Applying the second order two-scale approximation to a dispersive wave equation

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Abstract

The method of multiple scales is applied and the second order two-scale approximation is calculated for a linear dispersive wave equation with a small perturbation proportional to the amplitude cubed.

1. Introduction

Most of differential equations can't be solved explicitly, i. e. using elementary functions. For this reason, various approximate methods exist, including perturbation methods that are used when the equation to be solved is close to a solvable equation [1].

One of such methods is the method of multiple scales that comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent [2].

2. The equation and the first order approximation

J. Murdock in [1] applies a two-scale method to get an approximated solution of the following dispersive wave partial differential equation:

$$u_{tt} - u_{xx} + u + \varepsilon u^3 = 0 \qquad (1)$$

with the initial conditions

$$u(x,0) = \sin kx,$$

$$u_t(x,0) = \omega \cos kx$$

where

$$\omega = \sqrt{1+k^2}.$$

The solution is represented in the two-scale form as

$$u(x,t,\varepsilon) = u_0(x,t,\tau) + \varepsilon u_1(x,t,\tau) + \cdots. \qquad (2),$$

where

$$\tau = \varepsilon t.$$

The next usual differentiation rules are used:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau},$$
$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial \tau \partial t} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}. \qquad (3)$$

It is found that

$$u_{1tt} - u_{1xx} + u_1 = \frac{1}{4} \sin 3\left(kx + \omega t + \frac{3}{8\omega}\tau\right) \qquad (4)$$
$$u_1(x, 0,0) = 0, \qquad (5)$$
$$u_{1t}(x, 0,0) = -u_{0\tau}(x, 0,0), \qquad (6)$$

and the first order approximation is

$$u_0(x, t, \varepsilon t) = \sin\left(kx + \omega t + \frac{3}{8\omega}\tau\right) = \sin\left(kx + \omega t + \frac{3}{8\omega}\varepsilon t\right). \qquad (7)$$

3. The second order approximation

As for higher order approximations, it is stated [1, part 5.3] that this strategy requires solving certain differential equations that may not have explicit solutions, and for this reason such calculations are not always possible in practice.

In this paper we will find the second order approximation for equation (1) that is fully expressed in terms of elementary functions.

To find u_1, we need also equations for u_2. Substituting (2) to (1) and using (3), we get:

$$\varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \dots)^3 + u_{0tt} + 2\varepsilon u_{0\tau t} + \varepsilon^2 u_{0\tau\tau} - u_{0xx} + u_0$$
$$+ \varepsilon(u_{1tt} + 2\varepsilon u_{1\tau t} + \varepsilon^2 u_{1\tau\tau} - u_{1xx} + u_1) + \varepsilon^2(u_{2tt} + 2\varepsilon u_{2\tau t} + \varepsilon^2 u_{2\tau\tau} - u_{2xx} + u_2)$$
$$+ \varepsilon^3(u_{3tt} + 2\varepsilon u_{3\tau t} + \varepsilon^2 u_{3\tau\tau} - u_{3xx} + u_3) + \dots = 0.$$

Provided

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \dots)^3 = u_0^3 + 3u_0(\varepsilon u_1)^2 + 3u_0(\varepsilon^2 u_2)^2 + 6u_0 \varepsilon u_1 \varepsilon^2 u_2$$
$$+ 3u_0^2 \varepsilon u_1 + 3u_0^2 \varepsilon^2 u_2 + O(\varepsilon^3)$$

let us equate the coefficients of ε^2 members:

$$u_{2tt} - u_{2xx} + u_2 = -(3u_0^2 u_1 + u_{0\tau\tau} + 2u_{1\tau t}). \qquad (8)$$

Designating

$$z = kx + \omega t + \frac{3}{8\omega}\tau$$

we get

$$u_0 = \sin z,$$
$$u_{0\tau} = \frac{3}{8\omega} \cos z,$$

and from (4), (5),(6)

$$u_{1tt} - u_{1xx} + u_1 = \frac{1}{4}\sin 3z \quad , \qquad (9),$$

$$u_1(x, 0{,}0) = 0, \qquad (10),$$

$$u_{1t}(x, 0{,}0) = = -u_{0\tau}(x, 0{,}0) = -\frac{3}{8\omega}\cos kx \qquad (11)$$

From (9), (10), (11) with the additional condition that the right part of (8) does not contain resonant members we can find u_1.

We will search for a partial solution of (9) that is proportional to the right part:

$$\tilde{u}_1 = p \sin 3\, z,$$

substituting this to (9) we get

$$-(3\omega)^2 p \sin 3z + (3k)^2 p \sin 3z + p \sin 3z + \frac{1}{4}\sin 3z,$$

$$p = -\frac{1}{32} \ ,$$

$$\tilde{u}_1 = -\frac{1}{32}\sin 3z. \qquad (12)$$

Now we add solutions of the homogeneous equation. As it known, they are

$$\cos(lx)\ [A\cos(wt) + B\sin(wt)],$$

$$\sin(lx)\ [A\cos(wt) + B\sin(wt)]$$

that can be represented also as

$$A \sin lx\ \ \cos(\mathrm{w}t + \psi),$$

$$B \cos lx\ \ \sin(\mathrm{w}t + \psi),$$

for any A, B, ψ , l and $w = \sqrt{1 + l^2}$.

The first solution that we add is

$$S_1 = A \sin 3kx\ \ \cos(\omega_3 t + \psi_A \tau), \qquad (13)$$

where $\omega_3 = \sqrt{1 + (3k)^2}$.

Its purpose is to compensate the effect of $\tilde{u}_1$ on (10).

The second solution is

$$S_2 = B \cos 3kx\ \ \sin(\omega_3 t + \psi_B \tau), \qquad (14)$$

whose purpose is to fulfil (11).

Also we add solutions

$$S_3 = K_1 \cos kx \sin(\omega t + \varphi) +$$
$$+ K_2 \sin kx \cos(\omega t + \varphi) +$$
$$+ K_3 \sin kx \sin(\omega t + \varphi) +$$
$$+ K_4 \cos kx \cos(\omega t + \varphi), \qquad (15)$$

where $\varphi = \frac{3}{8\omega}\tau$.

These expressions are used to fulfil (11), because of (8) and especially of $u_0^2 u_1$ term.

K_1, K_2, K_3, K_4, ψ_A, ψ_B can depend on τ. A and B also can depend on τ, but at the end they will be found to be constant. The phases in (13), (14) can depend on τ in more general way, but the linear dependency was found to be enough. The phases in (15) are left like (7) because of interaction with $u_0^2 u_1$ term.

Now

$$u_1 = \tilde{u}_1 + S_1 + S_2 + S_3 \qquad (16)$$

and by differentiation we get

$$\begin{aligned} u_{1t} = & -\frac{1}{32} \cdot 3\omega \cos 3z \\ & -A\omega_3 \sin 3kx \sin(\omega_3 t + \psi_A \tau) \\ & +B\omega_3 \cos 3kx \cos(\omega_3 t + \psi_B \tau) \\ & + K_1\, \omega \cos kx \cos(\omega t + \varphi) + \\ & - K_2\, \omega \sin kx \sin(\omega t + \varphi) + \\ & +K_3\, \omega \sin kx \cos(\omega t + \varphi) + \\ & - K_4 \omega \cos kx \sin(\omega t + \varphi) \end{aligned} \qquad (17)$$

From (10) for t = τ = 0 we get

$$A = \frac{1}{32}. \qquad (18)$$

From (11), (17) for t = τ = 0 we get

$$B = \frac{3\omega}{32\omega_3}. \qquad (19)$$

Differentiating (17) by τ, we get

$$\begin{aligned} u_{1t\tau} = & \frac{27}{256} \sin 3z \\ & -A\psi_A \omega_3 \sin 3kx \cos(\omega_3 t + \psi_A \tau) \\ & -B\psi_B \omega_3 \cos 3kx \sin(\omega_3 t + \psi_B \tau) \\ & -\frac{3}{8} K_1 \cos kx \sin(\omega t + \varphi) \\ & +K_1'\, \omega \cos kx \cos(\omega t + \varphi) \\ & -\frac{3}{8} K_2 \sin kx \cos(\omega t + \varphi) \\ & -K_2'\, \omega \sin kx \sin(\omega t + \varphi) \\ & -\frac{3}{8} K_3 \sin kx \sin(\omega t + \varphi) \end{aligned}$$

$$+K_3'\,\omega \sin kx \cos(\omega t+\varphi)$$
$$-\frac{3}{8}K_4 \cos kx \cos(\omega t+\varphi)$$
$$-K_4'\,\omega \cos kx \sin(\omega t+\varphi).$$

Now we check all possible sources of resonant terms of kind *l = 3k* in the right part of (8).

Transform

$3u_0^2u_1 = 3\sin^2 z \cdot u_1 = 3 \cdot \frac{1-\cos 2z}{2} \cdot u_1 = \frac{3}{2}u_1 - \frac{3}{2}\cos 2z \cdot u_1$.

The multiplication of (13), (14) with *cos2z* cannot produce resonant members because these trigonometric functions are based on different values of *l* and hence the coefficients of *x* and *t* are not proportional. Hence the only possible sources of resonant terms are $2u_{1\tau t}$ and $\frac{3}{2}u_1$.

Let's *sc* be $\sin 3kx \cos(\omega_3 t + \psi_A\tau)$ and *cs* be $\cos 3kx \sin(\omega_3 t + \psi_B\tau)$.

All resonant members are shown in the next table:

coefficient	**source**	**sc**	**cs**
2	$u_{1\tau t}$	$-A\psi_A\omega_3$	$-B\psi_B\omega_3$
$\frac{3}{2}$	u_1	A	B

From this,

$$-2A\psi_A\omega_3 + \frac{3}{2}A = 0,$$
$$-2B\psi_B\omega_3 + \frac{3}{2}B = 0,$$
$$\psi_A = \psi_B = \frac{3}{4\omega_3}\ .$$

As for $\tilde{u}_1$, that was defined in (12), it can't create terms of *l=3k* type.

Now we will check (8) for *l=k* resonances to find K_1, K_2, K_3, K_4.

Let's *sc* be $\sin kx \cos(\omega t+\varphi)$ and we designate other similar expressions as *sc, ss, cc*.

All resonant members of this type are shown in the next table:

Source	**Coefficient**	**Resonant members**			
Interaction of *l=k*	$-\frac{3}{2}\cdot\frac{1}{4}$	*(-sc –cs)* K_1	*(-sc –cs)* K_2	*(-cc +ss)* K_3	*(cc -ss)* K_4

members with *cos2z*					
$\frac{3}{2}u_1$	$\frac{3}{2}$	$cs\ K_1$	$sc\ K_2$	$ss\ K_3$	$cc\ K_4$
$2u_{1\tau t}$	2	$K_1'\omega cc - K_1\omega\frac{3}{8\omega}cs$	$-K_2'\omega ss - K_2\omega\frac{3}{8\omega}sc$	$K_3'\omega sc - K_3\omega\frac{3}{8\omega}ss$	$K_4'\omega cs - K_4\omega\frac{3}{8\omega}cc$
$u_{0\tau\tau}$	$-\frac{9}{64\omega^2}$	*cs + sc*			
Interaction of $\tilde{u}_1$ with *cos2z*	$-\frac{3}{2}\cdot\left(-\frac{1}{32}\right)\cdot\frac{1}{2}$	*cs + sc*			

For checking interactions with *cos2z* we used identities like

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

and

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

and chose only potentially resonant members.

Now we separate members for cs, sc, ss, cc.

$$\frac{3}{8}K_1+\frac{3}{8}K_2+\frac{3}{2}K_1-\frac{2\cdot 3}{8}K_1-2K_4'\omega-\frac{9}{64\omega^2}+\frac{3}{128}=0 \text{ (for } cs\text{)},$$

$$\frac{3}{8}K_1+\frac{3}{8}K_2+\frac{3}{2}K_2-\frac{2\cdot 3}{8}K_2+2K_3'\omega-\frac{9}{64\omega^2}+\frac{3}{128}=0 \text{ (for } sc\text{)},$$

$$-\frac{3}{8}K_3+\frac{3}{8}K_4+\frac{3}{2}K_3-2K_2'\omega\ -\frac{2\cdot 3}{8}K_3=0 \text{ (for } ss\text{)},$$

$$\frac{3}{8}K_3-\frac{3}{8}K_4+\frac{3}{2}K_4+2K_1'\omega\ -\frac{2\cdot 3}{8}K_4=0 \text{ (for } cc\text{)},$$

From here

$$K_1'=-\frac{3}{16\omega}K_3-\frac{3}{16\omega}K_4,$$

$$K_2'=+\frac{3}{16\omega}K_3+\frac{3}{16\omega}K_4,$$

$$K_3'=-\frac{9}{16\omega}K_2-\frac{3}{16\omega}K_1-b,$$

$$K_4'=\frac{9}{16\omega}K_1+\frac{3}{16\omega}K_2+b,$$

$$\text{where } b=\left(\frac{3}{128}-\frac{9}{64\omega^2}\right)\cdot\frac{1}{2\omega}. \qquad (20)$$

Substituting (17) to (11) and equating like terms we find

$$K_1(0) = -\frac{3}{8\omega^2},$$

$$K_3(0) = 0\,,$$

and substituting (17) to (10) we find

$$K_2(0) = 0,$$

$$K_4(0) = 0. \qquad (21)$$

Now (20), (21) is a system of 1st order linear equations with constant coefficients. The solution of the system is:

$$K_1(\tau) = -\frac{3}{8\omega^2},$$

$$K_2(\tau) = 0,$$

$$K_3(\tau) = -\frac{3(\omega^2-24)\tau}{256\omega^3},$$

$$K_4(\tau) = \frac{3(\omega^2-24)\tau}{256\omega^3}.$$

Substituting (12), (13),(14),(15),(18),(19) to (16), we finally get

$$u_1(x,t,\tau)=-\frac{1}{32}\sin 3\left(kx+\omega t+\frac{3}{8\omega}\tau\right)$$

$$+\frac{1}{32}\sin 3kx\cos\left(\omega_3 t+\frac{3}{4\omega_3}\tau\right)$$

$$+\frac{3\omega}{32\omega_3}\cos 3kx\sin\left(\omega_3 t+\frac{3}{4\omega_3}\tau\right)$$

$$-\frac{3}{8\omega^2}\cos kx\sin\left(\omega t+\frac{3}{8\omega}\tau\right)$$

$$-\frac{3(\omega^2-24)\tau}{256\omega^3}\sin kx\sin\left(\omega t+\frac{3}{8\omega}\tau\right)$$

$$+\frac{3(\omega^2-24)\tau}{256\omega^3}\cos kx\cos\left(\omega t+\frac{3}{8\omega}\tau\right)=$$

$$=-\frac{1}{32}\sin 3\left(kx+\omega t+\frac{3}{8\omega}\tau\right)$$

$$+\frac{1}{32}\sin 3kx\cos\left(\omega_3 t+\frac{3}{4\omega_3}\tau\right)$$

$$+\frac{3\omega}{32\omega_3}\cos 3kx\sin\left(\omega_3 t+\frac{3}{4\omega_3}\tau\right)$$

$$-\frac{3}{8\omega^2}\cos kx\sin\left(\omega t+\frac{3}{8\omega}\tau\right)$$

$$+\frac{3(\omega^2-24)\tau}{256\omega^3}\cos\left(kx+\omega t+\frac{3}{8\omega}\tau\right),$$

where

$$\omega=\sqrt{1+k^2},\ \omega_3=\sqrt{1+(3k)^2}$$

and the full second order approximation according to (2) is

$$u(x, t, \varepsilon) = u_0(x, t, \tau) + \varepsilon u_1(x, t, \tau).$$

References

[1] James A. Murdock. *Perturbations: Theory and Methods*: SIAM, Philadelphia, 1999.

[2] “Multiple-scale analysis”. Wikipedia, https://en.wikipedia.org/wiki/Multiple-scale_analysis.

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