ANOTHER PROOF FOR FERMAT'S LAST THEOREM

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Abstract

In this paper we propose another proof for Fermat's Last Theorem (FLT). We found a simpler approach through Pythagorean Theorem, so our demonstration would be close to the times FLT was formulated. On the other hand it seems the Pythagoras' Theorem was the inspiration for FLT. It resulted one of the most difficult mathematical problem of all times, as it was considered. Pythagorean triples existence seems to support the claims of the previous phrase.

Key words: *Pythagorean triples; Fermat's Last Theorem; Pythagoras' Theorem; integer numbers.*

1. INTRODUCTION

Fermat's Last Theorem postulates that there are not three integers a, b, c which satisfy the Diophantine equation:

$$a^n + b^n = c^n, \quad a,b,c,n \notin Z^*, n > 2 \tag{1}$$

for any natural value of n greater than two. Apparently simple, this theorem statement has given much trouble to many generations of mathematicians. Although it was formulated in 1637, after several partial demonstrations, the successful proof comes later, in 1995, after some papers published by the British mathematician Wiles, [1] and [2]. A brief history of the unsuccessful efforts to solve FLT is found in ref. [3]. In our opinion, the demonstration difficultness of this theorem consists in its countless possibilities of approach. In this context we present, in the following, another demonstration of this theorem, in terms of Pythagorean Theorem.

2. PYTHAGORAS' THEOREM VS. FERMAT'S LAST THEOREM

In order to make an easy demonstration, which otherwise it would be reduced to an endless series of attempts by giving values to integers a, b and c, hence impossible to prove in fact, we must reduce our problem to another problem in which a, b and c must gain significance. The best way is to reduce equation (1) to a geometry problem. Consequently, a convenient way is to reduce the problem to a relation between sizes of a rectangular triangle. Therefore let us consider a, b and c the sizes of a rectangular triangle. It is known that the relation between them is, according to the Pythagorean Theorem:

$$a^2 + b^2 = c^2 \tag{2}$$

where $a,b,c \in R^*$ include both negative and positive values.
The sizes a and b can take positive or negative values if we convenient choose the reference system in which we built the rectangular triangle. Regarding the hypotenuse, given that it is a sum of squares:

$$c = \pm(a^2 + b^2)^{\frac{1}{2}}$$

it is obvious that it can also take negative values too.
Let us now define the trigonometric functions

$$\sin\alpha = a/c$$

and:

$$\cos\alpha = b/c$$

with $\alpha = (0,2\pi)$ because a and b must take both positive and negative values.
For a better understanding why a and b must take positive and negative values, we can imagine a circle like trigonometric circle in which we outlined in each quadrant a symmetric triangle. In quadrant I, a and b are positive and α is an acute angle. In quadrant II a is positive and b is negative, the angle α is already an obtuse angle. You might think now that sharp angles sum is not equal to π/2. The fulfilling condition for a rectangular triangle is broken. Beginning with quadrant II angle α would be already obtuse and this is the reason why triangles cannot be conceived in these quadrants. But this is not exactly so. In quadrant II, for example, the angle symmetrical with α will have the value π-α. If the opposed angle will have the value α-π / 2, then their sum will be exactly the condition that must be fulfilled by a rectangular triangle. Thus our circle will not be a simple reference for the measure of a and b, positive and negative. Hence we obtain four identical rectangular triangles, with identical sizes and angles. The only difference between them is that some of them are positive, other negative, something conventional, actually illustrating their perfect symmetry from the center of coordinate axes.

With these established notions we can think now a way to transform equation (2) in a relation close to equation (1). One way is to multiply equation (2) by c and using trigonometric functions defined above in order to obtain an equation in which the power of the unknowns a, b, and c to be multiplied by one order. Remaking the multiplication n times we have therefore:

$$c^n = a^n \sin^{2-n}\alpha + b^n \cos^{2-n}\alpha \tag{3}$$

It would have been easier to write

$$c^n = \left(a \cdot \sin^{\frac{2-n}{n}}\alpha\right)^n + \left(b \cdot \cos^{\frac{2-n}{n}}\alpha\right)^n \tag{4}$$

but we choose the form (3) because the functions formed by sine and cosine of high power in (4) cannot be defined if the trigonometric functions have negative values. We have therefore (3) in which we can already operate the restriction $a,b,c \in Z^*$.

We distinguish two main cases. The first case corresponds to the general situation in which:

$$n = 2p \tag{5}$$

Here p must be seen not necessarily an integer but rather as a number that make (5) an integer.
Considering (5), equation (3) becomes:

$$c^{2p} = a^{2p} \sin^{2-2p}\alpha + b^{2p} \cos^{2-2p}\alpha \tag{6}$$

Obviously, for p=1 it results equation (2) from which we started. a, b, and c are integers this time, and we can say that we have already demonstrate the viability of the general relation (2) for integer numbers. But we are interested in higher powers than two of n, respectively higher than one of p and therefore in an equation (6) of the form (1). Note that we can do this, and this is to write equation (6) of the form:

$$c^{2p} = d^{2p} + e^{2p} \tag{7}$$

with $d = a\sin\alpha$ and $e = b\cos\alpha$, $d,e \in Z^*$, only if the condition

$$2p = 2 - 2p$$

is fulfilled.

By replacing $p = 1/2$ in equation (6) we get immediately the equation:

$$c = a\sin\alpha + b\cos\alpha$$

which turns in equation (2) through trigonometric functions sine and cosine definition relations. So, our equation (6) can be written as equation (1), but only for p=1/2. For higher powers of p, $p > 1/2$, we always have:

$$2p > 2 - 2p$$

which leads to the conclusion that between three numbers $a, b, c \in Z^*$, for any p>1, there is only a relation of the form (6). The form (6) does not exclude therefore a relation like:

$$c^{2p} = d^k + e^m$$

with $k \neq m < 2p$, but with d and e real numbers. The quantities d and e cannot be integers because they are a product of an integer and a real number. Particular cases of extreme values of trigonometric functions are excluded because the quantities included in equation (6) are the sizes of a rectangular triangle for which the sum of the sharp angles is always equal to π / 2.

A special case is the one in which:

$$\sin\alpha = \cos\alpha = 2^{-\frac{1}{2}}$$

For $n = 2p$ we have $(2^{-1/2})^{2-n} = 2^{p-1}$, and equation (6) becomes

$$c^{2p} = a^{2p}2^{p-1} + b^{2p}2^{p-1}$$

Note that for p=1 the result is equation (2). But we are interested in higher values of p. Therefore, to have an equation of the form (7) it have to

$$p - 1 = 2p$$

Hence p=-1 for which

$$c^{-2} = d^{-2} + e^{-2}$$

equation that can be written as

$$\left(\frac{1}{c}\right)^2 = \left(\frac{1}{d}\right)^2 + \left(\frac{1}{e}\right)^2$$

or

$$c'^2 = d'^2 + e'^2$$

with $c', d', e' \notin Z^*$.

The second main case is corresponding to:

$$n = 2p + 1$$

and brings nothing special in the spectrum of results. We obtain the equation:

$$c^{2p+1} = a^{2p+1}\sin^{1-2p}\alpha + b^{2p+1}\cos^{1-2p}\alpha \quad (8)$$

for which we do the same kind of reasoning. Equation (8) becomes the type:

$$c^{2p+1} = d^{2p+1} + e^{2p+1} \quad (9)$$

with $d = a\sin\alpha$ and $e = b\cos\alpha$, $d, e \in Z^*$, under condition:

$$2p + 1 = 1 - 2p\,.$$

This condition easily leads to the only possible result, equation (2). We are interested in all situations but especially in

$$2p + 1 > 1 - 2p$$

But this leads to the conclusion that between the three integer numbers $a, b, c \in Z^*$ and p> 1 could not be any relation but that illustrated by the form (8).

The particular case in which the sine and cosine values are equal is described by the equation:

$$c^{2p+1} = a^{2p+1}2^{\frac{2p-1}{2}} + b^{2p+1}2^{\frac{2p-1}{2}} \quad (10)$$

The condition:

$$2p+1=\frac{2p-1}{2}$$

leads to equation:

$$c^{-2}=d^{-2}+e^{-2}$$

which can be written as:

$$c'^2=d'^2+e'^2$$

with $c',d',e'\notin Z^*$.

3. PARTICULAR CASE

A simplest case, whose solution is obvious, is when trigonometric functions are positive, $\alpha\in(0,\pi/2)$ In equation (3), written in the form (4):

$$c^n=\left(a\cdot\sin^{\frac{2-n}{n}}\alpha\right)^n+\left(b\cdot\cos^{\frac{2-n}{n}}\alpha\right)^n$$

we can do the restriction $a,b,c\in N^*$. For n=2 we can see that the resulting that equation (2) it results very easily. For n>2 we must see what happens with trigonometric functions within parentheses. Noting that:

$$\frac{2-n}{n}<1$$

and:

$$0<\sin\alpha<1,\ 0<\cos\alpha<1,\ 0<\sin^{\frac{2-n}{n}}\alpha<1,\ 0<\cos^{\frac{2-n}{n}}\alpha<1$$

also:

$$\sin^{\frac{2-n}{n}}\alpha,\cos^{\frac{2-n}{n}}\alpha\in Q$$

can arise only as:

$$c^n=d^n+e^n,\ c\in N^*,\ d,e\in R^+$$

where $d=a\cdot\sin^{\frac{2-n}{n}}\alpha$ and $e=b\cdot\cos^{\frac{2-n}{n}}\alpha$.
From equation (4) it might exist a situation. Consider:

$$n=3,\ \sin\alpha=0{,}1 \text{ and } a=xy...z0,$$

where x, y, z are component ciphers of the natural number a. It is possible, therefore, the situation in which $d\in N^*$. In this situation we also know that $e\in R^+$ and $c\in N^*$. As you can see, this situation cannot exist because a natural number cannot be the sum of an integer number and a real one. So this situation is impossible.

The same conclusion applies also for n=3k, for k natural number and sinα=0,0…01, with a=x0…0. As for the situation when sinα=cosα, It is treated the same way and lead to the same results as in the previous cases, n = 2p and n = 2p +1.

By summing, ultimately, all the conclusions of all cases in a single one there are no three integers a, b and c that satisfy equation (1), for n> 2. The solutions of this equation are only for n = 2.

4. DISCUSSIONS

Whereas equations (6) and (8) result from equation (2) by multiplying with c, you would think that we've proven nothing so far, these equations are equivalent to equation (2). We assume that equation (2) is valid from the very beginning and we got finally that only (2) may be valid. It is what you might call a trivial case. But things seem to stand exactly opposite.. We assume, in the hypothesis, that equation (2) is indeed valid, but a, b, c are real numbers. The result is that the equation (2) is indeed valid, but in case we are interested, when a, b, c are integers. Other cases, for higher powers of n, are impossible. It is a correct reasoning because in the FLT statement equation (2) is assumed valid, for any integer triplet, other cases being impossible. In our reasoning we start with what is known, but the numbers a, b and c are assumed to be not integers. This is the result we reached. We initially assumed that the equation (2) is valid, whatever be three real numbers a, b and c.

By using this reasoning, if we assume that the Pythagorean Theorem is of the form $c^3 = a^3 + b^3$, we have reached the same conclusion: there cannot be three integers a, b, c which satisfy the equation $c^3 = a^3 + b^3$. The same conclusion is if the Pythagorean Theorem is $c^{20} = a^{20} + b^{20}$. No matter what form has therefore the Pythagorean Theorem, the result is of the form of Pythagorean Theorem. This seems to be strange at first sight, but in our opinion, is due to the similarities of the two theorems form. Up to a point you might get the impression it is one and the same theorem. You might think that FLT is just an existence theorem for equation (2), and there is no a clear line of separation between them. And this is, perhaps, because Fermat's Theorem was inspired by the Pythagoras' Theorem, and tries to apply it only to some numerical values without any meaning. He succeeded to create, hence, the most difficult mathematical problem, according to some, of all times. By the same blur delimitation between the two theorems we use in our approach. We have shown that there can be no triples of integers satisfying the equation (1), for higher powers of n. The solutions are only for n=2.

5. Conclusions

Based on the similarities that exist, to certain extent, from Pythagorean Theorem and Fermat's Last Theorem, we attempted, in this paper, to demonstrate the later through the former. It has been shown that given equation (1), it has no solution a, b, c integers, but only for n=2. Case corresponding to Pythagorean Theorem.

6. REFERENCES

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