# Graded Lie Algebra of Quaternions and Superalgebra of $SO(3,1)$

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**Abstract**

In the present discussion we study the grading of Quaternion algebra($\mathbb{H}$) and Lorentz algebra of $O(3,1)$ group. Then we have made an attempt to make the whole Poincaré algebra of $SO(3,1)$ in terms of Quaternions. After this the supersymmetrization of this group has been done in a consistent manner. Finally the dimensional reduction from $D=4$ to $D=2$ has been studied.

## 1 Introduction

In order to unify the symmetry of Poincaré group with some internal group several attempts have been made, Coleman and Mandula in[1] 1967 set a restriction to the incorporation of Poincaré symmetry to the internal symmetry group upto which this unification is possible. But actually this doesn't offer any unification of Poincaré group with internal symmetry group. This theorem which is called no-go theorem of Coleman and Mandula based on the extension of symmetry of $S$-matrix under the assumption of physical conditions of locality, causility, positivity of energy and for finite numbers of particles. Wess and Zumino [2] realized that the unification of Poincaré group to internal symmetry group is possible by introducing anticommutation relations of supersymmetric charges into the theory, which relate the fermions to bosons. However it's proof has been estabilished by Haag, Lapuszanski and Sohnius [3] . Thus Supersymmetric field theories arises as the maximum symmetry of $S$-matrix that is possible. This is the largest extension of the Lie algebra of Poincaré group and internal symmetry group. Which has not only commutators but also anticommutators of supercharges which generate the supersymmetric transformation. Since this theory can be a possible answer for the most of the heirarchy problems [4] of the standard model, unification of gravitation and dark matter and dark energy several attempts have been made experimently in search of this symmetry. But still not confirmed yet. However it is always been a interesting theory searched by the theoretical physicists in attempt to unify the fundamental forces of nature.

In the work of L. Brink and J. H. Schwarz it is argued that [5] the supersymmetry is only possible for the cases of dimensions $D=2,4,6,10$ called critical dimensions. They show that the action is supersymmetric for $D=4,6,10$ dimensions without the inclusion of further fields. It has shown previously that [6] nonabelian Yang–Mills fields with minimal coupling to massless spinors are supersymmetric if and only if the dimension of spacetime is $3, 4, 6$ or 10. Before this the realization between the Lorentz group in various dimensions and the division algebras [7] has been established by Kugo-Townsend [8] which further extended and generalized by Jerzy Lukierski et. all. [9, 10] and many more authors [11, 12, 13] previously. It is summarized that the algebras $\mathbb{R}$,$\mathbb{C}$,$\mathbb{H}$,$\mathcal{O}$ can be useful for the description of supersymmetric field theories in higher dimensions [6]-[13]. Also the compactification of higher dimensional supersymmetry to lower dimensional supersymmetry theory has been studied by L. Brink et.all.[5] in a consistent manner.

Keeping in view the connection between the normed division algebras ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathcal{O}$) and supersymmetric theories in the present paper we made an explicit relation between the Poincaré group of $SO(3,1)$ and normed division algebra of Quaternions, then supersymmetrization of this Poincaré group has been done in a consistent manner. Finally the dimensional reduction of the supersymmetry in $D = 4$ to $D = 2$ dimensional space has been studied.

# 2 Quaternion Algebra

## 2.1 Quaternion algebra in matrix representation:

Quaternion algebra ($\mathbb{H}$) [7] is described over the field of real numbers by

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

where $q_j (\forall j = 1, 2, 3)$ are the real numbers and the basis elements $e_i$ satisfying the following relations

$$e_i e_j = -\,\delta_{ij} + \in_{ijk}\, e_k \quad (i, j, k = 1\, to\, 3) \tag{1}$$

$\in_{ijk}$ is Levi-Civita tensor, which is a totally antisymmetric tensor and having value $+1$ for the permutation $(ijk) = (123), (231), (312)$. In $4 \times 4$ real matrix representation the Quaternion elements [10] are

$$\begin{aligned} e_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = i\sigma_2 \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ e_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = I \otimes \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ e_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = i\sigma_2 \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{aligned} \tag{2}$$

These basis elements satify the following commutator bracket relations

$$[e_i, e_j] = 2 \in_{ijk}\, e_k \qquad (\forall i, j, k = 1\, to\, 3) \tag{3}$$

by substituting $\tau_i = \frac{e_i}{2}$ for these basis elements we have the Lie algebra of Quaternion as

$$[\tau_i, \tau_j] = \in_{ijk} \tau_k \qquad (\forall i, j, k = 1\, to\, 3) \tag{4}$$

and the $\in_{ijk}$ is the structure element of the Lie algebra of quaternion. Since there is a correspondence between the Pauli matrices and quaternion basis elements [7] as $e_i \to -i\sigma_i (\forall i = 1, 2, 3)$.

## 2.2 Grading of Quaternion algebra

The $Z_2$-graded algebra [14] $L$ is the direct sum of two algebras $L_0 \oplus L_1$having the following properties

(i) $L_0$ is an even lie algebra with degree $g(L_0) = 0$ and $L_0 \times L_0 \to L_0$

(ii) $L_1$is odd Lie algebra with degree $g(L_1) = 1$ and $L_0 \times L_1 \to L_0$ , $L_1 \times L_1 \to L_0$

(iii) Representation of $L_0$ in $dim\, L_1 \times dim\, L_1$

taking the 4x4 dimensional representation of algebra $L_0$ of quaternion we have

$$\begin{aligned} L_0 =& \{\tau_i \in L_0, \;\; [\tau_i, \tau_j] = \in_{ijk} \tau_k, \; (\forall i, j, k = 1\, to\, 3)\} \\ L_1 =& \{Q_a \in L_1, (\forall a = 1\, to\, 4) \;\; [Q_a, \tau_i] = (\tau_i)_{ab}\, Q_b \in L_1, \\ & and \; \{Q_a, Q_b\} = (\kappa_i)_{ab} \tau_i \in L_0 \; (\forall i = 1\, to\, 3, \forall a, b = 1\, to\, 4)\}. \end{aligned} \tag{5}$$

The $\kappa_i$'s must be symmetric. By the generalized Jacobbi identity we have

$$[\tau_i, \{Q_a, Q_b\}] + \{Q_b, [\tau_i, Q_a]\} + \{Q_a, [Q_b, \tau_i]\} = 0 \tag{6}$$

which reduces to the equation

$$\tau_i \kappa_j + (\tau_i \kappa_j)^T = \in_{ijk} \kappa_k \tag{7}$$

by which the matrices $\kappa_i$'s evaluated as

$$\begin{aligned} \kappa_1 =& \frac{1}{2}\sigma_3 \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ \kappa_2 =& -\frac{1}{2}\sigma_1 \otimes \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \kappa_3 =& -\frac{1}{2}\sigma_3 \otimes \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{8}$$

These $\kappa_i's$ matrices are symmetric matrices which are not closed in a sence that

$$[\kappa_i, \kappa_j] = - \in_{ijk} \tau_k \tag{9}$$

however together with matrices of (2) they span the gradded lie algebra space of quaternion. Similarly the other generalized Jacobbi's identity is satisfied also

$$\{Q_a, \{Q_b, Q_c\}\} + \{Q_b, \{Q_c, Q_a\}\} + \{Q_c, \{Q_a, Q_b\}\} = 0 \tag{10}$$

So the superalgebra of Quaternion(H) has:-

$Z_2$-graded algebra $L$ is the direct sum of two algebras $L_0 \oplus L_1$

The matrix representations of (2) and (8) together form a real representation of the group $O(3,1)$. So by gradding of quaternion algebra may be extend the rotaion group of to the Lorentz group of $O(3,1)$ group.

# 3 Lorentz Group and Poincare Algebra in $D = 4$ space:

The Lie algebra of quaternion

$$[\tau_i, \tau_j] = \in_{ijk} \tau_k \tag{11}$$

satisfy the Lie algebra of $O(3)$ group where the $\tau_i$'s are the generators of the rotation group $O(3)$. Similarly the $\kappa_i$'s follow the commutators relations

$$\begin{aligned} [\kappa_i, \kappa_j] &= - \in_{ijk} \tau_k \\ [\kappa_i, \tau_j] &= \in_{ijk} \kappa_k. \end{aligned} \tag{12}$$

As may be seen by the commutator algebra relations of (11) $\kappa_i$ satisfy the relations of the commutator relations of the generators of Lorentz boost. They form the real matrices representation of Lorentz boost generators. Further we may construct the Hermitian matrices of Angular momentum [14]

$$J_i = i\tau_i, \;\; J_i^\dagger = J_i \tag{13}$$

and anti-Hermitian matrices for the generators of Lorentz boost

$$K_i = i\kappa_i, \;\; K_i^\dagger = -K_i \tag{14}$$

The generators of Lorentz group follow the following commutation relations

$$
\begin{aligned}
[J_i, J_j] =& i \in_{ijk} J_k \\
[K_i, K_j] =& - i \in_{ijk} J_k \\
[K_i, J_j] =& i \in_{ijk} K_k
\end{aligned} \tag{15}
$$

The above generators of Lorentz group make $O(3,1)$ noncompact group of four dimensional rotation. These relations can be combined into an explicit form by writing

$$
\begin{aligned}
\in_{ijk} M_{lm} =& J_i \\
M_{0n} =& - K_n
\end{aligned} \tag{16}
$$

The $M_{\mu\nu}$'s are the generators of lorenz group $O(3,1)$ satisfy the relation

$$
[M_{\mu\nu}, M_{\rho\sigma}] = - i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \tag{17}
$$

Poincare Algebra contain the elements of Generators of four dimensional rotation and the generators of translation (linear momentum operator) which satisfy these commutation rules:

$$
\begin{aligned}
[P_\mu, P_\nu] =& 0 \\
[M_{\mu\nu}, P_\rho] =& i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \\
[M_{\mu\nu}, M_{\rho\sigma}] =& - i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma})
\end{aligned} \tag{18}
$$

### 3.1 Quaternionic realization of Poincare group in $SO(3,1)$ group:

The $2 \times 2$ dimensional quaternionic $\Gamma$-matrices in Weyl representation can be written as

$$
\Gamma_1 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}, \ \Gamma_2 = \begin{pmatrix} 0 & e_2 \\ -e_2 & 0 \end{pmatrix}, \ \Gamma_3 = \begin{pmatrix} 0 & e_3 \\ -e_3 & 0 \end{pmatrix}, \Gamma_0 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}
$$

which satisfy the relation of clifford algebra as

$$
\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} I_2 \tag{20}
$$

where $\eta_{\mu\nu} = \{-1, 1, 1, 1\}$. Then we can form second rank tensor quantity $\Sigma_{\mu\nu}$ as

$$
\Sigma_{\mu\nu} = \frac{i}{4}[\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu]. \tag{21}
$$

Which satisfy the folowing commutation relations

$$\begin{aligned}[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] =& - i(\eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma} + \eta_{\nu\sigma}\Sigma_{\mu\rho}) \\ [\Sigma_{\mu\nu}, \Gamma_{\sigma}] =& i(\eta_{\nu\sigma}\Gamma_{\mu} - \eta_{\mu\sigma}\Gamma_{\nu})\end{aligned} \tag{22}$$

The $4 \times 4$ Dirac $\gamma$-matrices in Weyl represantation may be written in quaternionic realization as

$$\gamma_{\mu} = \begin{pmatrix} 0 & \Gamma_{\mu} \\ \Gamma_{\mu} & 0 \end{pmatrix} \qquad (\forall \mu = 0 \; to \; 3). \tag{23}$$

the generators of Lorentz group are defined as

$$\Xi_{\mu\nu} = \frac{i}{4}[\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}] = \begin{pmatrix} \Sigma_{\mu\nu} & 0 \\ 0 & \Sigma_{\mu\nu} \end{pmatrix} \qquad (\forall \mu, \nu = 0 \; to \; 3) \tag{24}$$

The generators of translation or linear momentum may be written as

$$p_{\mu} = \left\{ \begin{pmatrix} 0 & 0 \\ \Gamma_{\mu} & 0 \end{pmatrix} \right\} \qquad (\forall \mu, \nu = 0 \; to \; 3) \tag{25}$$

Which together with $\Sigma_{\mu\nu}$ satisfy the following commutation rules

$$\begin{aligned}[p_{\mu}, p_{\nu}] =& 0 \\ [\Xi_{\mu\nu}, p_{\rho}] =& - i(\eta_{\mu\rho}p_{\nu} - \eta_{\nu\rho}p_{\mu}) \\ [\Xi_{\mu\nu}, \Xi_{\rho\sigma}] =& - i(\eta_{\mu\rho}\Xi_{\nu\sigma} - \eta_{\mu\sigma}\Xi_{\nu\rho} - \eta_{\nu\rho}\Xi_{\mu\sigma} + \eta_{\nu\sigma}\Xi_{\mu\rho})\end{aligned} \tag{26}$$

### 3.2 Quaternionic realization of Casimir Invariants of the Poincare Group in $SO(3,1)$:

The algebra of $SO(3,1)$ is non-compact group. However we know that $SO(4)$ group is it's compact group which is homomorphic to the $SU(2) \otimes SU(2)$ group [14] so there are two Casimir operator for the Poincare Algebra for $SO(3,1)$.

Since the matrix realization of generators of translation $P_{\mu}$is such that

$$P_{\mu} = p_{\mu} \tag{27}$$

The matrix realization of $P^{\nu}$ and $M^{\mu\nu}$is such that

$$\begin{aligned} P^{\nu} \to & p_{\mu}^{\dagger} \\ M^{\mu\nu} \to & \Sigma_{\mu\nu}^{\dagger}\end{aligned}$$

Where $\dagger$ is transpose of quaternionic conjugate of matrices. So we define the scalar product $P^{\mu}P_{\mu}$by

$$P^{\mu}P_{\mu} = \frac{1}{4}Tr(p_{\mu}p_{\mu}^{\dagger} + p_{\mu}^{\dagger}p_{\mu}) \tag{28}$$

where the $Trace$ of quaternion matrices is define as the real trace of the matices.The $P^{\mu}P_{\mu}$is the first casimir operator of Poincare algebra of $SO(4)$. The second one is constructed by the Pauli-Ljubanski polarization vector[14]. In this case we construct this vector by writing

$$W_{\mu} =\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\nu}M^{\rho\sigma} \tag{29}$$

Then second casimir operator is defined as $W_{\mu}W^{\mu}$ which commute with every element of the Poincare algebra of $SO(3,1)$.

# 4 Supersymmetrization of Poincare Group:

The supersymmetrization [14] of the Poincaré algebra in $SO(3,1)$ may be done in the following manner

$L=L_0\oplus L_1$with

$(i)L_0:$ (Poincaré algebra in $SO(3,1)$)

$(ii)L_1$ :$(Q_a,\forall a=1\,to\,4)$ $Q_a$are 4 dimensional two component spinor as

$$Q_a=\left(\begin{array}{c} Q_{\alpha}\\ \overline{Q}_{\dot{\alpha}}\end{array}\right), \qquad (\forall\alpha=1\,to\,2) \tag{30}$$

$Q_{\alpha}$is 2 dimensional spinor. For the $z_2$ grading of the Poincaré algebra in $SO(3,1)$ we define :

Defining the composition rule $\star$ in L such as

$$\begin{aligned} \star:L\times L\rightarrow & L\\ A\star B= & AB-(-1)^{g(L_r)g(L_s)}AB\qquad(\forall i,j=0,1) \end{aligned} \tag{31}$$

where $A\epsilon L_r$ $B\epsilon L_s$and $A\star B\epsilon L_{r+s\,mod\,2}$.$g(L_r)$, $g(L_s)$ are the order of grading for the sub -algebras $L_r$ and $L_s$ defined as

$$g(L_r)=\begin{array}{lll} 0, & & (for\,bosons)\\ 1, & & (for\,fermions)\ (\forall r=0,1)\end{array} \tag{32}$$

So, $g(L_0)=0$ and $g(L_1)=1$. Taking these considerations we get the commutation relations as

1. : $L_0\times L_0\rightarrow L_0$ whose commutation rules are obtained in equation .
2. : $L_0\times L_1\rightarrow L_1$ which enables the following commutation rules

$$\begin{aligned} [P_{\mu},Q_a]= & 0\\ [M_{\mu\nu},Q_a]= & -(\Xi_{\mu\nu})_{ab}Q_b,\ \{\forall a,b=1\,to\,4,\ and\,\forall\mu,\nu=0\,to\,3\} \end{aligned} \tag{33}$$

3. :$L_1\times L_1\rightarrow L_0$ gives rise the following anti commutation relations for spinors

$$\begin{aligned} \{Q_a,Q_b\}\epsilon L_o\\ \{Q_a,\bar{Q}_b\}\epsilon L_o. \end{aligned}$$

As such, the $L_0$ contain the generators of Poincare algebra of $D = 6$ space. So, there must be

$$\{Q_a, Q_b\} = \alpha^\mu P_\mu + \beta^{\mu\nu} M_{\mu\nu} \tag{34}$$

where $\alpha^\mu = -2(\Upsilon^\mu C)_{ab}$ and $\beta^{\mu\nu} = (\Xi^{\mu\nu} C)_{ab}$ , $C$ is charge conjugation matrix and $\Xi_{\mu\nu}$ are the representations of Lorentz algebra in $D = 10$ space. However, by the generalized Jacobi identity the second term $\beta^{\mu\nu}$ in equation vanishes and hence, we get the anticommutator rule

$$\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu. \tag{35}$$

Multiplying both side of the above equation by $C$ and imposing Majorana condition $\{(CQ)_a = -\bar{Q}_a\}$, we get

$$\{Q_a, \bar{Q}_b\} = 2(\gamma^\mu)_{ab} P_\mu \tag{36}$$

In quaternionic $4x4$ dim. representation:-

$L_0$ : Lie algebra of Poincare group $\{P_\mu, M_{\mu\nu}\}$ in quaternionic representation

$L_1$: Lie algebra of $\{Q_a\, [a = 1\, to\, 4]\}$

satisfy the following equations

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}\Xi_{\nu\sigma} - \eta_{\mu\sigma}\Xi_{\nu\rho} - \eta_{\nu\rho}\Xi_{\mu\sigma} + \eta_{\nu\sigma}\Xi_{\mu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[P_\mu, P_\nu] &= 0 \\
[P_\mu, Q_a] &= 0 \\
[M_{\mu\nu}, Q_a] &= -(\Xi_{\mu\nu})_{ab} Q_b \\
\{Q_a, \bar{Q}_b\} &= 2(\gamma^\mu)_{ab} P_\mu \\
\{Q_a, Q_b\} &= -2(\gamma^\mu C)_{ab} P_\mu \\
\{\bar{Q}_a, \bar{Q}_b\} &= 2(C^{-1}\gamma^\mu)_{ab} P_\mu
\end{aligned} \tag{37}$$

Where $C = i\gamma_2\gamma_0$

### 4.1 Quaternionic realization of Casimir invariants of Super-Poincare group of $SO(3,1)$:

Supersymmetric extension of Super-Poincare group involve the generators of Poincare group and anticommutating super charges also, which generate the symmetry between the particle and their super patners. By the (36) we have

$$\{Q_a, \bar{Q}_b\} = 2(\gamma^\mu)_{ab} P_\mu = 2\begin{pmatrix} 0 & \Gamma_\mu \\ \Gamma_\mu & 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} Q_\alpha \\ \overline{Q}_{\dot{\alpha}} \end{pmatrix}, \begin{pmatrix} Q_\beta & \overline{Q}_{\dot{\beta}} \end{pmatrix}\right\} = 2\begin{pmatrix} 0 & (\Gamma_\mu)_{\alpha\dot{\beta}} \\ (\Gamma_\mu)_{\dot{\alpha}\beta} & 0 \end{pmatrix}_{ab} P_\mu \tag{38}$$

which yield

$$
\begin{aligned}
\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} =&2\left(\Gamma_\mu\right)_{\alpha\dot{\beta}} P_\mu \\
\left\{Q_\alpha, Q_\beta\right\} =&0, \quad \left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\} = 0
\end{aligned} \tag{39}
$$

The matrix representation can take the form in Graded $8 \times 8$ matrix representation [10] for this group represented by

$$
M^s_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{pmatrix}, P^s_\mu = \begin{pmatrix} P_\mu & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{pmatrix}, Q_a = \begin{pmatrix} 0 & \vdots & Q_a \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{pmatrix} \tag{40}
$$

So one of the Casimir operator for this super group is still $P^\mu P_\mu$since $Str(p_\mu p^\dagger_\mu + p^\dagger_\mu p_\mu) = tr(p_\mu p^\dagger_\mu + p^\dagger_\mu p_\mu)$ is an identity element of the supergroup

$$
P^\mu P_\mu = \frac{1}{4} tr(p_\mu p^\dagger_\mu + p^\dagger_\mu p_\mu) = \begin{pmatrix} I & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & I \end{pmatrix} \tag{41}
$$

Whereas $W_\mu W^\mu = I_2$ is not a casimir operator since it remain not an identity element for the group

$$
W_\mu W^\mu = \begin{pmatrix} I_2 & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{pmatrix}. \tag{42}
$$

In this case the second Casimir operator of group is

$$
C^2 = C_{\mu\nu} C^{\mu\nu} \tag{43}
$$

where $C_{\mu\nu}$ is

$$
\begin{aligned}
C_{\mu\nu} =& B_\mu P_\nu - B_\nu P_\mu \\
B_\mu =& W_\mu + \frac{1}{2} \bar{Q} \Gamma_\mu \Gamma_5 Q
\end{aligned} \tag{44}
$$

# 5 Dimensional reduction from 4 to 2 dimensions:

The supersymmetric gauge invariant Lagrangian for gauge supermultiplets [5, 15] in on-shell case is

$$
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \mathrm{Re}\left(i \bar{\chi}^a \Gamma^\mu D_\mu \chi^a\right) \tag{45}
$$

Where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\mu$$
$$D_\mu \chi^a = \partial_\mu \chi^a + g f^{abc} A^b_\mu \chi^c \tag{46}$$

Supersymmetric action

$$S = \int d^4x \left( -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \mathrm{Re}\left( i\bar{\chi}^a \Gamma^\mu D_\mu \chi^a \right) \right) \tag{47}$$

Which is invariant under the supersymmetric transformation

$$\delta A^a_\mu = i\bar{\in}\Gamma_\mu \chi^a, \quad \delta\chi^a = \Sigma_{\mu\nu} F^{\mu\nu a} \tag{48}$$

Where $\chi^a$ and $\bar{\epsilon}$ are quaternionic spinors and $\Gamma^\mu$are 4-dimensional Dirac matrices in $D = 4$ space represented as

$$\Gamma_1 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}, \ \Gamma_2 = \begin{pmatrix} 0 & e_2 \\ -e_2 & 0 \end{pmatrix}, \ \Gamma_3 = \begin{pmatrix} 0 & e_3 \\ -e_3 & 0 \end{pmatrix}, \Gamma_0 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{49}$$

In $D = 4$ space the spinor is a 4-component spinor containing the structure of two 2-component Dirac spinors

$$\chi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{50}$$

satisfying the Majorana condition[5]

$$\chi = C\bar{\chi}^T \tag{51}$$

where C is charge conjugation matrix $C = i\Gamma_2\Gamma_0$. By this condition we have

$$\chi = \begin{pmatrix} \phi \\ \breve{\phi} \end{pmatrix} \tag{52}$$

where $\breve{\phi} = e_2\phi = -i\sigma_2\phi$.To reduce $D = 4$ into lower dimensional space $D = 2$ can be done by

$$\partial_0, \partial_3 \to 0, \ \partial_\mu = (\partial_1 \to \partial_t = \partial_{0'}, \partial_2 \to \partial_{1'}) \tag{53}$$

then the first term of the lagrangian reduces to

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{4} F^a_{\mu'\nu'} F^{a\mu'\nu'} - \frac{1}{2}(D_{\mu'} A^a_3)(D^{\mu'} A^{3a}) - \frac{1}{2}(D_{\mu'} A^a_0)(D^{\mu'} A^{0a}) - \frac{1}{2}(g f^{abc} A^{0b} A^{3c})^2 \tag{54}$$

the second term reduces to

$$\bar{\chi}^a\Gamma^\mu D_\mu\chi^a = \bar{\phi}^a\gamma^{\mu'}D_{\mu'}\phi^a + \bar{\phi}^a g f^{abc}A_0^b\phi^c + \bar{\phi}^a e_3\phi^c A_3^b g f^{abc} \tag{55}$$

Where the second term transform as a scalar and third term as a pseoscalar since $e_3 = -i\sigma_3 = \gamma_5$

The supersymmetric invariant action can be written as

$$\begin{aligned} S = \int d^2x[&-\frac{1}{4}F^a_{\mu'\nu'}F^{a\mu'\nu'} - \frac{1}{2}(D_{\mu'}A_3^a)(D^{\mu'}A^{3a}) - \frac{1}{2}(D_{\mu'}A_0^a)(D^{\mu'}A^{0a}) - \frac{1}{2}(gf^{abc}A^{0b}A^{3c})^2 \\ &+Re\, i(\bar{\phi}^a\gamma^{\mu'}D_{\mu'}\phi^a + i\bar{\phi}^a g f^{abc}A_0^b\phi^c + \bar{\phi}^a e_3\phi^c A_3^b g)] \end{aligned} \tag{56}$$

Which is invariant under the supersymmetric transformation

$$\begin{aligned} \delta A^{a\mu'} =& (\bar{\phi}^a\gamma^{\mu'}\epsilon - \bar{\epsilon}\gamma^{\mu'}\phi^a) \\ \delta A^{3a} =& \bar{\epsilon}\gamma_5\phi^a - \bar{\phi}^a\gamma_5\epsilon \\ \delta A^{0a} =& i(\bar{\epsilon}\phi^a - \bar{\phi}^a\epsilon) \\ \delta\phi^a =& (\Sigma_{\mu'\nu'}F^{a\mu'\nu'} + igf^{abc}A_4^bA_0^c + \gamma^{\mu'}D_{\mu'}A^{3a}gf^{abc} - \gamma^{\mu'}D_{\mu'}A^{0a}gf^{abc})\epsilon \end{aligned} \tag{57}$$

# 6 Conclusion:

The previous attempts of relating the division algebra to extradimensional space has been studied. In this present discussion first we made an attept for grading of Quaternion algebra in matrix representation. Then a natural transition from the grading of quaternion algebra to $O(3,1)$ Lorentz group has been developed. The Poincare algebra can be shown to be expressed in a consist manner in terms of quaternion algebra. Then supersymmetrization of this group has been done. Since it has been shown by earlier authers that the Poincare symmetry can be related to the internal symmetry only through the maximum extension of S matrix via supersymmetry which relates fermion states to bosonic states. Supersymmetry has potential to answer the hierarchy problem, dark matter, unification of fundamental interactions. Here we tried to make a clear relation between the normed division algebra of Quaternions and supersymmetry of $SO(3,1)$ group. At last we tried to compact this supersymmetry theory into $D = 2$ space. Where the potentials of compactified dimensions become a scalar and pseudoscalar parts of potential.

# References


[1] S. Coleman and J. mandula, Phys. Rev. **159**(1967)**1251.**

[2] J. Wess and B. Zumino , Phys. Lett.**49B**(1974)**52** & Nucl. Phys. **B70** (1974)**39.**

[3] R. Haag J. T. Lopuzanski and M. F. Sohnius, Nucl. Phys. **B 88**,(1975)**257**.

[4] M.F. Sohnius, Physics Reports (Review Section of Physics Latters) **128** Nos. **2& 3**(1985) **39-204**, North-Holland, Amsterdam.

[5] L. Brink, J.H. Schwarz, J. Scherk, Nuclear Physics **B121**(1977) **77-92**.

[6] J. C. Baez and J. Huerta, arXiv:**hep-th/0909.0551v2 (2009).**

[7] W.R. Hamilton, Elements of Quaternions, Vol. **I & II** Chelsea Publishing, New York (1969). Susumo Okubo, **"Introduction to Octonion and Other Non-Associative Algebras in Physics"**, Cambridge Univ. Press (1995).

[8] T. Kugo and Paul Townsend, Nuclear Physics **B221** (1983) **357-380**.

[9] Jerzy Lukierski and Francesco Toppan arXiv: **hep-th/0203149v1 (2002).**

[10] Jerzy Lukierski and Anatol Nowicki, Annals of Physics **166** (1986)**164-188**.

[11] J. M. Evans, Nuclear Physics **B298**(1988) **92-108.**

[12] Seema Rawat and O.P.S. Negi, Int. J. Theor. Phys. **48**(2009) **305** & Int. J. Theor. Phys. **48**(2009) **2222.**

[13] Bhupendra C. S. Chauhan, O. P. S. Negi, "**Quaternion Generalization of Super Poincaré group**", arXiv:**1508.05368** [physics.gen-ph](2015).

[14] H. J. W. Müller Kirstein and A. Wiedemann, **"Supersymmetry"**, Word Scientific (1987).

[15] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, JHEP **1408** (2014) **080.**