A GENERAL FORM FOR THE ELECTRIC FIELD LINES EQUATION CONCERNING AN AXIALLY SYMMETRIC CONTINUOUS CHARGE DISTRIBUTION

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Abstract..By using an unexpected approach it results a general form for the electric field lines equation. It is a general formula, a derivative-integral equation structured as a multi-pole expansion series. By solving this equation we can find the electric field lines expressions for any type of an axially symmetric multi-pole continuous electric charge distributions we interested in, without the need to take again the calculus from the beginning for each case particularly, for instance as in discrete charge distribution case.

Key words: electric field lines equation; multi-pole expansion series; axially symmetric continuous electric charge distribution.

1. Introduction

From an axially symmetric magnetic multi-pole of arbitrary degree n, (Jackson, 1975), we can derive the exact equation for the field lines, (Jeffreys, 1988). The method presented in (Jeffreys, 1988) deals with spherical harmonics in the most general way. Consequently the equation for the field lines is the expression of a general case. Another two exact equations for the field lines are given in (Willis & Gardiner, 1988). The equations are for two special magnetic multi-poles of arbitrary degree with no axial symmetry. These cases may be classified as either symmetric or anti-symmetric sectorial multi-poles.

By using the above considerations the aim of this paper is to find a general form for an exact equation for the field lines of an electric multi-pole with axial symmetry.

2. Theory

Let's consider now a continuous electrostatic charge distribution within a spatial volume. We must evaluate the electric potential in a point P outside the distribution, as we can see in figure below:

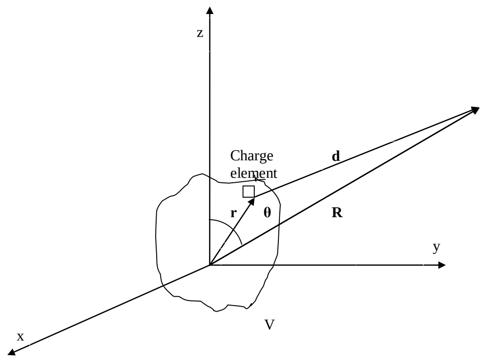

Fig.1

The electric field lines equation is the well known expression:

(1) $$\vec{E} \times d\vec{l} = 0$$

By assuming that we have a charge distribution with an axial symmetry with respect to z axis, we can explicit the length element and the electric field as:

(2) $$d\vec{l} = dR \cdot \vec{u}_R + R \cdot d\theta \cdot \vec{u}_\theta$$

and:

(3) $$\vec{E} = -\nabla V = -\frac{\partial V}{\partial R}\vec{u}_R - \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{u}_\theta$$

The cross product (1) leads after an elementary calculus to the well known field lines equation written in polar coordinates:

(4) $$\frac{dR}{R}\frac{\partial V}{\partial \theta} - R \cdot d\theta \cdot \frac{\partial V}{\partial R} = 0$$

For a continuous charge distribution the electric potential V can be expanded as a Legendre series, according to (Eyges, 1980):

$$V_{(R,\theta)} = \frac{1}{4\pi\varepsilon_0}\sum_{m=0}^{\infty}\frac{1}{R^{m+1}}\int P_m(\cos\theta)\rho(r)r^m d^3r$$

Consequently the potential derivatives from equation (3) can be written as:

$$\frac{\partial V}{\partial R} = -\frac{1}{4\pi\varepsilon_0}\sum_{m=0}^{\infty}\frac{m+1}{R^{m+2}}\int P_m(\cos\theta)\rho(r)r^m d^3r$$

and:

$$\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\varepsilon_0}\sum_{m=0}^{\infty}\frac{1}{R^{m+1}}\frac{\partial}{\partial \theta}\int P_m(\cos\theta)\rho(r)r^m d^3r$$

By introducing these results within equation (3) and considering the property:

$$\frac{\partial}{\partial \theta}\int P_m(\cos\theta)\rho(r)r^m d^3r = \int \frac{\partial}{\partial \theta}P_m(\cos\theta)\rho(r)r^m d^3r$$

the electric field lines equation can be expressed as:

$$\frac{dR}{R}\sum_{m=0}^{\infty}\frac{1}{R^{m+1}}\frac{\partial}{\partial\theta}P_m(\cos\theta)+Rd\theta\sum_{m=0}^{\infty}\frac{m+1}{R^{m+2}}P_m(\cos\theta)=0 \tag{5}$$

This is a general expression for the electric field lines equation under continuous charge distribution hypothesis. At first sight it exhibits a complicate form which requires for solving a derivative-integral equation method. Despite this appearance the solutions can be obtained in a simple and direct manner, as its show in the following examples.

It is useful for our calculations to consider the Rodrigues representation of Legendre polynomials:

$$P_m(\cos\theta)=\frac{1}{2^m m!}\cdot\frac{d^m}{d\cos^m\theta}\left(\cos^2\theta-1\right)^m \tag{6}$$

Under these circumstances equation (5) became more explicit and simple. The derivative with respect to θ of expression (6):

$$\frac{\partial P_m(\cos\theta)}{\partial\theta}=-\frac{m}{2^m m!}\frac{d^m}{d\cos^m\theta}\left[\left(\cos^2\theta-1\right)^{m-1}\cdot 2\cos\theta\sin\theta\right] \tag{7}$$

leads to an important observation that we can make the derivatives with respect to cosine before we make the integration, and thus the equation (5) became only an integral equation, more simpler to solve.
It is obvious that the case m=0 doesn't exist because the derivatives (7) don't exist. More interesting is the dipole case:

$$m=1$$

By taking into account the expressions (6) and (7), the equation (5) can be written as:

$$-\frac{dR}{R}\frac{1}{R^2}\frac{1}{2}\frac{d}{d\cos\theta}\left[\left(\cos^2\theta-1\right)^0\cdot 2\cos\theta\sin\theta\right]+$$
$$Rd\theta\frac{2}{R^3}\frac{1}{2}\frac{d}{d\cos\theta}\left(\cos^2\theta-1\right)=0$$

After trivial simplification and obvious derivatives we obtain the equation:

$$\frac{dR}{R}\sin\theta-2\cos\theta\cdot d\theta=0$$

which can be directly integrated as:

$$R=C\sin^2\theta \tag{8}$$

and it is the well-known expression, in polar coordinates, of the field lines for an electric dipole.

The mathematical treatment of the case $m=2$ is the same as the previous case. We obtain the equation:

$$-\frac{dR}{R}\frac{1}{R^3}\frac{2}{2^2 2}\frac{d^2}{d\cos^2\theta}[(\cos^2\theta-1)^1\cdot 2\cos\theta\sin\theta]+$$

$$Rd\theta\frac{3}{R^4}\frac{1}{2^2 2}\frac{d^2}{d\cos^2\theta}(\cos^2\theta-1)^2=0$$

from which is deduced the most simplest form:

(9) $$\frac{dR}{R}=\frac{3\cos^2\theta-1}{2\sin\theta\cos\theta}d\theta$$

Finally, after integrating equation (9), we are obtaining the following relation:

(10) $$R^2=k\sin^2\theta\cos\theta$$

which is the well-known expression of the field lines for an electric 4-pole.

Equation (5) is the direct consequence of the equation (3). If the electric field couldn't be an expression of a scalar potential, then all the above mathematical statement has no basis. The magnetic analog for V doesn't support sources. Subsequently the magnetic analog for equation (3) can be written only with the vector potential A. The vector potential is defined in terms of current density. Under axial symmetry and continuous distribution of current density hypothesis, A can also be expanded in Legendre series. But compared with the electric field this is the only similarity. The magnetic field lines equation appears in a double cross-product form. The solutions of this equation are more complicate than equation (5), (see (Jeffreys, 1988)).

3. Conclusions

The aim of this paper is to deduce a new form for the electric field lines equation. We obtain a general formula, a derivative-integral equation structured as a multi-pole expansion series. The equation has exact solutions corresponding to an axially symmetric electric multi-pole continuous charge distribution, without the need to consider special assumptions for $m>0$. Equation (5) can be the starting point of the entire section 2., because is valid in mentioned approximations, without the need to deduce it from equation (1) for each case from the beginning, for instance as in discrete charge distribution case.

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