Dynamics of satellite rotation

(revisiting the results of J.Wisdom, *et al.* (1984)).

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The main motivation of the current research is the analytical exploring of the dynamics of satellite rotation during the motion on the elliptic orbit around the planet.

We should discuss the revisited results of J.Wisdom, *et al.* (1984). By elegant change of variables (considering the true anomaly *f* as the *independent* variable), the governing equation of satellite rotation is presented in a form of the *Abel* ODE of the 2-nd type, a kind of generalization of *Riccati* ODE. We should also note that for the reason of a special character of the solutions of *Riccati*-type ODE, there exists a possibility for sudden *jumping* of the magnitude of a solution at some moment of time-parameter.

In physical sense, such the jumping of the *Riccati*-type solutions of the governing ODE could be associated with the effect of sudden acceleration/deceleration of the satellite rotation around the chosen principle axis at definite moment of parametric time. It means that there exists not only a chaotic regime of rotation of satellite (according to the results of J.Wisdom, *et al.* (1984)), but a kind of *gradient catastrophe* Arnold 1992 could occur during the process of satellite rotation. We should especially note that if gradient catastrophe *could occur*, it does not mean that it must occur: such a possibility depends on the initial conditions.

Besides, the asymptotical solutions have been obtained, manifesting a *quasi-periodic* character of the solution even at a strong simplifying assumptions $e \to 0$, $p = 1$, which reduces the governing equation of J.Wisdom, *et al.* (1984) to a kind of the Beletskii's equation.

Keywords: Beletskii's equation, satellite rotation, *Abel* ODE, *gradient catastrophe*.

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1. Introduction, the governing equation.

Planar oscillations and rotations of a satellite around its center of mass, moving along an elliptic orbit with eccentricity *e*, is described by the Beletskii's equation Beletskii 1959, Sadov 2006:

$$(1+e\cos f)\frac{d^2\delta}{d f^2} - 2e\sin f\,\frac{d\delta}{d f} + \omega_0^2 \sin\delta \;=\; 4e\sin f \qquad (1)$$

Here, δ is the *doubled* angle between the radius vector of the center of mass and one of the axes of inertia of the satellite, $\omega_0^2 = 3(B - A)/C$ is the inertial parameter of the satellite; *A* and *B* are the moments of inertia of the satellite with respect to the axes of inertia lying in the orbit plane, and *C* is the moment of inertia with respect to the axis perpendicular to the orbit plane. The true anomaly *f* is *the independent* variable. It is the angular distance of the radius vector from the pericenter of the orbit.

2. Revisiting the results of J.Wisdom, *et al.* (1984).

Referring to the results of J.Wisdom, *et al.* (1984), consider a satellite whose spin axis is normal to its orbital plane. The satellite is assumed to be a *triaxial* ellipsoid with principal moments of inertia $A < B < C$, and *C* is the moment about the spin axis. The orbit is assumed to be a fixed ellipse with semi-major axis *a*, eccentricity *e*, true anomaly *f*, instantaneous radius *r*, and longitude of *periapse* ϖ, which is taken as the origin of longitudes. The orientation of the satellite's long axis is specified by ϑ and thus ϑ - *f* measures the orientation of the satellite's long axis relative to the planet-satellite center line. This notation is the same as that of J.Wisdom, *et al.* (1984), Goldreich and Peale (1966). Without external tidal torques, the equation for motion for ϑ is

$$\frac{d^2\vartheta}{dt^2} + \frac{\omega_0^2}{2r^3}\sin(2(\vartheta - f)) \;=\; 0 \qquad (2)$$

- where $\omega_0^2 = 3(B - A)/C$ and units have been chosen so that the orbital period of the satellite is 2π and its semimajor axis is equal to one. Thus the dimensionless time t is equal to the mean longitude.

If we define $\psi_p = \vartheta - p{\cdot}f$ and Eq. (2) is rewritten with the true anomaly f as the *independent* variable, the equation of motion for ψ_p becomes (*see the proper derivation in Appendix, with only the resulting formulae left in the main text below*):

$$(1+e\cos f)\cdot\frac{d^2\psi_p}{d f^2} - 2e\sin f\cdot\left(p+\frac{d\psi_p}{d f}\right) + \frac{\omega_0^2}{2}\cdot\sin\Big(2(\psi_p+(p-1)\cdot f)\Big) = 0 \qquad (3)$$

Equation (3) could be transformed by the change of variables $(d\psi_p/df) = \Psi(\psi_p)$ as below (note that $(d^2\psi_p/df^2) = (d\Psi/d\psi_p)\cdot\Psi(\psi_p)$), as first approximation:

$$\Big(D\cdot\Psi\Big)\cdot\left(\frac{d\Psi}{d\psi_p}\right) = E\cdot\Psi + F(\psi_p) \qquad (4)$$

$$D=(1+e\cos f),\ E=2e\sin f,\ F(\psi_p)=2e\sin f\cdot p-\frac{\omega_0^2}{2}\cdot\sin\Big(2(\psi_p+(p-1)\cdot f)\Big)$$

- recall that the true anomaly f is supposed to be an *independent* variable, including being independent from ψ_p. So, true anomaly f could be considered as *a slowly variable* parameter in Eq.(4), which is known to be changing periodically for elliptical motion of the satellite on its orbit around the planet (or *quasi*-periodically).

The last Eq. (4) is known to be the *Abel* ODE of the 2-nd type, a kind of generalization of *Riccati* ODE. Such a type of equations has no analytical solution in general case Kamke 1971. We should note also that a modern methods exist for obtaining of the solution of *Riccati* equations with a good approximation Bender and Orszag (1999), Rosu, *et al*. (2012).

3. The revolving scheme for resolving of cascade of Abel Eqs. for satellite rotation.

Let us clarify the essential assumption which was made at transformation of Eq. (3) to the Eq. (4). Namely, we assumed that the true anomaly f could be considered as *a slowly variable* parameter in Eq.(4), which is known to be changing periodically for elliptical motion of the satellite on its orbit around the planet (or *quasi*-periodically).

Indeed, we have neglected by the tidal torque during derivation of (1)-(2), which could be estimated as below Goldreich and Peale (1966), Khan, *et al.* (1988):

$$T = -\frac{3}{2}\frac{GM^2}{C}\frac{R^5}{r^6}\frac{k_2}{Q} \tag{5}$$

- here G is the gravitational constant, M is the mass of the planet, C is the *maximal* principal moment of inertia of satellite, R is the mean volumetric radius of the satellite, k_2 is the *Love* number (which is describing the response of the potential of the distorted body in regard to the experiencing tides), Q is the quality factor (which is inversely proportional to the amount of energy dissipated essentially as heat by tidal friction). Besides, recall that instantaneous radius r for Keplerian motion of satellite is given as below

$$r = a \cdot \left(\frac{1 - e^2}{1 + e \cdot \cos f} \right) \tag{6}$$

For the satellite which is supposed to be *Maclaurin* spheroid, $C = 0.4 m \cdot R^2$ (m is the mass of satellite), we obtain

$$T = -\frac{15}{4}\frac{GM^2}{m}\frac{R^3}{r^6}\frac{k_2}{Q} \tag{7}$$

So, using Eqs. (5)-(7), Eq. (2) should be presented in a complete form Goldreich and Peale (1966) (*let us assume GM = 1 below just for simplicity of computations*):

$$\frac{d^2\vartheta}{dt^2} + \frac{\omega_0^2}{2}\frac{GM}{r^3}\sin(2(\vartheta - f)) = T \tag{8}$$

All in all, using the procedure of averaging of tidal torque (7) for the period of satellite evolving on its orbit, Beletskii 2007 has obtained for Eq. (8) (see Melnikov 2016):

$$(1+e\cos f)\frac{d^2\delta}{d f^2} + \left(\alpha\cdot(1+e\cos f)^5 - 2e\sin f\right)\frac{d\delta}{d f} + \omega_0^2\cdot\sin\delta = 4e\sin f\,, \qquad (9)$$

$$\alpha = \frac{15}{4}\frac{M}{m}\left(\frac{R}{a}\right)^3\frac{k_2}{Q}(1-e^2)^{-\frac{9}{2}}$$

- here δ is the *doubled* angle between the radius vector of the center of mass and appropriate axis of inertia of the satellite.

We could estimate now from Eq. (9) the changing Δ*f* of the true anomaly *f* during the motion of the satellite on its orbit, for which *f* could be considered as *a constant* parameter when we are resolving Eq.(4). Inside of such short period Δ*f*, the changing of the term (2$e\cdot$*sin f*) in Eq. (9) should be less than the changing of the previously neglected tidal torque during the derivation of (1)-(2):

$$\alpha\cdot(1+e\cos f)^5 \cong 2e\cdot\Delta f\,, \quad \Rightarrow \quad \Delta f \le \frac{\alpha\cdot(1-e)^5}{2e}\,, \qquad (10)$$

$$\alpha = \frac{15}{4}\frac{M}{m}\left(\frac{R}{a}\right)^3\frac{k_2}{Q}(1-e^2)^{-\frac{9}{2}}$$

- or in case of $e \to 0$

$$\Delta f \cong \frac{\alpha}{2e}\,, \qquad (11)$$

$$\alpha = \frac{15}{4}\frac{M}{m}\left(\frac{R}{a}\right)^3\frac{k_2}{Q}$$

- here k_2 ≅0.03, Q ≅ 100 Melnikov 2016 (*see the proper estimations in Appendix*).

Thus, we obtain the revolving scheme for resolving of the cascade of Abel Eqs. for calculating of satellite rotation during the motion of the satellite on its orbit: variable parameter of true anomaly f should be divided by n steps ($n = f/\Delta f$), the size of each is assumed to be Δf (10)-(11); during the motion of the satellite on its orbit within the each of the interval Δf, the absolute meaning of the true anomaly f could be considered as *a constant* parameter when we are resolving Eq.(4). Besides, at each next step of calculations, the final values of solution at previous step should be considered as the *initial conditions* for the next step of calculations.

4. Asymptotical solution of the governing ODE (case $e \to 0$).

We should note that in case $p = 1$, equation (3) could be transformed to a kind of the Beletskii's equation (1) by change of variables $\psi_p \to \delta$, where δ is the *doubled* angle between the radius vector of the center of mass and appropriate axis of inertia of the satellite. So, we should multiply each of the Eqs. (2-3) by 2 for transforming it to the Beletskii's equation along with the accordingly chosen the inertial parameter of the satellite, ω_0^2 (as well as along with the accordingly chosen the moments of inertia of the satellite with respect to the axis of inertia lying in the orbit plane).

Let us transform Eq. (4) from the *Abel* ODE of the 2-nd type to the *Abel* ODE of the 1-st type Kamke 1971, by change of variables $\Psi = 1/\Omega$:

$$\left(D\cdot\frac{1}{\Omega}\right)\cdot\left(-\frac{1}{\Omega^2}\cdot\frac{d\Omega}{d\psi_p}\right) = E\cdot\frac{1}{\Omega} + F(\psi_p) \quad \Rightarrow$$

$$\frac{d\Omega}{d\psi_p} = \left(-\frac{F(\psi_p)}{D}\right)\cdot\Omega^3 - \frac{E}{D}\cdot\Omega^2 \tag{12}$$

$$D = (1 + e\cos f),\; E = 2e\sin f,\; F(\psi_p) = 2e\sin f\cdot p - \frac{\omega_0^2}{2}\cdot\sin\Big(2(\psi_p + (p-1)\cdot f)\Big)$$

Despite Eq. (12) has no analytical solution in general case Kamke 1971, let us explore the asymptotic solution under assumption $e \to 0$:

$$\frac{d\Omega}{d\psi_p} = \left(-\frac{F(\psi_p)}{D} \right) \cdot \Omega^3 - \frac{E}{D} \cdot \Omega^2$$

$$D \cong 1,\ E \cong 0,\ F(\psi_p, f) \cong -\frac{\omega_0^2}{2} \cdot \sin\left(2(\psi_p + (p-1) \cdot f) \right) \ \Rightarrow$$

$$\frac{1}{\Omega^2} = \int 2F(\psi_p) d\psi_p \ \Rightarrow \Psi = \pm\sqrt{\left(\int 2F(\psi_p, f) d\psi_p \right)} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \sqrt{C_0 + \cos\left(2(\psi_p + (p-1) \cdot f) \right)},$$

- here $C_o = const$, $C_o > 1$; besides, the last equation yields as below:

$$\frac{d\psi_p}{d f} = \Psi \quad \Rightarrow \quad \frac{d\psi_p}{d f} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \sqrt{C_0 + \cos\left(2(\psi_p + (p-1) \cdot f) \right)} \tag{13}$$

Obviously, Eq. (13) could be resolved analytically only at $p = 1$:

$$\frac{d\psi_p}{d f} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \sqrt{C_0 + \cos 2\psi_p} \quad \Rightarrow \quad \int \frac{d\psi_p}{\sqrt{C_0 + \cos 2\psi_p}} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \int d f \tag{14}$$

Nevertheless, we can see from Eq. (14) that even at the appropriate simplifying assumptions $e \to 0$, $p = 1$, the asymptotical solution for the Beletskii's equation (1) seems not to be resolved analytically ($2\psi_p \to \delta$, where δ is the *doubled* angle between the radius vector of the center of mass and appropriate axis of inertia of the satellite).

Meanwhile, we obtain from Eq. (14):

$$\cos 2\psi_p = u \quad \Rightarrow \quad \frac{1}{2} \int \frac{d(\arccos u)}{\sqrt{C_0 + u}} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \int d f \quad \Rightarrow$$

$$-\frac{1}{2} \int \frac{d u}{\sqrt{1 - u^2} \cdot \sqrt{C_0 + u}} = \pm\frac{\omega_0}{\sqrt{2}} \cdot \int d f \tag{15}$$

The left part of Eq. (15) is known to be the proper *elliptical* integral in regard to the variable u, see Lawden 1989. But the elliptical integral is known to be a generalization of a class of inverse periodic functions. Thus, by the obtaining of re-inverse dependence for the expression (15), we could present the solution *as a set of quasi-periodic cycles*: it means a quasi-periodic character of the evolution of the double angle ψ_P, which is the angle between the radius vector of the center of mass and appropriate axis of inertia of the satellite.

Discussion

The last but not least, we should discuss the revisited results of J.Wisdom, *et al.* (1984). The Eq. (4) is known to be the *Abel* ODE of the 2-nd type, a kind of generalization of *Riccati* ODE. We should also note that for the reason of a special character of the solutions of *Riccati*-type ODE Kamke 1971, there exists a possibility for sudden *jumping* of the magnitude of a solution at some meaning of time-parameter (*see Fig.1 with the aim just to schematically imagine the type of such the jumping of a solution*):

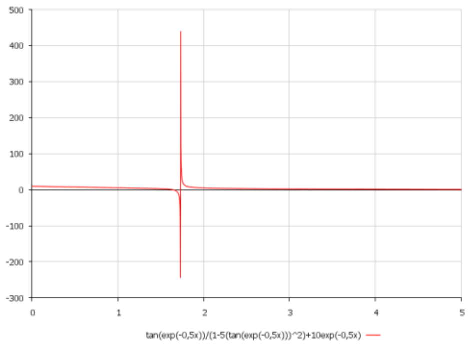

Fig.1. Schematically imagined the jumping of the *Riccati*-type solutions.

In physical sense, such the jumping of the *Riccati*-type solutions of Eq. (4) could be associated with the effect of sudden acceleration/deceleration of the satellite rotation around the chosen principle axis at definite moment of parametric time. It means that there exists not a chaotic regime of rotation of satellite, but a kind of *gradient catastrophe* could occur Arnold 1992 (depending on the initial conditions).

Conclusion

The main motivation of the current research is the analytical exploring of the dynamics of satellite rotation during the motion on the elliptic orbit around the planet.

We should discuss the revisited results of J.Wisdom, *et al.* (1984). By elegant change of variables (considering the true anomaly f as the *independent* variable), the governing equation of satellite rotation is presented in a form of the *Abel* ODE of the 2-nd type, a kind of generalization of *Riccati* ODE. We should also note that for the reason of a special character of the solutions of *Riccati*-type ODE, there exists a possibility for sudden *jumping* of the magnitude of a solution at some moment of time-parameter.

In physical sense, such the jumping of the *Riccati*-type solutions of the governing ODE could be associated with the effect of sudden acceleration/deceleration of the satellite rotation around the chosen principle axis at definite moment of parametric time. It means that there exists not only a chaotic regime of rotation of satellite (according to the results of J.Wisdom, *et al.* (1984)), but a kind of gradient catastrophe Arnold 1992 could occur during the process of satellite rotation. We should especially note that if gradient catastrophe *could occur*, it does not mean that it must occur: such a possibility depends on the initial conditions.

Besides, the asymptotical solutions have been obtained, manifesting a *quasi-periodic* character of the solution even at a strong simplifying assumptions $e \to 0$, $p = 1$, which reduces the governing equation of J.Wisdom, *et al.* (1984) to a kind of the Beletskii's equation.

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this article.

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Appendix.

A.1. The derivation of Eq. (3).

We recall the relation between the true anomaly f and the mean anomaly τ (in accordance with the Kepler's law of orbital motion):

$$\frac{d\,f}{d\,\tau} = \frac{(1+e\cdot\cos f)^2}{(1-e^2)^{\frac{3}{2}}} \tag{16}$$

- here $\tau = n{\cdot}t$, but n being the mean motion of satellite:

$$n^2 = \frac{G\,(M+m_0)}{a^3}, \tag{17}$$

- where units have been chosen so that the orbital period of the satellite is 2π and its semimajor axis in (17) is equal to one (the mean motion is defined by the formula $n = 2\pi/P$, where P is the period of the body motion in orbit). Thus the dimensionless time t is equal to the mean anomaly τ.

We could derive from Eqs. (2) and (16) as below:

$$\frac{d\,\vartheta}{d\,t} = \frac{d\,\psi_p}{d\,f}\cdot\frac{d\,f}{d\,t} + p\cdot\frac{d\,f}{d\,t}$$

$$\frac{d^2\vartheta}{d\,t^2} = \frac{d^2\psi_p}{d\,f^2}\cdot\left(\frac{d\,f}{d\,t}\right)^2 + \frac{d\,\psi_p}{d\,f}\cdot\frac{d^2 f}{d\,t^2} + p\cdot\frac{d^2 f}{d\,t^2} \quad \Rightarrow$$

$$\left(\frac{d\,f}{d\,t} = \frac{(1+e\cdot\cos f)^2}{(1-e^2)^{\frac{3}{2}}}, \quad \frac{d^2 f}{d\,t^2} = \frac{2(1+e\cdot\cos f)\cdot(-e\cdot\sin f)}{(1-e^2)^{\frac{3}{2}}}\cdot\frac{d\,f}{d\,t}\right)$$

$$\Rightarrow \quad \frac{d^2\psi_p}{d\,f^2}\cdot\left(\frac{(1+e\cdot\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\right)^2 + \left(\frac{d\,\psi_p}{d\,f} + p\right)\cdot\frac{2(1+e\cdot\cos f)\cdot(-e\cdot\sin f)}{(1-e^2)^{\frac{3}{2}}}\cdot\frac{(1+e\cdot\cos f)^2}{(1-e^2)^{\frac{3}{2}}} +$$
$$+ \frac{\omega_0^2}{2r^3}\sin(2(\psi_p + (p-1)\cdot f)) = 0 \tag{18}$$

So, Eq. (18) could be transformed properly and should yield as below:

$$(1+e\cdot\cos f)\cdot\frac{d^2\psi_p}{d\,f^2} - 2\,e\cdot\sin f\cdot\left(\frac{d\,\psi_p}{d\,f} + p\right) +$$
$$+ \frac{\omega_0^2}{2r^3}\left(\frac{1-e^2}{1+e\cdot\cos f}\right)^3 \sin(2(\psi_p + (p-1)\cdot f)) = 0 \tag{19}$$

Just compare Eq. (19) with the Eq. (3) to observe the difference:

$$(1+e\cos f)\cdot\frac{d^2\psi_p}{d\,f^2} - 2e\sin f\cdot\left(p + \frac{d\,\psi_p}{d\,f}\right) +$$
$$+ \frac{\omega_0^2}{2}\cdot\sin\left(2(\psi_p + (p-1)\cdot f)\right) = 0,$$

- recall (6) as well as that units have been chosen so that the expression below is equal to one (*a* = *1*)

$$\frac{1}{r^3}\left(\frac{1-e^2}{1+e\cdot\cos f}\right)^3 = \frac{1}{r^3}\left(\frac{r}{a}\right)^3 = 1 \qquad (*)$$

A.2. Estimations in the revolving scheme for resolving of cascade of Abel Eqs.

Let us estimate the required interval Δf, for which the absolute meaning of the true anomaly f could be considered as *a constant* parameter when we are resolving Eq.(4).

For example, let us consider the case of Titania satellite of Uranus (the most massive satellite of Uranus). According to the formulae (10)-(11), we obtain (using the actual data for calculations from the astrometric observations, see Emelyanov 2016):

$$\Delta f \;\cong\; \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00180148)},$$

$$\alpha \;=\; \frac{15}{4}\frac{GM}{Gm}\left(\frac{R}{a}\right)^3\frac{k_2}{Q} = \frac{15}{4}\frac{5\,793\,939.3}{235.3}\left(\frac{788.9}{436\,281.94}\right)^3\frac{0.03}{100}$$

$$\Rightarrow \quad \Delta f \;\cong\; \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00180148)} = 4.55\cdot10^{-5} \cong 9.4'' \qquad (20)$$

- here we choose k_2 ≅0.03, Q ≅ 100 Melnikov 2016. The proper interval Δf (20), required for calculations during the resolving of cascade of Abel Eqs. (4) for satellite rotation, corresponds circa to the path 125 km on elliptical orbit in case of Titania.

Let us also consider the cases of other massive satellites of Uranus:

1) The case of satellite Ariel:

$$\alpha = \frac{15}{4}\frac{GM}{Gm}\left(\frac{R}{a}\right)^3\frac{k_2}{Q} = \frac{15}{4}\frac{5\,793\,939.3}{90.3}\left(\frac{577.9}{190\,929.79}\right)^3\frac{0.03}{100}$$

$$\Rightarrow \quad \Delta f \cong \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00136551)} = 73.3\cdot 10^{-5} \cong 151.2'' \qquad (21)$$

- which corresponds to path 879 km on its quasi-circle orbit (orbital velocity is 5.5 km/s).

2) The case of satellite Umbriel:

$$\alpha = \frac{15}{4}\frac{GM}{Gm}\left(\frac{R}{a}\right)^3\frac{k_2}{Q} = \frac{15}{4}\frac{5\,793\,939.3}{78.2}\left(\frac{584.7}{265984.01}\right)^3\frac{0.03}{100}$$

$$\Rightarrow \quad \Delta f \cong \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00424068)} = 10.44\cdot 10^{-5} \cong 21.5'' \qquad (22)$$

- which corresponds circa to the path 174 km on its elliptical orbit.

3) The case of satellite Oberon:

$$\alpha = \frac{15}{4}\frac{GM}{Gm}\left(\frac{R}{a}\right)^3\frac{k_2}{Q} = \frac{15}{4}\frac{5\,793\,939.3}{201.1}\left(\frac{761.4}{583449.53}\right)^3\frac{0.03}{100}$$

$$\Rightarrow \quad \Delta f \cong \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00140798)} = 2.56\cdot 10^{-5} \cong 5.3'' \qquad (23)$$

- which corresponds circa to the path 94 km on its quasi-circle orbit.

4) The case of satellite Miranda:

$$\alpha = \frac{15}{4}\frac{GM}{Gm}\left(\frac{R}{a}\right)^3\frac{k_2}{Q} = \frac{15}{4}\frac{5\,793\,939.3}{4.4}\left(\frac{234.2}{129848.11}\right)^3\frac{0.03}{100}$$

$$\Rightarrow \quad \Delta f \cong \frac{\alpha}{2e} = \frac{\alpha}{2\cdot(0.00132732)} = 327.4\cdot 10^{-5} \cong 675.4'' \qquad (24)$$

- which corresponds circa to the path 2671 km on its quasi-circle orbit (orbital velocity is circa 6.7 km/s).

As we can see, the best candidates for calculations (by the revolving scheme for resolving of cascade of *Abel* Eqs. for satellite rotation) are the Ariel and Miranda: time-interval for calculations in case of Ariel is circa (879 km/5.5 km/s) ≅ 160 s, in case of Miranda it appears to be circa (2671 km/6.7 km/s) ≅ 400 s. It means 8,573 steps of iterations of calculations for Ariel (by the revolving scheme for resolving of cascade of *Abel* Eqs. for satellite rotation) per 1 period of rotation and 1,919 steps of iterations for Miranda.

References:

Arnold V.I.: Catastrophe Theory, 3-rd Ed. Berlin: Springer-Verlag (1992).

Beletskii, V.V.: Libration of a Satellite, in Iskusstvennye sputniki Zemli. Vyp. **3** (Earth's Artificial Satellites, Issue **3**), Moscow: Akad. Nauk SSSR, pp.13–31 (1959).

Beletskii, V.V.: Regular and Chaotic motions of rigid bodies. M. – Izhevsk: IKM, 132 pages (2007).

Carl M. Bender, Steven A. Orszag: Advanced Mathematical Methods for Scientists and Engineers. Origin.published by McGraw Hill, 1978, XIV. pp.20-22 (1999).

Goldreich, P., and Peale, S. J.: Spin-orbit coupling in the Solar system. Astron.J., **71**, pp.425-438 (1966).

Emelyanov, N.V.: Erratum: Ephemerides of the main Uranian satellites. MNRAS Advance Access published June 14 (2016).

Kamke E.: Hand-book for Ordinary Differential Eq. Moscow: Science (1971).

Khan, A., Sharma, R., Saha, L. M.: Chaotic motion of an ellipsoidal satellite. I. Astron.J., Vol. **116, No.** 4, p.2058 (1988).

Lawden D.: Elliptic Functions and Applications. Springer-Verlag (1989).

Melnikov A.V.: Resonant and chaotic phenomena in the dynamics of celestial bodies. Spb, GAO Pulkovo Observatory, dissertation thesises for the degree of the Doctor of Science: 238 pages (2016).

Rosu H.C., Cornejo-Pérez O., and Ojeda-May P.: Traveling kinks in cubic nonlinear Ginzburg-Landau equations, Phys. Rev. E **85**, 037102 (2012).

Sadov S.Yu.: Stability of Resonance Rotation of a Satellite with Respect to Its Center of Mass in the Orbit Plane Cosmic Research, Vol. **44**, No. 2, pp. 160–171. Pleiades Publishing, Inc (2006).

Wisdom, J., and Peale, S. J., and Mignard F.: The chaotic rotation of Hyperion. Icarus, **58**, No.2, pp.137-152 (1984).