Mass shift due to the nonlinear Lorentz Group.

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Abstract

We determine nonlinear Lorentz transformations between coordinate systems which are mutually in a constant symmetrical accelerated motion. The maximal acceleration as an analogue of the maximal velocity in special relativity follows from the nonlinear Lorentz group of transformtion. The mass formula was derived by the same method as the Thomas precession formula by author. It can play crucial role in particle physics and cosmology.

1 Introduction

In physics, mass is a property of a physical body. According to Ernst Mach, it is a measure of an resistance to acceleration i.e. a change in its state of motion, represented by the relationship $F = ma$, when a force is applied.

It also determines the strength of its mutual gravitational attraction to other bodies. Although some theorists have speculated that some of these phenomena could be independent of each other, current experiments have found no difference in results, whatever way is used to measure mass.

Active gravitational mass measures the gravitational force exerted by an object. Passive gravitational mass measures the gravitational force exerted on an object in a known gravitational field. Mass-energy measures the total amount of energy contained in a body.

Origin of mass and mass generation mechanism are the crucial problems of particle physics. A mass generation mechanism in particle physics is a theory which attempts to explain the origin of mass from the most fundamental laws of physics. To date, a number of different models have been proposed which advocate different views of the origin of mass. The problem is complicated by the fact that the notion of mass is strongly related to the gravitational interaction but a theory of the latter has not been yet reconciled with the currently popular model of particle physics, known as the Standard Model.

We derive here the dependence of mass on acceleration as an analogue phenomenon in special theory of relativity. The derived formulas are based on the non-linear Lorentz transformation for accelerated systems and can play crucial role in particle physics and cosmology.

2 Acceleration in special theory of relativity

The problem of acceleration of charged particles or systems of particles is the permanent and the most prestige problem in the accelerator physics. Particles can be accelerated by different ways. Usually by the classical electromagnetic fields, or, by light pressure of the laser fields (Baranova et al., 1994; Pardy, 1998, 2001, 2002). The latter method is the permanent problem of the laser physics for many years.

Here, we determine transformations between coordinate systems which moves mutually with the same acceleration. We determine transformations between non relativistic and relativistic uniformly accelerated systems.

Let us remind the special theory of relativity velocity and acceleration The Lorentz transformation between two inertial coordinate systems $S(0, x, y, z)$ and $S'(0, x', y', z')$ (where system S' moves in such a way that x-axes converge, while y and z-axes run parallel and at time $t = t' = 0$ for the origin of the systems O and O' it is $O \equiv O'$) is as follows:

$$x' = \gamma(v)(x - vt), \quad y' = y, \quad z' = z', \quad t' = \gamma(v)\left(t - \frac{v}{c^2}x\right), \tag{1}$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2}$$

The infinitesimal form of this transformation is evidently given by differentiation of the every equation. Or,

$$dx' = \gamma(v)(dx - vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(v)\left(dt - \frac{v}{c^2}dx\right). \tag{3}$$

It follows from eqs. (3) that if v_1 is velocity of the inertial system 1 with regard to S and v_2 is the velocity of the inertial systems 2 with regard to 1, then the relativistic sum of the two velocities is

$$u_2 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \tag{4}$$

The mathematic object called four-velocity follows from the Lorentz transformation after some additional operations. From the ordinary three-dimensional velocity vector one can form a four-vector. This four-dimensional velocity (four-velocity) of a particle is the vector

$$u^\mu = \frac{dx^\mu}{ds}, \tag{5}$$

where, according to Landau et al. (1987)

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}} \tag{6}$$

with v being the ordinary three-dimensional velocity of the particle and c being the velocity of light. Thus

$$u^1 = \frac{dx^1}{ds} = \frac{dx}{cdt\sqrt{1 - \frac{v^2}{c^2}}} = \frac{v_x}{c\sqrt{1 - \frac{v^2}{c^2}}}. \tag{7}$$

Then,

$$u^\mu = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{c\sqrt{1 - \frac{v^2}{c^2}}}\right). \tag{8}$$

Note, that the four-velocity is a dimensionless quantity. The components of the four-velocity are not independent. Noting that $dx^\mu dx_\mu = ds^2$, we have

$$u^\mu u_\mu = 1. \tag{9}$$

Geometrically, u^μ is a unit four-vector tangent to the world line of the particle.

Similarly to the definition of the four-velocity, the second derivative

$$a^\mu = \frac{d^2 x^\mu}{ds^2} = \frac{du^\mu}{ds} \tag{10}$$

may be called the four-acceleration. Differentiating formula (9), we find:

$$u^\mu a_\mu = 0, \tag{11}$$

i.e. the four-vectors of velocity and acceleration are "mutually perpendicular".

Now, let us determine the relativistic uniformly accelerated motion, i.e. the rectilinear motion for which the acceleration a^μ in the proper reference frame (at each instant of time) remains constant. We proceed as follows.

In the reference frame in which the particle velocity is $v = 0$, the components of the four-acceleration $a^\mu = (0, a/c^2, 0, 0)$ (where $\mathbf{a}$ is the ordinary three-dimensional acceleration, which is directed along the x axis). The relativistically invariant condition for uniform acceleration must be expressed by the constancy of the four-scalar which coincides with a^2 in the proper reference frame:

$$a^\mu a_\mu = const = -\frac{a^2}{c^4}. \tag{12}$$

In the "fixed" frame, with respect to which the motion is observed, writing out the expression for $a^\mu a_\mu$ gives the equation:

$$\frac{d}{dt}\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = a, \tag{13}$$

or,

$$\frac{v}{c\sqrt{1-\frac{v^2}{c^2}}} = at + const. \tag{14}$$

Setting $v = 0$ for $t = 0$, we find that $const = 0$, so that

$$v = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}, \tag{15}$$

Integrating once more and setting $x = 0$ for $t = 0$, we find:

$$x = \frac{c^2}{a}\left(\sqrt{1+\frac{a^2t^2}{c^2}} - 1\right). \tag{16}$$

For $at \ll c$, these formulas go over the classical expressions $v = at, x = \frac{a}{2}t^2$. For $at \to \infty$, the velocity tends toward the constant value c.

The proper time of a uniformly accelerated particle is given by the integral (Landau et al., 1987)

$$\int_0^t \sqrt{1+\frac{v^2(t)}{c^2}}dt = \frac{c}{a}\text{arcsinh}\frac{at}{c}. \tag{17}$$

At the limit $t \to \infty$ it increases much more slowly than t, according to the law

$$\frac{c}{a}\ln\frac{2at}{c}. \tag{18}$$

The infinitesimal form of Lorentz transformation (3) evidently gives the Lorentz length contraction and time dilation. Namely, if we put $dt = 0$ in the first equation of system

(3), then the Lorentz length contraction follows in the infinitesimal form $dx' = \gamma(v)dx$. Or, in other words, if in the system S' the infinitesimal length is dx', then the relative length with regard to the system S is $\gamma^{-1}dx'$. Similarly, from the last equation of (3) it follows the time dilatation for $dx = 0$. Historical view on this effect is in the Selleri article (Selleri, 1997).

3 Uniformly accelerated frames with space-time symmetry

Let us take two systems $S(0, x, y, z)$ and $S'(0, x', y', z')$, where system S' moves in such a way that x-axes converge, while y and z-axes run parallel and at time $t = t' = 0$ for the beginning of the systems O and O' it is $O \equiv O'$. Let us suppose that system S' moves relative to some basic system B with acceleration $a/2$ and system S' moves relative to system B with acceleration $-a/2$. It means that both systems moves one another with acceleration a and are equivalent because in every system it is possibly to observe the force caused by the acceleration $a/2$. In other words no system is inertial.

Now, let us consider the formal transformation equations between two systems. At this moment we give to this transform only formal meaning because at this time, the physical meaning of such transformation is not known. On the other hand, the consequences of the transformation will be shown very interesting. The first published derivation of such transformation by the standard way was given by author (Pardy, 2003; 2004; 2005), and the same transformations were submitted some decades ago (Pardy, 1974). The old results can be obtained if we perform transformation

$$t \to t^2, \quad t' \to t'^2, \quad v \to \frac{1}{2}a, \quad c \to \frac{1}{2}\alpha \tag{19}$$

in the original Lorentz transformation (1). We get:

$$x' = \Gamma(a)(x - \frac{1}{2}at^2), \quad y' = y, \quad z' = z, \quad t'^2 = \Gamma(a)\left(t^2 - \frac{2a}{\alpha^2}x\right) \tag{20}$$

with

$$\Gamma(a) = \frac{1}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{21}$$

We used practically new denotation of variables in order to get the transformation (20) between accelerated systems.

The transformations (20) form the one-parametric group with the parameter a. The proof of this mathematical statement can be easy performed if we perform the transformation T_1 from S to S', transformation T_2 from S' to S'' and transformation T_3 from S to S''. Or,

$$x' = x'(x, t; a_1), \quad t' = t'(x, t; a_1), \tag{22}$$

$$x'' = x''(x', t'; a_2), \quad t'' = t''(x', t'; a_2), \tag{23}$$

After insertion of transformations (22) into (23), we get

$$x'' = x''(x, t; a_3), \quad t'' = t''(x, t; a_3), \tag{24}$$

where parameter a_3 is equal to

$$a_3 = \frac{a_1 + a_2}{1 + \frac{a_1 a_2}{\alpha^2}}. \tag{25}$$

The inverse parameter is $-a$ and parameter for identity is $a = 0$ It may be easy to verify that the final relation for the definition of the continuous group transformation is valid for our transformation. Namely (Eisenhart, 1943):

$$(T_3 T_2)\, T_1 = T_3\, (T_2 T_1)\,. \tag{26}$$

The physical interpretation of the nonlinear Lorentz transformation is the same as in the case of the Lorentz transformation in STR, only the physical interpretation of the invariant function $x = (1/2)\alpha t^2$ is different. Namely it can be expressed by the statement. If there is a physical signal in the system S with the law $x = (1/2)\alpha t^2$, then in the system S' the law of the process is $x' = (1/2)\alpha t'^2$, where α is the constant of maximal acceleration. It is new constant and cannot be defined by the game with known physical constants.

Let us remark, that it follows from history of physics, that Lorentz transformation was taken first as physically meaningless mathematical object by Larmor, Voigt and Lorentz and later only Einstein decided to put the physical meaning to this transformation and to the invariant function $x = ct$. We hope that the derived nonlinear Lorentz transformation will appear as physically meaningful.

Using relations $t \leftarrow t^2, \quad t' \leftarrow t'^2, \quad v \leftarrow \frac{1}{2}a, \quad c \leftarrow \frac{1}{2}\alpha$, the nonlinear transformation is expressed as the Lorentz transformation forming the one-parametric group. This proof is equivalent to the proof by the above direct calculation. The integral part of the group properties is the so called addition theorem for acceleration.

$$a_3 = \frac{a_1 + a_2}{1 + \frac{a_1 a_2}{\alpha^2}}. \tag{27}$$

where a_1 is the acceleration of the S' with regard to the system S, a_2 is the acceleration of the system S'' with regard to the system S' and a_3 is the acceleration of the system S'' with regard to the system S. The relation (27), expresses the law of acceleration addition theorem on the understanding that the events are marked according to the relation (20).

If $a_1 = a_2 = a_3 = ... + a_n = a$, for n accelerated carts which rolls in such a way that the first cart rolls on the basic cart, the second rolls on the first cart and so on, then we get for the sum of n accelerated carts the following formula

$$a_{sum} = \frac{1 - \left(\frac{1-a/\alpha}{1+a/\alpha}\right)^n}{1 + \left(\frac{1-a/\alpha}{1+a/\alpha}\right)^n}, \tag{28}$$

which is an analogue of the formula for the inertial systems (Lightman et al., 1975).

In this formula as well as in the transformation equation (20) appears constant α which cannot be calculated from the theoretical considerations, or, constructed from the known physical constants (in analogy with the velocity of light). What is its magnitude can be established only by experiments. The notion maximal acceleration was introduced some decades ago by author (Pardy, 1974). Caianiello (1981) introduced it as some consequence of quantum mechanics and Landau theory of fluctuations. Revisiting view on the maximal acceleration was given by Papini (2003). At present time it was argued by Lambiase et al. (1999) that maximal acceleration determines the upper limit of the Higgs boson and that it gives also the relation which links the mass of W-boson with the mass of the Higgs boson. The LHC and HERA experiments presented different answer to this problem.

4 Dependence of mass on acceleration

If the maximal acceleration is the physical reality, then it should have the similar consequences in a dynamics as the maximal velocity of motion has consequences in the dependence of mass on velocity. We can suppose in analogy with the special relativity that mass depends on the acceleration for small velocities, in the similar way as it depends on velocity in case of uniform motion. Of course such assumption must be experimentally verified and in no case it follows from special theory of relativity, or, general theory of relativity (Fok, 1961). So, we postulate ad hoc, in analogy with special theory of relativity:

$$m(a) = \frac{m_0}{\sqrt{1 - \frac{a^2}{\alpha^2}}}; \quad v \ll c, \quad a = \frac{dv}{dt}. \tag{29}$$

Let us derive as an example the law of motion when the constant force F acts on the body with the rest mass m_0. Then, the Newton law reads (Landau et al., 1997):

$$F = \frac{dp}{dt} = m_0 \frac{d}{dt} \frac{v}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{30}$$

The first integral of this equation can be written in the form:

$$\frac{Ft}{m_0} = \frac{v}{\sqrt{1 - \frac{a^2}{\alpha^2}}}; \quad a = \frac{dv}{dt}, \quad F = const.. \tag{31}$$

Let us introduce quantities

$$v = y, \quad a = y', \quad A(t) = \frac{F^2 t^2}{m_0^2 \alpha^2}. \tag{32}$$

Then, the quadratic form of the equation (31) can be written as the following differential equation:

$$A(t)y'^2 + y^2 - A(t)\alpha^2 = 0, \tag{33}$$

which is nonlinear differential equation of the first order. The solution of it is of the form $y = Dt$, where D is some constant, which can be easily determined. Then, we have the solution in the form:

$$y = v = Dt = \frac{t}{\sqrt{\frac{m_0^2}{F^2} + \frac{1}{\alpha^2}}}. \tag{34}$$

For F $\to \infty$, we get $v = \alpha t$. This relation can play substantial role at the beginning of the big-bang, where the accelerating forces can be considered as infinite, however the law of acceleration has finite nonsingular form.

At this moment it is not clear if the dependence of the mass on acceleration can be related to the energy dependence on acceleration similarly to the Einstein relation uniting energy, mass and velocity (Okun, 2001, 2002; Sachs, 1973).

The infinitesimal form of author transformation (20) evidently gives the length contraction and time dilation. Namely, if we put $dt = 0$ in the first equation of system (20), then the length contraction follows in the infinitesimal form $dx' = \Gamma(a)dx$. Or, in other words, if in the system S' the infinitesimal length is dx', then the relative length with regard to the system S is $\Gamma^{-1}dx'$. Similarly, from the last equation of (20) it follows the time dilatation for $dx = 0$.

5 Discussion

The maximal acceleration constant which was derived here is kinematical one and it differs from the Caianiello (1981) definition following from quantum mechanics. Our constant cannot be determined by the system of other physical constants. It is an analogue of the numeric velocity of light which cannot be composed from others physical constants, or, the Heisenberg fundamental length in particle physics. The nonlinear transformations (20) changes the Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{35}$$

to the new metric with the Riemann form. Namely:

$$ds^2 = \alpha^2 t^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{36}$$

and it can be investigated by the methods of differential geometry. So, equations (20) and (36) can form the preamble to investigation of accelerated systems.

If some experiment will confirm the existence of kinematical maximal acceleration α, then it will have certainly crucial consequences for Einstein theory of gravity because this theory does not involve this factor. Also the cosmological theories constructed on the basis of the original Einstein equations will require modifications. The so called Hubble constant will be changed and the scenario of the accelerating universe modified.

Also the standard model of particle physics and supersymmetry theory will require generalization because they does not involve the maximal acceleration constant. It is not excluded that also the theory of parity nonconservation will be modified by the maximal acceleration constant. In such a way the particle laboratories have perspective programes involving the physics with maximal acceleration. Many new resuls can be obtained from the old relativistic results having the form of the mathematical objects involving function $f(v/c)$.

The prestige problem in the modern theoretical physics - the theory of the Unruh effect, or, the existence of thermal radiation detected by accelerated observer - is in the development (Fedotov et al., 2002) and the serious statement, or comment to the relation of this effect to the maximal acceleration must be elaborated. The analogical statement is valid for the Hawking effect in the theory of black holes.

It is not excluded that the maximal acceleration constant will be discovered by ILC. The unique feature of the International Linear Collider (ILC) is the fact that its CM energy can be increased gradually simply by extending the main linac. The mass formula was derived by the same method as the Thomas precession formula by author (Pardy, 2014a, 2014b). It can play crucial role in particle physics and cosmology.

Let us remark in conclusion that it is possible to extend and modify quantum field theory by maximal acceleration. It is not excluded that the kinematical maximal acceleration constant will enable to reformulate the theory of renormalization.

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