Electromagnetic-Power-based Characteristic Mode Theory for Metal-Material Combined Objects

Renzun Lian

***Abstract*—As a companion to the ElectroMagnetic-Power-based Characteristic Mode Theory (EMP-CMT) for PEC systems (PEC-EMP-CMT) and the EMP-CMT for Material bodies (Mat-EMP-CMT), an EMP-CMT for Metal-Material combined objects (MM-EMP-CMT) is established in this paper, and then some power-based Characteristic Mode (CM) sets are constructed for depicting the inherent power characteristics of metal-material combined objects. The MM-EMP-CMT is valuable for analyzing and designing the metal-material combined electromagnetic structures, such as the microstrip antennas and the Dielectric Resonant Antennas (DRAs) mounted on metal platforms etc. In addition, a variational formulation for the scattering problem of metal-material combined objects is provided based on the conservation law of energy.**

***Index Terms*—Characteristic Mode (CM), Electromagnetic Power, Input Power, Interaction, Metal-Material Combined Object, Output Power.**

I. INTRODUCTION

THE modal theories and methods can efficiently reveal the inherent characteristics of physical systems, so they are widely applied in both theoretical research and engineering practice. The original works on modal problems can be dated back to Euler era. After a nearly 300-year study, it has evolved into a relatively complete theoretical system, and is now called as Eigen-Mode Theory (EMT). In mathematical physics, the EMT is divided into two categories, normal EMT and singular EMT [1]. The EMT corresponding to open electromagnetic structures belongs to the singular EMT, and some modal methods for analyzing the open electromagnetic structures, such as the Singularity Expansion Method (SEM) [2], determinant root seeking method [3], and model-based modal methods (e.g., cavity model method [4]), etc., have been developed under the EMT framework.

Around 1970, a new electromagnetic modal theory called as Characteristic Mode Theory (CMT) was introduced by Robert J. Garbacz [5]-[6], and subsequently refined by Roger F. Harrington and Joseph R. Mautz under the MoM framework [7]-[9]. Recently, the Poynting's theorem-based derivations for some traditional MoM-based CMTs are provided in [10]-[11], such that the physical pictures of these MoM-based CMTs become clearer. In addition, a series of ElectroMagnetic-Power-based CMTs (EMP-CMTs) are developed in [12]-[14], such that the applicable range of the traditional MoM-based CMT is expanded.

As a companion to the EMP-CMT for PEC systems (PEC-EMP-CMT) [12] and the EMP-CMT for Material bodies (Mat-EMP-CMT) [13]-[14], an EMP-CMT for Metal-Material combined objects (MM-EMP-CMT) is established in this paper. The main destination of MM-EMP-CMT is the same as the PEC-EMP-CMT and Mat-EMP-CMT, i.e., to construct the power-based Characteristic Mode (CM) sets which have abilities to reveal the inherent power characteristics of the metal-material combined objects. The MM-EMP-CMT is valuable for many engineering applications related to the metal-material combined structures, such as the microstrip antennas [4] and the Dielectric Resonant Antennas (DRAs) mounted on metal platforms [15] etc. In addition, a variational formulation for the scattering problem of metal-material combined objects is provided in this paper based on the conservation law of energy [16].

This paper is organized as follows. The fundamental principles and formulations related to MM-EMP-CMT are provided in Secs. II-V, and Sec. VI concludes this paper. In what follows, the $e^{j\omega t}$ convention is used throughout.

II. SCATTERING SOURCES AND BASIC VARIABLE

In this paper, the metal-material combined object is simply called as scatterer. When an external excitation $\bar{F}^{inc}$ incidents on scatterer, some scattering sources will be excited on the scatterer, and then the scattering field $\bar{F}^{sca}$ is generated by the scattering sources. The summation of $\bar{F}^{inc}$ and $\bar{F}^{sca}$ is the total field $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$, here $F = E, H$. In addition, it is restricted in this paper that the source of $\bar{F}^{inc}$ doesn't distribute on scatterer.

A. Scattering sources.

For the metal-material combined objects, the scattering currents include the following kinds: the line electric current $\bar{J}^{l}$ on the metal line part, the surface electric current $\bar{J}^{s}_{met,surf}$ on the metal surface part, the surface electric current $\bar{J}^{s}_{met,vol}$ on the boundary of the metal volume part, the volume ohmic electric current $\bar{J}^{vo}$ on the material volume part, the volume polarized electric current $\bar{J}^{vp}$ on the material volume part, and the volume magnetized magnetic current $\bar{M}^{vm}$ on the material

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volume part [17]-[19]. In addition, the summation of the $\bar{J}^{vo}$ and $\bar{J}^{vp}$ is denoted as $\bar{J}^{vop}$ in this paper. Various scattering charges are related to the corresponding scattering currents by current continuity equations, so the scattering field can be uniquely determined by the scattering currents mentioned above [17]-[19].

The domains occupied by the metal line part, the metal surface part, the metal volume part, and the material volume part are respectively denoted as $D^{met,line}$, $D^{met,surf}$, $D^{met,vol}$, and $D^{mat,vol}$, and their boundaries are correspondingly denoted as $\partial D^{met,line}$, $\partial D^{met,surf}$, $\partial D^{met,vol}$, and $\partial D^{mat,vol}$ respectively. In the three-dimensional Euclidean space $\mathbb{R}^3$, it is obvious that [20]

$$D^{met,line} = \partial D^{met,line} \tag{1.1}$$
$$D^{met,surf} = \partial D^{met,surf} \tag{1.2}$$

To simplify the symbolic system of this paper and to efficiently distinguish the different domains from each other, the $D^{met,line}$ and $D^{met,surf}$ are respectively denoted as L^{met} and S^{met}, and their boundaries have the same symbolic representations as themselves because of (1); the $D^{met,vol}$, $D^{mat,vol}$, $\partial D^{met,vol}$, and $\partial D^{mat,vol}$ are respectively denoted as V^{met}, V^{mat}, ∂V^{met}, and ∂V^{mat}; the term "material volume part" is simply called as "material part", because there is no need to distinguish it from the material line and surface parts, which are not considered in this paper.

When the magnetized magnetic current model is utilized, there doesn't exist the material-based surface electric current on ∂V^{mat} [17]-[19], so the metal-based surface currents $\bar{J}^{s}_{met,surf}$ and $\bar{J}^{s}_{met,vol}$ can be uniformly denoted as $\bar{J}^{s}$, i.e.,

$$\bar{J}^{s}(\bar{r}) = \begin{cases} \bar{J}^{s}_{met,surf}(\bar{r}) & , \quad (\bar{r} \in S^{met}) \\ \bar{J}^{s}_{met,vol}(\bar{r}) & , \quad (\bar{r} \in \partial V^{met}) \\ 0 & , \quad (\bar{r} \notin S^{met} \cup \partial V^{met}) \end{cases} \tag{2}$$

In fact, the domain $S^{met} \cup \partial V^{met}$ in (2) can be equivalently rewritten as follows [20]

$$S^{met} \cup \partial V^{met} = \partial\left(S^{met} \cup V^{met}\right) = \partial\left(D^{met,surf} \cup D^{met,vol}\right) \tag{3}$$

so the domain $S^{met} \cup \partial V^{met}$ can also be simply denoted as $\partial D^{met,sv}$.

Some typical examples of the scattering currents, domains, and boundaries mentioned above are illustrated in Fig. 1.

Various scattering currents satisfy the following relations [17]-[19].

$$\bar{J}^{l}(\bar{r}) = \begin{cases} \oint_{\bar{r}' \to \bar{r}} \bar{H}^{tot}(\bar{r}') \cdot d\bar{l}' & , \quad (\bar{r} \in L^{met}) \\ 0 & , \quad (\bar{r} \notin L^{met}) \end{cases} \tag{4}$$

for the line electric current, and

$$\bar{J}^{s}(\bar{r}) = \begin{cases} \hat{n}_{+}(\bar{r}) \times \left[\bar{H}^{tot}(\bar{r}_{+}) - \bar{H}^{tot}(\bar{r}_{-})\right]_{\bar{r}_{\pm} \to \bar{r}} & , \quad (\bar{r} \in \partial D^{met,sv}) \\ 0 & , \quad (\bar{r} \notin \partial D^{met,sv}) \end{cases} \tag{5}$$

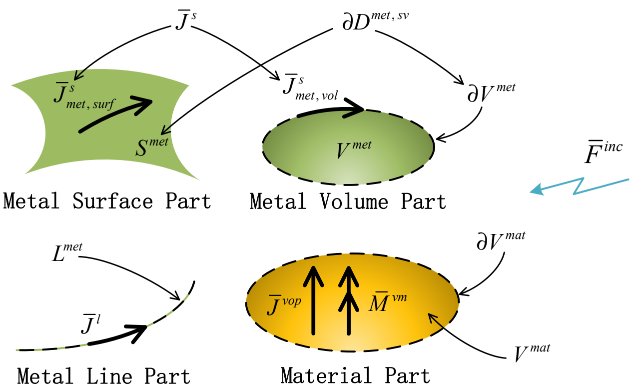

Fig. 1. The metal-material object excited by incident field.

for the surface electric current, and

$$\bar{J}^{vop}(\bar{r}) = \begin{cases} \bar{J}^{vo}(\bar{r}) + \bar{J}^{vp}(\bar{r}) = j\omega\Delta\varepsilon_c \bar{E}^{tot}(\bar{r}) & , \quad (\bar{r} \in V^{mat}) \\ 0 & , \quad (\bar{r} \notin V^{mat}) \end{cases} \tag{6.1}$$

$$\bar{M}^{vm}(\bar{r}) = \begin{cases} j\omega\Delta\mu \bar{H}^{tot}(\bar{r}) & , \quad (\bar{r} \in V^{mat}) \\ 0 & , \quad (\bar{r} \notin V^{mat}) \end{cases} \tag{6.2}$$

for the volume currents.

In (4), the symbol "$\oint_{\bar{r}' \to \bar{r}}$" represents that the integral domain is a closed line which encloses $\bar{J}^{l}$, and that all points $\bar{r}'$ on the integral line approach to the point $\bar{r} \in L^{met}$, and then that the area enclosed by the integral line approaches to zero. In (5), the subscripts "$+$" and "$-$" respectively represent the two sides of surface $\partial D^{met,sv}$; the subscript "$\bar{r}_{\pm} \to \bar{r}$" represents that the point $\bar{r}_{\pm}$ approaches to the point $\bar{r}$ from the "$\pm$" side; the vector $\hat{n}_{+}$ is the normal unit vector pointing to the "$+$" side; the symbol "$\times$" is the cross product of field vectors. In (6), $\bar{J}^{vo} = \sigma \bar{E}^{tot}$, and $\bar{J}^{vp} = j\omega\Delta\varepsilon \bar{E}^{tot}$; $\Delta\mu = \mu - \mu_0$, $\Delta\varepsilon = \varepsilon - \varepsilon_0$, and $\Delta\varepsilon_c = \varepsilon_c - \varepsilon_0$; the $\varepsilon_c = \varepsilon + \sigma / j\omega$ is complex permittivity; the ε and ε_0 are the permitivities in material part and vacuum; the μ and μ_0 are the permeabilities in material part and vacuum; the σ is the electric conductivity in material part, and its vacuum version is zero; all these material parameters can be the functions about spatial position $\bar{r}$, except the ε_0 and μ_0; the $\omega = 2\pi f$ is angle frequency, and the f is frequency.

B. *Basic variable.*

It has been pointed out in [10]-[11], [13]-[14] that to express various related scattering sources as the functions of a single variable is indispensable for the CMT, and the single variable is specifically called as the *basic variable* in [13].

In this paper, the basic variable used to express various volume scattering currents on V^{mat} is selected as $\bar{H}^{tot}(\bar{r})$, here $\bar{r} \in V^{mat}$, and the reason can be easily found out from the following discussions. Based on this, the incident and total fields on V^{mat} and the scattering currents on V^{mat} can be expressed as the following linear operator forms [13].

$$\bar{F}^{X}(\bar{r}) = \mathcal{F}^{X}\left(\bar{H}^{tot}; \bar{r}\right) \quad , \quad \left(\bar{r} \in V^{mat}\right) \tag{7}$$

$$\bar{J}^{vop}(\bar{r}) = \bar{J}^{vop}(\bar{H}^{tot};\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{8.1}$$
$$\bar{M}^{vm}(\bar{r}) = \bar{M}^{vm}(\bar{H}^{tot};\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{8.2}$$

In (7), $X = inc,tot$, and $F = E,H$, and the definition domain represents that the operator in (7) is not defined for the points $\bar{r}\notin V^{mat}$ [13]. When $\bar{r}\notin V^{mat}$, the currents $\bar{J}^{vop}(\bar{r})$ and $\bar{M}^{vm}(\bar{r})$ can also be viewed as the functions of the $\bar{H}^{tot}$ on V^{mat}, because they are zeros for any $\bar{r}\notin V^{mat}$, and this is just the reason why the definition domains of the operators in (8) are written as $\mathbb{R}^3$. The mathematical expressions for the operators in (7)-(8) can be found in [13].

It can be found out from (5) that the $\bar{J}^s(\bar{r})$ is automatically determined, if the $\bar{H}^{tot}(\bar{r}_+\to\bar{r})$ and $\bar{H}^{tot}(\bar{r}_-\to\bar{r})$ are simultaneously determined, here $\bar{r}\in\partial D^{met,sv}$. Considering of this, the boundary points $\bar{r}$ on $\partial D^{met,sv}$ are divided into two categories in this paper, based on whether or not $\bar{r}\in \text{int}\left(V^{met}\cup\partial D^{met,sv}\cup V^{mat}\cup\partial V^{mat}\right)$. The one kind is the free surface boundary points corresponding to that $\bar{r}\notin \text{int}\left(V^{met}\cup\partial D^{met,sv}\cup V^{mat}\cup\partial V^{mat}\right)$, and the other kind is the unfree surface boundary points corresponding to that $\bar{r}\in \text{int}\left(V^{met}\cup\partial D^{met,sv}\cup V^{mat}\cup\partial V^{mat}\right)$. The symbol "$\text{int}\, S$" represents the interior of set S. The set constituted by all free surface boundary points on $\partial D^{met,sv}$ is denoted as $\partial D^{met,sv}_{free}$, and the set constituted by all unfree surface boundary points on $\partial D^{met,sv}$ is denoted as $\partial D^{met,sv}_{unfree}$. It is obvious that

$$\partial D^{met,sv}_{free} \cup \partial D^{met,sv}_{unfree} = \partial D^{met,sv} \tag{9.1}$$
$$\partial D^{met,sv}_{free} \cap \partial D^{met,sv}_{unfree} = \varnothing \tag{9.2}$$

Based on (9), the following free surface electric current $\bar{J}^s_{free}$ and unfree surface electric current $\bar{J}^s_{unfree}$ are introduced.

$$\bar{J}^s_{free}(\bar{r}) = \begin{cases} \bar{J}^s(\bar{r}) & , \quad (\bar{r}\in\partial D^{met,sv}_{free}) \\ 0 & , \quad (\bar{r}\in\partial D^{met,sv}_{unfree}) \\ 0 & , \quad (\bar{r}\notin\partial D^{met,sv}) \end{cases} \tag{10.1}$$

$$\bar{J}^s_{unfree}(\bar{r}) = \begin{cases} 0 & , \quad (\bar{r}\in\partial D^{met,sv}_{free}) \\ \bar{J}^s(\bar{r}) & , \quad (\bar{r}\in\partial D^{met,sv}_{unfree}) \\ 0 & , \quad (\bar{r}\notin\partial D^{met,sv}) \end{cases} \tag{10.2}$$

so

$$\bar{J}^s(\bar{r}) = \bar{J}^s_{free}(\bar{r}) + \bar{J}^s_{unfree}(\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{11}$$

Some typical cases are plotted in Figs. 2 and 3. Based on above discussions and considering of (5) and that $\bar{H}^{tot}(\bar{r}_\pm) = 0$ for any $\bar{r}_\pm\in \text{int}\, V^{met}$, it is easy to find out that the $\bar{J}^s_{unfree}$ is only related to the $\bar{H}^{tot}$ on V^{mat}, so the $\bar{J}^s_{unfree}$ can be expressed as the function of the $\bar{H}^{tot}$ on V^{mat} as follows

$$\bar{J}^s_{unfree}(\bar{r}) = \bar{J}^s_{unfree}(\bar{H}^{tot};\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{12}$$

and this is the reason why the $\bar{J}^s_{unfree}$ is called as unfree surface current.

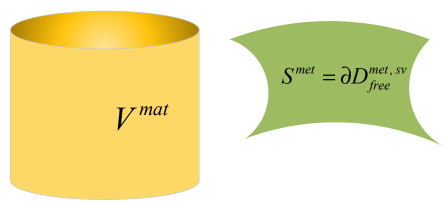

Fig. 2 (a). The metal surface and material parts don't contact with each other.

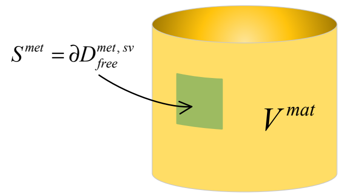

Fig. 2 (b). The metal surface and material parts contact with each other.

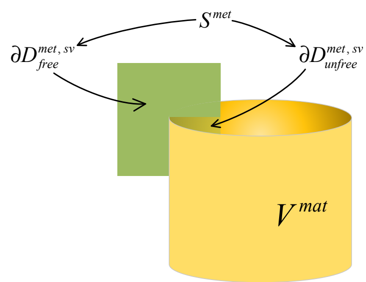

Fig. 2 (c). The metal surface part is partially immerged into the material part.

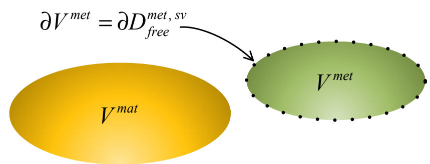

Fig. 3 (a). The metal volume and material parts don't contact with each other.

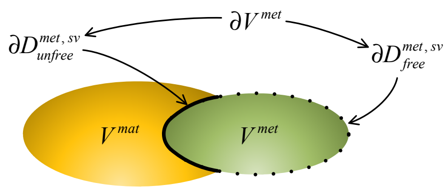

Fig. 3 (b). The metal volume and material parts contact with each other.

Similarly to the above discussions for $\bar{J}^s(\bar{r})$, the points $\bar{r}$ on L^{met} are divided into two categories, based on whether or not $\bar{r}\in \text{int}\left(V^{mat}\cup\partial V^{mat}\right)$. The set constituted by the points corresponding to $\bar{r}\notin \text{int}\left(V^{mat}\cup\partial V^{mat}\right)$ is denoted as L^{met}_{free}, and the set constituted by the points corresponding to $\bar{r}\in \text{int}\left(V^{mat}\cup\partial V^{mat}\right)$ is denoted as L^{met}_{unfree}. It is obvious that [20]

$$L^{met}_{free} \cup L^{met}_{unfree} = L^{met} \tag{13.1}$$
$$L^{met}_{free} \cap L^{met}_{unfree} = \varnothing \tag{13.2}$$

Based on (13), the following two kinds of line electric currents are introduced.

$$\bar{J}^{l}_{free}\left(\bar{r}\right) = \begin{cases} \bar{J}^{l}\left(\bar{r}\right) & , \quad \left(\bar{r}\in L^{met}_{free}\right) \\ 0 & , \quad \left(\bar{r}\in L^{met}_{unfree}\right) \\ 0 & , \quad \left(\bar{r}\notin L^{met}\right) \end{cases} \tag{14.1}$$

$$\bar{J}^{l}_{unfree}\left(\bar{r}\right) = \begin{cases} 0 & , \quad \left(\bar{r}\in L^{met}_{free}\right) \\ \bar{J}^{l}\left(\bar{r}\right) & , \quad \left(\bar{r}\in L^{met}_{unfree}\right) \\ 0 & , \quad \left(\bar{r}\notin L^{met}\right) \end{cases} \tag{14.2}$$

and then

$$\bar{J}^{l}\left(\bar{r}\right) = \bar{J}^{l}_{free}\left(\bar{r}\right)+\bar{J}^{l}_{unfree}\left(\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{15}$$

Some typical cases are plotted in Fig. 4. Based on (4), it can be concluded that the $\bar{J}^{l}_{unfree}$ is only related to the $\bar{H}^{tot}$ on V^{mat}, so the $\bar{J}^{l}_{unfree}$ can be expressed as the function of the $\bar{H}^{tot}$ on V^{mat} as follows

$$\bar{J}^{l}_{unfree}\left(\bar{r}\right) = \bar{J}^{l}_{unfree}\left(\bar{H}^{tot};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{16}$$

Based on (8), (11), (12), (15), and (16), and considering of that the scattering field is the one generated by all scattering sources in vacuum [17]-[19], the scattering field on whole $\mathbb{R}^{3}$ can be expressed as follows

$$\begin{aligned} \bar{F}^{sca}\left(\bar{r}\right) &= \bar{F}\left(\bar{J}^{l},\bar{J}^{s},\bar{J}^{vop},\bar{M}^{vm};\bar{r}\right) \\ &= \bar{F}\left(\bar{J}^{l}_{free}+\bar{J}^{l}_{unfree},\bar{J}^{s}_{free}+\bar{J}^{s}_{unfree},\bar{J}^{vop},\bar{M}^{vm};\bar{r}\right) \\ &= \bar{F}\left(\bar{J}^{l}_{free},\bar{J}^{s}_{free},0,0;\bar{r}\right)+\bar{F}\left(\bar{J}^{l}_{unfree},\bar{J}^{s}_{unfree},\bar{J}^{vop},\bar{M}^{vm};\bar{r}\right) \\ &= \bar{F}\left(\bar{J}^{l}_{free},\bar{J}^{s}_{free},0,0;\bar{r}\right)+\bar{F}'\left(\bar{H}^{tot};\bar{r}\right) \end{aligned} \tag{17}$$

here $\bar{r}\in\mathbb{R}^{3}$. In (17), the operator $\bar{F}\left(\bar{J}^{l},\bar{J}^{s},\bar{J}^{v},\bar{M}^{v};\bar{r}\right)$ represents the field generated by line current $\bar{J}^{l}$, surface current $\bar{J}^{s}$, volume electric current $\bar{J}^{v}$, and volume magnetic current $\bar{M}^{v}$, and its mathematical expression can be found in [17]-[18]; the operator $\bar{F}'\left(\bar{H}^{tot};\bar{r}\right)$ is the composite of the operator $\bar{F}\left(\bar{J}^{l},\bar{J}^{s},\bar{J}^{v},\bar{M}^{v};\bar{r}\right)$ and the operators (8), (12), and (16); the third equality in (17) is based on the linear superposition principle [16]. For simplifying the symbolic system of this paper, the (17) is simply denoted as the following linear operator form.

$$\bar{F}^{sca}\left(\bar{r}\right) = \bar{F}^{sca}\left(\bar{J}^{l}_{free},\bar{J}^{s}_{free},\bar{H}^{tot};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{18}$$

Based on the above discussions, the basic variable can be expressed as follows

$$\text{Basic Variable } \bar{V}\left(\bar{r}\right): \begin{cases} \bar{J}^{l}_{free}\left(\bar{r}\right) & , \quad \left(\bar{r}\in L^{met}_{free}\right) \\ \bar{H}^{tot}\left(\bar{r}'\right) & , \quad \left(\bar{r}\in L^{met}_{unfree}\right) \\ \bar{J}^{s}_{free}\left(\bar{r}\right) & , \quad \left(\bar{r}\in \partial D^{met,sv}_{free}\right) \\ \bar{H}^{tot}\left(\bar{r}_{\pm}\right) & , \quad \left(\bar{r}\in \partial D^{met,sv}_{unfree}\right) \\ \bar{H}^{tot}\left(\bar{r}\right) & , \quad \left(\bar{r}\in V^{mat}\right) \\ 0 & , \quad \left(\bar{r}\notin L^{met}\cup\partial D^{met,sv}\cup V^{mat}\right) \end{cases} \tag{19}$$

here $\bar{r}',\bar{r}_{\pm}\in V^{mat}$, and $\bar{r}',\bar{r}_{\pm}\to\bar{r}$.

Inserting the (19) into the (7), (8), (12), (16), and (18), the various fields and currents can be further written as the following linear operator forms.

$$\bar{F}^{X}\left(\bar{r}\right) = \bar{F}^{X}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in V^{mat}\right) \tag{20}$$

$$\bar{F}^{sca}\left(\bar{r}\right) = \bar{F}^{sca}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{21}$$

$$\bar{J}^{l}\left(\bar{r}\right) = \bar{J}^{l}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{22}$$

$$\bar{J}^{s}\left(\bar{r}\right) = \bar{J}^{s}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{23}$$

$$\bar{J}^{vop}\left(\bar{r}\right) = \bar{J}^{vop}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{24.1}$$

$$\bar{M}^{vm}\left(\bar{r}\right) = \bar{M}^{vm}\left(\bar{V};\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^{3}\right) \tag{24.2}$$

here $X = inc,tot$, and $F = E,H$.

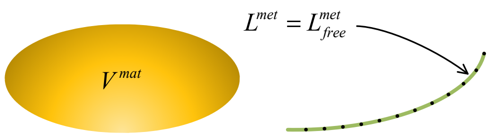

Fig. 4 (a). The metal line and material parts don't contact with each other.

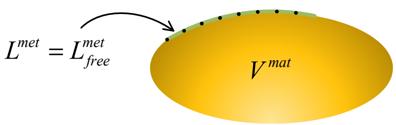

Fig. 4 (b). The metal line and material parts contact with each other.

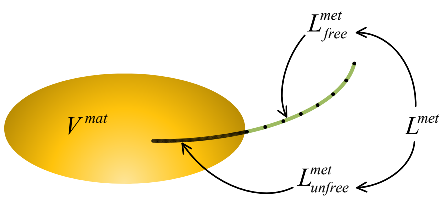

Fig. 4 (c). The metal line part is partially immerged into the material part.

III. Interaction, Input Power, and Output Power

The interaction between incident field and scatterer is just the interaction between $\bar{F}^{inc}$ and $\{\bar{J}^{l},\bar{J}^{s},\bar{J}^{vop},\bar{M}^{vm}\}$, and its mathematical expression is as follows

$$\mathcal{I} = \mathcal{I}^{met,line}+\mathcal{I}^{met,sv}+\mathcal{I}^{mat} \tag{25}$$

The $\mathcal{I}^{met,line}$ in (25) is the interaction between $\bar{F}^{inc}$ and $\bar{J}^{l}$, and

$$\mathcal{I}^{met,line} = \frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{inc}\right\rangle_{L^{met}} = -\frac{1}{2}\left\langle \bar{J}^{l},\bar{E}^{sca}\right\rangle_{L^{met}} \tag{26.1}$$

The $\mathcal{I}^{met,sv}$ in (25) is the interaction between $\bar{F}^{inc}$ and $\bar{J}^{s}$, and [12]

$$\mathcal{I}^{met,sv} = \frac{1}{2}\left\langle \bar{J}^{s}, \bar{E}^{inc} \right\rangle_{\partial D^{met,sv}} = -\frac{1}{2}\left\langle \bar{J}^{s}, \bar{E}^{sca} \right\rangle_{\partial D^{met,sv}} \tag{26.2}$$

The $\mathcal{I}^{mat}$ in (25) is the interaction between $\bar{F}^{inc}$ and $\{\bar{J}^{vop}, \bar{M}^{vm}\}$, and [13]-[14]

$$\begin{aligned} \mathcal{I}^{mat} &= (1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ &= (1/2)\left\langle \bar{J}^{vop}, \bar{E}^{tot} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{sca} \right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca}, \bar{M}^{vm} \right\rangle_{V^{mat}} \end{aligned} \tag{26.3}$$

The inner product in (26) is defined as $\langle \bar{g}, \bar{h} \rangle_{\Omega} \triangleq \int_{\Omega} \bar{g}^{*} \cdot \bar{h} \, d\Omega$, and the symbol "$*$" denotes the complex conjugate of relevant quantity, and the symbol "$\cdot$" is the scalar product for field vectors. The second equality in (26.2) is based on the surface EFIE [17]-[18], and the second equality in (26.3) is due to that $\bar{F}^{inc} = \bar{F}^{tot} - \bar{F}^{sca}$. Inserting the (26) into (25), the interaction $\mathcal{I}$ can be written as follows

$$\begin{aligned} \mathcal{I} &= -(1/2)\left\langle \bar{J}^{l}, \bar{E}^{sca} \right\rangle_{L^{met}} \\ &\quad -(1/2)\left\langle \bar{J}^{s}, \bar{E}^{sca} \right\rangle_{\partial D^{met,sv}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{sca} \right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{tot} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot}, \bar{M}^{vm} \right\rangle_{V^{mat}} \end{aligned} \tag{27}$$

Based on the source Poynting's theorem [18], the first three lines in (27) can be rewritten as follows

$$\begin{aligned} & -(1/2)\left\langle \bar{J}^{l}, \bar{E}^{sca} \right\rangle_{L^{met}} \\ & -(1/2)\left\langle \bar{J}^{s}, \bar{E}^{sca} \right\rangle_{\partial D^{met,sv}} \\ & -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{sca} \right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ = \ & P^{sca,rad} + j\, P^{sac,react,vac} \end{aligned} \tag{28}$$

here

$$P^{sca,rad} = \frac{1}{2} \oiint_{S_{\infty}} \left[\bar{E}^{sca} \times \left(\bar{H}^{sca}\right)^{*} \right] \cdot d\bar{S} \tag{29.1}$$

$$P^{sac,react,vac} = 2\omega \left[\frac{1}{4}\left\langle \bar{H}^{sca}, \mu_0 \bar{H}^{sca} \right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0 \bar{E}^{sca}, \bar{E}^{sca} \right\rangle_{\mathbb{R}^3} \right] \tag{29.2}$$

here the symbol S_{∞} represents a spherical surface at infinity. Based on (6), the fourth line in (27) can be rewritten as follows

$$\begin{aligned} & (1/2)\left\langle \bar{J}^{vop}, \bar{E}^{tot} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ = \ & P^{tot,loss} + j\, P^{tot,react,mat} \end{aligned} \tag{30}$$

here

$$P^{tot,loss} = \frac{1}{4}\left\langle \sigma \bar{E}^{tot}, \bar{E}^{tot} \right\rangle_{V^{mat}} \tag{31.1}$$

$$P^{tot,react,mat} = 2\omega \left[\frac{1}{4}\left\langle \bar{H}^{tot}, \Delta\mu \bar{H}^{tot} \right\rangle_{V^{mat}} - \frac{1}{4}\left\langle \Delta\varepsilon \bar{E}^{tot}, \bar{E}^{tot} \right\rangle_{V^{mat}} \right] \tag{31.2}$$

Inserting the (28) and (30) into the (27), the interaction $\mathcal{I}$ can be written as follows

$$\mathcal{I} = P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \tag{32}$$

Based on the conclusions given in [12]-[14], the output power P^{out} and input power P^{inp} are respectively as follows

$$\begin{aligned} P^{out} &= P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \\ &= -(1/2)\left\langle \bar{J}^{l}, \bar{E}^{sca} \right\rangle_{L^{met}} \\ &\quad -(1/2)\left\langle \bar{J}^{s}, \bar{E}^{sca} \right\rangle_{\partial D^{met,sv}} \\ &\quad -(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{sca} \right\rangle_{V^{mat}} - (1/2)\left\langle \bar{H}^{sca}, \bar{M}^{vm} \right\rangle_{V^{mat}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{tot} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{tot}, \bar{M}^{vm} \right\rangle_{V^{mat}} \end{aligned} \tag{33}$$

and

$$\begin{aligned} P^{inp} &= \mathcal{I} \\ &= (1/2)\left\langle \bar{J}^{l}, \bar{E}^{inc} \right\rangle_{L^{met}} \\ &\quad +(1/2)\left\langle \bar{J}^{s}, \bar{E}^{inc} \right\rangle_{\partial D^{met,sv}} \\ &\quad +(1/2)\left\langle \bar{J}^{vop}, \bar{E}^{inc} \right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_{V^{mat}} \end{aligned} \tag{34}$$

The conservation law of energy [16] corresponding to the electromagnetic power version is as follows

$$P^{out} = \mathcal{I} = P^{inp} \tag{35}$$

Inserting the (20)-(24) into the (33)-(34), the output and input powers can also be written as the following operator forms.

$$P^{out} = P^{out}\left(\bar{V}\right) \tag{36}$$

$$P^{inp} = P^{inp}\left(\bar{V}\right) \tag{37}$$

IV. The Matrix Form for Output Power

Similarly to the PEC-EMP-CMT [12] and the Mat-EMP-CMT [13]-[14], many power-based CM sets can be constructed for the metal-material combined objects. As a typical example, the Output power CM (OutCM) set is discussed in this paper.

The basic variable $\bar{V}$ is expanded in terms of the basis function set $\{\bar{b}_{\xi}(\bar{r})\}$ as follows

$$\bar{V}(\bar{r}) = \sum_{\xi=1}^{\Xi} a_{\xi} \bar{b}_{\xi}(\bar{r}) = \bar{B} \cdot \bar{a} \quad , \quad \left(\bar{r} \in \mathbb{R}^3\right) \tag{38}$$

here $\bar{B} = \left[\bar{b}_1(\bar{r}), \bar{b}_2(\bar{r}), \cdots, \bar{b}_{\Xi}(\bar{r})\right]$, and $\bar{a} = \left[a_1, a_2, \cdots, a_{\Xi}\right]^{T}$, and the superscript "$T$" represents matrix transposition; the basis functions are zeros for the points $\bar{r} \notin L^{met} \cup \partial D^{met,sv} \cup V^{mat}$; the symbol "$\cdot$" in (38) represents matrix multiplication.

Inserting (38) into (37) and employing (35), the matrix form for output power P^{out} can be written as follows

$$P^{out}(\bar{a}) = \bar{a}^H \cdot \bar{\bar{P}}^{out} \cdot \bar{a} \tag{39.1}$$

here $\bar{\bar{P}}^{out} = \left[p_{\xi\zeta}^{out}\right]_{\Xi\times\Xi}$, and

$$\begin{aligned} p_{\xi\zeta}^{out} = & -\frac{1}{2}\left\langle \bar{J}^l\left(\bar{b}_\xi\right), \bar{E}^{sca}\left(\bar{b}_\zeta\right)\right\rangle_{L^{met}} \\ & -\frac{1}{2}\left\langle \bar{J}^s\left(\bar{b}_\xi\right), \bar{E}^{sca}\left(\bar{b}_\zeta\right)\right\rangle_{\partial D^{met,sv}} \\ & +\frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{b}_\xi\right), \bar{E}^{inc}\left(\bar{b}_\zeta\right)\right\rangle_{V^{mat}} + \frac{1}{2}\left\langle \bar{H}^{inc}\left(\bar{b}_\xi\right), \bar{M}^{vm}\left(\bar{b}_\zeta\right)\right\rangle_{V^{mat}} \end{aligned} \tag{39.2}$$

In (39.2), the relevant operators are given in (20)-(24).

The matrix $\bar{\bar{P}}^{out}$ in (39.1) can be decomposed as [12]-[14]

$$\bar{\bar{P}}^{out} = \bar{\bar{P}}_+^{out} + j\,\bar{\bar{P}}_-^{out} \tag{40.1}$$

here

$$\begin{aligned} \bar{\bar{P}}_+^{out} &= \frac{1}{2}\left[\bar{\bar{P}}^{out} + \left(\bar{\bar{P}}^{out}\right)^H\right] \\ \bar{\bar{P}}_-^{out} &= \frac{1}{2j}\left[\bar{\bar{P}}^{out} - \left(\bar{\bar{P}}^{out}\right)^H\right] \end{aligned} \tag{40.2}$$

Obviously, the matrices $\bar{\bar{P}}_+^{out}$ and $\bar{\bar{P}}_-^{out}$ are Hermitian, so the $\bar{a}^H \cdot \bar{\bar{P}}_+^{out} \cdot \bar{a}$ and $\bar{a}^H \cdot \bar{\bar{P}}_-^{out} \cdot \bar{a}$ are always real numbers for any vector $\bar{a}$ [21], and then

$$\begin{aligned} \mathrm{Re}\left\{P^{out}(\bar{a})\right\} &= \bar{a}^H \cdot \bar{\bar{P}}_+^{out} \cdot \bar{a} \\ &= P^{sca,rad}(\bar{a}) + P^{tot,loss}(\bar{a}) \end{aligned} \tag{41.1}$$

$$\begin{aligned} \mathrm{Im}\left\{P^{out}(\bar{a})\right\} &= \bar{a}^H \cdot \bar{\bar{P}}_-^{out} \cdot \bar{a} \\ &= P^{sca,react,vac}(\bar{a}) + P^{tot,react,mat}(\bar{a}) \end{aligned} \tag{41.2}$$

V. OutCM Set and OutCM-Based Modal Expansion

The OutCM set and corresponding modal expansion method are discussed in this section. The fundamental principles and procedures to construct the OutCM set are similar to the PEC-EMP-CMT [12] and Mat-EMP-CMT [13]-[14], so only some important conclusions and formulations corresponding to the MM-EMP-CMT are simply given as follows.

A. Output power CM (OutCM) set.

When the matrix $\bar{\bar{P}}_+^{out}$ is positive definite at frequency f, the OutCM set can be obtained by solving the following generalized characteristic equation [12]-[14], [21].

$$\bar{\bar{P}}_-^{out}(f)\cdot\bar{a}_\xi(f) = \lambda_\xi(f)\,\bar{\bar{P}}_+^{out}(f)\cdot\bar{a}_\xi(f) \tag{42.1}$$

When the matrix $\bar{\bar{P}}_+^{out}$ is positive semi-definite at frequency f_0, the modal vectors can be obtained by using the following limitations for any $\xi = 1,2,\cdots,\Xi$ [12]-[14].

$$\bar{a}_\xi(f_0) = \lim_{f\to f_0}\bar{a}_\xi(f) \tag{42.2}$$

The modal basic variables are as follows for any $\xi = 1,2,\cdots,\Xi$

$$\bar{V}_\xi(\bar{r}) = \bar{B}\cdot\bar{a}_\xi \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{43}$$

The modal scattering currents are as follows

$$\bar{J}_\xi^l(\bar{r}) = \bar{J}^l\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{44}$$

$$\bar{J}_\xi^s(\bar{r}) = \bar{J}^s\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{45}$$

$$\bar{J}_\xi^{vop}(\bar{r}) = \bar{J}^{vop}\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{46.1}$$

$$\bar{M}_\xi^{vm}(\bar{r}) = \bar{M}^{vm}\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{46.2}$$

here the relevant operators are defined as (22)-(24). Various modal fields corresponding to above modal currents are as follows

$$\bar{F}_\xi^X(\bar{r}) = \bar{F}^X\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in V^{mat}\right) \tag{47}$$

$$\bar{F}_\xi^{sca}(\bar{r}) = \bar{F}^{sca}\left(\bar{V}_\xi;\bar{r}\right) \quad , \quad \left(\bar{r}\in\mathbb{R}^3\right) \tag{48}$$

here $X = inc, tot$, and $F = E, H$; the relevant operators are defined as (20) and (21). In addition, the following relation (49) exists.

$$\bar{F}_\xi^{inc}(\bar{r}) = \bar{F}_\xi^{tot}(\bar{r}) - \bar{F}_\xi^{sca}(\bar{r}) \quad , \quad \left(\bar{r}\in V^{mat}\right) \tag{49}$$

Above modal currents and modal fields satisfy the following power orthogonality [12]-[14].

$$\begin{aligned} P_\xi^{out}\delta_{\xi\zeta} &= P_{\xi\zeta}^{out} \\ &= P_{\xi\zeta;\,sca,rad}^{out} + P_{\xi\zeta;\,tot,loss}^{out} + j\left(P_{\xi\zeta;\,sca,react,vac}^{out} + P_{\xi\zeta;\,tot,react,mat}^{out}\right) \end{aligned} \tag{50}$$

In (50), the $\delta_{\xi\zeta}$ is Kronecker delta symbol, and

$$\begin{aligned} P_\xi^{out} &= \mathrm{Re}\left\{P_\xi^{out}\right\} + j\,\mathrm{Im}\left\{P_\xi^{out}\right\} \\ &= P_{\xi;\,sca,rad}^{out} + P_{\xi;\,tot,loss}^{out} + j\left(P_{\xi;\,sca,react,vac}^{out} + P_{\xi;\,tot,react,mat}^{out}\right) \end{aligned} \tag{51}$$

and

$$\begin{aligned} P_{\xi\zeta}^{out} = & -(1/2)\left\langle \bar{J}_\xi^l, \bar{E}_\zeta^{sca}\right\rangle_{L^{met}} \\ & -(1/2)\left\langle \bar{J}_\xi^s, \bar{E}_\zeta^{sca}\right\rangle_{\partial D^{met,sv}} \\ & +(1/2)\left\langle \bar{J}_\xi^{vop}, \bar{E}_\zeta^{inc}\right\rangle_{V^{mat}} + (1/2)\left\langle \bar{H}_\xi^{inc}, \bar{M}_\zeta^{vm}\right\rangle_{V^{mat}} \end{aligned} \tag{52.1}$$

$$P_{\xi\zeta;\,sca,rad}^{out} = (1/2)\oint\!\!\!\oint_{S_\infty}\left[\bar{E}_\zeta^{sca}\times\left(\bar{H}_\xi^{sca}\right)^*\right]\cdot d\bar{S} \tag{52.2}$$

$$P_{\xi\zeta;\,tot,loss}^{out} = (1/2)\left\langle \sigma\bar{E}_\xi^{tot}, \bar{E}_\zeta^{tot}\right\rangle_{V^{mat}} \tag{52.3}$$

$$P_{\xi\zeta;\,sca,react,vac}^{out} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}_\xi^{sca}, \mu_0\bar{H}_\zeta^{sca}\right\rangle_{\mathbb{R}^3} - \frac{1}{4}\left\langle \varepsilon_0\bar{E}_\xi^{sca}, \bar{E}_\zeta^{sca}\right\rangle_{\mathbb{R}^3}\right] \tag{52.4}$$

$$P_{\xi\zeta;\,tot,react,mat}^{out} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}_\xi^{tot}, \Delta\mu\bar{H}_\zeta^{tot}\right\rangle_{V^{mat}} - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}_\xi^{tot}, \bar{E}_\zeta^{tot}\right\rangle_{V^{mat}}\right] \tag{52.5}$$

In (51), $P^{out}_{\xi;sca,rad} = P^{out}_{\xi\xi;sca,rad}$, $P^{out}_{\xi;tot,loss} = P^{out}_{\xi\xi;tot,loss}$, $P^{out}_{\xi;sca,react,vac} = P^{out}_{\xi\xi;sca,react,vac}$, and $P^{out}_{\xi;tot,react,mat} = P^{out}_{\xi\xi;tot,react,mat}$.

B. OutCM-based modal expansion method.

Because of the completeness of the OutCM set [12]-[14], [21], the basic variable $\bar{V}$, the scattering currents $\{\bar{J}^l, \bar{J}^s, \bar{J}^{vop}, \bar{M}^{vm}\}$, the scattering fields $\{\bar{E}^{sca}, \bar{H}^{sca}\}$, and the fields $\{\bar{E}^{inc}, \bar{H}^{inc}\}$ and $\{\bar{E}^{tot}, \bar{H}^{tot}\}$ on V^{mat} can be expanded in terms of the OutCM set as follows

$$\bar{V}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{V}_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{53}$$

and

$$\bar{J}^l(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{J}^l_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{54}$$

$$\bar{J}^s(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{J}^s_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{55}$$

$$\bar{J}^{vop}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{J}^{vop}_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{56.1}$$

$$\bar{M}^{vm}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{M}^{vm}_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{56.2}$$

and

$$\bar{F}^{sca}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{F}^{sca}_\xi(\bar{r}) \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{57}$$

$$\bar{F}^{X}(\bar{r}) = \sum_{\xi=1}^{\Xi} c_\xi \bar{F}^{X}_\xi(\bar{r}) \quad , \quad (\bar{r} \in V^{mat}) \tag{58}$$

here $X = inc, tot$, and $F = E, H$.

Based on the power orthogonality for OutCM set, the output power can be expanded in terms of the modal powers as follows

$$\begin{aligned} P^{out} &= \sum_{\xi=1}^{\Xi} |c_\xi|^2 P^{out}_\xi \\ &= \left(\sum_{\xi=1}^{\Xi} |c_\xi|^2 P^{out}_{\xi;sca,rad} + \sum_{\xi=1}^{\Xi} |c_\xi|^2 P^{out}_{\xi;tot,loss} \right) \\ &\quad + j\left(\sum_{\xi=1}^{\Xi} |c_\xi|^2 P^{out}_{\xi;sca,react,vac} + \sum_{\xi=1}^{\Xi} |c_\xi|^2 P^{out}_{\xi;tot,react,mat} \right) \end{aligned} \tag{59}$$

In (59), the terms corresponding to loss will disappear, if the material part is lossless.

C. Expansion coefficients.

When the external excitation is given, the interaction $\mathcal{I}$ and output power P^{out} can be respectively written as the following (60) and (61) based on the discussions in Sec. III.

$$\begin{aligned} \mathcal{I}(\bar{V}) = {} & (1/2)\langle \bar{J}^l(\bar{V}), \bar{E}^{inc}\rangle_{L^{met}} + (1/2)\langle \bar{J}^s(\bar{V}), \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \\ & + (1/2)\langle \bar{J}^{vop}(\bar{V}), \bar{E}^{inc}\rangle_{V^{mat}} + (1/2)\langle \bar{H}^{inc}, \bar{M}^{vm}(\bar{V})\rangle_{V^{mat}} \end{aligned} \tag{60}$$

$$P^{out} = P^{out}(\bar{V}) \tag{61}$$

The $\bar{E}^{inc}$ and $\bar{H}^{inc}$ in (60) are known, and the operator (61) is the same as (36).

Based on the conservation law of energy (35) and the variational principle [22], the $\bar{V}$ will make the following functional be zero and stationary.

$$\mathcal{F}(\bar{V}) = \mathcal{I}(\bar{V}) - P^{out}(\bar{V}) \tag{62}$$

Inserting the (53) and (60)-(61) into (62) and employing the Ritz's procedure [23], the following simultaneous equations for the expansion coefficients $\{c_\xi\}$ in (53)-(59) are derived for any $\xi = 1, 2, \cdots, \Xi$.

$$\begin{aligned} & (1/2)\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} + (1/2)\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \\ & + (1/2)\langle \bar{J}^{vop}_\xi, \bar{E}^{inc}\rangle_{V^{mat}} + (1/2)\langle \bar{H}^{inc}, \bar{M}^{vm}_\xi\rangle_{V^{mat}} \\ = {} & -(1/2)\left\langle \bar{J}^l_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{sca}_\zeta \right\rangle_{L^{met}} - (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^l_\zeta, \bar{E}^{sca}_\xi \right\rangle_{L^{met}} \\ & -(1/2)\left\langle \bar{J}^s_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{sca}_\zeta \right\rangle_{\partial D^{met,sv}} - (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^s_\zeta, \bar{E}^{sca}_\xi \right\rangle_{\partial D^{met,sv}} \\ & +(1/2)\left\langle \bar{J}^{vop}_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{inc}_\zeta \right\rangle_{V^{mat}} + (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^{vop}_\zeta, \bar{E}^{inc}_\xi \right\rangle_{V^{mat}} \\ & +(1/2)\left\langle \bar{H}^{inc}_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{M}^{vm}_\zeta \right\rangle_{V^{mat}} + (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{H}^{inc}_\zeta, \bar{M}^{vm}_\xi \right\rangle_{V^{mat}} \end{aligned} \tag{63.1}$$

and

$$\begin{aligned} & -(1/2)\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} - (1/2)\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \\ & -(1/2)\langle \bar{J}^{vop}_\xi, \bar{E}^{inc}\rangle_{V^{mat}} + (1/2)\langle \bar{H}^{inc}, \bar{M}^{vm}_\xi\rangle_{V^{mat}} \\ = {} & (1/2)\left\langle \bar{J}^l_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{sca}_\zeta \right\rangle_{L^{met}} - (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^l_\zeta, \bar{E}^{sca}_\xi \right\rangle_{L^{met}} \\ & +(1/2)\left\langle \bar{J}^s_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{sca}_\zeta \right\rangle_{\partial D^{met,sv}} - (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^s_\zeta, \bar{E}^{sca}_\xi \right\rangle_{\partial D^{met,sv}} \\ & -(1/2)\left\langle \bar{J}^{vop}_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{E}^{inc}_\zeta \right\rangle_{V^{mat}} + (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{J}^{vop}_\zeta, \bar{E}^{inc}_\xi \right\rangle_{V^{mat}} \\ & -(1/2)\left\langle \bar{H}^{inc}_\xi, \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{M}^{vm}_\zeta \right\rangle_{V^{mat}} + (1/2)\left\langle \sum\nolimits_{\zeta=1}^{\Xi} c_\zeta \bar{H}^{inc}_\zeta, \bar{M}^{vm}_\xi \right\rangle_{V^{mat}} \end{aligned} \tag{63.2}$$

$$c_\xi = \begin{cases} \dfrac{1}{P^{out}_\xi} \cdot \left[\dfrac{1}{2}\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \right] & , \quad (\Delta\mu, \Delta\varepsilon_c = 0) \\ \dfrac{1}{P^{out}_\xi} \cdot \left[\dfrac{1}{2}\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} \right] = \left[\dfrac{1}{P^{out}_\xi} \cdot \dfrac{1}{2}\langle \bar{H}^{inc}, \bar{M}^{vm}_\xi\rangle_{V^{mat}} \right]^* & , \quad (\Delta\mu \neq 0, \Delta\varepsilon_c = 0) \\ \dfrac{1}{P^{out}_\xi} \cdot \left[\dfrac{1}{2}\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} + \dfrac{1}{2}\langle \bar{J}^{vop}_\xi, \bar{E}^{inc}\rangle_{V^{mat}} \right] & , \quad (\Delta\mu = 0, \Delta\varepsilon_c \neq 0) \\ \dfrac{1}{P^{out}_\xi} \cdot \left[\dfrac{1}{2}\langle \bar{J}^l_\xi, \bar{E}^{inc}\rangle_{L^{met}} + \dfrac{1}{2}\langle \bar{J}^s_\xi, \bar{E}^{inc}\rangle_{\partial D^{met,sv}} + \dfrac{1}{2}\langle \bar{J}^{vop}_\xi, \bar{E}^{inc}\rangle_{V^{mat}} \right] = \left[\dfrac{1}{P^{out}_\xi} \cdot \dfrac{1}{2}\langle \bar{H}^{inc}, \bar{M}^{vm}_\xi\rangle_{V^{mat}} \right]^* & , \quad (\Delta\mu, \Delta\varepsilon_c \neq 0) \end{cases} \tag{64}$$

In (63), the relation (49) has been utilized.

By solving (63), the coefficient $\{c_\xi\}$ can be determined. If the orthogonality of (52.1) is utilized in (63), the coefficient $\{c_\xi\}$ can be concisely written as the (64) for any $\xi = 1, 2, \cdots, \Xi$.

VI. CONCLUSIONS

In this paper, an electromagnetic-power-based CMT for the metal-material combined objects is established. The material part can be lossy and inhomogeneous, and the metal part can include the line, surface, and volume structures.

The physical essence of MM-EMP-CMT is the same as the PEC-EMP-CMT and Mat-EMP-CMT. The MM-EMP-CMT has many engineering applications, such as the analysis and design for the microstrip antennas and the DRAs mounted on a metal platform.

In addition, a variational formulation for the scattering problem of metal-material combined objects is provided based on the conservation law of energy.

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