ON A QUESTION CONCERNING THE LITTLEWOOD VIOLATIONS (1. 0. 0)

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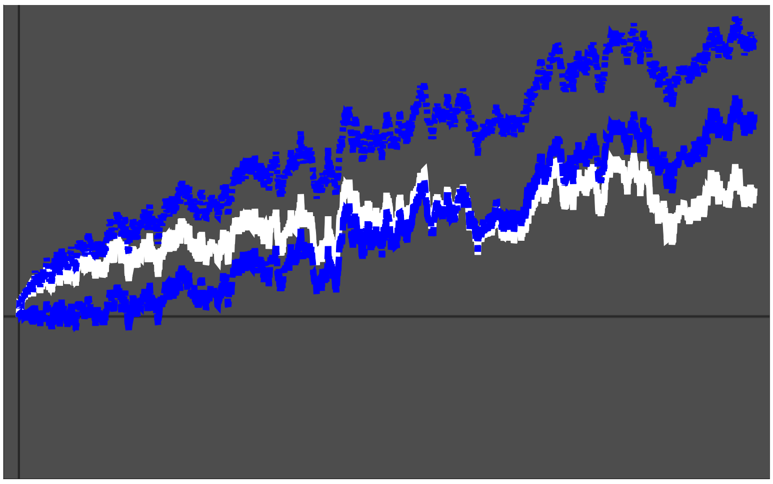
Riemann's prime-counting function $R(x) = \sum_{n}^{\infty} \frac{\mu(n)\,\mathrm{li}(x^{1/n})}{n}$ looks good for every value of x we can compute, but in the light of Littlewood's result its superiority over li(x) is illusory: Ingram (1938) pointed out that 'for special values of x (as large as we please), the one approximation will deviate as widely as the other from the true value'. This note introduces a type of prime-counting function that is *always* better than li(x)...

PART I

Compare li(x) to Legendre's function $(\frac{x}{\mathrm{Log}[x]-1.08366})$. For small values of x, the latter is superior, but this superiority is short-lived. However, if we *modify* li(x) slightly, we find that this modification has the same defect as $(\frac{x}{\mathrm{Log}[x]-1.08366})$ - it stays with the target function for a few million x and then diverges from it in a dramatic fashion.



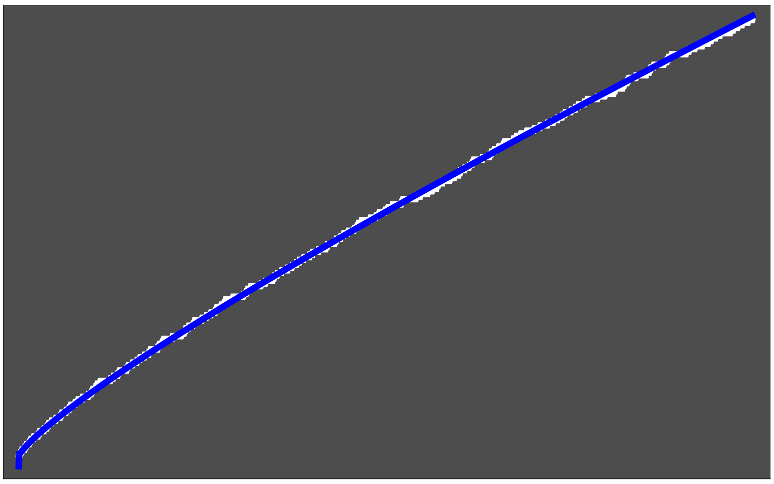
$\mathrm{li}(x) - \pi(x)$

$\frac{x}{\log(x)-1.08366} - \pi(x)$

$\mathrm{li}(x)\,\zeta(11) - \pi(x)$

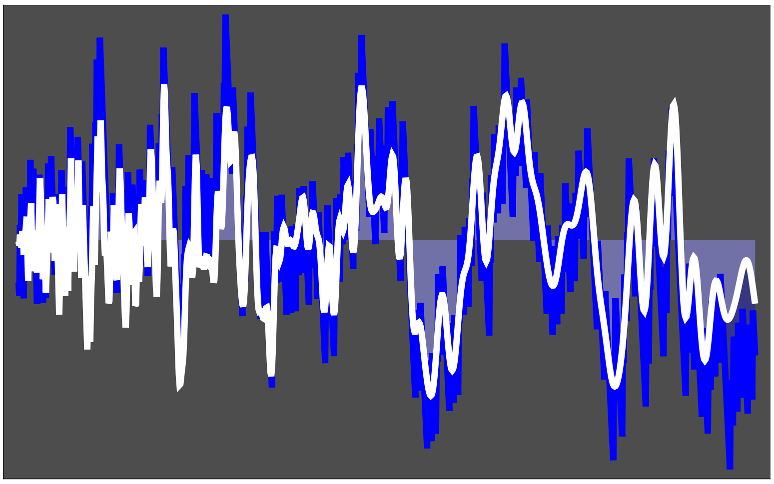
It was shown by Littlewood (1914) that, while li(x) overestimates π(x) for any value of x we can presently compute, it eventually underestimates it, and thereafter changes signs infinitely many times. These considerations raise a natural question: do the differences between these other functions and π(x) increase without bound, or do they oscillate?

While the traditional prime-counting and prime-simulation functions, are logarithmic, the relative smallness of the difference between log(x) and H_x implies the existence of *harmonic* functions, functions based on harmonic numbers, rather than logarithms. These harmonic functions will for certain values of x possess a similar degree of accuracy. A cursory examination reveals that they are naturally *more* accurate than their logarithmic counterparts for small x, and their error terms exhibit oscillatory behavior that we are able to survey. Reimann's function is a *logarithmic* function, but we can readily construct comparatively effective prime-counting, and prime-simulation functions, which involve summing logarithmic *and* harmonic functions



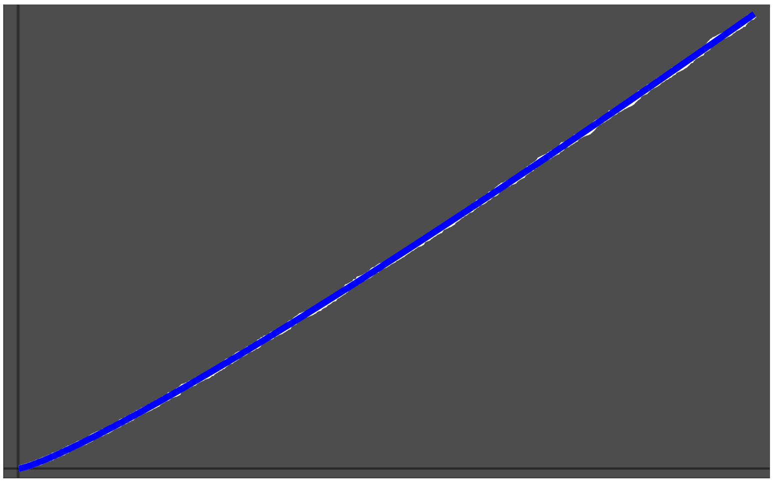
$\pi(x)$

$$\frac{\int_1^x \left(\sum_{n=2}^{x} \frac{a1}{H_x a1}\right) dn + \int_1^x \left(\sum_{n=2}^{x} \frac{a2}{\log(a2\, x)}\right) dn}{x}$$



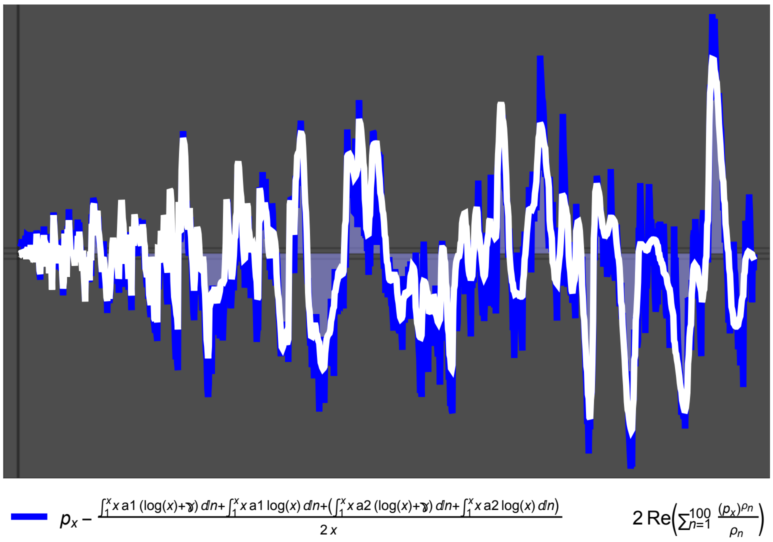
$$\pi(x) - \frac{\int_1^x \left(\sum_{n=2}^{x} \frac{a1}{H_x a1}\right) dn + \int_1^x \left(\sum_{n=2}^{x} \frac{a2}{\log(a2\, x)}\right) dn}{x}$$

$$-2\,\mathrm{Re}\left(\sum_{n=1}^{100} \mathrm{Ei}(\rho_{-n} \log(x))\right)$$



p_x

$$\frac{\int_1^x x\, a1\,(\log(x)+\gamma)\, dn + \int_1^x x\, a1 \log(x)\, dn + \left(\int_1^x x\, a2\,(\log(x)+\gamma)\, dn + \int_1^x x\, a2 \log(x)\, dn\right)}{2\, x}$$



Consider now the possibility that some of these functions -both logarithmic and harmonic- are waves resembling the waves of the Littlewood Violations in that they a) oscillate infinitely about the *primes* (or what is the equivalent thing that the primes oscillate infinitely about them) and that b) their wave lengths are so long that it is difficult if not impossible to survey these oscillatory properties directly. This possibility is consistent with an error term that a) takes the form of a wave, and that b) seems to be converging, function by function, to a wave that *does* oscillate infinitely. That these waves appear to be fractals suggests the possibility that they are self-similar at every scale.

PART II

The deepest underlying concept of this inquiry is 'singularity'. There are various definitions, all of which have some connection to infinity: in mathematics, singularity is the inevitable result of division by 0, and any such division involves an infinite answer. In physics, singularity is the result of some quantity taking an infinite value. If for example the curvature of space-time is infinite, the result is a *gravitational* singularity, a consequence of which is that an infinite quantity of potential energy is 'contained' within a zero amount of space (an idea that is danger of making no sense), and you get the distinction between the singularity at the beginning of time ('the singularity at The Big Bang') and that at the end of time (the singularities at the centers of 'Black Holes'). A simplifying definition derives from the concept of a circle of area 1. If E/1 = E, then the total energy of any system governed by a circle of area 1 should have the same intensity from center to circumference. This uniform intensity is what we would expect if there was no radius (a point), or if the radius was infinite (a line). Let E be the total amount of energy in the universe, and E/0 is a state of infinite contraction, while E/1 is a state of infinite expansion. That neither of these extremes is compatible with arithmetic -and nor therefore with physics- is suggested by the radius of a circle of area 1 arising from the π r^2 formula - $\frac{1}{\sqrt{\pi}}$.

There is something mysterious about the presence of π in this radius, since π is a quantity that on the face of it seems too static to yield the potentially infinite number of energy levels a system poised between infinite contraction and infinite expansion presumably requires. Re-expressing the traditional formula for the area of a circle of area 1 as a limit, apparently solves this mystery, and provides an interesting way to look at the error term of the prime-counting and prime-simulation functions:

$$\pi \frac{1}{\sqrt{\pi}} = 1$$

becomes first

$$e^{2\gamma} (e^{-\gamma})^2 = 1$$

and then

$$e^{2\left(\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n^1}\,dn\right)} \left(e^{-\left(\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n}\,dn\right)}\right)^2 = 1$$

and finally infinite flexibility is obtained in this way

$$\lim_{x\to\infty} (e^{-\gamma})^2 \, e^{2\left(\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n} \, dn\right)} = 1$$

or in this way

$$\lim_{x\to\infty} \left(\frac{e^{2\gamma}}{e^{\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n} \, dn}} \right)^2 = 1$$

or in a wide variety of ways... A circle may be described as a spiral of zero growth, a line may be described as a circle of infinite radius, and so a circle -and a line- may be described also as a spiral of infinite growth, and so these limits don't describe a circle so much as they describe a spiral whose limit is a circle. There is always a 'gap' that stops the area from achieving the value of 1, and true circularity. No matter how close to 1, the area is always a fraction of unity. If it were equal 1, then either the energy of the point source would be undistributed (E/0) or fully distributed over the area (E/1).

As the radius of a traditional unit circle grows arithmetically, its area grows quadratically, and Newton's law of gravity (1726) is based on the inference that, since gravity is governed by the geometry of the unit circle, gravitational attraction must be inversely proportional to the distance from the center of gravity. A unit circle based on the *area* rather than radius reminds us that, while the inverse square law focuses on quantities that vary inversely as a square of the radius, there is a related inverse *square-root* law that concerns quantities that vary inversely as a *square-root* of the area. If we regard the area of a circle of area 1 in this way

$$\lim_{x\to\infty} e^{2\gamma} \sqrt{\frac{1}{e^{2\left(\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n} \, dn\right)}}^{\,2} = 1$$

we get

$$\lim_{x\to\infty} e^{2\gamma} \sqrt{\frac{n}{e^{2\left(\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n} \, dn\right)}}^{\,2} = n$$

As the area grows arithmetically, so that

$$e^{2\gamma} \sqrt{\frac{n}{e^{2\gamma}}}^{\,2} = A$$

the radius grows inverse-quadratically so that

$$\sqrt{\frac{n}{e^{2\gamma}}} = R$$

We might on this basis regard the difference between an ideal logarithmic + harmonic count of the primes and the primes -the 'error term'- as a wave whose amplitude grows at the rate $\sqrt{\frac{x}{e^{2\gamma}}}$ ...

If we re-arrange the equation $e^{2\gamma} (e^{-\gamma})^2 = 1$ to get

$$\lim_{x\to\infty} (e^{-\gamma})^2 \, e^{2\left(\sum_{n=1}^{x} \frac{1}{n^s} - \int_1^x \frac{1}{n^s} \, dn\right)} = e^{2\gamma} \left(e^{-\left(\zeta(s) - \frac{1}{s-1}\right)}\right)^2$$

we can go to

$$\sqrt[s]{\frac{e^{2\gamma} \left(e^{-\left(\zeta(s) - \frac{1}{s-1}\right)}\right)^2}{e^{2\gamma} \left(e^{-\left(\zeta(s) - \frac{1}{s-1}\right)}\right)^2 - e^{2\gamma} \left(\exp\left(-\left(\sum_{n=1}^{p_x} \frac{1}{n^s} - \int_1^{p_x} \frac{1}{n^s} \, dn\right)\right)\right)^2}}$$

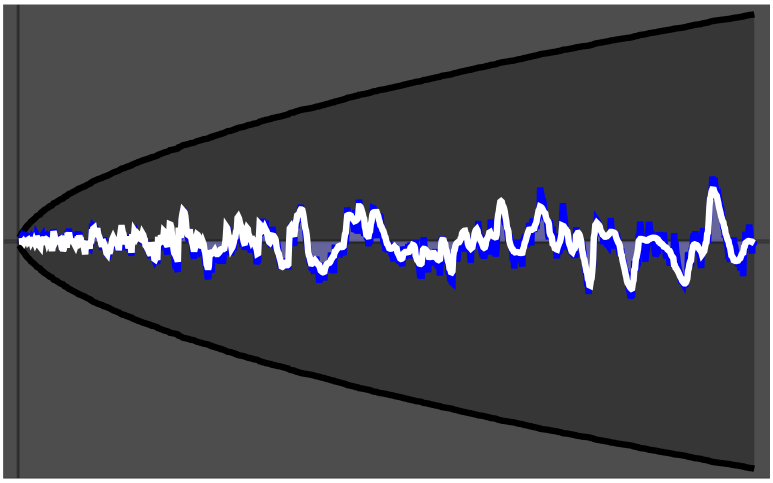
And from here to

$$\sqrt[s]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^s} - \int_1^{p_x}\frac{1}{n^s}\,dn\right)\right)\right)^2}} - \frac{\int_1^x \mathrm{a1}\,x\,H_x\,dn + \int_1^x \mathrm{a1}\,x\log(x)\,dn + \left(\int_1^x \mathrm{a2}\,x\,H_x\,dn + \int_1^x \mathrm{a2}\,x\log(x)\,dn\right)}{2x}$$

If and only if $s = 1$, in which case

$$\sqrt[s]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^s} - \int_1^{p_x}\frac{1}{n^s}\,dn\right)\right)\right)^2}} = \frac{1}{1 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^1} - \int_1^{p_x}\frac{1}{n^1}\,dn\right)\right)\right)^2}$$

we see this picture



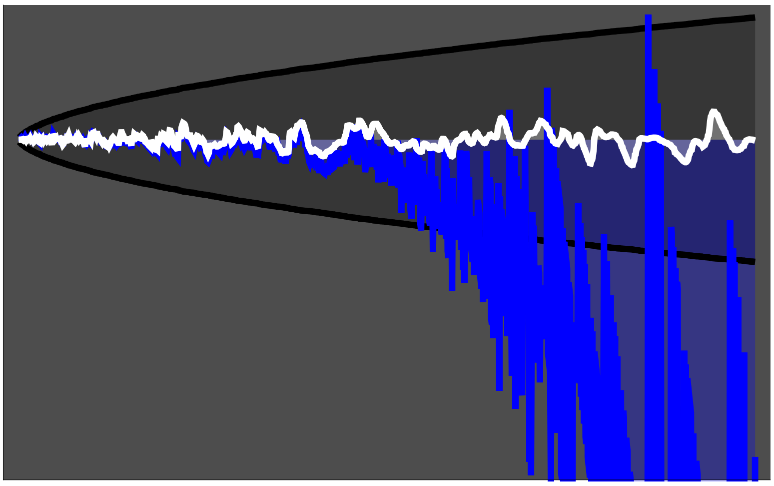
$\sqrt{e^{2\gamma} p_x}$

$\frac{1}{1-e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^1}-\int_1^{p_x}\frac{1}{n^1}dn\right)\right)\right)^2} - \frac{\int_1^x x\,\mathrm{a1}\,(\log(x)+\gamma)\,dn + \int_1^x x\,\mathrm{a1}\log(x)\,dn + \left(\int_1^x x\,\mathrm{a2}\,(\log(x)+\gamma)\,dn + \int_1^x x\,\mathrm{a2}\log(x)\,dn\right)}{2x}$

$2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$

However, if $s \neq 1$ then -given that we are concerned here with real-valued functions- we see a wave form that rapidly breaks free of the square root bound once the ratio of the point source to the difference between partial sum/integral and the limit reaches a critical minimum.

If $s = 4$

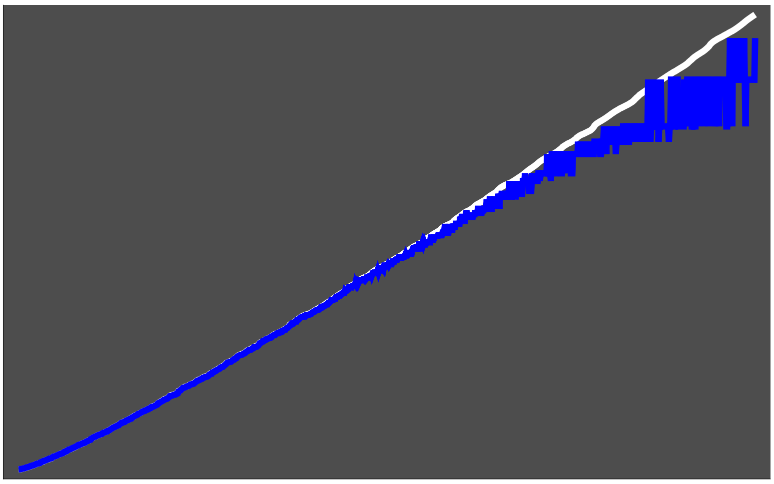


$\sqrt{e^{2\gamma} p_x}$

$$\sqrt[4]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(4)-\frac{1}{4-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(4)-\frac{1}{4-1}\right)}\right)^2-e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^4}-\int_1^{p_x}\frac{1}{n^4}\,dn\right)\right)\right)^2}} - \frac{\int_1^x x\,\mathrm{a1}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a1}\log(x)\,dn+\left(\int_1^x x\,\mathrm{a2}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a2}\log(x)\,dn\right)}{2x}$$

$$2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$

The same phenomenon from another point of view

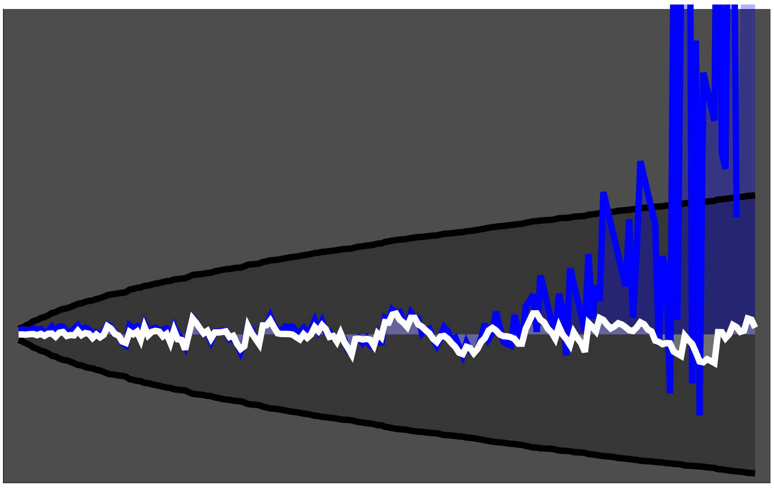


$$\frac{\int_1^x x\,\mathrm{a1}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a1}\log(x)\,dn+\left(\int_1^x x\,\mathrm{a2}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a2}\log(x)\,dn\right)}{2x}+2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$

$$\sqrt[4]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(4)-\frac{1}{4-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(4)-\frac{1}{4-1}\right)}\right)^2-e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^4}-\int_1^{p_x}\frac{1}{n^4}\,dn\right)\right)\right)^2}}$$

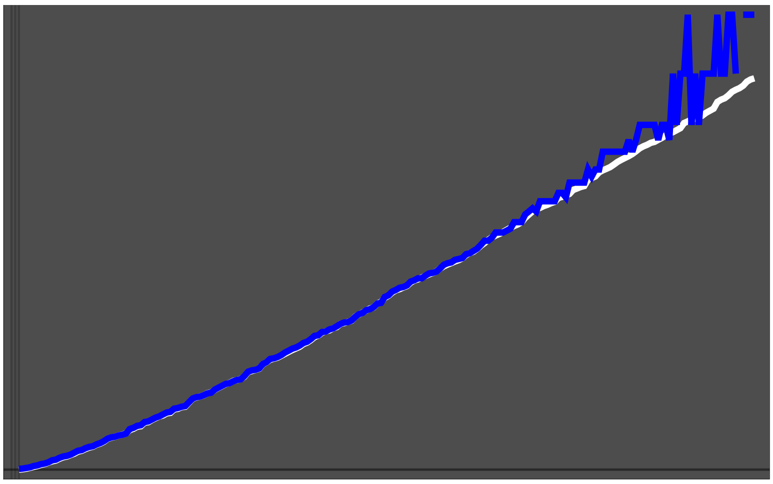
We can observe that if s is a real number greater than 1, then as the value of s increases in size, the wave decreases in length.

If $s = 5$



$$\sqrt{e^{2\gamma} p_x}$$

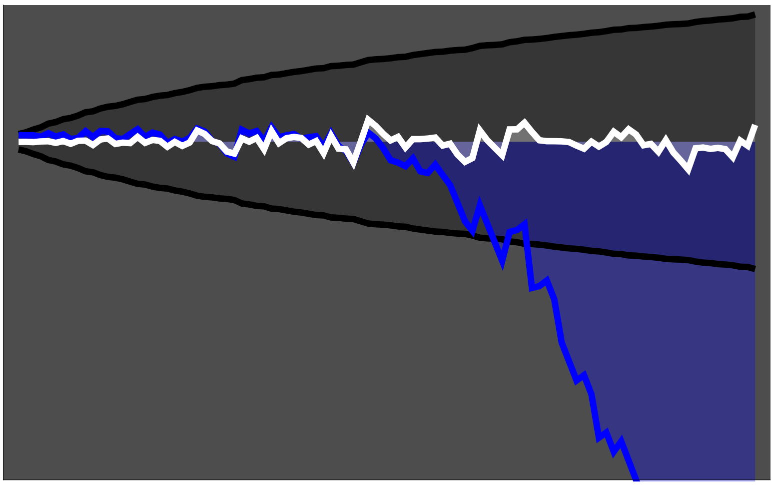
$$\sqrt[5]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(5)-\frac{1}{5-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(5)-\frac{1}{5-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^5}-\int_1^{p_x}\frac{1}{n^5}\,dn\right)\right)\right)^2}} - \frac{\int_1^x x\,\mathrm{a1}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a1}\log(x)\,dn+\left(\int_1^x x\,\mathrm{a2}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a2}\log(x)\,dn\right)}{2x} \qquad 2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$



$$\frac{\int_1^x x\,\mathrm{a1}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a1}\log(x)\,dn+\left(\int_1^x x\,\mathrm{a2}\,(\log(x)+\gamma)\,dn+\int_1^x x\,\mathrm{a2}\log(x)\,dn\right)}{2x} + 2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$

$$\sqrt[5]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(5)-\frac{1}{5-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(5)-\frac{1}{5-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^5}-\int_1^{p_x}\frac{1}{n^5}\,dn\right)\right)\right)^2}}$$

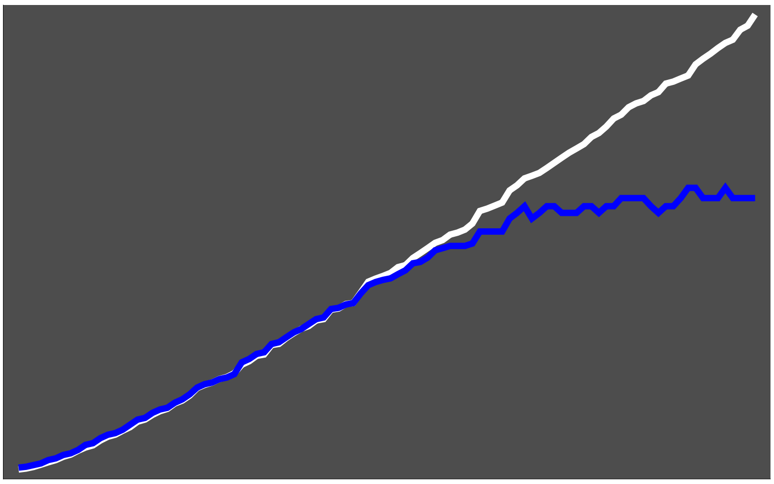
If $s = 6$



$$\sqrt{e^{2\gamma}\, p_x}$$

$$\sqrt[6]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(6)-\frac{1}{6-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(6)-\frac{1}{6-1}\right)}\right)^2-e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^6}-\int_1^{p_x}\frac{1}{n^6}\,dn\right)\right)\right)^2}}-\frac{\int_1^x x\,a1\,(\log(x)+\gamma)\,dn+\int_1^x x\,a1\log(x)\,dn+\left(\int_1^x x\,a2\,(\log(x)+\gamma)\,dn+\int_1^x x\,a2\log(x)\,dn\right)}{2x}$$

$$2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$



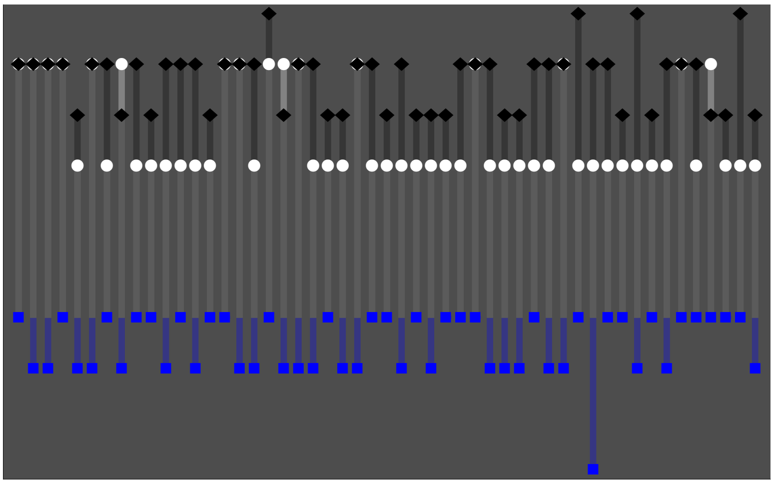
$$\frac{\int_1^x x\,a1\,(\log(x)+\gamma)\,dn+\int_1^x x\,a1\log(x)\,dn+\left(\int_1^x x\,a2\,(\log(x)+\gamma)\,dn+\int_1^x x\,a2\log(x)\,dn\right)}{2x}+2\,\mathrm{Re}\left(\sum_{n=1}^{100}\frac{(p_x)^{\rho_n}}{\rho_n}\right)$$

$$\sqrt[6]{\frac{e^{2\gamma}\left(e^{-\left(\zeta(6)-\frac{1}{6-1}\right)}\right)^2}{e^{2\gamma}\left(e^{-\left(\zeta(6)-\frac{1}{6-1}\right)}\right)^2-e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^6}-\int_1^{p_x}\frac{1}{n^6}\,dn\right)\right)\right)^2}}$$

These ‘short-ranged’ waves come to the end of their range as

$$e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^s} - \int_1^{p_x}\frac{1}{n^s}\,dn\right)\right)\right)^2$$

tends to 0, when the following kind of pattern presents itself



$$e^{2\gamma}\left(e^{-\left(\zeta(4)-\frac{1}{4-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^4} - \int_1^{p_x}\frac{1}{n^4}\,dn\right)\right)\right)^2$$

■ $$e^{2\gamma}\left(e^{-\left(\zeta(5)-\frac{1}{5-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^5} - \int_1^{p_x}\frac{1}{n^5}\,dn\right)\right)\right)^2$$

◆ $$e^{2\gamma}\left(e^{-\left(\zeta(6)-\frac{1}{6-1}\right)}\right)^2 - e^{2\gamma}\left(\exp\left(-\left(\sum_{n=1}^{p_x}\frac{1}{n^6} - \int_1^{p_x}\frac{1}{n^6}\,dn\right)\right)\right)^2$$

But this loss of arithmetic growth is something that can occur only if $s \neq 1$: if $s = 1$, then arithmetic growth is potentially infinite, and there can be no solutions to

$$e^{2\gamma}\left(e^{-\gamma}\right)^2 - e^{2\gamma}\left(e^{-\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x\frac{1}{n}\,dn\right)}\right)^2 = 0$$

If and only if $s \neq 1$, there are infinitely many solutions to

$$e^{2\gamma}\left(e^{-\left(\zeta(s)-\frac{1}{s-1}\right)}\right)^2 - e^{2\gamma}\left(e^{-\left(\sum_{n=1}^{x}\frac{1}{n^s}-\int_1^x\frac{1}{n^s}\,dn\right)}\right)^2 = 0$$

PART III

$e^{2\gamma}\left(e^{-\gamma}\right)^2 = 1$ implies that the primes are distributed in a special way. In particular, it implies that the growth rate of the spiral governing their global distribution is limited by

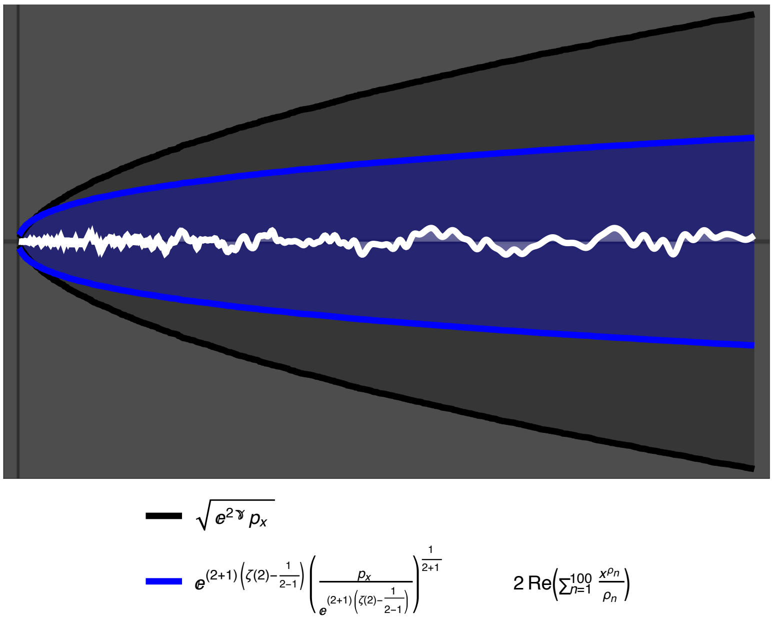
$$e^{2\gamma}\sqrt{\frac{A}{e^{2\gamma}}}^{\,2} = A$$

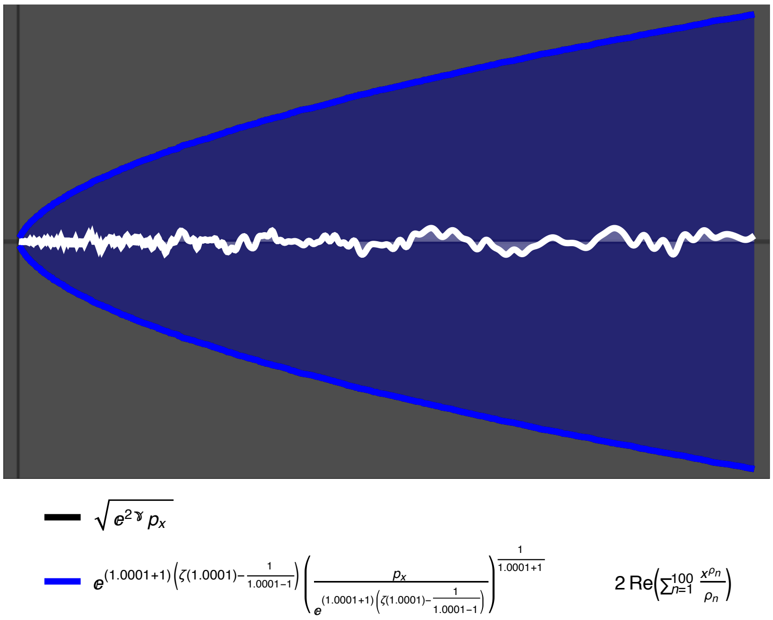
Yet the existence of an infinite number of prime-spirals growing at greater or lessor rates is a consequence of our starting equation, from which we get

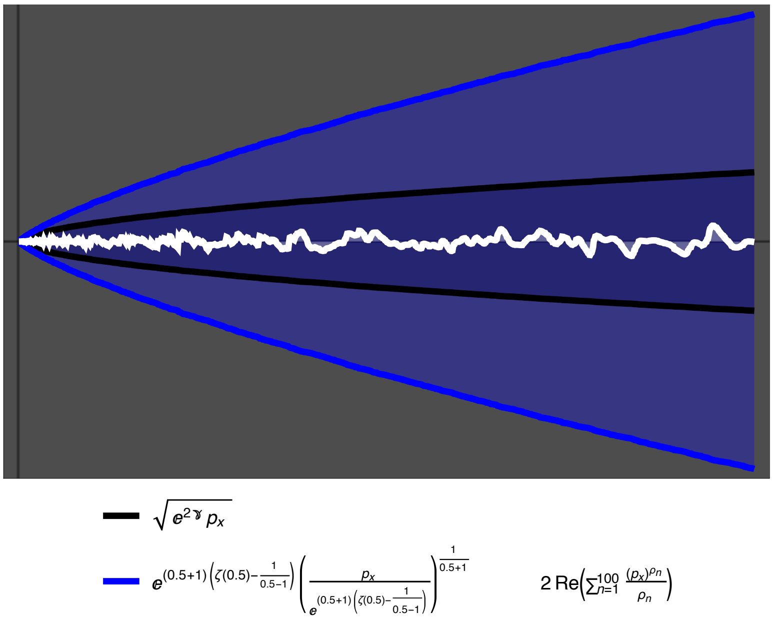
$$e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}\left(\left(\frac{A}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = A$$

If and only if $s = 1$, then

$$e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}\left(\left(\frac{A}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = e^{2\gamma}\sqrt{\frac{A}{e^{2\gamma}}}^{\,2} = A$$



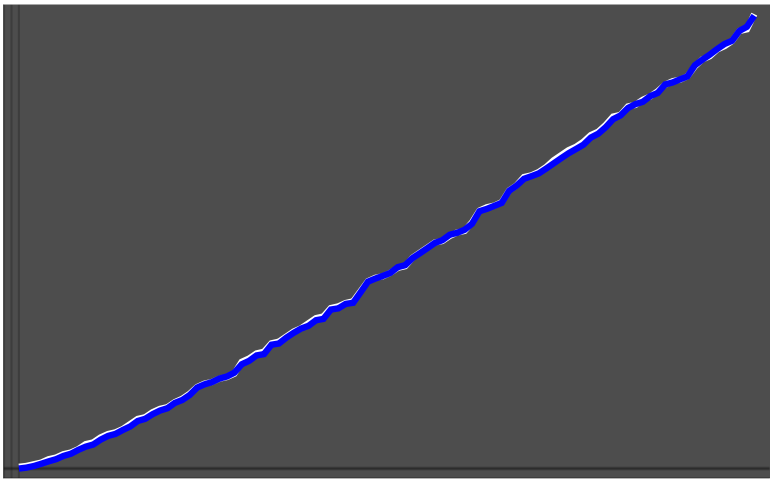




And the analysis above indicates that any prime-spiral whose growth rate is greater or less than the growth rate of $\lim_{x\to\infty} e^{2\gamma}\sqrt{\frac{A}{e^{2\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x \frac{1}{n}dn\right)}}}^{2} = A$ ultimately ceases its unique development/begins to intersect itself, and that such spirals correspond to strictly finite amounts of prime-density.

If we flip a fair coin multiple times, we arrive at balance of heads and tails, give or take a margin or error that has square root size. We can describe this situation in terms of a spiral, and indeed *any* random sequence of 1 and -1s, or 1s and 0s etc. is associated with a spiral in the sense that the balance of alternatives is limited by the dimensions of a spiral growing at the rate limited by $e^{s+1}(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)\left(\left(\frac{A}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = e^{2\gamma}\sqrt{\frac{A}{e^{2\gamma}}}^{2} = A$.

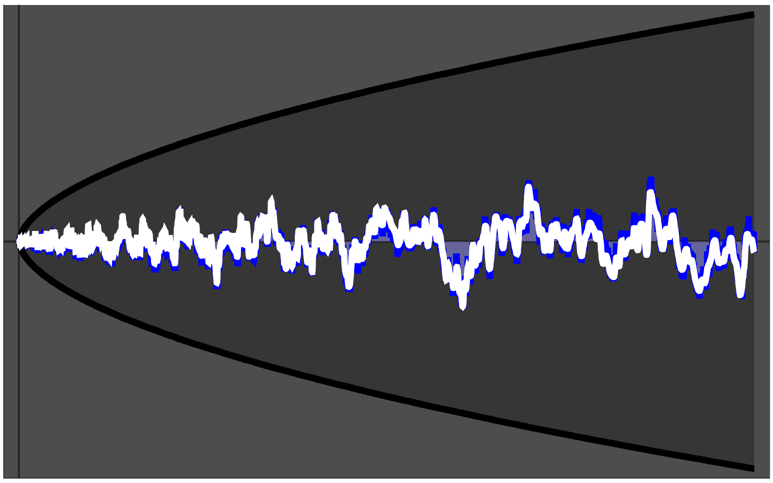
Consider in this light the following figure.



p_x

$\frac{\int_1^x x\, \text{a1}\, (\log(x)+\gamma)\, dn + \int_1^x x\, \text{a1} \log(x)\, dn + \left(\int_1^x x\, \text{a2}\, (\log(x)+\gamma)\, dn + \int_1^x x\, \text{a2} \log(x)\, dn\right)}{2\, x} + 2\, \text{Re}\left(\sum_{n=1}^{100} \frac{(p_x)^{\rho_n}}{\rho_n}\right)$

As far as we can observe, the upward and downward spikes follow the sum $2\,\text{Re}\left(\sum_{n=1}^{x} \frac{(p_x)^{\rho_n}}{\rho_n}\right)$, and so as far as we can observe a upward spike is as likely as a downward spike, i.e. although prime-density necessarily decreases from a global perspective, it is equally likely to be increasing or decreasing from a local perspective.



$\sqrt{e^{2\gamma}\, p_x}$

$p_x - \frac{\int_1^x x\, \text{a1}\, (\log(x)+\gamma)\, dn + \int_1^x x\, \text{a1} \log(x)\, dn + \left(\int_1^x x\, \text{a2}\, (\log(x)+\gamma)\, dn + \int_1^x x\, \text{a2} \log(x)\, dn\right)}{2\, x}$ $\quad 2\, \text{Re}\left(\sum_{n=1}^{500} \frac{(p_x)^{\rho_n}}{\rho_n}\right)$

Assume now for sake of argument that there is some n such that the real part of p_n is other than 1/2. Then the local fluctuations of the primes are ultimately *not* random, and are governed by spirals growing at a greater or lessor rate than $e^{2\gamma} \sqrt{\frac{A}{e^{2\gamma}}}^{2}$. Any such spiral we have seen is capable of but a finite amount of unique development. Since the primes are infinite and unique, it is not possible that their global distribution be governed by one of these finite spirals, from which it follows that the assumption is false. Arithmetic growth, it follows, is dependent on a quantity -prime-density- that is globally time-irreversible (the real part of p_n is not equal to 1), but is locally time-reversible (the real part of p_n is equal to *1/2*).

We can express this argument with reference to the concept of smooth surfaces. If $s = 1$, the surface areas corresponding to our spiral have holes in them corresponding to the differences between spirals and circles. The holes in these surfaces area can be continuously contracted whilst the surface remains constant, or equivalently, the surfaces can be continuously expanded while the holes remain constant. A hole cannot be made infinitely small, or a surface infinitely large, because of the constraint imposed by the difference between the difference between the partial sum/integral $\sum_{n=1}^{x} \frac{1}{n} - \int_1^x \frac{1}{n} dn$ and γ -there is a gap whose size is at least $\frac{1}{2x}$ - but such surfaces are as smooth as possible. The smoothness of these surfaces leaves us free to assume that the radius, or 'π', or some other dimension in the formula $e^{2\gamma}(e^{-\gamma})^2 = 1$ is constant and let the area vary continuously, or to assume the area is constant and let the radius, or π, or some other dimension vary continuously. Early on we saw that

$$\lim_{x\to\infty} e^{2\gamma}\left(e^{-\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x\frac{1}{n}dn\right)}\right)^2 = 1$$

which can be expressed as

$$\lim_{x\to\infty} e^{2\gamma}\left(\frac{1}{e^{2\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x\frac{1}{n}dn\right)}}\right)^2 = 1$$

but other possible expressions include

$$\lim_{x\to\infty} \frac{1}{\sqrt{e^{2\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x\frac{1}{n}dn\right)}}} = e^{-\gamma}$$

and

$$\lim_{x\to\infty} \frac{1}{\sqrt{\frac{\left(e^{-\left(\sum_{n=1}^{x}\frac{1}{n}-\int_1^x\frac{1}{n}dn\right)}\right)^2}{e^{2\gamma}}}} = e^{2\gamma}$$

We could go on, but the point is that these variations are continuous if and only $s = 1$. It is easy to see that the jointly continuous and symmetrical relationship between a surface and a hole in the surface depends on $s = 1$. If $s \neq 1$, then the stage is ultimately reached when the invertible contraction/expansion process comes to a halt. These 'sticking points' show that non-smooth surfaces involve global asymmetries - global biases- in favour of contraction. Moreover, they show that non-smooth surfaces are sub-units, parts of unitary surfaces rather than a unitary surfaces in their own right. Assume now for sake of argument that there is some n such that the real part of p_n is other than 1/2. From the connection between spirals developing at the rate $e^{2\gamma}\sqrt{\frac{A}{e^{2\gamma}}}^{-2}$ and $s = 1$, it follows that the primes form a surface associated with $s \neq 1$... The argument that arises from these considerations is this

1) Every smooth surface is associated with $s = 1$;

2) If there exists some n such that the real part of p_n is not equal to 1/2, then there exists some smooth surface -the surface formed by the distribution of the primes- that is not associated with s = 1 ;

3) Therefore every n is such that the real part of p_n is equal to 1/2.

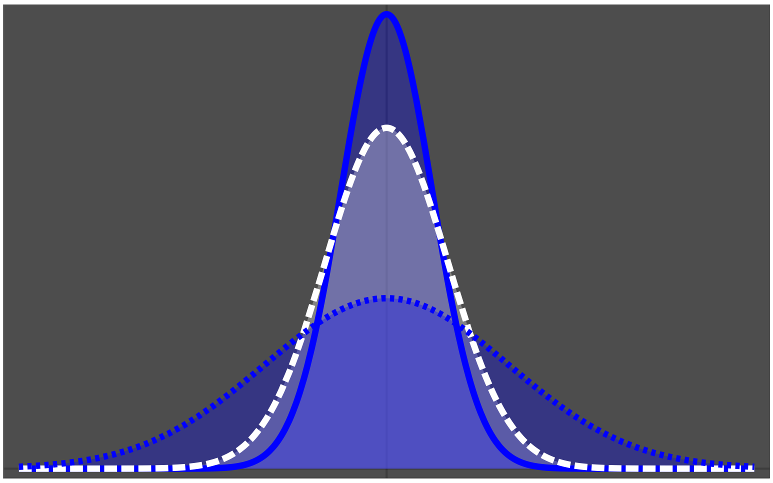
Let's express the argument with reference to the double-slit experiment, in which particles are shot, particle by particle, through 1 or other of 2 slits and their arrival registered on a screen. If there is '2-path information' -if for example detectors are attached to the slits, or if the particles are 'measured' in some manner- then the particles behave like fair coins. Rather than 1/2 + 1/2 = 1, we should write

$$\left(\frac{1}{2}+m\right)+\left(\frac{1}{2}-m\right)=1$$

to describe the probability of a fair coin turning up heads or tails (to take account of the error). But having taken account of the error, we can attribute it either to the count or to the counting of the coins. By tradition we attribute it to the count of the coins (the counted units), but could equally well attribute it to the *counting* of the coins (the counting units), i.e. we assume without justification that the balance of flips and integers to be exactly 1/2, but we might assume instead that balance of heads and tails is exactly 1/2, thus upsetting the balance of flips and integers, but the end result is the same in either case.

The probability of a particle going through the 1st or the 2nd slit in presence of 2-path information can be described by the equation $\left(\frac{1}{2}+m\right)+\left(\frac{1}{2}-m\right)=1$, or in the language of quantum mechanics

$$\left(\sqrt{\frac{1}{2}}^{\,2}+i\,m\right)+\left(\sqrt{\frac{1}{2}}^{\,2}-i\,m\right)=1.$$



"Number of Heads/Number of Particles passing through Slit-1"

But in the *absence* of 2-path information, the particles go through both slits at once and interfere with each other.... This requires us to identify a probability calculus which has two related but distinct modes. In both modes, probable outcomes are weighted and summed, and the sum is 1: in the one mode -the classical mode- there is an error between the counted and the counting units whose growth rate is continuous and symmetrical; in the other -quantum- mode, the growth rate of the error is discontinuous and asymmetrical. In the absence of 2-path information, we cannot describe the state of the particles before they reach the screen in this way: $\left(\sqrt{\frac{1}{2}}^{\,2}+i\,m\right)+\left(\sqrt{\frac{1}{2}}^{\,2}-i\,m\right)=1$. If we *could*, there would be no interference pattern on the screen, no difference between the presence and absence of 2-path information, and no such thing as quantum mechanics as something distinct from classical mechanics. In the presence of 2-path information, the probability of find of a particle going through the one slit is equal to that it going though the other, but in the absence of 2-path information, there is a *bias* toward one slit or the other, captured by

$$\left(\sqrt{\frac{1}{n}}^{\,2}+i\,m\right)+\left(\sqrt{\frac{n-1}{n}}^{\,2}-i\,m\right)=1$$

where $n>2$. We can identify both modes with

$$e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}\left(\left(\frac{1}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1}=1$$

and we can distinguish them because in the case of the first mode, but not the second, $s=1$ and we would write $e^{2\gamma}\sqrt{\frac{1}{e^{2\gamma}}}^{\,2}=1$.

Assume now for the sake of argument that there is some n such that real part of ρ_n in the sum $2\,\mathrm{Re}\sum_{n=1}^{x}\frac{(p_x)^{\rho_n}}{\rho_n}$ is not equal to 1/2. This rogue ρ_n is a solution to the equation $\left(\sqrt{\frac{1}{n}}^{\,2}+i\,m\right)+\left(\sqrt{\frac{n-1}{n}}^{\,2}-i\,m\right)=1$ where $n>2$, is associated with some quantum state, and with $1-e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}\left(\left(\frac{1}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1}$ where $s\neq 1$. If $s\neq 1$, then the last is in its turn associated with an *imbalance* of prime-density, i.e. a distribution of the primes through space that is *not* in accordance with $e^{2\gamma}\sqrt{\frac{A}{e^{2\gamma}}}^{2} = A$. In particular, these distributions involve a finite number of primes. Every *well-behaved* ρ_n, by contrast, is a solution to the equation $\left(\sqrt{\frac{1}{2}}^{2} + i\,m\right) + \left(\sqrt{\frac{1}{2}}^{2} - i\,m\right) = 1$, is associated with some *classical* state, and with an *infinite* number of primes. If and only if our assumption is true, classical states are macroscopic quantum states, but our assumption also implies that there are finite primes... Argument:

1) Every classical state is associated with $s = 1$;

2) If there exists some classical state that is a macroscopic quantum state, then this state is not associated with $s = 1$;

3) Therefore there are no macroscopic quantum states.

Still another expression of the same argument is as follows. Suppose there is a computer program or ‘algorithm’ that can determine of any algorithm that it terminates, or as they say ‘halts’; then create an algorithm that halts if and only if the algorithm under investigation *doesn’t* halt; if the algorithm under investigation *is* this created algorithm, it halts if and only if it doesn’t; thus some outputs cannot be computed. We can modify this argument by supposing that there is an algorithm that can determine of any algorithm that it halts in ‘polynomial time’; then create an algorithm that halts in polynomial time if and only if the algorithm under investigation halts in exponential time; again, if the algorithm under investigation *is* this created algorithm, it halts in polynomial time if and only if it doesn’t; thus it is arguable that some computable outputs cannot be computed in polynomial time. In the light of these preliminary arguments, consider the following matrices, in which outputs are matched to the algorithms that compute them.

□	a1	a2	a3	a4	a5
o1	□	True but Non – Provable	True but Non – Provable	True but Non – Provable	True but Non – Provable
*o*2	Efficiently Provable	□	True but Non – Provable	True but Non – Provable	True but Non – Provable
o3	Efficiently Provable	Efficiently Provable	□	True but Non – Provable	True but Non – Provable
o4	Efficiently Provable	Efficiently Provable	Efficiently Provable	□	True but Non – Provable
*o*5	Efficiently Provable	Efficiently Provable	Efficiently Provable	Efficiently Provable	□

□	a1	a2	a3	a4	a5
01	□	False but Non – Disprovable	False but Non – Disprovable	False but Non – Disprovable	False but Non – Disprovable
02	Efficiently Disprovable	□	False but Non – Disprovable	False but Non – Disprovable	False but Non – Disprovable
03	Efficiently Disprovable	Efficiently Disprovable	□	False but Non – Disprovable	False but Non – Disprovable
04	Efficiently Disprovable	Efficiently Disprovable	Efficiently Disprovable	□	False but Non – Disprovable
05	Efficiently Disprovable	Efficiently Disprovable	Efficiently Disprovable	Efficiently Disprovable	□

To the right of the center diagonal lie those matches such that the complexity of the output is greater than that of the adjoining algorithm. This illustrates the first argument above, and it is clear that the algorithm can’t compute the output in finite time. On the center diagonal itself, lie those outputs whose complexity is the same as that of the adjoining algorithm. If the complexity of an output is the same as that of the adjoining algorithm, the algorithm is effectively re-producing itself, and it therefore takes exponential time to run the input. If for example the output is Shakespeare’s plays, and the algorithm consists simply of Shakespeare’s plays, then the algorithm is re-producing itself, and it takes exponential time to run the input ‘write Shakespeare’s plays’. This illustrates the *second* argument above. Only if the match is on the *left* of the center diagonal is the output computable by the adjoining algorithm in polynomial’ time.

Consider next that, abstractly viewed, these matrices involve a balances of opposites (1’s and -1’s, 1’s and 0’s) and thus can be seen as representations of arithmetic progressions, random sequences... If we work with $e^{s+1}(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)\left(\left(\frac{1}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = 1$ rather than $\pi\sqrt{\frac{1}{\pi}}^{2} = 1$ (which is how we avoid the paradox of the unit circle), we see that these balances are maintained, and these matrices are infinite matrices, if and only if $s = 1$ and

$$e^{s+1}(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)\left(\left(\frac{1}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = e^{2\gamma}\sqrt{\frac{1}{e^{2\gamma}}}^{2} = 1$$

If $s \neq 1$, the matrices are finite, in which case the limit imposed by the self-output is reached. Arguments:

1) Every algorithm is associated with an output whose complexity is greater than the algorithm that produces it -with a greater-than-self-output- and thus every algorithm is associated with $s \neq 1$;

2) If there exists an algorithm for every output, then there is an algorithm associated with $s = 1$;
3) Therefore some outputs are such that no algorithm can compute them.

1) Every algorithm is associated with an output whose complexity is the same as the algorithm that produces it -with a self-output- and thus every algorithm is associated with $s \neq 1$;
2) If there exists a polynomial-time algorithm for every output then there is an algorithm associated with $s = 1$;
3) Therefore some outputs are such that no polynomial-time algorithm can compute them.

In the same way that the square root limit on the ratio of heads and tails tells of a fluid balance thereof ($\left(\frac{1}{2} + m\right) + \left(\frac{1}{2} - m\right) = 1$) -can be attributed to the coins or to the coin flips- the square root limit there is on the sum

$$2 \operatorname{Re} \sum_{n=1}^{x} \frac{(p_x)^{\rho_n}}{\rho_n}$$

tells of a fluid balance of prime-density and prime-sparsity ($\left(\frac{1}{2} + m\right) + \left(\frac{1}{2} - m\right) = 1$) that can be attributed to the primes or to the count of the primes. What matters is that the difference, or the margin of error, be the same from either perspective. The value 1/2 lies midway between these extremes... This balance *and* the imbalance can be associated with

$$e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)} \left(\left(\frac{1}{e^{(s+1)\left(\zeta(s)-\frac{1}{s-1}\right)}}\right)^{\frac{1}{s+1}}\right)^{s+1} = 1$$

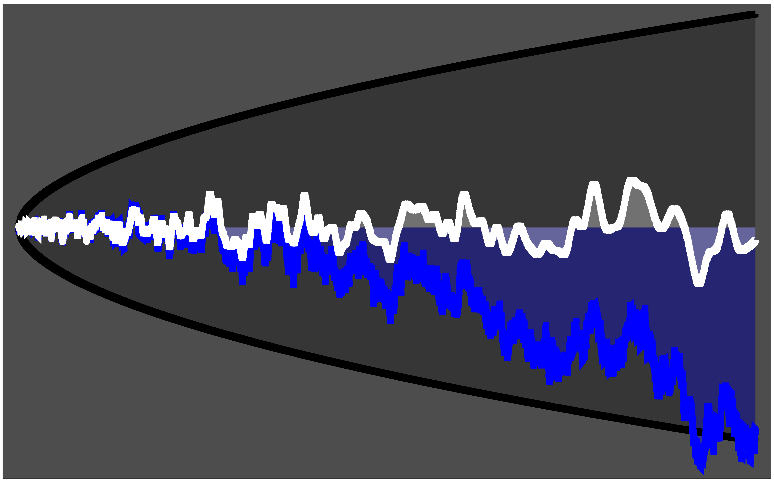
where in the one case $s = 1$, and in the other $s \neq 1$, and

$$e^{2\left(\zeta(s)-\frac{1}{s-1}\right)} \sqrt{\frac{1}{e^{2\gamma}}}^{2} \neq 1$$

There is an equal chance of a coin turning up heads or tails if and only if the series of coin flips grows continuously. Similarly, there is an equal chance of a particle going through slit-1 or slit-2 in a double-slit experiment if and only if the series of particle-shoots grows continuously - in the absence of 2-path information, there is *not* an equal chance. From the equation whose simplest form is $e^{2\gamma}(e^{-\gamma})^2 = 1$, we learn that there is an equal chance of a loss (+) or a gain in prime-density (-) from unit to unit if and only if the units grow continuously. Where this symmetry -this great *harmony*- between the world and the mind that counts it exists, we find on the one hand that there is an equal chance of a loss (+) or a gain in prime-density (-) as an arithmetic progression grows continuously, and on the other that an arithmetic progression grows continuously, in which case there is an equal chance of a loss (+) or a gain in prime-density (-). The instigator of this discussion, Euclid, argued that there are infinite primes on the basis of the consideration that it is always possible to add 1 to an arithmetic progression, a premise that he assumed but failed to justify. And unbeknownst to Euclid, there is a significant sense in which it is *not* always possible to add 1 to an arithmetic progression. A general conclusion to which various premises of our argument point us is this: in so far as it is always possible to add 1 to an arithmetic progression, that progression involves infinite primes, and in so far as an arithmetic progression involves infinite primes, the balance of prime density and sparsity is in accordance with the generalized inverse square law and the measure $(1/2 + i\, m) + (1/2 - i\, m) = 1$, a measure which can be interpreted as lying midway between infinite prime-density and infinite prime-sparsity.

PART IV

Good enough up to 20, 000 x, $p_x - \frac{\int_1^x a1\, x\, H_x\, dn + \int_1^x a1\, x \log(x)\, dn + \left(\int_1^x a2\, x\, H_x\, dn + \int_1^x a2\, x \log(x)\, dn\right)}{2\,x}$, breaks the bound $e^{2\gamma}\sqrt{\frac{p_x}{e^{2\gamma}}}$ by 100, 000



And although we're using only handful of functions and coefficients, it is clear that no matter how many functions we use, and no matter the care with which we choose the coefficients, it will at some point break free of these bounds. But compare this bound-breaking behavior to that of $p_x - x\log(x)$, which *never* returns to the x axis (Rosser 1938)... $p_x - x\log(x)$ is not associated with a wave, or at least it is associated with a wave whose frequency is fundamental. We see now that p_x − x log(x) is the same as

$$p_x - a1\, x \log(x)$$

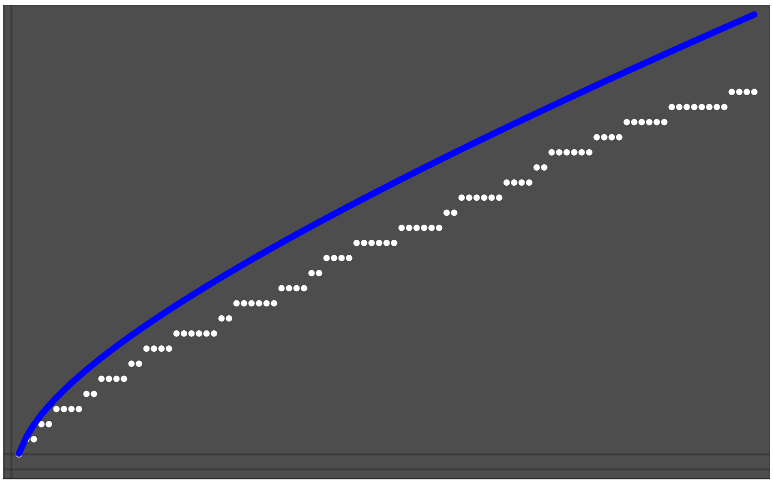
in the case that $a1 = e^{2\gamma}\left(e^{\frac{1}{s-1}-\zeta(s)}\right)^2$ and $s = 1$, and indeed what we did with both $\frac{\int_1^x\left(\sum_{n=2}^x \frac{a1}{H_{x\, a1}}\right) dn + \int_1^x\left(\sum_{n=2}^x \frac{a2}{\log(a2\, x)}\right) dn}{x}$ and $\frac{\int_1^x x\, a1\, H_x\, dn + \left(\int_1^x x\, a1\, H_x\, dn + \int_1^x x\, a2 \log(x)\, dn\right) + \int_1^x x\, a2 \log(x)\, dn}{2\,x}$ was to set the coefficients a1 and a2 equal to for a pair of real values of s *other* than 1. The existence of some n such that the real part of ρ_n is other than 1/2 implies that, like the difference between $\pi(x)$ - li(x) , the differences between $\pi(x) - \frac{x}{\log(x)}$ and $p_x - x\log(x)$ change sign infinitely many times. That there is no such n implies, as we know, that the differences increase without bound... It also implies something somewhat less obvious, i.e. that any correctly formulated prime-counting function oscillates infinitely about the primes within the appropriate bounds. In particular, if s ≠1 and coefficients corresponding to $e^{2\gamma}\left(e^{\frac{1}{s-1}-\zeta(s)}\right)^2$ are chosen

$$\lim_{x\to\infty}\left(p_x - \frac{\int_1^x a1\, x\, H_x\, dn + \int_1^x a1\, x \log(x)\, dn + \left(\int_1^x a2\, x\, H_x\, dn + \int_1^x a2\, x \log(x)\, dn\right)}{2\,x}\right) = 2\,C\,\mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{(p_x)^{\rho_n}}{\rho_n}\right)$$

and more generally

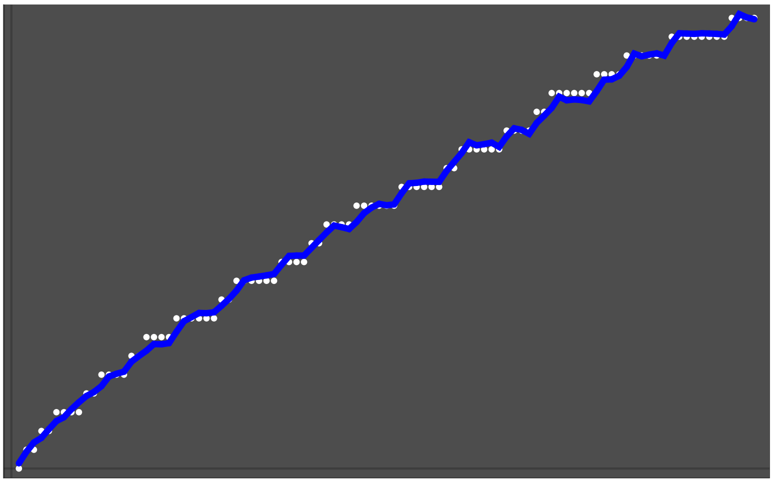
$$\lim_{x\to\infty}\left(p_x - \frac{\int_1^x a1\, x\, H_x\, dn + \int_1^x a1\, x\, (x \log)\, dn + \left(\int_1^x a2\, x\, H_x\, dn + \int_1^x a2\, x\, (x \log)\, dn\right) \ldots}{n\,x}\right) = 2\,\mathrm{Re}\sum_{n=1}^{\infty} \frac{(p_x)^{\rho_n}}{\rho_n}$$

Gauss counted primes in his spare time, and speculated on the basis of numerical evidence alone that the probability of an integer being prime is approximately $\frac{1}{\log(n)}$, which seems to imply that the primes thin out. On *this* account, the probability of an integer being prime is also approximately $\frac{1}{H_n}$: $\frac{1}{\log(n)}$ is a measure of sparsity, while $\frac{1}{H_n}$ is a measure of density, and the primes may be regarded as a fundamental form of energy governed by the inverse square law, which implies that sparsity and density are in balance, and that they *spread out* rather than thin out. Yes there are fewer of them further out, but they have as it were more ground to cover. Gauss gave us this picture



$\pi(x)$ • $\text{li}(x)$

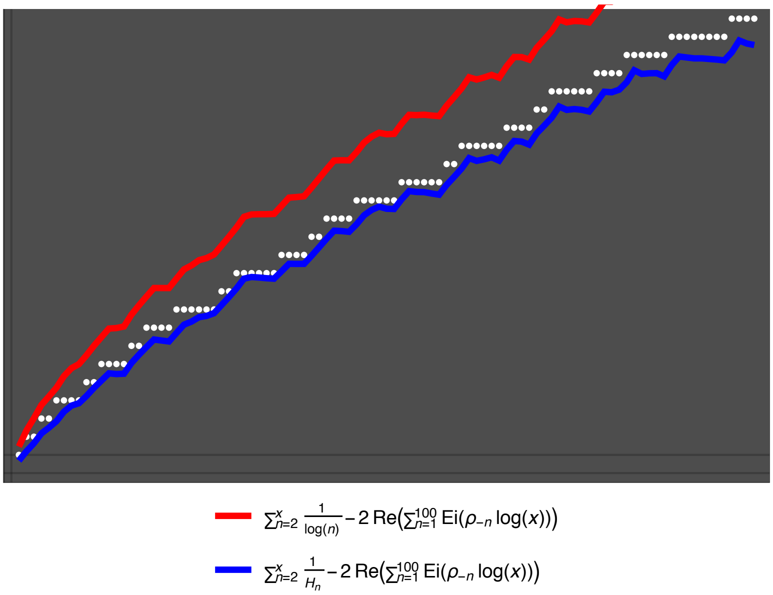
Riemann gave us this picture



$\pi(x)$

• $\sum_n^{10} \frac{\mu(n)\,\text{li}\left(x^{1/n}\right)}{n} - 2\,\text{Re}\left(\sum_{n=2}^{100} \text{Ei}(\rho_{-n} \log(x))\right)$

The following picture is one that emerges from all that we have considered above



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