A NEWTONIAN EQUIVALENT FOR THE COSMOLOGICAL CONSTANT

BY

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Abstract. Following some older ideas, an equivalent for the cosmological constant in classical mechanics it was found. In our development, this Newtonian equivalent for the cosmological constant appears in a natural way into Friedmann first equation.

Key words::cosmological constant; Newtonian mechanics; algebraic potential .

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1. Introduction

Empirically introduced by Einstein in 1917 [1], the cosmological constant, after the discovery of expanding universe [2], was repelled by its author. However, in Friedmann model [3], Λ appears from general relativity equations of field deduced by Lemaitre in 1927, [4, 5]. They had used general relativity, for an unbounded homogeneous universe, to show the validity of the Friedmann equations.

The Newtonian derivation of the Friedmann equations by Milne [6] and McCrea [7] came after Friedmann and Lamaitre demonstrations. Milne and McCrea, using a Newtonian formalism, had obtained the same results like relativistic cosmology theory. Nevertheless this Newtonian cosmology can not explain all the observational data since it does not contain a theory of light propagation. Even so, this approach is quite legitimate, since the structural similarity of general relativity and Newtonian celestial mechanics were pointed out by Cartan [8, 9]. Following his ideas, [10] showed a correct derivation of the Friedmann equations from Newtonian theory.

We presently develop a follow-up of Milne ideas by introducing in a natural manner the cosmological constant Λ .

2. Theoretical treatment

If two compact spheres attract each other with a force proportional with $\frac{1}{r^2}$, as demonstrated by Newton's theorem, then we can replace the two spheres with points that have the mass of the associated spheres. The same idea is valid if we have instead the two spheres, two spherical shells. More than this, we can expand the spherical shell thickness in vicinity of its center, so that finally the difference between a compact sphere and a spherical shell is the vicinity of central point, undetermined. In this case one can study, the general form of the gravitational potential and the limits of this approximation.

Consider a homogenous spherical shell with α its thickness, density σ and its center O, situated at distance r from an arbitrary exterior point P. If note $\Phi_{(r)}$ the gravitational potential generated by an unit mass placed at distance r, one observes that the gravitational potential in P is equivalent to one generated by the point mass $m_{(\alpha)}$ placed in O. Consequently the gravitational potential of the point mass at distance r must be equal to the gravitational potential of the spherical shell at the same distance:

$$m_{(\alpha)}\Phi_{(r)} + 2\pi\sigma\alpha\gamma_{(\alpha)} = \frac{2\pi\sigma\alpha}{r}\int_{r-\alpha}^{r+\alpha}\beta\Phi_{(\beta)}d\beta \tag{1}$$

where the term containing $\gamma_{(\alpha)}$ is a constant which can be added to the potential without altering its associated force law.

The general solution of equation (1) is given by:

$$\Phi = \sum_{i=1}^{4}\Phi_i \tag{2}$$

where the potentials Φ_i are the independent solutions of eq. (1):

$$\Phi_1 = \frac{A}{r},\ \Phi_2 = \frac{B}{r}e^{-\xi\cdot r},$$

$$\Phi_3 = \frac{C}{r}e^{\xi\cdot r},\ \Phi_4 = Dr^2$$

Eq. (1) must satisfy the relation:

$$m\cdot\Phi = \sum_{i=1}^{4}m_i\Phi_i$$

where the equivalent masses are:

$$m_1 = m_4 = 4\pi\sigma\alpha^2,\ m_2 = m_3 = 4\pi\sigma\alpha\frac{sh(\xi\cdot\alpha)}{\xi}$$

(3) $$\gamma(\alpha) = 2\alpha E + 2K\alpha^2$$

is a constant which can be added to the potential without alter its associated force law.

The solutions Φ_i were deduced independently in [11], ($\Phi = \Phi_2 + \Phi_3$) and in [12], ($\Phi = \Phi_1 + \Phi_4$). Because the first solution describes an interaction at small scale we concentrate to the second one. This solution is actually the algebraic potential:

(4) $$\Phi_{(r)} = \frac{B_1}{r} + B_2 r^2 + B_3$$

with the same equivalent mass:

$$m_{(\alpha)} = 4\pi\sigma\alpha^2$$

B_1, B_2, B_3 are arbitrary real constants and:

$$\gamma_{(\alpha)} = 2B_3\alpha + 2B_2\alpha^2$$

is a constant with the form (2).

The well known Newtonian potential $\Phi \approx r^{-1}$ is deduced by considering another property of inverse square radius force. For the potential (4) the interior of the spherical shell must be an equipotent region. In general $\Phi_{(r)}$ will have this property if:

(5) $$\gamma_{(\alpha)} \cdot r = \int_{\alpha-r}^{\alpha+r} \beta \cdot \Phi_{(\beta)} d\beta \text{, for } r < \alpha$$

Equation (5) has the unique solution:

(6) $$\Phi_{(r)} = \frac{B}{r} + C$$

where the constant C can be zero without it's associated force law being altered.

A potential with a form close to (6) has been used by Milne to derive the first Friedmann equation from the energy integral:

(7) $$H^2 = \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

where the term containing the cosmological constant has been introduced by postulating a so-called expansion force. Nevertheless if we observe that equation (4) includes the term $B_2 r^2$ which contains the Newtonian equivalent of a cosmological constant, one may reconsider Milne's derivation.

First thing we must do is to presume valid the potential (4). In other words to consider as valid the hypothesis that the interior region of the supposed homogenous spherical shell will not be an equipotent region. Consequently we have:

(8)

$$\Phi_{(r)} = \frac{B_1}{r} + B_2 r^2$$

which is the potential (4) with $B_3 = 0$, an operation which simplifies (4) without the associated force law being altered.

Then we have:

$$B_1 = GM,$$

where G is the Newton's gravitational constant and M is the entire mass within the sphere, a constant with respect to time,

$$M = \frac{4\pi}{3}\rho \cdot r^3$$

and ρ is the mass density. We chose $B_1 = GM = B$, to be in accordance with Milne's derivation. Thus we expand the thickness of the spherical shell in vicinity of its center, in order to have a good approximation between our spherical shell and a compact sphere. The constant B_2 is presumed positive, it correspond to a repulsive force.

The force applied on a particle of mass m in motion within the potential (8) is:

(8')
$$m(d^2 r / dt^2) = -\frac{GMm}{r^2} + B_2 r^2 m$$

This equation will lead us, by integration, to an equation of form (7). Multiplying it with the first derivative of 2r one will observe very easily that the left term is the derivative with respect to time of square first derivative. The last right term is obvious the derivative with respect to time of the repulsive potential. To calculate the first right term we need the expression:

$$\rho r \frac{dr}{dt} = -\rho r^3 \frac{d}{dt}\left(\frac{1}{r}\right) = \frac{d}{dt}\left[\rho r^3 \frac{1}{r}\right]$$

which result by the fact that the entire mass within the sphere, M, is a constant with respect to time and this leads to:

$$r^3 \frac{d\rho}{dt} + 3\rho r^2 \frac{dr}{dt} = 0$$

After we integrate (8'), introducing the scale parameter and replacing the integration constant with another constant proportional with the ratio r/a it results, after an elementary calculus, the energy integral:

(9)
$$H^2 = \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2} + 2B_2$$

with k a dimensionless constant given by:

$$k = \frac{2E}{mc^2}$$

There is a physical equivalence between the two equations, observed directly from similarities between expressions (7) and (9). The only significant difference between them is the fact that one is deduced naturally from equation (8), the other is deduced from a postulated so-called expansion force.

In conclusion if we neglect equation (7) and set:

$$2B_2 = \frac{\Lambda c^2}{3} \tag{10}$$

we have been found a Newtonian equivalent for the cosmological constant.

3. Discussions

The equivalence relation (10) doesn't solve the problem of the cosmological constant physical nature. We can't conceive a Newtonian cosmology pure and simple based on this theory. To establish a concordance with observational data we must exceed the Newtonian theory and make the assumptions which lead us to Λ's physical nature. Thus the constant B_2 will result from other theories, as until now. Some early generalizations of general relativity theory found that the cosmological constant arose naturally from the mathematics, [13, 14]. But these ideas don't have a natural support and were soon abandoned. Inevitable we must consider B_2 as an intrinsic energy density of the vacuum in which case the form (10) it will be conserved. A positive cosmological constant is generated by a negative vacuum pressure. The sign plus of the constant B and the cosmological constant legitimate this idea. A positive repelling force corresponds only to a negative pressure. Hence the cosmological constant problem is occurring, which is known like the most difficult situation of fine-tuning in physics. There is no coherent procedure to derive the cosmological constant from particle physics, and also the modern field theories are pessimistic regarding this matter.

A solution to the cosmological constant physical nature matter is provided by the gravitational potential (8) itself. The existence of a gravitational potential in the form (8) seems to legitimate the idea that the cosmological constant is the natural consequence of it. So, the physical nature of cosmological constant must be searched in the matter density not in vacuum energy density as it was by now. To solve the cosmological constant physical nature problem it is equivalent to consider Λ as an expression of the observational lack of matter and energy in the entire universe, the dark matter and the dark energy.

4. Conclusions

In this paper we obtain a Newtonian equivalent for the cosmological constant by using the first Friedmann's equation. The Newtonian equivalent was introduced in a natural way, as it resulted from (8), by calculus not by postulation. It is discussed then the material nature of Λ, which seems to be the lack of matter and energy in the entire universe.

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