsigma) + i\varsigma\partial_\pi\psi^{2s+1}(s,\varsigma)] \end{bmatrix} \\ \partial_x\psi^{2s+1}(s,\varsigma) + i\partial_y\psi^{2s+1}(s,\varsigma) = 0, \partial_z\psi^{2s+1}(s,\varsigma) - i\varsigma\partial_\pi\psi^{2s+1}(s,\varsigma) = 0 \end{cases}$$

3.6 Spin-s particle switch spin equation with sources

Corollary 3.6.1. $[(s+\phi)D_a + S_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)$

$\Leftrightarrow \exists\tilde{J}_\phi(s,\varsigma), (\sigma \otimes I_{2s}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma) = \varsigma[\tilde{J}(s,\varsigma) + \tilde{J}_\phi(s,\varsigma)], -\phi D_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}_\phi(s,\varsigma)$

Proposition 3.6.1. $(\sigma \otimes I_{2s}, -i\varsigma)^a\partial_a\tilde{\psi}(s,\varsigma) = \varsigma[\tilde{J}(s,\varsigma) + \tilde{J}_\phi(s,\varsigma)], -\phi\partial_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}_\phi(s,\varsigma), \phi \neq 0$

$\Leftrightarrow (\phi + 2s + 1)\tilde{J}_\phi(s,\varsigma) = -\phi\tilde{J}(s,\varsigma), -\phi\partial_a\psi(s,\varsigma) = \mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}_\phi(s,\varsigma), \phi \neq 0$

$\Leftrightarrow (\phi + 2s + 1)\tilde{J}_\phi(s,\varsigma) = -\phi\tilde{J}(s,\varsigma), \psi(s,\varsigma) = -\int \phi^{-1}\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}_\phi(s,\varsigma)dx^a, \phi \neq 0$

$\Rightarrow \psi(s,\varsigma) = \int (\phi + 2s + 1)^{-1}\mathbb{Z}_a(s,\varsigma)\varsigma\tilde{J}(s,\varsigma)dx^a, \phi \neq 0, \phi \neq -(2s+1)$

4 Synchronous representation transformation

4.1 Electromagnetic field synchronous representation transformation

Corollary 4.1.1. $[D_a + S_{ab}(1,\varsigma)D^b]\psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)\varsigma\tilde{J}(1,\varsigma) \Leftrightarrow [D_a + S_{ab}(1,\varsigma)_m D^b]\Psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)_m\varsigma\tilde{\mathcal{J}}(1,\varsigma)$

Proof: $[D_a + S_{ab}(1,\varsigma)D^b]\psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)\varsigma\tilde{J}(1,\varsigma)$

$\Leftrightarrow [D_a + S_m(1)S_{ab}(1,\varsigma)S_m^+(1)D^b]S_m(1)\psi(1,\varsigma) = S_m(1)\mathbb{Z}_a(1,\varsigma)S_{em}^+(\varsigma)S_{em}(\varsigma)\varsigma\tilde{J}(1,\varsigma)$

$\Leftrightarrow [D_a + S_{ab}(1,\varsigma)_m D^b]\Psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)_m\varsigma\tilde{\mathcal{J}}(1,\varsigma)$ □

Corollary 4.1.2. $(\sigma \otimes I, -i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = \varsigma\tilde{J}(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = \varsigma\tilde{\mathcal{J}}(1,\varsigma)$

Proof: $(\sigma \otimes I, -i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = \varsigma\tilde{J}(1,\varsigma)$

$\Leftrightarrow (S_{em}(\varsigma)\sigma \otimes IS_{em}^+(\varsigma), -i\varsigma)^a D_a S_{em}(\varsigma)\tilde{\psi}(1,\varsigma) = \varsigma S_{em}(\varsigma)\tilde{J}(1,\varsigma)$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = \varsigma\tilde{\mathcal{J}}(1,\varsigma)$ □

Corollary 4.1.3. $[D_a + S_{ab}(1,\varsigma)_m D^b]\Psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)_m\varsigma\tilde{\mathcal{J}}(1,\varsigma) \Leftrightarrow (D_a + S_{ab}(1,\varsigma)_m D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\varsigma\sigma_\varsigma^{\beta_\varsigma}{}_{ab}J^b$

A summary of electromagnetic field synchronous representation transformation:

Corollary 4.1.4. $[D_a + S_{ab}(1,\varsigma)D^b]\psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)\varsigma\tilde{J}(1,\varsigma) \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = \varsigma\tilde{J}(1,\varsigma)$
$\Leftrightarrow [D_a + S_{ab}(1,\varsigma)_m D^b]\Psi(1,\varsigma) = \mathbb{Z}_a(1,\varsigma)_m\varsigma\tilde{\mathcal{J}}(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}(1,\varsigma) = \varsigma\tilde{\mathcal{J}}(1,\varsigma)$

Corollary 4.1.5. $$\begin{cases}(\sigma \otimes I, -i\varsigma)^a D_a\hat{\psi}(1,\varsigma) = \varsigma\hat{J}(1,\varsigma) \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\hat{\Psi}(1,\varsigma) = \varsigma\hat{\mathcal{J}}(1,\varsigma) \\ \hat{\Psi}(1,\varsigma) = S_{em}(\varsigma)\hat{\psi}(1,\varsigma), \hat{\mathcal{J}}(1,\varsigma) = S_{em}(\varsigma)\hat{J}(1,\varsigma) \\ \hat{\psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \Leftrightarrow \hat{\Psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot R} \\ \hat{J}(1,\varsigma) \sim e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \Leftrightarrow \hat{\mathcal{J}}(1,\varsigma) \sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\end{cases}$$

4.2 Gravitational field synchronous representation transformation

Definiton 4.2.1. $\bar{J}^{bcd} \equiv R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{;d]}, \bar{J}^{b\beta_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\beta_\varsigma}_{\varsigma cd}\bar{J}^{bcd}, J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} \equiv \frac{1}{2}(\sigma, -i\varsigma)^b{}_{A'_\varsigma A_\varsigma}\bar{\varepsilon}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma D_\varsigma}\bar{J}_b{}^{\beta_\varsigma}$

Corollary 4.2.1. $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\alpha_\varsigma}C^{\alpha_\varsigma\gamma_\varsigma}(1,\varsigma) \equiv -\varsigma\sigma^{\beta_\varsigma}_\varsigma{}_{ab}\bar{J}^{b\gamma_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\hat{C}^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_b{}^{\beta_\varsigma}$

Corollary 4.2.2. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\hat{C}^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_b{}^{\beta_\varsigma} \Leftrightarrow (\sigma_{-\varsigma} \otimes I_4, -i\varsigma)^a D_a\hat{\Psi}(2,\varsigma) \equiv \varsigma\hat{\mathcal{J}}(2,\varsigma)$

The components are written to a matrix, then it can be proved.

Corollary 4.2.3. $(\sigma_{-\varsigma} \otimes I_4, -i\varsigma)^a D_a\hat{\Psi}(2,\varsigma) \equiv \varsigma\hat{\mathcal{J}}(2,\varsigma) \Leftrightarrow (\sigma \otimes I_8, -i\varsigma)^a D_a\hat{\psi}(2,\varsigma) \equiv \varsigma\hat{J}(2,\varsigma)$

Proof: $(\sigma_{-\varsigma} \otimes I_4, -i\varsigma)^a D_a\hat{\Psi}(2,\varsigma) \equiv \varsigma\hat{\mathcal{J}}(2,\varsigma)$
$\Leftrightarrow S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma)(\sigma_{-\varsigma} \otimes I_4, -i\varsigma)^a S_{em}(\varsigma) \otimes S_{em}(\varsigma)D_a S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma)\hat{\Psi}(2,\varsigma) \equiv \varsigma S^+_{em}(\varsigma) \otimes S^+_{em}(\varsigma)\hat{\mathcal{J}}(2,\varsigma)$
$\Leftrightarrow (\sigma \otimes I_8, -i\varsigma)^a D_a\hat{\psi}(2,\varsigma) \equiv \varsigma\hat{J}(2,\varsigma)$ □

Corollary 4.2.4. $(\sigma \otimes I_8, -i\varsigma)^a D_a\hat{\psi}(2,\varsigma) \equiv \varsigma\hat{J}(2,\varsigma) \Leftrightarrow (\sigma \otimes I_4, -i\varsigma)^a D_a\tilde{\psi}(2,\varsigma) \equiv \varsigma\tilde{J}(2,\varsigma)$

To expand and arrange, it can be proved.

Corollary 4.2.5. $(\sigma \otimes I_4, -i\varsigma)^a D_a\tilde{\psi}(2,\varsigma) \equiv \varsigma\tilde{J}(2,\varsigma) \Leftrightarrow [2D_a + S_{ab}(2,\varsigma)D^b]\psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)\varsigma\tilde{J}(2,\varsigma)$

Corollary 4.2.6. $[D_a + S_{ab}(2,\varsigma)D^b]\psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)\varsigma\tilde{J}(2,\varsigma) \Leftrightarrow [D_a + S_{ab}(2,\varsigma)_m D^b]\Psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)_m\varsigma\tilde{\mathcal{J}}(2,\varsigma)$

Proof: $[2D_a + S_{ab}(2,\varsigma)D^b]\psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)\varsigma\tilde{J}(2,\varsigma)$
$\Leftrightarrow [2D_a + S_m(2)S_{ab}(2,\varsigma)S^+_m(2)D^b]S_m(2)\psi(2,\varsigma) \equiv S_m(2)\mathbb{Z}_a(2,\varsigma)S^+_{em}(\varsigma) \otimes S^+_{em}(\frac{1}{2})S_{em}(\varsigma) \otimes S_{em}(\frac{1}{2})\varsigma\tilde{J}(2,\varsigma)$
$\Leftrightarrow [2D_a + S_{ab}(2,\varsigma)_m D^b]\Psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)_m\varsigma\tilde{\mathcal{J}}(2,\varsigma)$ □

Corollary 4.2.7. $(\sigma \otimes I_4, -i\varsigma)^a D_a\tilde{\psi}(2,\varsigma) \equiv \varsigma\tilde{J}(2,\varsigma) \Leftrightarrow (\sigma_{-\varsigma} \otimes I, -i\varsigma)^a D_a\tilde{\Psi}(2,\varsigma) \equiv \varsigma\tilde{\mathcal{J}}(2,\varsigma)$

Proof: $(\sigma \otimes I_4, -i\varsigma)^a D_a\tilde{\psi}(2,\varsigma) \equiv \varsigma\tilde{J}(2,\varsigma)$
$\Leftrightarrow S_{em}(\varsigma) \otimes S_{em}(\frac{1}{2})(\sigma \otimes I_4, -i\varsigma)^a S^+_{em}(\varsigma) \otimes S^+_{em}(\frac{1}{2})D_a S_{em}(\varsigma) \otimes S_{em}(\frac{1}{2})\tilde{\psi}(2,\varsigma)$
$\equiv \varsigma S_{em}(\varsigma) \otimes S_{em}(\frac{1}{2})\tilde{J}(2,\varsigma)$
$\Leftrightarrow (\sigma_{-\varsigma} \otimes I, -i\varsigma)^a D_a\tilde{\Psi}(2,\varsigma) \equiv \varsigma\tilde{\mathcal{J}}(2,\varsigma)$ □

Corollary 4.2.8. $[D_a + S_{ab}(2,\varsigma)_m D^b]\Psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)_m\varsigma\tilde{\mathcal{J}}(2,\varsigma) \Leftrightarrow (\sigma_{-\varsigma} \otimes I, -i\varsigma)^a D_a\tilde{\Psi}(2,\varsigma) \equiv \varsigma\tilde{\mathcal{J}}(2,\varsigma)$

Corollary 4.2.9. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\hat{C}^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_b{}^{\beta_\varsigma} \Leftrightarrow (\gamma, -i\varsigma)^a{}_{l\alpha_\varsigma}D_a C^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_l{}^{\beta_\varsigma}$

To expand and arrange, it can be proved. A summary of various equivalent forms of gravitational field identity:

Corollary 4.2.10. $(D_a + S_{ab}D^b)^{\beta_\varsigma}{}_{\alpha_\varsigma}C^{\alpha_\varsigma\gamma_\varsigma}(1,\varsigma) \equiv -\varsigma\sigma^{\beta_\varsigma}_\varsigma{}_{ab}\bar{J}^{b\gamma_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\hat{C}^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_b{}^{\beta_\varsigma} \Leftrightarrow (\gamma, -i\varsigma)^a{}_{l\alpha_\varsigma}D_a C^{\alpha_\varsigma\beta_\varsigma} \equiv \varsigma\bar{J}_l{}^{\beta_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma} \otimes I_4, -i\varsigma)^a D_a\hat{\Psi}(2,\varsigma) \equiv \varsigma\hat{\mathcal{J}}(2,\varsigma) \Leftrightarrow (\sigma \otimes I_8, -i\varsigma)^a D_a\hat{\psi}(2,\varsigma) \equiv \varsigma\hat{J}(2,\varsigma)$
$\Leftrightarrow [D_a + S_{ab}(2,\varsigma)D^b]\psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)\varsigma\tilde{J}(2,\varsigma) \Leftrightarrow (\sigma \otimes I_4, -i\varsigma)^a D_a\tilde{\psi}(2,\varsigma) \equiv \varsigma\tilde{J}(2,\varsigma)$
$\Leftrightarrow [D_a + S_{ab}(2,\varsigma)_m D^b]\Psi(2,\varsigma) \equiv \mathbb{Z}_a(2,\varsigma)_m\varsigma\tilde{\mathcal{J}}(2,\varsigma) \Leftrightarrow (\sigma_{-\varsigma} \otimes I, -i\varsigma)^a D_a\tilde{\Psi}(2,\varsigma) \equiv \varsigma\tilde{\mathcal{J}}(2,\varsigma)$

Corollary 4.2.11.
$$\begin{cases}(\sigma\otimes I_8,-i\varsigma)^a D_a\hat{\psi}(1,\varsigma)=\varsigma\hat{J}(2,\varsigma)\Leftrightarrow(\sigma_{-\varsigma}\otimes I_4,-i\varsigma)^a D_a\hat{\Psi}(2,\varsigma)=\varsigma\hat{\mathcal{J}}(2,\varsigma)\\ \hat{\Psi}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\varsigma)\hat{\psi}(2,\varsigma),\hat{\mathcal{J}}(2,\varsigma)=S_{em}(\varsigma)\otimes S_{em}(\varsigma)\hat{J}(2,\varsigma)\\ \hat{\psi}(2,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\\ \Leftrightarrow\hat{\Psi}(2,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot R}\otimes e^{(i\omega+\varsigma\epsilon)\cdot R}\\ \hat{J}(2,\varsigma)\sim e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\\ \Leftrightarrow\hat{\mathcal{J}}(2,\varsigma)\sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\otimes e^{(i\omega+\varsigma\epsilon)\cdot R}\end{cases}$$

Chapter 5

Analysis of Bargmann-Wigner equation

1 Bargmann-Wigner equation

1.1 Bargmann-Wigner equation [20, 24]

$$[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2s}} = J^{\overbrace{\kappa_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2s}}, \psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2s}}, J^{\overbrace{\kappa_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2s}}{}_{\kappa_\varsigma} \tag{5.1}$$

2 Complete expansion of second-order matrices

2.1 Complete Pauli basis expansion of second-order matrices

Complete Pauli basis of second-order matrices: $\Gamma_a(\varsigma) = \{\sigma, i\varsigma\}$

Proposition 2.1.1. $x^a\Gamma_a(\varsigma) = 0 \Rightarrow x^a = 0$

Proof: $x^a\Gamma_a(\varsigma) = 0$
$\Rightarrow x^a(\sigma, i\varsigma)_a = 0$
$\Rightarrow \{x^a(\sigma, i\varsigma)_a, (\sigma, -i\varsigma)_b\} = 0$
$\Rightarrow x^a(2\delta_{ab}) = 0$
$\Rightarrow x^a = 0$ □

Corollary 2.1.1. $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$

Proposition 2.1.2. $X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in$ *second-order matrices.*

Proof: $X = X^{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + X^{12}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + X^{21}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + X^{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \forall X \in$ second-order matrices.
$\Leftrightarrow X = \frac{1}{2}[X^{11}(I+\sigma_z) + X^{12}(\sigma_x + i\sigma_y) + X^{21}(\sigma_x - i\sigma_y) + X^{22}(I - \sigma_z)], \forall X \in$ second-order matrices.
$\Leftrightarrow X = \frac{1}{2}(X^{12} + X^{21})\sigma_x + \frac{i}{2}(X^{12} - X^{21})\sigma_y + \frac{1}{2}(X^{11} - X^{22})\sigma_z - i\varsigma\frac{1}{2}(X^{11} + X^{22})i\varsigma I, \forall X \in$ second-order matrices.
$\Leftrightarrow X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in$ second-order matrices. □

Corollary 2.1.2. $X = x^a\Gamma_a(\varsigma), x^a = tr[\Gamma^a(-\varsigma)X], \forall X \in$ *second-order matrices.*

Properties of second-order matrices complete basis:

Orthogonality: $\Gamma_a(-\varsigma)\Gamma_a(\varsigma) = I, tr[\Gamma_a(-\varsigma)\Gamma_b(\varsigma)] = 2\delta_{ab}$ (5.2)

Linear independence: $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$ (5.3)

Completeness: $X = x^a\Gamma_a, \forall X \in$ second-order matrices. (5.4)

Expansion uniqueness: $X = x^a\Gamma_a \Leftrightarrow x^a = \dfrac{1}{2}tr[\Gamma_a(-\varsigma)X], \forall X \in$ second-order matrices. (5.5)

2.2 Symmetric and antisymmetric basis expansion of second-order matrices

Symmetric and antisymmetric basis of second-order matrices: $\Gamma_a(\varsigma)\bar{\varepsilon} = \{\sigma, i\varsigma\}\bar{\varepsilon}$, $[\Gamma_a(\varsigma)\bar{\varepsilon}]^T = \{\sigma, -i\varsigma\}\bar{\varepsilon}$
$\sigma\bar{\varepsilon}$ is a symmetric basis, $i\varsigma\bar{\varepsilon}$ is an antisymmetric basis.

Proposition 2.2.1. $x^a\Gamma_a(\varsigma)\bar{\varepsilon} = 0 \Leftrightarrow x^a = 0$

Proposition 2.2.2. $X = \frac{1}{2}tr[\varepsilon\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\bar{\varepsilon}, \forall X \in$ *second-order matrices.*

Proof: $X\varepsilon = \frac{1}{2}tr[\Gamma^a(-\varsigma)X\varepsilon]\Gamma_a(\varsigma), \forall X \in$ second-order matrices.

$\Leftrightarrow X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X\varepsilon]\Gamma_a(\varsigma)\bar{\varepsilon}, \forall X \in$ second-order matrices.

$\Leftrightarrow X = \frac{1}{2}tr[\bar{\varepsilon}\varepsilon\Gamma^a(-\varsigma)X\varepsilon]\Gamma_a(\varsigma)\bar{\varepsilon}, \forall X \in$ second-order matrices.

$\Leftrightarrow X = \frac{1}{2}tr[\varepsilon\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\bar{\varepsilon}, \forall X \in$ second-order matrices. □

3 Complete expansion of fourth-order matrices

3.1 Double Pauli basis expansion of fourth-order matrices

Proposition 3.1.1. $X = \frac{1}{4}tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X]\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma), \forall X \in$ *fourth-order matrices.*

Proof: $X = \frac{1}{2}\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}]\Gamma_a(\varsigma) & tr[\Gamma^a(-\varsigma)X_{12}]\Gamma_a(\varsigma) \\ tr[\Gamma^a(-\varsigma)X_{21}]\Gamma_a(\varsigma) & tr[\Gamma^a(-\varsigma)X_{22}]\Gamma_a(\varsigma) \end{bmatrix}, \forall X \in$ fourth-order matrices.

$\Leftrightarrow X = \frac{1}{2}\Gamma_a(\varsigma)\otimes\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}, \forall X \in$ fourth-order matrices.

$\Leftrightarrow X = \frac{1}{4}tr\{\Gamma^b(-\varsigma)\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\}\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma), \forall X \in$ fourth-order matrices.

$\Leftrightarrow X = \frac{1}{4}tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X]\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma), \forall X \in$ fourth-order matrices. □

Corollary 3.1.1. $tr\{\Gamma^b(-\varsigma)\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\} = tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X], \forall X \in$ *fourth-order matrices.*

3.2 Charge conjugate matrix C [7,12]

Definiton 3.2.1. *Charge conjugate matrix* $C \Leftrightarrow \bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = -C, C^+ = \bar{C}$

Corollary 3.2.1. $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T$

Proof: $\gamma_a(\varsigma)C = C\bar{C}\gamma_a(\varsigma)C = -C\gamma_a^T(\varsigma) = C^T\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)C]^T$ □

Corollary 3.2.2. $\bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T$

Proof: $\bar{C}\gamma_a(\varsigma) = \bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_a^T(\varsigma)\bar{C} = -[C^*\gamma_a(\varsigma)]^T = [\bar{C}\gamma_a(\varsigma)]^T$ □

Corollary 3.2.3. $S_{ab}(e,\varsigma)C = [S_{ab}(e,\varsigma)C]^T$

Proof: $S_{ab}(e,\varsigma)C = \frac{1}{4}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]C$

$= \frac{1}{4}[C\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C - C\bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C]$

$= \frac{1}{4}C[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)] = -\frac{1}{4}C^T[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T$

$= C^TS_{ab}^T(e,\varsigma) = [S_{ab}(e,\varsigma)C]^T$ □

Corollary 3.2.4. $\bar{C}S_{ab}(e,\varsigma) = [\bar{C}S_{ab}(e,\varsigma)]^T$

Proof: $\bar{C}S_{ab}(e,\varsigma) = \frac{1}{4}\bar{C}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]$

$= \frac{1}{4}[\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C\bar{C} - \bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C}]$

$= \frac{1}{4}[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)]\bar{C} = -\frac{1}{4}[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T\bar{C}^T$

$= S_{ab}^T(e,\varsigma)\bar{C}^T = [\bar{C}S_{ab}(e,\varsigma)]^T$ □

Corollary 3.2.5. $\bar{C}\gamma_5(\varsigma)C = \gamma_5^T(\varsigma)$

Proof: $\bar{C}\gamma_5(\varsigma)C = \bar{C}\gamma_x(\varsigma)\gamma_y(\varsigma)\gamma_z(\varsigma)\gamma_\pi(\varsigma)C$

$= \bar{C}\gamma_x(\varsigma)C\bar{C}\gamma_y(\varsigma)C\bar{C}\gamma_z(\varsigma)C\bar{C}\gamma_\pi(\varsigma)C$

$= \gamma_x^T(\varsigma)\gamma_y^T(\varsigma)\gamma_z^T(\varsigma)\gamma_\pi^T(\varsigma) = [\gamma_\pi(\varsigma)\gamma_z(\varsigma)\gamma_y(\varsigma)\gamma_x(\varsigma)]^T = \gamma_5^T(\varsigma)$ □

Corollary 3.2.6. $C = -C^T, \bar{C} = -\bar{C}^T,$

Corollary 3.2.7. $\gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T$

Corollary 3.2.8. $\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$

Proof: $\gamma_5(\varsigma)\gamma_a(\varsigma)C = C\bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C = -C\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)$
$= C^T\gamma_5^T(\varsigma)\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)\gamma_5(\varsigma)C]^T = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$ □

Corollary 3.2.9. $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$

Proof: $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = \bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}$
$= \gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}^T = [\bar{C}\gamma_a(\varsigma)\gamma_5(\varsigma)]^T = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$ □

summary:

Symmetric basis: $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T, \bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T, S_{ab}(e,\varsigma)C = [S_{ab}(e,\varsigma)C]^T, \bar{C}S_{ab}(e,\varsigma) = [\bar{C}S_{ab}(e,\varsigma)]^T$

Antisymmetric basis: $C = -C^T, \bar{C} = -\bar{C}^T, \gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, , \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T,$

$\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$

3.3 Dirac matrices under special representation [7,12]

Take Dirac matrices of special representation: $[\gamma_a(\varsigma), \gamma_5(\varsigma)] = [(\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x), \varsigma I\otimes\sigma_z]$

Expansion in detail:

$[\gamma_a(\varsigma), \gamma_5(\varsigma)] = [(\sigma_x\otimes\sigma_y, \sigma_y\otimes\sigma_y, \sigma_z\otimes\sigma_y, \varsigma I\otimes\sigma_x), \varsigma I\otimes\sigma_z]$

$[\gamma_a(\varsigma), \gamma_5(\varsigma)]\gamma_5(\varsigma) = i\varsigma[(\sigma_x\otimes\sigma_x, \sigma_y\otimes\sigma_x, \sigma_z\otimes\sigma_x, -\varsigma I\otimes\sigma_y), -i\varsigma I\otimes I]$

$$S_{ab}(e,\varsigma) = \frac{1}{4}[\gamma_a(\varsigma), \gamma_b(\varsigma)] = \frac{i}{2}\begin{bmatrix} 0 & \sigma_z\otimes I & -\sigma_y\otimes I & -\varsigma\sigma_x\otimes\sigma_z \\ -\sigma_z\otimes I & 0 & \sigma_x\otimes I & -\varsigma\sigma_y\otimes\sigma_z \\ \sigma_y\otimes I & -\sigma_x\otimes I & 0 & -\varsigma\sigma_z\otimes\sigma_z \\ \varsigma\sigma_x\otimes\sigma_z & \varsigma\sigma_y\otimes\sigma_z & \varsigma\sigma_z\otimes\sigma_z & 0 \end{bmatrix}$$

Charge conjugation matrix under special representation:

$C = \bar{\varepsilon}\otimes\sigma_z = \varsigma\gamma_y(\varsigma)\gamma_\pi(\varsigma), \bar{C} = -C = \varepsilon\otimes\sigma_z = -\varsigma\gamma_y(\varsigma)\gamma_\pi(\varsigma)$

3.4 Complete Dirac basis expansion of fourth-order matrices [7,12]

Complete Dirac basis of fourth-order matrices: $\Gamma_A(\varsigma) = [\gamma_a(\varsigma), -2iS_{ab}(e,\varsigma), -I_4, -i\gamma_a(\varsigma)\gamma_5(\varsigma), -\gamma_5(\varsigma)]$

Proposition 3.4.1. $X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\begin{cases}\phi = -\frac{1}{4}trX \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, F^{ab} = -\frac{i}{2}tr[S^{ab}(e,\varsigma)X]$$

Proof: $X = \frac{1}{4}tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X]\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma), \forall X$

$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\begin{cases} im\mathbf{A}^i = \frac{i\varsigma}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^x(-\varsigma)X] = \frac{1}{4}tr[\gamma^i(\varsigma)\gamma^5(\varsigma)X] \\ imA^\pi = \frac{i}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^x(-\varsigma)X] = \frac{1}{4}tr[\gamma^\pi(\varsigma)X]\end{cases}$$

$$\begin{cases} imA^i = \frac{1}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^y(-\varsigma)X] = \frac{1}{4}tr[\gamma^i(\varsigma)X] \\ im\mathbf{A}^\pi = \frac{\varsigma}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^y(-\varsigma)X] = \frac{1}{4}tr[\gamma^\pi(\varsigma)\gamma^5(\varsigma)X]\end{cases}$$

$$\begin{cases} F^{i\pi} = -F^{\pi i} = -\frac{\varsigma}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^z(-\varsigma)X] = -\frac{i}{2}tr[S^{i\pi}(e,\varsigma)X] \\ \Phi = -\frac{i}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^z(-\varsigma)X] = -\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases}$$

$$\begin{cases} F^{yz} = -F^{zy} = \frac{i\varsigma}{4}tr[\Gamma^x(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{yz}(e,\varsigma)X] \\ F^{zx} = -F^{xz} = \frac{i\varsigma}{4}tr[\Gamma^y(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{zx}(e,\varsigma)X] \\ F^{xy} = -F^{yx} = \frac{i\varsigma}{4}tr[\Gamma^z(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{xy}(e,\varsigma)X] \\ \phi = \frac{1}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X] = -\frac{1}{4}trX\end{cases}$$

$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\begin{cases}\phi = -\frac{1}{4}trX \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, F^{ab} = -\frac{i}{2}tr[S^{ab}(e,\varsigma)X]$$ □

Corollary 3.4.1. $X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$$

$$\begin{cases}\phi = -\frac{1}{4}trX \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases}, \begin{cases}imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, F^{ab} = -\frac{i}{2}tr[S^{ab}(e,\varsigma)X]$$

3.5 Symmetric and antisymmetric basis expansion of fourth-order matrices

Symmetric and antisymmetric basis of second-order matrices: $\Gamma_A(\varsigma) = [\gamma_a(\varsigma), -2iS_{ab}(e,\varsigma), |-I_4, -i\gamma_a(\varsigma)\gamma_5(\varsigma), -\gamma_5(\varsigma)]C$

Proposition 3.5.1. $X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X], \begin{cases}imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = -\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, \begin{cases}\phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X]\end{cases}$$

Proof: $X\bar{C} = \frac{1}{4}tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X\bar{C}]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

$$\Leftrightarrow X\bar{C} = [im\gamma_a(\varsigma)A^a - iS_{ab}(e,\varsigma)F^{ab}] - [\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]$$

$$\begin{cases}\phi = -\frac{1}{4}tr[X\bar{C}] \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X\bar{C}]\end{cases}, \begin{cases}imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X\bar{C}] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X\bar{C}]\end{cases}, F^{ab} = -\frac{i}{2}tr[S^{ab}(e,\varsigma)X\bar{C}]$$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi]$$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X], \begin{cases}imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = -\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, \begin{cases}\phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X]\end{cases} \quad \square$$

Corollary 3.5.1. $X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X], \begin{cases}imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = -\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases}, \begin{cases}\phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X]\end{cases}$$

3.6 Expansion of symmetric fourth-order matrices

Fourth-order matrices symmetric basis:

$\Gamma_A(\varsigma) = [\gamma_a(\varsigma), -2iS_{ab}(e,\varsigma)]C, \ \bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = \bar{C} = -C, C^+(\varsigma) = \bar{C}$

Proposition 3.6.1. $G = im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}, G = G^T, iF^{ab} = tr[\bar{C}S^{ab}(e,\varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G]$

Proof: $G = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi]$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)G], G = G^T, \begin{cases}imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G] \\ im\mathbf{A}^a = -\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)G] = 0\end{cases}, \begin{cases}\phi = -\frac{1}{4}tr[\bar{C}G] = 0 \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)G] = 0\end{cases}$$

$$\Leftrightarrow G = im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}, G = G^T, iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G] \quad \square$$

4 Spin-1 Bargmann-Wigner equation [20, 24]

4.1 Analysis of spin-1 Bargmann-Wigner equation with mass

Lemma 4.1.1. $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a{}^\sigma - iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma] = J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$,

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], im[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}] \\ imD^aA_a{}^\sigma = \frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], iD^b{*}F_{ab}{}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\end{cases}$$

Proof: $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a{}^\sigma - iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma] = J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$,

$$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_cA_a{}^\sigma - i\gamma^c(\varsigma)S^{ab}(e,\varsigma)D_cF_{ab}{}^\sigma + im^2\gamma^a(\varsigma)A_a{}^\sigma - imS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C}$$

$$\Leftrightarrow im[\delta^{ca} + 2S^{ca}(e,\varsigma)]D_cA_a{}^\sigma - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma$$
$$+ im^2\gamma^a(\varsigma)A_a{}^\sigma - imS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C},$$

$$\Leftrightarrow im[D^aA_a{}^\sigma + 2S^{ab}(e,\varsigma)D_aA_b{}^\sigma] - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma$$
$$+ im^2\gamma^a(\varsigma)A_a{}^\sigma - imS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C}$$

$$\Leftrightarrow i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma)\gamma^a(\varsigma)C - im[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)]S^{ab}(e,\varsigma)C$$

$$+imD^aA_a{}^\sigma C+iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$$
$$\Leftrightarrow\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}],im[F_{ab}{}^\sigma-(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}],0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}],iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\end{cases}\quad\square$$

Lemma 4.1.2.
$$\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\im[F_{ab}{}^\sigma-(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0\\imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0,0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0\end{cases}$$
$$\Leftrightarrow\begin{cases}D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0\\J^{[\kappa_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\end{cases}$$

Proof:
$$\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\im[F_{ab}{}^\sigma-(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0\\imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0,0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0\end{cases}$$
$$\Leftrightarrow\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}],D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0\\J^{[\kappa_\varsigma\mu_\varsigma]\sigma}=\begin{bmatrix}0&J^{[1_\varsigma2_\varsigma]\sigma}\\J^{[2_\varsigma1_\varsigma]\sigma}&0\end{bmatrix},\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]=0\end{cases}$$
$$\Leftrightarrow\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}],J^{[\kappa_\varsigma\mu_\varsigma]\sigma}=\begin{bmatrix}0&J_a{}^\sigma\Gamma^a(\varsigma)\bar{\varepsilon}\\J_a{}^\sigma\Gamma^a(-\varsigma)\bar{\varepsilon}&0\end{bmatrix}\\D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0\end{cases}$$
$$\Leftrightarrow\begin{cases}D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0\\J^{[\kappa_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\end{cases}\quad\square$$

Corollary 4.1.1. $[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$$\Leftrightarrow D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0$$

Corollary 4.1.2. $[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$$\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$$

Corollary 4.1.3.
$$\begin{cases}[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$$
$$\Leftrightarrow D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0$$

Corollary 4.1.4.
$$\begin{cases}[\gamma^c(\varsigma)\partial_c+m][im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)C\partial_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma\end{cases}$$
$$\Leftrightarrow\partial^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma,\partial^aJ_a{}^\sigma=0$$

Corollary 4.1.5.
$$\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma\end{cases}\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\\F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$$

Corollary 4.1.6.
$$\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\\psi^{\lambda_\varsigma\mu_\varsigma\sigma}=\psi^{\mu_\varsigma\lambda_\varsigma\sigma}\end{cases}\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\end{cases}$$

Corollary 4.1.7. $\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi^{\lambda_\varsigma\mu_\varsigma\sigma}=\psi^{\mu_\varsigma\lambda_\varsigma\sigma}\end{cases}$

$$\Leftrightarrow\begin{cases}D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma=0, F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma, D^aA_a{}^\sigma=0\\ \psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\end{cases}$$

Corollary 4.1.8. $\begin{cases}[\gamma^c(\varsigma)\partial_c+m]\psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi^{\lambda_\varsigma\mu_\varsigma\sigma}=\psi^{\mu_\varsigma\lambda_\varsigma\sigma}\end{cases}$

$$\Leftrightarrow\begin{cases}\partial^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma, \partial^aJ_a{}^\sigma=0, F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma\\ \psi^{[\lambda_\varsigma\mu_\varsigma]\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)C\partial_b]A_a{}^\sigma\end{cases}$$

4.2 Spin-1 Bargmann-Wigner equation with mass

Theorem 4.2.1. $\begin{cases}[\gamma^c(\varsigma)\partial_c+m]\psi^{[\lambda_\varsigma]\mu_\varsigma\sigma}=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi^{\lambda_\varsigma\mu_\varsigma\sigma}=\psi^{\mu_\varsigma\lambda_\varsigma\sigma}\end{cases}\Leftrightarrow\begin{cases}(-\partial^b\partial_b+m^2)A_a{}^\sigma=-J_a{}^\sigma\\ \partial^aA_a{}^\sigma=0, \partial^aJ_a{}^\sigma=0\\ \psi^{\lambda_\varsigma\mu_\varsigma\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^\sigma\end{cases}$

4.3 Analysis of spin-1 Bargmann-Wigner equation without mass

Lemma 4.3.1. $[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$,

$$\Leftrightarrow\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], im[F_{ab}{}^\sigma-(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\ imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], 0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\end{cases}$$

Proof: $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$

$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_cA_a{}^\sigma-i\gamma^c(\varsigma)S^{ab}(e,\varsigma)D_cF_{ab}{}^\sigma=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C}$

$\Leftrightarrow im[\delta^{ca}+2S^{ca}(e,\varsigma)]D_cA_a{}^\sigma-\frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma)-\gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C}$

$\Leftrightarrow im[D^aA_a{}^\sigma+2S^{ab}(e,\varsigma)D_aA_b{}^\sigma]-\frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma)-\gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}\bar{C}$

$\Leftrightarrow iD^bF_{ab}{}^\sigma\gamma^a(\varsigma)C+im[(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]S^{ab}(e,\varsigma)C+imD^aA_a{}^\sigma C+iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C=J^{[\kappa_\varsigma\mu_\varsigma]\sigma}$

$$\Leftrightarrow\begin{cases}iD^bF_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], -im(D_aA_b{}^\sigma-D_bA_a{}^\sigma)=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\\ imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], 0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}], iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J^{[\kappa_\varsigma\mu_\varsigma]\sigma}]\end{cases}$$ □

4.4 Spin-1 Bargmann-Wigner equation without mass

Proposition 4.4.1. $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$

$\Leftrightarrow D^bF_{ab}{}^\sigma=-J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma=0, D_aA_b{}^\sigma-D_bA_a{}^\sigma=0, D^aA_a{}^\sigma=0$

Proof: $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma-iS^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$

$\Leftrightarrow iD^bF_{ab}{}^\sigma\gamma^a(\varsigma)C+im(D_aA_b{}^\sigma-D_bA_a{}^\sigma)S^{ab}(e,\varsigma)C+imD^aA_a{}^\sigma C+iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C=-iJ_a{}^\sigma\gamma^a(\varsigma)C$

$\Leftrightarrow D^bF_{ab}{}^\sigma=-J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma=0, D_aA_b{}^\sigma-D_bA_a{}^\sigma=0, D^aA_a{}^\sigma=0$ □

Proposition 4.4.2. $\gamma^c(\varsigma)\partial_c[im\gamma^a(\varsigma)CA_a-iS^{ab}(e,\varsigma)CF_{ab}]=-iJ_a\gamma^a(\varsigma)C$

$\Leftrightarrow\partial^bF_{ab}=-J_a, \partial^b*F_{ab}=0, \partial^a\partial_a\phi=0, A_a=\partial_a\phi$

In massless cases due to $A_a{}^\sigma$ and $F_{ab}{}^\sigma$ is each other completely independent, we can't get a more concise and meaningful conclusion. And there are redundant equations. It's not simple enough. It can't be naturally extended to high spin cases. So Bargmann-Wigner equation seems not to be suitable for describing massless particles. Penrose spinorial equation or Spin Equation is more suitable to describe massless particles.

5 Spin-$\frac{3}{2}$, 2 Bargmann-Wigner equation [20, 24]

5.1 Analysis of spin-$\frac{3}{2}$ Bargmann-Wigner equation with mass

Proposition 5.1.1. $\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma}=[im\gamma^a(\varsigma)C+2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma}\qquad tr[\bar{C}\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}]=0$

$\Rightarrow[im\gamma^a(\varsigma)+2iS^{ab}(e,\varsigma)D_b]A_a{}^{[\eta_\varsigma]\sigma}=0$

Proposition 5.1.2. $\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \qquad tr[\bar{C}\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0$

$\Rightarrow [im\gamma^a(\varsigma) - 2iS^{ab}(e,\varsigma)D_b]A_a{}^{[\eta_\varsigma]\sigma} = 0$

Proposition 5.1.3. $\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \qquad tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0,$

$\Rightarrow [im\gamma^a(\varsigma)\gamma^c(\varsigma) - 2iS^{ab}(e,\varsigma)\gamma^c(\varsigma)D_b]A_a{}^{[\eta_\varsigma]\sigma} = 0$

Corollary 5.1.1. $\begin{cases}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ tr[\bar{C}\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0\end{cases}$

$$\Leftrightarrow \begin{cases}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0, D^aA_a{}^{[\eta_\varsigma]\sigma} = 0\end{cases}$$

Corollary 5.1.2. $\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\lambda_\varsigma\eta_\varsigma\mu_\varsigma\sigma}$

$\Leftrightarrow tr[\bar{C}\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0$

Corollary 5.1.3. $\begin{cases}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\lambda_\varsigma\eta_\varsigma\mu_\varsigma\sigma}\end{cases}$

$$\Leftrightarrow \begin{cases}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)D_b + m]A_a{}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0\end{cases}$$

Corollary 5.1.4. $\begin{cases}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0\end{cases}$

$$\Leftrightarrow \begin{cases}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma = A_{ba}{}^\sigma, \partial^aA_{ab}{}^\sigma = 0\end{cases}$$

Proof: $A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0$

$$\Leftrightarrow \begin{cases}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ im[\delta^{ab} + 2S^{ab}(e,\varsigma)]A_{ab}{}^\sigma - i\gamma^d(\varsigma)\gamma^5(\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}{}^\sigma + i\gamma^z(\varsigma)(\delta^{ab}\partial_zA_{ab}{}^\sigma - \partial^aA_{az}{}^\sigma) = 0\end{cases}$$

$$\Leftrightarrow \begin{cases}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma - A_{ba}{}^\sigma = 0, (\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}{}^\sigma = 0, (\delta^{ab}\partial_zA_{ab}{}^\sigma - \partial^aA_{az}{}^\sigma) = 0\end{cases}$$

$$\Leftrightarrow \begin{cases}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma = A_{ba}{}^\sigma, \partial^aA_{ab}{}^\sigma = 0\end{cases}$$ □

5.2 Spin-$\frac{3}{2}$ Bargmann-Wigner equation with mass in curved spacetime

Theorem 5.2.1. $\begin{cases}[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma\sigma} = -iJ_a{}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} \textit{ is full symmetry except } \sigma.\end{cases}$

$$\Leftrightarrow \begin{cases}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ D^bF_{ab}{}^{\eta_\varsigma\sigma} + m^2A_a{}^{\eta_\varsigma\sigma} = -J_a{}^{\eta_\varsigma\sigma}, D^b{*}F_{ab}{}^{\eta_\varsigma\sigma} = 0, F_{ab}{}^{\eta_\varsigma\sigma} = D_aA_b{}^{\eta_\varsigma\sigma} - D_bA_a{}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)D_b + m]A_a{}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0\end{cases}$$

Proof: $[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma\sigma} = -iJ_a{}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C, \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma}$ is full symmetry except σ.

$$\Leftrightarrow \begin{cases}[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = -iJ_a{}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C, \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\mu_\varsigma\lambda_\varsigma\eta_\varsigma\sigma} \\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0\end{cases}$$

$$\Leftrightarrow \begin{cases}[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = -iJ_a{}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C, \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi^{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]\sigma}] = 0\end{cases}$$

$$\Leftrightarrow \begin{cases}[\gamma^a(\varsigma)D_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = -iJ_a{}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C, \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)D_b + m]A_a{}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0, D^aA_a{}^{[\eta_\varsigma]\sigma} = 0\end{cases}$$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ D^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = -{J_a}^{\eta_\varsigma\sigma}, D^b{*F_{ab}}^{\eta_\varsigma\sigma} = 0, {F_{ab}}^{\eta_\varsigma\sigma} = D_a{A_b}^{\eta_\varsigma\sigma} - D_b{A_a}^{\eta_\varsigma\sigma}, D^a{A_a}^{\eta_\varsigma\sigma} = 0 \\ [\gamma^b(\varsigma)D_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0, D^a{A_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)CD_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ D^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = -{J_a}^{\eta_\varsigma\sigma}, D^b{*F_{ab}}^{\eta_\varsigma\sigma} = 0, {F_{ab}}^{\eta_\varsigma\sigma} = D_a{A_b}^{\eta_\varsigma\sigma} - D_b{A_a}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)D_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){J_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases} \qquad \square$$

In curved spacetime, the equation can't be further simplified. So it can't get more concise and more meaningful conclusions.

5.3 Source item requirement of spin-$\frac{3}{2}$ Bargmann-Wigner equation with mass in flat spacetime

Theorem 5.3.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma\sigma} = -i{J_a}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} \textit{ is full symmetry except } \sigma. \end{cases}$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ \partial^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = 0, {J_a}^{\eta_\varsigma\sigma} = 0, {F_{ab}}^{\eta_\varsigma\sigma} = \partial_a{A_b}^{\eta_\varsigma\sigma} - \partial_b{A_a}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)\partial_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases}$$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma\sigma} = -i{J_a}^{\eta_\varsigma\sigma}\gamma^a(\varsigma)C \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} \text{ is full symmetry except } \sigma. \end{cases}$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ \partial^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = -{J_a}^{\eta_\varsigma\sigma}, \partial^b{*F_{ab}}^{\eta_\varsigma\sigma} = 0, {F_{ab}}^{\eta_\varsigma\sigma} = \partial_a{A_b}^{\eta_\varsigma\sigma} - \partial_b{A_a}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)\partial_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ \partial^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = -{J_a}^{\eta_\varsigma\sigma}, {F_{ab}}^{\eta_\varsigma\sigma} = \partial_a{A_b}^{\eta_\varsigma\sigma} - \partial_b{A_a}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)\partial_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ \partial^b{F_{ab}}^{\eta_\varsigma\sigma} + m^2{A_a}^{\eta_\varsigma\sigma} = 0, {J_a}^{\eta_\varsigma\sigma} = 0, {F_{ab}}^{\eta_\varsigma\sigma} = \partial_a{A_b}^{\eta_\varsigma\sigma} - \partial_b{A_a}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)\partial_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0 \end{cases} \qquad \square$$

Comparing with curved spacetime case, the equation is further simplified. We get a more concise and more meaningful conclusion. The equation itself also automatically requires that the source term must be zero.

5.4 Spin-$\frac{3}{2}$ Bargmann-Wigner equation with mass in flat spacetime [23]

Theorem 5.4.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} \textit{ is full symmetry except } \sigma. \end{cases} \Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]{A_a}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma){A_a}^{[\eta_\varsigma]\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \end{cases}$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} \text{ is full symmetry except } \sigma. \end{cases}$

$$\Leftrightarrow \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\mu_\varsigma\lambda_\varsigma\eta_\varsigma\sigma} \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\lambda_\varsigma\eta_\varsigma\mu_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2){A_a}^{\sigma} = 0, \partial^a{A_a}^{\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}{A_a}^{\eta_\varsigma\sigma} \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = \psi^{\lambda_\varsigma\eta_\varsigma\mu_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)A_a{}^\sigma = 0, \partial^a A_a{}^\sigma = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \\ [\gamma^b(\varsigma)\partial_b + m]A_a{}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_a{}^{[\eta_\varsigma]\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\sigma} \end{cases} \qquad \square$$

5.5 Spin-2 Bargmann-Wigner equation with mass in curved spacetime

Theorem 5.5.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} \textit{ is full symmetry except } \sigma. \end{cases}$

$$\Leftrightarrow \begin{cases} (-\partial^z\partial_z + m^2)A_{ab}{}^\sigma = 0, \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma = A_{ba}{}^\sigma, \partial^a A_{ab}{}^\sigma = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}[im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \end{cases}$$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} \text{ is full symmetry except } \sigma. \end{cases}$

$$\Leftrightarrow \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi^{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} \text{ is full symmetry except } \xi_\varsigma\sigma. \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = \psi^{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0, \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = \psi^{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0, A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = A_a{}^{\xi_\varsigma\eta_\varsigma\sigma} \\ \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^z\partial_z + m^2)A_{ab}{}^\sigma = 0, \partial^b A_{ab}{}^\sigma = 0 \\ A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \gamma^a(\varsigma)A_a{}^{[\eta_\varsigma]\xi_\varsigma\sigma} = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^z\partial_z + m^2)A_{ab}{}^\sigma = 0, \partial^b A_{ab}{}^\sigma = 0 \\ A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \\ \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma = A_{ba}{}^\sigma, \partial^a A_{ab}{}^\sigma = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}A_a{}^{\eta_\varsigma\xi_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^z\partial_z + m^2)A_{ab}{}^\sigma = 0, \delta^{ab}A_{ab}{}^\sigma = 0, A_{ab}{}^\sigma = A_{ba}{}^\sigma, \partial^a A_{ab}{}^\sigma = 0 \\ \psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\sigma} = [im\gamma^a(\varsigma)C + 2iS^{ab}(e,\varsigma)C\partial_b]^{\lambda_\varsigma\mu_\varsigma}[im\gamma^b(\varsigma)C + 2iS^{bz}(e,\varsigma)C\partial_z]^{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma \end{cases} \qquad \square$$

6 Arbitrary spin Bargmann-Wigner equation in flat spacetime [20, 24]

6.1 Spin-n Bargmann-Wigner equation with mass in flat spacetime [25]

Definiton 6.1.1. $\mathbb{X}_a \equiv [im\gamma_a(\varsigma) + 2iS_{ab}(e,\varsigma)\partial^b]C$

Theorem 6.1.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi^{\overbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n}\sigma}=0\\ \psi^{\overbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n}\sigma\sigma}\end{cases}\Leftrightarrow\begin{cases}(-\partial^z\partial_z+m^2)A^{\overbrace{abc\cdots}^{n}\sigma}=0\\ \delta_{ab}A^{\overbrace{abc\cdots}^{n}\sigma}=0,\partial_a A^{\overbrace{abc\cdots}^{n}\sigma}=0\\ A^{\overbrace{abc\cdots}^{n}\sigma}\ \textit{is full symmetry except}\ \sigma.\\ \psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n}\sigma}=\overbrace{\mathbb{X}_a{}^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}_b{}^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}A^{\overbrace{abc\cdots}^{n}\sigma}\end{cases}$

6.2 Spin-n+$\frac{1}{2}$ Bargmann-Wigner equation with mass in flat spacetime [21]

Theorem 6.2.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi^{\overbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n+1}\sigma}=0\\ \psi^{\overbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n+1}\sigma}\ \textit{is full symmetry except}\ \sigma.\end{cases}\Leftrightarrow\begin{cases}[\gamma^z(\varsigma)\partial_z+m]A^{\overbrace{abc\cdots}^{n}[\zeta_\varsigma]\sigma}=0\\ \delta_{ab}A^{\overbrace{abc\cdots}^{n}[\zeta_\varsigma]\sigma}=0,\gamma_a(\varsigma)A^{\overbrace{abc\cdots}^{n}[\zeta_\varsigma]\sigma}=0\\ A^{\overbrace{abc\cdots}^{n}[\zeta_\varsigma]\sigma}\ \textit{is full symmetry except}\ \zeta_\varsigma\sigma.\\ \psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\zeta_\varsigma}^{2n+1}\sigma}=\overbrace{\mathbb{X}_a{}^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}_b{}^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}A^{\overbrace{abc\cdots}^{n}[\zeta_\varsigma]\sigma}\end{cases}$

Using mathematical induction method and using $s=\frac{3}{2}$ and $s=2$ reasoning skills we can easily and strictly prove the above two theorems. I won't dwell on it.

6.3 Review of Bargmann-Wigner equation with mass

From the above we can know that Bargmann-Wigner equation is equivalent to Rarita-Schwinger equation [21] in half integer spin cases in flat spacetime. It's equivalent to Klein-Gordon equation [25] in integer spin cases in flat spacetime. It reveals profound and rich physical connotations of Bargmann-Wigner equation. However, if the source term is considered, the equivalent result can't be obtained. Only satisfying certain conditions of the source term, it can be established. And only for spin $s=\frac{1}{2}$ or $s=1$ cases it can take a source term. For spin $s=\frac{3}{2}$ or above cases, due to the intrinsic self requirement of the equation it must be zero. In curved spacetime due to the existence of generalized covariant derivative term, this equivalence is no longer valid. This situation is not as good as Penrose spinorial equation or spin equation. Generally speaking, Penrose spinorial equation or spin equation is more suitable to describe particles without mass. Bargmann-Wigner equation is more suitable to describe particles with mass.

7 Antisymmetry Dirac equation [7]

7.1 Analysis of antisymmetry Dirac equation with mass

Theorem 7.1.1. $[\gamma^c(\varsigma)\partial_c+m]F^{[\lambda_\varsigma\mu_\varsigma]}=J,F^{\lambda_\varsigma\mu_\varsigma}=-F^{\mu_\varsigma\lambda_\varsigma}$

$$\Leftrightarrow\begin{cases}[2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b-\gamma_a(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)]C+[m(\Phi+i\partial_a\mathbf{A}^a)+m\gamma_5(\varsigma)\phi-\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C=-\gamma_5(\varsigma)J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

Proof: $[\gamma^a(\varsigma)\partial_a+m]F^{[\lambda_\varsigma\mu_\varsigma]}=J,F^{\lambda_\varsigma\mu_\varsigma}=-F^{\mu_\varsigma\lambda_\varsigma}$

$$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m]F^{[\lambda_\varsigma\mu_\varsigma]}=J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m][C\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a+\gamma_5(\varsigma)C\Phi]=-J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m][\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

$$\Leftrightarrow\begin{cases}m\phi+\gamma_a(\varsigma)\partial^a\phi+im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b+\gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)+m\gamma_5(\varsigma)\Phi=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

$$\Leftrightarrow\begin{cases}m\phi+\gamma_a(\varsigma)\partial^a\phi+2imS_{ab}(e,\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b+\gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)+m\gamma_5(\varsigma)(\Phi+i\partial_a\mathbf{A}^a)=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$$

$$\Leftrightarrow \begin{cases} m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi + 2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi) + m(\Phi + i\partial_a\mathbf{A}^a) = -\gamma_5(\varsigma)J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} [2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$ □

7.2 Analysis of antisymmetry Dirac equation without mass

Theorem 7.2.1. $\gamma^c(\varsigma)\partial_c F^{[\lambda_\varsigma\mu_\varsigma]} = J, F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma}$

$$\Leftrightarrow \begin{cases} [2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi]C + [im\partial_a\mathbf{A}^a - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

Proof: $\gamma^a(\varsigma)\partial_a F^{[\lambda_\varsigma\mu_\varsigma]} = J, F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma}$

$$\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b F^{[\lambda_\varsigma\mu_\varsigma]} = J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b[C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi] = -J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi] = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi + im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi + 2imS_{ab}(e,\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi + im\gamma_5(\varsigma)\partial_a\mathbf{A}^a = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} -\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi + 2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi + im\partial_a\mathbf{A}^a = -\gamma_5(\varsigma)J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} [2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi]C + [im\partial_a\mathbf{A}^a - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$ □

7.3 Pseudo scalar field equation with mass

Theorem 7.3.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F^{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma} \end{cases} \Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F^{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma} \end{cases}$

$$\Leftrightarrow \begin{cases} [2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\frac{j}{m}C \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, (im^2\mathbf{A}^a + \partial^a\Phi) = 0, \phi = 0, \partial^a\phi = 0 \\ m^2(\Phi + i\partial_a\mathbf{A}^a) = -j \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ \mathbf{A}^a = im^{-2}\partial^a\Phi, \phi = 0 \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$ □

Corollary 7.3.1. $[\gamma^a(\varsigma)\partial_a + m][\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi = j\gamma_5(\varsigma)C \Leftrightarrow (-\partial^a\partial_a + m^2)\Phi = -j$

7.4 Pseudo scalar field equation without mass

Corollary 7.4.1. $\gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_b\gamma_5(\varsigma)C\Phi] = j\gamma_5(\varsigma)C \Leftrightarrow \gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_b C\Phi] = jC \Leftrightarrow \partial^a\partial_a\Phi = j$

Corollary 7.4.2. $\gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\Phi] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\Phi] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\Phi = J^a$

7.5 Pseudo vector field equation with mass

Theorem 7.5.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a + m]F^{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma}\end{cases} \Leftrightarrow \begin{cases}(-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a\end{cases}$

Proof: $\begin{cases}[\gamma^a(\varsigma)\partial_a + m]F^{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F^{\lambda_\varsigma\mu_\varsigma} = -F^{\mu_\varsigma\lambda_\varsigma}\end{cases}$

$\Leftrightarrow \begin{cases}[2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = i\gamma_a(\varsigma)CJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow \begin{cases}\partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0, \partial^a\phi = 0 \\ (im^2\mathbf{A}^a + \partial^a\Phi) = -iJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow \begin{cases}(-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a \\ \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0 \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a\end{cases}$

$\Leftrightarrow \begin{cases}(-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a\end{cases}$ □

Corollary 7.5.1. $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$

Proof: $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$

$\Leftrightarrow 2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - i\gamma_a(\varsigma)(m^2\mathbf{A}^a - \partial^a\partial_b\mathbf{A}^b) = i\gamma_a(\varsigma)J^a$

$\Leftrightarrow -\partial^a\partial_b\mathbf{A}^b + m^2\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$

$\Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$ □

7.6 Pseudo vector field equation without mass

Corollary 7.6.1. $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$

Proof: $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$

$\Leftrightarrow [2imS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b + i\gamma_a(\varsigma)\partial^a\partial_b\mathbf{A}^b] + im\partial_a\mathbf{A}^a = i\gamma_a(\varsigma)J^a$

$\Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$

$\Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$ □

Corollary 7.6.2. $\gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a$

Corollary 7.6.3. $\gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a] = \gamma_5(\varsigma)[jC + J^{ab}S_{ab}(e,\varsigma)]$

$\Leftrightarrow \gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)C\mathbf{A}^a] = jC + J^{ab}S_{ab}(e,\varsigma)$

$\Leftrightarrow \partial^a\mathbf{A}^b - \partial^b\mathbf{A}^a = J^{ab}, \partial_a\mathbf{A}^a = j$

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