“SOME EVIDENCE THAT THE GOLDBACH CONJECTURE COULD BE PROVED OR PROVED FALSE”

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Abstract: The present study is an effort for giving some evidence that the goldbach conjecture is not true, by showing that not all even natural numbers greater than two can be expressed as a sum of two primes. This conclusion can be drawn by showing that prime numbers are not enough –in population- so that, when added in couples, to give all the even numbers.

1. INTRODUCTION

In the present chapter, the necessary theoretical background is presented, on which the proving procedure of the next chapter will be based.

Christian Goldbach conjectured, in a letter to L. Euler in the year 1742, that every even integer is the sum of two numbers that are either primes or 1. This can be formulated to the general statement that "every even integer greater than 4 can be written as a sum of two odd prime numbers". Hardy and Littlewood made some progress in 1922; based on the unproved generalised Riemann hypothesis, they showed that every sufficiently large odd number is the sum of three odd primes. In 1937, I.M. Vinogradov removed this dependence on the generalised Riemann hypothesis, and established that all odd integers greater than some effectively computable –and sufficiently large- n, can be expressed as the sum of three odd primes. It is currently known that every even integer is the sum of six or fewer primes (Burton, 1998).

The first basis, upon which this study is based, concerns estimation of the population of prime numbers among all the natural numbers; more specifically, how many primes there are less than a given natural number. The answer can be given by the Prime Number Theorem, which was proved (and not for the first time) by both Alte Selberg (1949) and Paul Erdoes (1949). It asserts that, for a given natural number x, there are about x / logx primes before it:

$$\lim_{x \to \infty} \pi(x) / (x / \log x) = 1 \qquad \textbf{(1)}$$

where $\pi(x)$ is expressing the population of those prime numbers not exceeding the natural number x. Thus, the following approximation is expected:

$$\pi(x) \approx x / \log x \qquad \textbf{(2)}$$

As equation (1) implies, better approximations are achieved by this formula as x increases.

The second basis, upon which this study is based, concerns estimation of the population of twin primes (primes differing by two, of the form p, p+2), the population of cousin primes (of the form p, p+4), the population of those primes differing by six (p, p+6), then of those differing by eight (p, p+8), etc.

Let $\pi 2(x)$ denote the number of prime pairs not exceeding x, which is the number of primes p, for which $p + 2 \leq x$ is also a prime. Based on the unpredictable and random occurrence of the twin prime pairs, Hardy and Littlewood conjectured the following: The chance of two numbers p and p+2, both being prime, acts like the chance of getting a head on two successive tosses of a coin. It would follow from the prime number theorem that there are about $x / (\log x)^2$ twin primes less than or equal to x.

$$\pi 2(x) \approx x / (\log x)^2 \qquad \textbf{(3)}$$

For reasons involving the dependence of p+2 being prime on the supposition that p is already prime, Hardy and Littlewood estimated that $\pi 2(x)$ increases much like the function

$$F(x) = 1.32032 \int_2^x du / (\log u)^2 \qquad \textbf{(4)}$$

where the number 1.32032 is twice the value of the known twin-prime constant.
Following the same pattern, the population of prime pairs differing by four (p and p+4) was estimated to be of the same magnitude –according to equation (4)- while the population of prime pairs differing by six (p and p+6) was estimated to have twice this magnitude.

Due to the fact that the procedure, which follows in the next chapter, is partly based on these unproved hypotheses of Hardy and Littlewood, the degree of their approximations is illustrated with tables (1) and (2).

Table 1. Actual (counted) population size of prime pairs differing by two, four and six and are not exceeding the value 10^8

Number of pairs counted	10^5	10^6	10^7	10^8
p, p+2	1224	8169	58980	440312
p, p+4	1216	8144	58622	440258
p, p+6	2447	16386	117207	879905

(SOURCE: Burton 1998, and www.mathworld.wolfram.com)

Table 2. The population size of prime pairs differing by two, four and six (and are not exceeding the value 10^8), as predicted by the Hardy and Littlewood formulas.

Number of pairs predicted	10^5	10^6	10^7	10^8
p, p+2	1249	8248	58754	440368
p, p+4	1249	8248	58754	440368
p, p+6	2497	16496	117508	880736

(SOURCE: Burton 1998, and www.mathworld.wolfram.com)

2. PROVING PROCEDURE; RESULTS AND DISCUSSION

The proving strategy, that will be used, is an effort for showing the following: for a positive integer x, where x tends to infinity, the population of those primes not exceeding x, (and when added in couples not resulting an even that exceeds x), is not big enough so that –when these primes are added in couples- to result all the evens which do not exceed x.

For this reason, estimation of all the comparable quantities is needed.

The following table (table 3.) is illustrating the mathematical concepts from a rather geometrical perspective.

Table 3. An illustration of all the combinations of possible additions of the prime numbers p and q, which take every prime value starting by 3 and ending to x, where x tends to infinity.

p q +	**3**	**5**	**7**	**11**	**13**	**17**	**19**	**23**	**29**	**31**	**37**	**41**	**43**	**47**	**53**	**...**	**X**
3	6																
5	8	**10**															
7	**10**	12	14														
11	14	16	18	22													
13	16	18	20	24	26												
17	20	22	24	28	30	34											
19	22	24	26	30	32	36	38										
23	26	28	30	34	36	40	42	46									
29	32	34	36	40	42	46	48	52	58								
31	34	36	38	42	44	48	50	54	60	62							
37	40	42	44	48	50	54	56	60	66	68	74						
41	44	46	48	52	54	58	60	64	70	72	78	82					
43	46	48	50	54	56	60	62	66	72	74	80	84	86				
47	50	52	54	58	60	64	66	70	76	78	84	88	90	94			
53	56	58	60	64	66	70	72	76	82	84	90	94	96	100	106		
…																	
x																	2x

The population of the even numbers not exceeding x is x / 2 or (x-1) / 2, depending on whether x is chosen to be even or odd respectively. Table (3.) implies that the integer x is an odd prime.

The population of all the combinations of possible additions of the prime numbers (e.g. p and q, which take all the prime values starting by 3 and ending to x, where x tends to infinity) is estimated to be [(x / logx)(x / logx)] / 2. The estimation is based on the prime number theorem. Table (3.) can show the size of this population from a more simple and geometrical point of view (equal to the population of integers which make the triangle that is created inside the table). Due to the fact, though, that the results of the possible additions must not exceed x, the estimated population should be smaller (an example is given on table 3., where the expected population is above the drawn line, leaving us with almost the half of the originally estimated quantity).

Taking the worse possible case for the argument of the present study, the approximate quantity will be considered as originally estimated. Thus, the population of all the possible addition schemes between two primes is estimated by equation (5):

$$\text{No. of all possible additions} \approx x^2 / 2 (\log x)^2 \quad \textbf{(5)}$$

Several combinations of possible additions appear to be 'ineffective', in the sense that they yield an already existent result. For example, the results of the additions between 3+7 and 5+5 coincide (as it can be seen by the highlighted results on table 3.). In order to have comparable quantities, the 'ineffective' combinations of additions will be removed, so as only clear combinations of additions to remain (i.e. additions that give results which appear only once). So, an equation of the following form should be expected:

$$\text{No. of clear possible additions} = \text{No. of all possible add.} - \text{ineffective additions} \quad \textbf{(6)}$$

Before estimating the number of those 'ineffective' combinations of additions, another condition on which this study is based should be presented. The estimation of the populations of primes differing by two, four, six, etc. was made in the previous chapter, and in some cases there were differences between them. To overcome this inconsistency, the approximate quantity $x / (\log x)^2$ will be used to represent all the following populations: those primes differing by two, those differing by four, those differing by six, those differing by eight, and those differing by any other even number. The approximation is based on equation (3) and is the smallest possible approximation from all the appeared formulas (Gepner, 2005). Once again, the approximation represents the worse possible case for the argument of this study.

Now, the number of the ineffective combinations of additions should be estimated. When applying the first two columns of table (3.), i.e. the additions of the prime 3 with every prime ≥ 3 not exceeding x, and the additions of the prime 5 with every prime ≥ 5 not exceeding x, the following 'ineffective' results appear: the difference between the two primes (5 – 3) is two. So, every time they are combined with a couple of twin primes, they will give a common result. The interaction of the second column of table (3.) with the first one yields as many 'ineffective' results as the number of twin prime couples (that not exceed x); more specifically, about $x / (\log x)^2$ are expected. Following the same pattern, the interactions between the first three columns of the table yield as many 'ineffective' results as the following prime couples (that not exceed x): number of twin couples (due to the difference between 3 and 5)

+ number of twin couples (due to the difference between 5 and 7)

+ number of cousin couples (due to the difference between 3 and 7)

So, using the approximation $x / (\log x)^2$ for each case (as stated before), $3x / 2(\log x)^2$ 'ineffective' results are expected.

Concluding, the interactions between all the expected $x / (\log x)$ columns are yielding;

Number of 'ineffective' additions =

$$[x / (\log x)^2] [x^2 / 2(\log x)^2] / 2 \qquad \textbf{(7)}$$

Replacing the information of equations (5) and (7), equation (6) takes its final form:

Number of all 'clear' possible additions =

$$x^2 / 2 (\log x)^2 - [x / (\log x)^2] [x^2 / 2(\log x)^2] / 2 \qquad \textbf{(8)}$$

The final step of the procedure concerns the desired inequality, which supports the argument of the present study. Thus, the following inequality should hold:

Number of all 'clear' possible additions < Number of all evens not exceeding x

And replacing the quantities according to equation (8), the inequality becomes:

$$x^2 / 2 (\log x)^2 - [x / (\log x)^2] [x^2 / 2(\log x)^2] / 2 < x / 2 \qquad \textbf{(9)}$$

Equation (9) can be proved true, using mathematical induction. If not true, then the reverse inequality holds, giving evidence that goldbach conjecture can be proved.

Additional notes:

- The symbol logx represents the symbol lnx.
- The integer 2 is neglected from both sides of the inequality (9), since it was not one of the possible primes or one of the possible evens that take place in the procedure.
- The choice of the quantity x / 2 to represent the population of the even numbers (and not (x-1) / 2) is made arbitrarily, without violating the generality.
- Due to the unpredictable gaps between prime numbers, an exact estimation of the number of all combinations of possible additions was not achievable (eq.5). The fraction x^2 / 2 $(\log x)^2$ is an approximate quantity, which overestimates the real magnitude of the quantity, and is expressing the worse possible case, concerning what the present study tries to show (see equation8). Even if half of the population of the additions were considered to be used (e.g. the integers above the drawn line of table 3.), the inequality (9) would still remain valid; the right-hand side of the inequality would be replaced by a smaller fraction, something that enforces the argument.

3. CONCLUSIONS

It appears that, although the sequence of prime numbers shows unpredictable gaps and irregularities of detail, some trends can be identified 'in the large'. So, at least at gross terms, and as x gets larger values, the estimations could become more accurate, enforcing the argument of the present study that: 'for a positive integer x, where x tends to infinity, the population of those primes not exceeding x, (and when added in couples not resulting an even that exceeds x), is not big enough so that –when these primes are added in couples- to result all the evens which do not exceed x.' It should be noted that, when the estimations of the compared quantities had not been exactly predicted, the worse possible cases –as far as the argument is concerned- were taken into account.

Clearly, one of the limitations of the present study is that the concepts are based on the unproved Hardy and Littlewood conjecture (as summarised by equation 4), and on the hypothetical estimation of the populations of twin primes, of primes differing by four, six, eight etc. An unconditional proof, which could remove the dependence on these hypotheses, would be ideal.

At this point it should be mentioned, that the numerical data suggesting the truth of the Goldbach's conjecture is overwhelming. Many are convinced about the validity of the conjecture; Vinogradov showed that

$$\lim_{x \to \infty} G(x) / x = 0 \qquad \textbf{(10)}$$

where G(x) is the number of even integers $n \leq x$, which are not the sum of two primes. E. Landau also believed that almost all even integers satisfy the conjecture, saying that: "The Goldbach conjecture is false for at most 0% of all even integers; this *at most 0%* does not exclude, of course, the possibility that there are infinitely many exceptions." (Landau, 1992)

4. GENERAL REFERENCES

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