

Using the stress function in the flow of generalized Newtonian fluids through pipes and slits

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Abstract

We use a generic and general numerical method to obtain solutions for the flow of generalized Newtonian fluids through circular pipes and plane slits. The method, which is simple and robust can produce highly accurate solutions which virtually match any analytical solutions. The method is based on employing the stress, as a function of the pipe radius or slit thickness dimension, combined with the rate of strain function as represented by the fluid rheological constitutive relation that correlates the rate of strain to stress. Nine types of generalized Newtonian fluids are tested in this investigation and the solutions obtained from the generic method are compared to the analytical solutions which are obtained from the Weissenberg-Rabinowitsch-Mooney-Schofield method. Very good agreement was obtained in all the investigated cases. All the required quantities of the flow which include local viscosity, rate of strain, flow velocity profile and volumetric flow rate, as well as shear stress, can be obtained from the generic method. This is an advantage as compared to some traditional methods which only produce some of these quantities. The method is also superior to the numerical meshing techniques which may be used for resolving the flow in these systems. The method is particularly useful when analytical solutions are not available or when the available analytical solutions do not yield all the flow parameters.

Keywords: fluid dynamics; rheology; pipe flow; slit flow; Newtonian; power law; Ellis; Ree-Eyring; Carreau; Cross; Bingham; Herschel-Bulkley; Casson.

1 Introduction

There are many applications for the flow of generalized Newtonian fluids in circular pipes and plane slits. These include biological flow in living organisms and fluid shipping in transport and process industries. Hence various analytical and numerical methods have been developed and used to obtain the flow parameters which include the stress function, rate of shear strain, flow velocity profile, local viscosity and volumetric flow rate as functions of the conduit geometry, pressure drop and fluid rheology [1, 2].

A very simple numerical method, which is very natural to use, is to employ the stress function, which is defined in terms of the conduit geometry and pressure drop, directly. The method is based on a simple combination of the easily obtained stress function with the fluid rheology to obtain the rate of strain as a function of the velocity-varying spatial dimension of the conduit, i.e. the radius for the pipe flow and the thickness dimension for the slit flow. The flow velocity profile and the volumetric flow rate can then be obtained from simple numerical integrations. The local viscosity can also be obtained from the fluid rheological relation as soon as the stress and rate of strain are obtained.

The method is very generic, as well as being more convenient to implement and use, and hence the solutions obtained from it can be more reliable than the solutions obtained from more sophisticated methods. It is also very general since it can be applied whenever the rate of strain can be expressed as an explicit or implicit function of the shear stress; a relation that usually can be obtained from the fluid rheological equation. Hence, the method can be used to solve almost all the generalized Newtonian flow problems in pipes and slits since the stress function for these systems can be easily obtained and the appropriate rheological equations can be readily expressed in the desired form. The method is particularly useful when the flow parameters, or some of which, cannot be obtained analytically. Hence, it

can provide a good alternative to the classical methods for solving these problems which are traditionally solved by employing rather complicated methods such as numerical discretization techniques like finite element and finite difference.

In section § 2 we give a general description of the method and its theoretical justification as well as the type of flow systems to which the method applies and the restrictions and assumptions that should be observed in its application. In section § 3 we discuss practical issues about the implementation of the method. We also present a sample of the volumetric flow rate solutions that were obtained from the method with comparison to similar results obtained from the Weissenberg-Rabinowitsch-Mooney-Schofield (WRMS) method. The paper is ended in section § 4 with general discussions and summary of the main objectives and accomplishments of this investigation.

2 Method

We assume an incompressible, laminar, isothermal, steady, pressure-driven, fully-developed flow of purely-viscous, time-independent fluids that are properly characterized by the following generalized Newtonian fluid model

$$\tau = \mu\gamma \tag{1}$$

where the shear viscosity, μ , and shear stress, τ , depend only on the contemporary rate of strain, γ , with no memory for the fluid of its deformation history. The effects of external body forces, like gravitational attraction or electromagnetic interaction, as well as the edge effects at the entry and exit zones of the conduit are assumed insignificant. Dependencies on physical factors like temperature, which are not related to deformation, are also ignored assuming fixed conditions or negligible contribution from these factors. The flow is also assumed to be in shear mode with no significant extensional contributions.

As for the boundary conditions, a no-slip at the conduit wall is assumed and hence a zero velocity condition at the fluid-solid interface is maintained. For the investigated types of rheology and flow system, the flow velocity profile has a stationary derivative point at the symmetry center line of the pipe and symmetry center plane of the slit which means zero stress and rate of strain at these loci. For viscoplastic fluids, these stationary zones extend to include all the points at the forefront of the flow profile whose stress falls below the fluid yield stress.

Concerning the type of conduit, we use circular pipe geometry with length L and radius r , and plane slit geometry with length L , width W and thickness $2B$ where the latter dimension is the smallest of the three. A pressure drop Δp is imposed along the conduit length dimension which defines the flow direction. The pipe is assumed straight with a cross sectional area that is uniform in shape and size while

the slit is assumed straight, long and thin with a uniform cross section. For both types of conduit, rigid mechanical properties of the conduit wall are assumed and hence the conduit wall is not deformable under the considered range of pressure drop. It is also assumed that the slit is positioned symmetrically in its thickness dimension, z , with respect to the plane $z = 0$.

The generic method is based on the proportionality between the stress and the spatial coordinate in the flow profile dimension, s , which stands for the radius r in the case of pipe and for the thickness dimension z in the case of slit. Since the stress τ as a function of s is known from the above mentioned proportionality, then the rate of strain, γ , can be easily obtained from the fluid rheological constitutive relation which correlates the rate of shear strain to the shear stress as long as it can be put in the form $\gamma = \gamma(\tau)$ where the dependency of γ on τ can be explicit or implicit. If γ is an explicit function of τ , as it is the case for example for Ellis fluids (refer to Table 1), then γ can be obtained directly by a simple substitution in the rheological relation. If, on the other hand, γ is an implicit function of τ , as it is the case for example for Cross fluids (refer to Table 1), then γ can be obtained numerically using a simple numerical solver based for instance on a bisection method. In both cases, the obtained rate of strain as a function of s can be used to obtain the velocity profile and subsequently the volumetric flow rate by consecutive numerical integrations.

To be more specific, the stress in the investigated flow systems of circular pipes and plane slits is a function of the conduit geometry and pressure drop. The shear stress as a function of the velocity-varying dimension s can be obtained by several methods which can be found in many fluid mechanics and rheology textbooks, e.g. [1–3], and hence we are not going to give a formal derivation; instead we give the final results with a sketch of how it can be obtained. From a simple force balance argument based on applying the Newton second law of mechanics to the

non-accelerating steady flow, where the normal force exerted on the conduit cross section is balanced by the shear stress force, it can be established that the stress is proportional to s as summarized in the following equations for pipe and slit respectively

$$\tau = \frac{\tau_R}{R}r \quad \text{and} \quad \tau = \frac{\tau_B}{B}z \quad (2)$$

where τ_R and τ_B are, respectively, the pipe and slit wall shear stress as given by

$$\tau_R = \frac{R\Delta p}{2L} \quad \text{and} \quad \tau_B = \frac{B\Delta p}{L} \quad (3)$$

where R is the pipe radius, B is the slit half thickness, and L is the conduit length across which a pressure drop Δp is exerted.

Since the spatial dependence of the stress function is known, the rate of strain as a function of s can be obtained from the rheological relation expressed in the form $\gamma(\tau(s))$, as given for a sample of generalized Newtonian fluids in Table 1, and hence $\gamma(s)$ is obtained. The obtained strain rate function is then integrated numerically with respect to the spatial coordinate of the flow profile, s , to obtain the flow velocity profile where the no-slip at wall condition provides an initial value for the flow velocity, $v = 0$, which is then incremented in moving from the wall to the center during the integration process. The numerically obtained flow velocity is then integrated numerically with respect to the conduit cross sectional area perpendicular to the flow direction to obtain the volumetric flow rate. For viscoplastic fluids, the zero stress condition at the pipe center line and slit center plane is extended to include all the regions at the forefront of the flow profile whose shear stress falls below the fluid yield stress. In Table 1 the rheological constitutive relations for nine fluid models employed in this study are presented. For Carreau and Cross models, γ is given as an implicit function of τ and hence a simple numerical solver like

bisection is required to obtain γ as a function of $\tau(s)$ and hence as a function of s .

Table 1: The rate of shear strain, γ , as a function of shear stress, τ , for the nine fluid models used in this investigation [1, 2, 4–6]. For Carreau and Cross models, γ is given as an implicit function of τ . The meaning of the symbols can be obtained from Nomenclature § 5.

Model	Rate of Shear Strain
Newtonian	$\gamma = \frac{\tau}{\mu_o}$
Power Law	$\gamma = \sqrt[n]{\frac{\tau}{k}}$
Ellis	$\gamma = \frac{\tau}{\mu_e} \left[1 + \left(\frac{\tau}{\tau_h} \right)^{\alpha-1} \right]$
Ree-Eyring	$\gamma = \frac{\tau_c}{\mu_r} \sinh \left(\frac{\tau}{\tau_c} \right)$
Carreau	$\gamma \left[\mu_i + (\mu_o - \mu_i) (1 + \lambda^2 \gamma^2)^{(n-1)/2} \right] = \tau$
Cross	$\gamma \left[\mu_i + \frac{\mu_o - \mu_i}{1 + \lambda^m \gamma^m} \right] = \tau$
Bingham	$\gamma = \frac{\tau - \tau_0}{C'}$
Herschel-Bulkley	$\gamma = \sqrt[n]{\frac{\tau - \tau_0}{C}}$
Casson	$\gamma = \frac{(\tau^{1/2} - \tau_0^{1/2})^2}{K}$

3 Implementation, Results and Assessment

The generic method, as described in the last section, was implemented in a computer code using standard numerical integration and bisection solver techniques. The method was then employed to obtain solutions for the flow of nine types of generalized Newtonian fluid through circular pipes and plane slits. The nine types of fluid are: Newtonian, power law, Ellis, Ree-Eyring, Carreau, Cross, Bingham, Herschel-Bulkley and Casson. The numerical results of the volumetric flow rate which are obtained from the generic method using wide ranges of fluid and conduit parameters were thoroughly compared to the results obtained from the WRMS analytical method.

A representative sample of the results obtained from the two methods are presented in Figures 1 and 2 where the fluid and conduit parameters of these examples are given in Table 2. As seen in these examples, the solutions obtained from the generic method agree very well with the WRMS analytical solutions. The minor departure between the two methods is due mainly to the nature of the generic method as it heavily relies on numerical techniques, i.e. bisection solvers and successive numerical integrations, which can accumulate numerical errors.

The big advantage of using the generic method in the flow problems through circular pipes and plane slits is that it is very general as it can be applied to almost any generalized Newtonian fluid model regardless of the complexity of its constitutive relation as long as the rheological relation can be casted as an explicit or implicit function of the form $\gamma(\tau(s))$. Moreover, as the method only involves very simple and reliable numerical techniques, like bisection solvers and numerical integration techniques, high accuracy of the solution can be achieved. It also has no convergence difficulties under normal conditions, as it does not require matrix solvers and similar sophisticated computational techniques, and hence it can be very reliable. Furthermore, the method is very simple and hence it is easy and

convenient to implement and use; which is an advantage on its own plus being a further contributing factor to the reliability of the solutions. Hence, the generic method can produce virtually exact solutions that match in their accuracy any actual or potential analytical solutions even for the fluids with complex rheology. The generic method can therefore offer a better alternative to the commonly used numerical methods which employ more sophisticated computational techniques like finite difference.

Although the generic method is presented here as a numerical technique, it can also be applied analytically where the stress function, $\tau(s)$, is substituted in the rheological relation to obtain the rate of strain, $\gamma(s)$, which can be integrated successively to obtain the flow velocity profile and volumetric flow rate. This can provide alternative analytical forms that can be useful in some cases for verification or other purposes.

Table 2: Fluid and conduit parameters for the examples of Figures 1 and 2. The ‘R’ column applies to pipes and the ‘B’ column applies to slits; the other columns are common to both. SI units apply to all dimensional quantities as given in Nomenclature § 5, and ‘HB’ stands for Herschel-Bulkley.

Model	Fluid Properties	R	B	L
Newtonian	$\mu_o = 0.013$	0.05	0.007	0.45
Power Law	$k = 0.023, \quad n = 0.63$	0.015	0.008	0.55
Ellis	$\mu_e = 0.026, \quad \tau_h = 8, \quad \alpha = 1.6$	0.005	0.002	0.22
Ree-Eyring	$\mu_r = 0.41, \quad \tau_c = 43$	0.03	0.004	1.65
Carreau	$\mu_0 = 0.37, \quad \mu_i = 0.0079, \quad \lambda = 0.65, \quad n = 0.62$	0.043	0.0086	0.95
Cross	$\mu_0 = 0.42, \quad \mu_i = 0.0093, \quad \lambda = 0.77, \quad m = 0.58$	0.032	0.01	1.26
Bingham	$C' = 0.033, \quad \tau_0 = 5.7$	0.01	0.01	0.25
HB	$C = 0.042, \quad \tau_0 = 7.9, \quad n = 1.34$	0.005	0.002	0.13
Casson	$K = 0.71, \quad \tau_0 = 3.2$	0.08	0.011	1.27

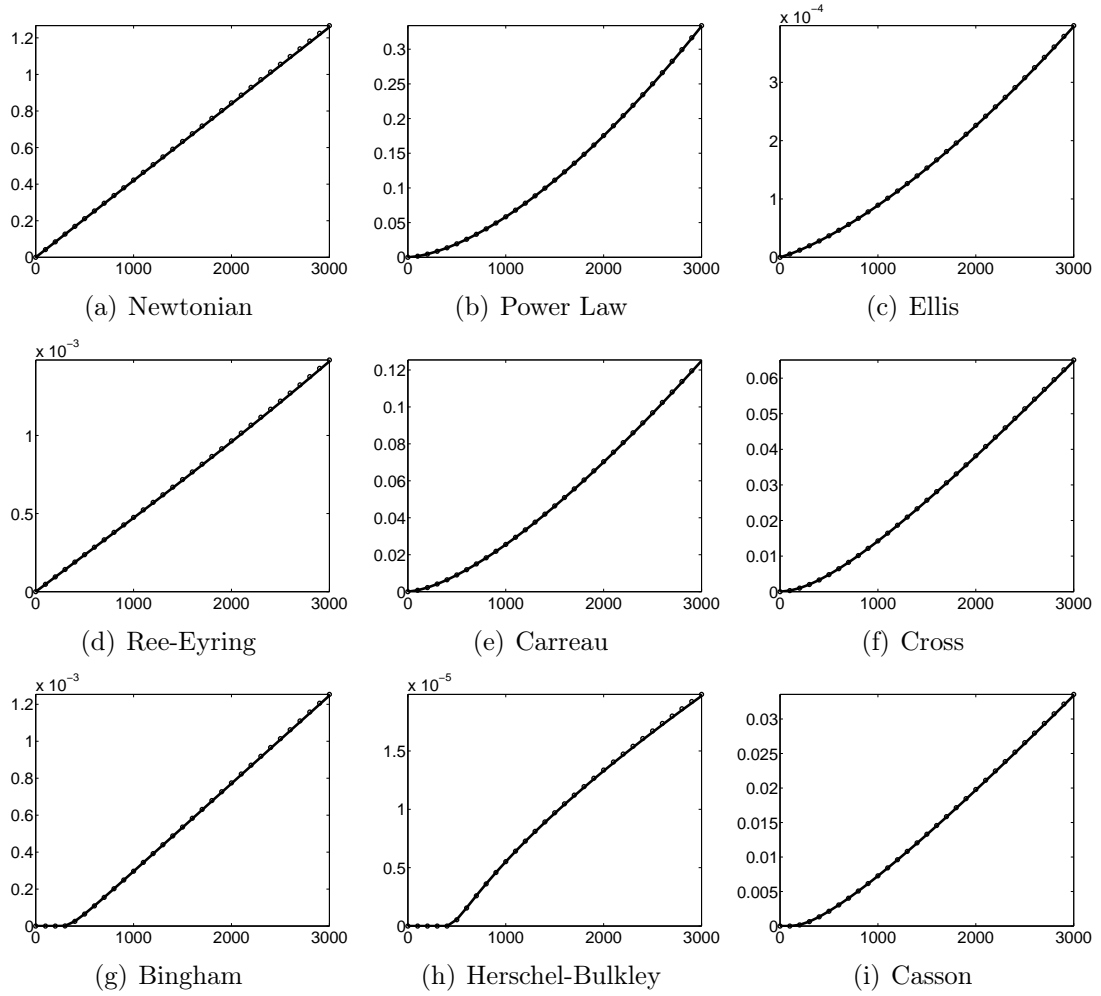


Figure 1: Comparing the WRMS analytical solutions (solid line) with the numerical solutions from the generic method (circles) of Q in $\text{m}^3 \cdot \text{s}^{-1}$ (vertical axis) versus Δp in Pa (horizontal axis) for the flow of the nine fluid models in pipes. The pipe and fluid parameters are given in Table 2.

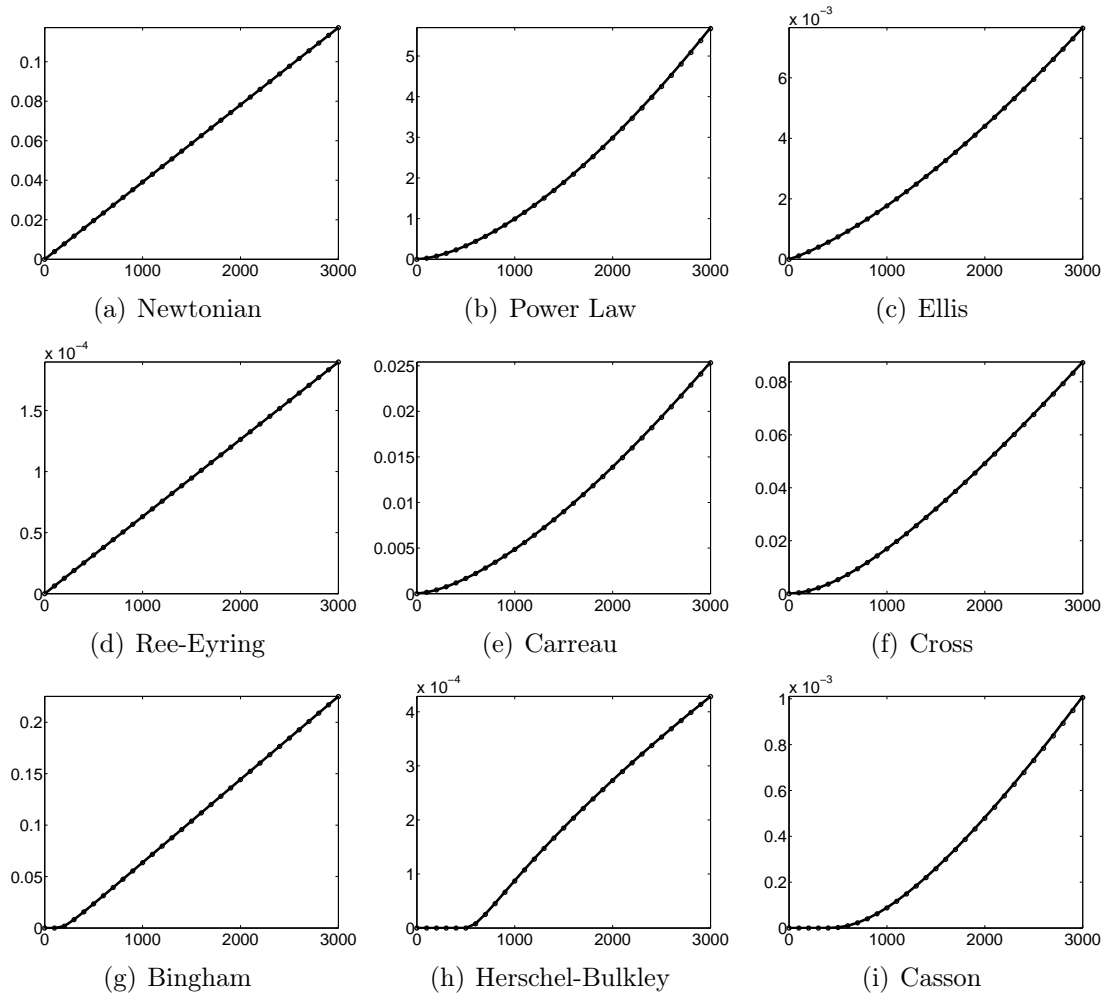


Figure 2: Comparing the WRMS analytical solutions (solid line) with the numerical solutions from the generic method (circles) of Q in $\text{m}^3 \cdot \text{s}^{-1}$ (vertical axis) versus Δp in Pa (horizontal axis) for the flow of the nine fluid models in slits. The slit and fluid parameters are given in Table 2. In all these examples $W = 1.0$ m.

4 Conclusions

In this study we presented a generic and general numerical method for finding the flow solutions of generalized Newtonian fluids in one dimensional flow problems that can be applied easily to circular pipes and plane slits. The method can be used to obtain all the required flow parameters which include shear stress, local viscosity, shear rate, flow velocity profile and volumetric flow rate.

Thorough comparisons were made between the results of the generic method and the results of the analytical solutions obtained from the Weissenberg-Rabinowitsch-Mooney-Schofield method. In all cases, the two methods produced virtually identical results considering the numerical errors introduced by the heavy use of numerical methods, like numerical integration and bisection solvers, in the generic method.

The generic method enjoys several advantages over the competing numerical methods like finite element and finite difference. These advantages include ease and convenience to implement and use, generality, reliability and accuracy. The method may also be useful to apply analytically in some cases.

The generic method can be particularly useful when no analytical solutions can be obtained, or the analytical solutions can provide only some of the flow parameters, e.g. volumetric flow rate, but not others, e.g. flow velocity profile.

5 Nomenclature

B	slit half thickness (m)
C	viscosity coefficient in Herschel-Bulkley model (Pa.s ^{n})
C'	viscosity coefficient in Bingham model (Pa.s)
k	viscosity coefficient in power law model (Pa.s ^{n})
K	viscosity coefficient in Casson model (Pa.s)
L	conduit length (m)
m	indicial parameter in Cross model
n	flow behavior index in power law, Carreau and Herschel-Bulkley models
Δp	pressure drop across the conduit length (Pa)
Q	volumetric flow rate (m ³ .s ⁻¹)
r	radius (m)
R	pipe radius (m)
s	spatial coordinate that represents r for pipe and z for slit (m)
v	fluid velocity in the flow direction (m.s ⁻¹)
W	slit width (m)
z	spatial coordinate of slit thickness dimension (m)
α	indicial parameter in Ellis model
γ	rate of shear strain (s ⁻¹)
λ	characteristic time constant in Carreau and Cross models (s)

μ	fluid shear viscosity (Pa.s)
μ_0	zero-shear viscosity in Carreau and Cross models (Pa.s)
μ_e	low-shear viscosity in Ellis model (Pa.s)
μ_i	infinite-shear viscosity in Carreau and Cross models (Pa.s)
μ_o	Newtonian viscosity (Pa.s)
μ_r	characteristic viscosity in Ree-Eyring model (Pa.s)
τ	shear stress (Pa)
τ_0	yield stress in Bingham, Herschel-Bulkley and Casson models (Pa)
τ_B	shear stress at slit wall (Pa)
τ_c	characteristic shear stress in Ree-Eyring model (Pa)
τ_h	shear stress when viscosity equals $\frac{\mu_e}{2}$ in Ellis model (Pa)
τ_R	shear stress at pipe wall (Pa)

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