

Introduction to Tensor Calculus

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Preface

These are general notes on tensor calculus originated from a collection of personal notes which I prepared some time ago for my own use and reference when I was studying the subject. I decided to put them in the public domain hoping they may be beneficial to some students in their effort to learn this subject. Most of these notes were prepared in the form of bullet points like tutorials and presentations and hence some of them may be more concise than they should be. Moreover, some notes may not be sufficiently thorough or general. However this should be understandable considering the level and original purpose of these notes and the desire for conciseness. There may also be some minor repetition in some places for the purpose of gathering similar items together, or emphasizing key points, or having self-contained sections and units.

These notes, in my view, can be used as a short reference for an introductory course on tensor algebra and calculus. I assume a basic knowledge of calculus and linear algebra with some commonly used mathematical terminology. I tried to be as clear as possible and to highlight the key issues of the subject at an introductory level in a concise form. I hope I have achieved some success in reaching these objectives at least for some of my target audience. The present text is supposed to be the first part of a series of documents about tensor calculus for gradually increasing levels or tiers. I hope I will be able to finalize and publicize the document for the next level in the near future.

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1 Notation, Nomenclature and Conventions

- In the present notes we largely follow certain conventions and general notations; most of which are commonly used in the mathematical literature although they may not be universally adopted. In the following bullet points we outline these conventions and notations. We also give initial definitions of the most basic terms and concepts in tensor calculus; more thorough technical definitions will follow, if needed, in the forthcoming sections.

- Scalars are algebraic objects which are uniquely identified by their magnitude (absolute value) and sign (\pm), while vectors are broadly geometric objects which are uniquely identified by their magnitude (length) and direction in a presumed underlying space.

- At this early stage in these notes, we generically define “tensor” as an organized array of mathematical objects such as numbers or functions.

- In generic terms, the rank of a tensor signifies the complexity of its structure. Rank-0 tensors are called scalars while rank-1 tensors are called vectors. Rank-2 tensors may be called dyads although this, in common use, may be restricted to the outer product of two vectors and hence is a special case of rank-2 tensors assuming it meets the requirements of a tensor and hence transforms as a tensor. Like rank-2 tensors, rank-3 tensors may be called triads. Similar labels, which are much less common in use, may be attached to higher rank tensors; however, none will be used in the present notes. More generic names for higher rank tensors, such as polyad, are also in use.

- In these notes we may use “tensor” to mean tensors of all ranks including scalars (rank-0) and vectors (rank-1). We may also use it as opposite to scalar and vector (i.e. tensor of rank- n where $n > 1$). In almost all cases, the meaning should be obvious from the context.

- Non-indexed lower case light face Latin letters (e.g. f and h) are used for scalars.

- Non-indexed (lower or upper case) bold face Latin letters (e.g. \mathbf{a} and \mathbf{A}) are used for vectors. The exception to this is the basis vectors where indexed bold face lower or upper case symbols are used. However, there should be no confusion or ambiguity about the

meaning of any one of these symbols.

- Non-indexed upper case bold face Latin letters (e.g. **A** and **B**) are used for tensors (i.e. of rank > 1).
- Indexed light face italic symbols (e.g. a_i and B_i^{jk}) are used to denote tensors of rank > 0 in their explicit tensor form (index notation). Such symbols may also be used to denote the components of these tensors. The meaning is usually transparent and can be identified from the context if not explicitly declared.
- Tensor indices in this document are lower case Latin letters usually taken from the middle of the Latin alphabet like (i, j, k) . We also use numbered indices like (i_1, i_2, \dots, i_k) when the number of tensor indices is variable.
- The present notes are largely based on assuming an underlying orthonormal Cartesian coordinate system. However, parts of which are based on more general coordinate systems; in these cases this is stated explicitly or made clear by the content and context.
- Mathematical identities and definitions may be denoted by using the symbol ‘ \equiv ’. However, for simplicity we will use in the present notes the equality sign “=” to mark identities and mathematical definitions as well as normal equalities.
- We use 2D, 3D and n D for two-, three- and n -dimensional spaces. We also use Eq./Eqs. to abbreviate Equation/Equations.
- Vertical bars are used to symbolize determinants while square brackets are used for matrices.
- All tensors in the present notes are assumed to be real quantities (i.e. have real rather than complex components).
- Partial derivative symbol with a subscript index (e.g. i) is frequently used to denote the i^{th} component of the Cartesian gradient operator ∇ :

$$\partial_i = \nabla_i = \frac{\partial}{\partial x_i} \quad (1)$$

- A comma preceding a subscript index (e.g. $_{,i}$) is also used to denote partial differentiation with respect to the i^{th} spatial coordinate in Cartesian systems, e.g.

$$A_{,i} = \frac{\partial A}{\partial x_i} \quad (2)$$

- Partial derivative symbol with a spatial subscript, rather than an index, are used to denote partial differentiation with respect to that spatial variable. For instance

$$\partial_r = \nabla_r = \frac{\partial}{\partial r} \quad (3)$$

is used for the partial derivative with respect to the radial coordinate in spherical coordinate systems identified by (r, θ, ϕ) spatial variables.

- Partial derivative symbol with repeated double index is used to denote the Laplacian operator:

$$\partial_{ii} = \partial_i \partial_i = \nabla^2 = \Delta \quad (4)$$

The notation is not affected by using repeated double index other than i (e.g. ∂_{jj} or ∂_{kk}).

The following notations:

$$\partial_{\bar{i}i}^2 \quad \partial^2 \quad \partial_i \partial^i \quad (5)$$

are also used in the literature of tensor calculus to symbolize the Laplacian operator.

However, these notations will not be used in the present notes.

- We follow the common convention of using a subscript semicolon preceding a subscript index (e.g. $A_{kl;i}$) to symbolize covariant differentiation with respect to the i^{th} coordinate (see § 6). The semicolon notation may also be attached to the normal differential operators to indicate covariant differentiation (e.g. $\nabla_{;i}$ or $\partial_{;i}$ to indicate covariant differentiation with respect to the index i).

- All transformation equations in these notes are assumed continuous and real, and all

derivatives are continuous in their domain of variables.

- Based on the continuity condition of the differentiable quantities, the individual differential operators in the mixed partial derivatives are commutative, that is

$$\partial_i \partial_j = \partial_j \partial_i \quad (6)$$

- A permutation of a set of objects, which are normally numbers like $(1, 2, \dots, n)$ or symbols like (i, j, k) , is a particular ordering or arrangement of these objects. An even permutation is a permutation resulting from an even number of single-step exchanges (also known as transpositions) of neighboring objects starting from a presumed original permutation of these objects. Similarly, an odd permutation is a permutation resulting from an odd number of such exchanges. It has been shown that when a transformation from one permutation to another can be done in different ways, possibly with different numbers of exchanges, the parity of all these possible transformations is the same, i.e. all even or all odd, and hence there is no ambiguity in characterizing the transformation from one permutation to another by the parity alone.

- We normally use indexed square brackets (e.g. $[\mathbf{A}]_i$ and $[\nabla f]_i$) to denote the i^{th} component of vectors, tensors and operators in their symbolic or vector notation.

- In general terms, a transformation from an n D space to another n D space is a correlation that maps a point from the first space (original) to a point in the second space (transformed) where each point in the original and transformed spaces is identified by n independent variables or coordinates. To distinguish between the two sets of coordinates in the two spaces, the coordinates of the points in the transformed space may be notated with barred symbols, e.g. $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ or $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ where the superscripts and subscripts are indices, while the coordinates of the points in the original space are notated with unbarred symbols, e.g. (x^1, x^2, \dots, x^n) or (x_1, x_2, \dots, x_n) . Under certain conditions,

such a transformation is unique and hence an inverse transformation from the transformed to the original space is also defined. Mathematically, each one of the direct and inverse transformation can be regarded as a mathematical correlation expressed by a set of equations in which each coordinate in one space is considered as a function of the coordinates in the other space. Hence the transformations between the two sets of coordinates in the two spaces can be expressed mathematically by the following two sets of independent relations:

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^n) \quad \& \quad x^i = x^i(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) \quad (7)$$

where $i = 1, 2, \dots, n$. An alternative to viewing the transformation as a mapping between two different spaces is to view it as being correlating the same point in the same space but observed from two different coordinate frames of reference which are subject to a similar transformation.

- Coordinate transformations are described as “proper” when they preserve the handedness (right- or left-handed) of the coordinate system and “improper” when they reverse the handedness. Improper transformations involve an odd number of coordinate axes inversions through the origin.
- Inversion of axes may be called improper rotation while ordinary rotation is described as proper rotation.
- Transformations can be active, when they change the state of the observed object (e.g. translating the object in space), or passive when they are based on keeping the state of the object and changing the state of the coordinate system from which the object is observed. Such distinction is based on an implicit assumption of a more general frame of reference in the background.
- Finally, tensor calculus is riddled with conflicting conventions and terminology. In this text we will try to use what we believe to be the most common, clear or useful of all of these.

2 Preliminaries

2.1 Introduction

- A tensor is an array of mathematical objects (usually numbers or functions) which transforms according to certain rules under coordinates change. In a d -dimensional space, a tensor of rank- n has d^n components which may be specified with reference to a given coordinate system. Accordingly, a scalar, such as temperature, is a rank-0 tensor with (assuming 3D space) $3^0 = 1$ component, a vector, such as force, is a rank-1 tensor with $3^1 = 3$ components, and stress is a rank-2 tensor with $3^2 = 9$ components.
- The term “tensor” was originally derived from the Latin word “tensus” which means tension or stress since one of the first uses of tensors was related to the mathematical description of mechanical stress.
- The d^n components of a tensor are identified by n distinct integer indices (e.g. i, j, k) which are attached, according to the commonly-employed tensor notation, as superscripts or subscripts or a mix of these to the right side of the symbol utilized to label the tensor (e.g. A_{ijk} , A^{ijk} and A_i^{jk}). Each tensor index takes all the values over a predefined range of dimensions such as 1 to d in the above example of a d -dimensional space. In general, all tensor indices have the same range, i.e. they are uniformly dimensioned.
- When the range of tensor indices is not stated explicitly, it is usually assumed to have the values (1, 2, 3). However, the range must be stated explicitly or implicitly to avoid ambiguity.
- The characteristic property of tensors is that they satisfy the principle of invariance under certain coordinate transformations. Therefore, formulating the fundamental physical laws in a tensor form ensures that they are form-invariant; hence they are objectively-representing the physical reality and do not depend on the observer. Having the same form in different coordinate systems may be labeled as being “covariant” but this word is

also used for a different meaning in tensor calculus as explained in § 2.5.1.

- “Tensor term” is a product of tensors including scalars and vectors.
- “Tensor expression” is an algebraic sum (or more generally a linear combination) of tensor terms which may be a trivial sum in the case of a single term.
- “Tensor equality” (symbolized by ‘=’) is an equality of two tensor terms and/or expressions. A special case of this is tensor identity which is an equality of general validity (the symbol ‘ \equiv ’ may be used for identity as well as for definition).
- The order of a tensor is identified by the number of its indices (e.g. A_{jk}^i is a tensor of order 3) which normally identifies the tensor rank as well. However, when contraction (see § 2.6.4) takes place once or more, the order of the tensor is not affected but its rank is reduced by two for each contraction operation.¹
- “Zero tensor” is a tensor whose all components are zero.
- “Unit tensor” or “unity tensor”, which is usually defined for rank-2 tensors, is a tensor whose all elements are zero except the ones with identical values of all indices which are assigned the value 1.
- While tensors of rank-0 are generally represented in a common form of light face non-indexed symbols, tensors of rank ≥ 1 are represented in several forms and notations, the main ones are the index-free notation, which may also be called direct or symbolic or Gibbs notation, and the indicial notation which is also called index or component or tensor notation. The first is a geometrically oriented notation with no reference to a particular reference frame and hence it is intrinsically invariant to the choice of coordinate systems, whereas the second takes an algebraic form based on components identified by indices and hence the notation is suggestive of an underlying coordinate system, although being a tensor makes it form-invariant under certain coordinate transformations and therefore

¹In the literature of tensor calculus, rank and order of tensors are generally used interchangeably; however some authors differentiate between the two as they assign order to the total number of indices, including repetitive indices, while they keep rank to the number of free indices. We think the latter is better and hence we follow this convention in the present text.

it possesses certain invariant properties. The index-free notation is usually identified by using bold face symbols, like \mathbf{a} and \mathbf{B} , while the indicial notation is identified by using light face indexed symbols such as a^i and B_{ij} .

2.2 General Rules

- An index that occurs once in a tensor term is a “free index”.
- An index that occurs twice in a tensor term is a “dummy” or “bound” index.
- No index is allowed to occur more than twice in a legitimate tensor term.²
- A free index should be understood to vary over its range (e.g. $1, \dots, n$) and hence it can be interpreted as saying “for all components represented by the index”. Therefore a free index represents a number of terms or expressions or equalities equal to the number of allowed values of its range. For example, when i and j can vary over the range $1, \dots, n$ the following expression

$$A_i + B_i \tag{8}$$

represents n separate expressions while the following equation

$$A_i^j = B_i^j \tag{9}$$

represents $n \times n$ separate equations.

- According to the “summation convention”, which is widely used in the literature of tensor calculus including in the present notes, dummy indices imply summation over their

²We adopt this assertion, which is common in the literature of tensor calculus, as we think it is suitable for this level. However, there are many instances in the literature of tensor calculus where indices are repeated more than twice in a single term. The bottom line is that as long as the tensor expression makes sense and the intention is clear, such repetitions should be allowed with no need in our view to take special precaution like using parentheses. In particular, the summation convention will not apply automatically in such cases although summation on such indices can be carried out explicitly, by using the summation symbol \sum , or by special declaration of such intention similar to the summation convention. Anyway, in the present text we will not use indices repeated more than twice in a single term.

range, e.g. for an n D space

$$A^i B_i \equiv \sum_{i=1}^n A^i B_i = A^1 B_1 + A^2 B_2 + \dots + A^n B_n \quad (10)$$

$$\delta_{ij} A^{ij} \equiv \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} A^{ij} \quad (11)$$

$$\epsilon_{ijk} A^{ij} B^k \equiv \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \epsilon_{ijk} A^{ij} B^k \quad (12)$$

- When dummy indices do not imply summation, the situation must be clarified by enclosing such indices in parentheses or by underscoring or by using upper case letters (with declaration of these conventions) or by adding a clarifying comment like “no summation on repeated indices”.
- Tensors with subscript indices, like A_{ij} , are called covariant, while tensors with superscript indices, like A^k , are called contravariant. Tensors with both types of indices, like A_{lk}^{mn} , are called mixed type. More details about this will follow in § 2.5.1.
- Subscript indices, rather than subscripted tensors, are also dubbed “covariant” and superscript indices are dubbed “contravariant”.
- Each tensor index should conform to one of the variance transformation rules as given by Eqs. 20 and 21, i.e. it is either covariant or contravariant.
- For orthonormal Cartesian coordinate systems, the two variance types (i.e. covariant and contravariant) do not differ because the metric tensor is given by the Kronecker delta (refer to § 5 and 3.1) and hence any index can be upper or lower although it is common to use lower indices in such cases.
- For tensor invariance, a pair of dummy indices should in general be complementary in their variance type, i.e. one covariant and the other contravariant. However, for or-

thonormal Cartesian systems the two are the same and hence when both dummy indices are covariant or both are contravariant it should be understood as an indication that the underlying coordinate system is orthonormal Cartesian if the possibility of an error is excluded.

- As indicated earlier, tensor order is equal to the number of its indices while tensor rank is equal to the number of its free indices; hence vectors (terms, expressions and equalities) are represented by a single free index and rank-2 tensors are represented by two free indices. The dimension of a tensor is determined by the range taken by its indices.
- The rank of all terms in legitimate tensor expressions and equalities must be the same.
- Each term in valid tensor expressions and equalities must have the same set of free indices (e.g. i, j, k).
- A free index should keep its variance type in every term in valid tensor expressions and equations, i.e. it must be covariant in all terms or contravariant in all terms.
- While free indices should be named uniformly in all terms of tensor expressions and equalities, dummy indices can be named in each term independently, e.g.

$$A_{ik}^i + B_{jk}^j + C_{lmk}^{lm} \quad (13)$$

- A free index in an expression or equality can be renamed uniformly using a different symbol, as long as this symbol is not already in use, assuming that both symbols vary over the same range, i.e. have the same dimension.
- Examples of legitimate tensor terms, expressions and equalities:

$$A_{ij}^{ij}, \quad A_m^{im} + B_{nk}^{ink}, \quad C_{ij} = A_{ij} - B_{ij}, \quad a = B_j^j \quad (14)$$

- Examples of illegitimate tensor terms, expressions and equalities:

$$B_i^{ii}, \quad A_i + B_{ij}, \quad A^i + B^j, \quad A_i - B^i, \quad A_i^i = B_i, \quad (15)$$

- Indexing is generally distributive over the terms of tensor expressions and equalities, e.g.

$$[\mathbf{A} + \mathbf{B}]_i = [\mathbf{A}]_i + [\mathbf{B}]_i \quad (16)$$

and

$$[\mathbf{A} = \mathbf{B}]_i \iff [\mathbf{A}]_i = [\mathbf{B}]_i \quad (17)$$

- Unlike scalars and tensor components, which are essentially scalars in a generic sense, operators cannot in general be freely reordered in tensor terms, therefore

$$fh = hf \quad \& \quad A_i B^i = B^i A_i \quad (18)$$

but

$$\partial_i A_i \neq A_i \partial_i \quad (19)$$

- Almost all the identities in the present notes which are given in a covariant or a contravariant or a mixed form are similarly valid for the other forms unless it is stated otherwise. The objective of reporting in only one form is conciseness and to avoid unnecessary repetition.

2.3 Examples of Tensors of Different Ranks

- Examples of rank-0 tensors (scalars) are energy, mass, temperature, volume and density. These are totally identified by a single number regardless of any coordinate system and hence they are invariant under coordinate transformations.

- Examples of rank-1 tensors (vectors) are displacement, force, electric field, velocity and acceleration. These need for their complete identification a number, representing their magnitude, and a direction representing their geometric orientation within their space. Alternatively, they can be uniquely identified by a set of numbers, equal to the number of dimensions of the underlying space, in reference to a particular coordinate system and hence this identification is system-dependent although they still have system-invariant properties such as length.
- Examples of rank-2 tensors are Kronecker delta (see § 3.1), stress, strain, rate of strain and inertia tensors. These require for their full identification a set of numbers each of which is associated with two directions.
- Examples of rank-3 tensors are the Levi-Civita tensor (see § 3.2) and the tensor of piezoelectric moduli.
- Examples of rank-4 tensors are the elasticity or stiffness tensor, the compliance tensor and the fourth-order moment of inertia tensor.
- Tensors of high ranks are relatively rare in science and engineering.
- Although rank-0 and rank-1 tensors are, respectively, scalars and vectors, not all scalars and vectors (in their generic sense) are tensors of these ranks. Similarly, rank-2 tensors are normally represented by matrices but not all matrices represent tensors.

2.4 Applications of Tensors

- Tensor calculus is very powerful mathematical tool. Tensor notation and techniques are used in many branches of science and engineering such as fluid mechanics, continuum mechanics, general relativity and structural engineering. Tensor calculus is used for elegant and compact formulation and presentation of equations and identities in mathematics, science and engineering. It is also used for algebraic manipulation of mathematical expressions and proving identities in a neat and succinct way (refer to § 4.6).

- As indicated earlier, the invariance of tensor forms serves a theoretically and practically important role by allowing the formulation of physical laws in coordinate-free forms.

2.5 Types of Tensors

- In the following subsections we introduce a number of tensor types and categories and highlight their main characteristics and differences. These types and categories are not mutually exclusive and hence they overlap in general; moreover they may not be exhaustive in their classes as some tensors may not instantiate any one of a complementary set of types such as being symmetric or anti-symmetric.

2.5.1 Covariant and Contravariant Tensors

- These are the main types of tensor with regard to the rules of their transformation between different coordinate systems.
- Covariant tensors are notated with subscript indices (e.g. A_i) while contravariant tensors are notated with superscript indices (e.g. A^{ij}).
- A covariant tensor is transformed according to the following rule

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad (20)$$

while a contravariant tensor is transformed according to the following rule

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad (21)$$

where the barred and unbarred symbols represent the same mathematical object (tensor or coordinate) in the transformed and original coordinate systems respectively.

- An example of covariant tensors is the gradient of a scalar field.
- An example of contravariant tensors is the displacement vector.

- Some tensors have mixed variance type, i.e. they are covariant in some indices and contravariant in others. In this case the covariant variables are indexed with subscripts while the contravariant variables are indexed with superscripts, e.g. A_i^j which is covariant in i and contravariant in j .
- A mixed type tensor transforms covariantly in its covariant indices and contravariantly in its contravariant indices, e.g.

$$\bar{A}^l{}_m{}^n = \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^k} A^i{}_j{}^k \quad (22)$$

- To clarify the pattern of mathematical transformation of tensors, we explain step-by-step the practical rules to follow in writing tensor transformation equations between two coordinate systems, unbarred and barred, where for clarity we color the symbols of the tensor and the coordinates belonging to the unbarred system with blue while we use red to mark the symbols belonging to the barred system. Since there are three types of tensors: covariant, contravariant and mixed, we use three equations in each step. In this demonstration we use rank-4 tensors as examples since this is sufficiently general and hence adequate to elucidate the rules for tensors of any rank. The demonstration is based on the assumption that the transformation is taking place from the unbarred system to the barred system; the same rules should apply for the opposite transformation from the barred system to the unbarred system. We use the sign ‘ $\stackrel{\circ}{=}$ ’ for the equality in the transitional steps to indicate that the equalities are under construction and are not complete.

We start by the very generic equations between the barred tensor \bar{A} and the unbarred

tensor A for the three types:

$$\begin{aligned}
 \bar{A} &\stackrel{\circ}{=} A && \text{(covariant)} \\
 \bar{A} &\stackrel{\circ}{=} A && \text{(contravariant)} \\
 \bar{A} &\stackrel{\circ}{=} A && \text{(mixed)}
 \end{aligned} \tag{23}$$

We assume that the barred tensor and its coordinates are indexed with $ijkl$ and the unbarred are indexed with $npqr$, so we add these indices in their presumed order and position (lower or upper) paying particular attention to the order in the mixed type:

$$\begin{aligned}
 \bar{A}_{ijkl} &\stackrel{\circ}{=} A_{npqr} \\
 \bar{A}^{ijkl} &\stackrel{\circ}{=} A^{npqr} \\
 \bar{A}^{ij}_{kl} &\stackrel{\circ}{=} A^{np}_{qr}
 \end{aligned} \tag{24}$$

Since the barred and unbarred tensors are of the same type, as they represent the same tensor in two coordinate systems,³ the indices on the two sides of the equalities should match in their position and order. We then insert a number of partial differential operators on the right hand side of the equations equal to the rank of these tensors, which is 4 in our example. These operators represent the transformation rules for each pair of corresponding coordinates one from the barred and one from the unbarred:

$$\begin{aligned}
 \bar{A}_{ijkl} &\stackrel{\circ}{=} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} A_{npqr} \\
 \bar{A}^{ijkl} &\stackrel{\circ}{=} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} A^{npqr} \\
 \bar{A}^{ij}_{kl} &\stackrel{\circ}{=} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} A^{np}_{qr}
 \end{aligned} \tag{25}$$

Now we insert the coordinates of the barred system into the partial differential operators

³Similar basis vectors are assumed.

noting that (i) the positions of any index on the two sides should match, i.e. both upper or both lower, since they are free indices in different terms of tensor equalities, (ii) a superscript index in the denominator of a partial derivative is in lieu of a covariant index in the numerator⁴, and (iii) the order of the coordinates should match the order of the indices in the tensor:

$$\begin{aligned}
 \bar{A}_{ijkl} &\stackrel{\circ}{=} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} A_{npqr} \\
 \bar{A}^{ijkl} &\stackrel{\circ}{=} \frac{\partial x^i}{\partial} \frac{\partial x^j}{\partial} \frac{\partial x^k}{\partial} \frac{\partial x^l}{\partial} A^{npqr} \\
 \bar{A}^{ij}_{kl} &\stackrel{\circ}{=} \frac{\partial x^i}{\partial} \frac{\partial x^j}{\partial} \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} A^{np}_{qr}
 \end{aligned} \tag{26}$$

For consistency, these coordinates should be barred as they belong to the barred tensor; hence we add bars:

$$\begin{aligned}
 \bar{A}_{ijkl} &\stackrel{\circ}{=} \frac{\partial}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{x}^j} \frac{\partial}{\partial \bar{x}^k} \frac{\partial}{\partial \bar{x}^l} A_{npqr} \\
 \bar{A}^{ijkl} &\stackrel{\circ}{=} \frac{\partial \bar{x}^i}{\partial} \frac{\partial \bar{x}^j}{\partial} \frac{\partial \bar{x}^k}{\partial} \frac{\partial \bar{x}^l}{\partial} A^{npqr} \\
 \bar{A}^{ij}_{kl} &\stackrel{\circ}{=} \frac{\partial \bar{x}^i}{\partial} \frac{\partial \bar{x}^j}{\partial} \frac{\partial}{\partial \bar{x}^k} \frac{\partial}{\partial \bar{x}^l} A^{np}_{qr}
 \end{aligned} \tag{27}$$

Finally, we insert the coordinates of the unbarred system into the partial differential operators noting that (i) the positions of the repeated indices on the same side should be opposite, i.e. one upper and one lower, since they are dummy indices and hence the position of the index of the unbarred coordinate should be opposite to its position in the unbarred tensor, (ii) an upper index in the denominator is in lieu of a lower index in the numerator, and (iii) the order of the coordinates should match the order of the indices in

⁴The use of upper indices in the denominator of partial derivatives, which is common in this type of equations, is to indicate the fact that the coordinates and their differentials transform contravariantly.

the tensor:

$$\begin{aligned}
 \bar{A}_{ijkl} &= \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^l} A_{npqr} \\
 \bar{A}^{ijkl} &= \frac{\partial \bar{x}^i}{\partial x^n} \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial \bar{x}^k}{\partial x^q} \frac{\partial \bar{x}^l}{\partial x^r} A^{npqr} \\
 \bar{A}^{ij}_{kl} &= \frac{\partial \bar{x}^i}{\partial x^n} \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^l} A^{np}_{qr}
 \end{aligned} \tag{28}$$

We also replaced the ‘ \doteq ’ sign in the final set of equations with the strict equality sign ‘=’ as the equations now are complete.

- A tensor of m contravariant indices and n covariant indices may be called type (m, n) tensor, e.g. A^k_{ij} is a type $(1, 2)$ tensor. When one or both variance types are absent, zero is used to refer to the absent type in this notation, e.g. B^{ik} is a type $(2, 0)$ tensor.
- The covariant and contravariant types of a tensor are linked through the metric tensor (refer to § 5).
- For orthonormal Cartesian systems there is no difference between covariant and contravariant tensors, and hence the indices can be upper or lower.
- The vectors providing the basis set (not necessarily of unit length or mutually orthogonal) for a coordinate system are of covariant type when they are tangent to the coordinate axes, and they are of contravariant type when they are perpendicular to the local surfaces of constant coordinates. These two sets are identical for orthonormal Cartesian systems.
- Formally, the covariant and contravariant basis vectors are given respectively by:

$$\mathbf{E}_i = \frac{\partial \mathbf{r}}{\partial u^i} \quad \& \quad \mathbf{E}^i = \nabla u^i \tag{29}$$

where \mathbf{r} is the position vector in Cartesian coordinates and u^i is a generalized curvilinear coordinate. As indicated already, a superscript in the denominator of partial derivatives is equivalent to a subscript in the numerator.

- In general, the covariant and contravariant basis vectors are not mutually orthogonal

or of unit length; however the two sets are reciprocal systems and hence they satisfy the following reciprocity relation:

$$\mathbf{E}_i \cdot \mathbf{E}^j = \delta_i^j \quad (30)$$

where δ_i^j is the Kronecker delta (refer to § 3.1).

- A vector can be represented either by covariant components with contravariant coordinate basis vectors or by contravariant components with covariant coordinate basis vectors. For example, a vector \mathbf{A} can be expressed as

$$\mathbf{A} = A_i \mathbf{E}^i \quad \text{or} \quad \mathbf{A} = A^i \mathbf{E}_i \quad (31)$$

where \mathbf{E}^i and \mathbf{E}_i are the contravariant and covariant basis vectors respectively. The use of the covariant or contravariant form of the vector representation is a matter of choice and convenience.

- More generally, a tensor of any rank (≥ 1) can be represented covariantly using contravariant basis tensors of that rank, or contravariantly using covariant basis tensors, or in a mixed form using a mixed basis of opposite type. For example, a rank-2 tensor \mathbf{A} can be written as:

$$\mathbf{A} = A_{ij} \mathbf{E}^i \mathbf{E}^j = A^{ij} \mathbf{E}_i \mathbf{E}_j = A_i^j \mathbf{E}^i \mathbf{E}_j \quad (32)$$

where $\mathbf{E}^i \mathbf{E}^j$, $\mathbf{E}_i \mathbf{E}_j$ and $\mathbf{E}^i \mathbf{E}_j$ are dyadic products (refer to § 2.6.3).

2.5.2 True and Pseudo Tensors

- These are also called polar and axial tensors respectively although it is more common to use the latter terms for vectors. Pseudo tensors may also be called tensor densities.
- True tensors are proper (or ordinary) tensors and hence they are invariant under coordinate transformations, while pseudo tensors are not proper tensors since they do not transform invariantly as they acquire a minus sign under improper orthogonal transfor-

mations which involve inversion of coordinate axes through the origin with a change of system handedness.

- Because true and pseudo tensors have different mathematical properties and represent different types of physical entities, the terms of consistent tensor expressions and equations should be uniform in their true and pseudo type, i.e. all terms are true or all are pseudo.
- The direct product (refer to § 2.6.3) of even number of pseudo tensors is a proper tensor, while the direct product of odd number of pseudo tensors is a pseudo tensor. The direct product of true tensors is obviously a true tensor.
- The direct product of a mix of true and pseudo tensors is a true or pseudo tensor depending on the number of pseudo tensors involved in the product as being even or odd respectively.
- Similar rules to those of direct product apply to cross products (including curl operations) involving tensors (usually of rank-1) with the addition of a pseudo factor for each cross product operation. This factor is contributed by the permutation tensor (see § 3.2) which is implicit in the definition of the cross product (see Eqs. 121 and 146).
- In summary, what determines the tensor type (true or pseudo) of the tensor terms involving direct⁵ and cross products is the parity of the multiplicative factors of pseudo type plus the number of cross product operations involved since each cross product contributes an ϵ factor.
- Examples of true scalars are temperature, mass and the dot product of two polar or two axial vectors, while examples of pseudo scalars are the dot product of an axial vector and a polar vector and the scalar triple product of polar vectors.
- Examples of polar vectors are displacement and acceleration, while examples of axial vectors are angular velocity and cross product of polar vectors in general (including curl operation on polar vectors) due to the involvement of the permutation symbol which is

⁵Inner product (see § 2.6.5) is the result of a direct product operation followed by a contraction (see § 2.6.4) and hence it is a direct product in this context.

a pseudo tensor (refer to § 3.2). The essence of this distinction is that the direction of a pseudo vector depends on the observer choice of the handedness of the coordinate system whereas the direction of a proper vector is independent of such choice.

- Examples of proper tensors of rank-2 are stress and rate of strain tensors, while examples of pseudo tensors of rank-2 are direct products of two vectors: one polar and one axial.
- Examples of proper tensors of higher ranks are piezoelectric moduli tensor (rank-3) and elasticity tensor (rank-4), while examples of pseudo tensors of higher ranks are the permutation tensor of these ranks.

2.5.3 Absolute and Relative Tensors

- Considering an arbitrary transformation from a general coordinate system to another, a relative tensor of weight w is defined by the following tensor transformation:

$$\bar{A}^{ij\dots k}_{lm\dots n} = \left| \frac{\partial x}{\partial \bar{x}} \right|^w \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \cdots \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial x^d}{\partial \bar{x}^l} \frac{\partial x^e}{\partial \bar{x}^m} \cdots \frac{\partial x^f}{\partial \bar{x}^n} A^{ab\dots c}_{de\dots f} \quad (33)$$

where $\left| \frac{\partial x}{\partial \bar{x}} \right|$ is the Jacobian of the transformation between the two systems. When $w = 0$ the tensor is described as an absolute or true tensor, while when $w = -1$ the tensor is described as a pseudo tensor. When $w = 1$ the tensor may be described as a tensor density.⁶

- As indicated earlier, a tensor of m contravariant indices and n covariant indices may be called type (m, n) . This may be generalized to include the weight as a third entry and hence the type of the tensor is identified by (m, n, w) .
- Relative tensors can be added and subtracted if they are of the same variance type and have the same weight; the result is a tensor of the same type and weight. Also, relative tensors can be equated if they are of the same type and weight.
- Multiplication of relative tensors produces a relative tensor whose weight is the sum of

⁶Some of these labels are used differently by different authors.

the weights of the original tensors. Hence, if the weights are added up to a non-zero value the result is a relative tensor of that weight; otherwise it is an absolute tensor.

2.5.4 Isotropic and Anisotropic Tensors

- Isotropic tensors are characterized by the property that the values of their components are invariant under coordinate transformation by proper rotation of axes. In contrast, the values of the components of anisotropic tensors are dependent on the orientation of the coordinate axes. Notable examples of isotropic tensors are scalars (rank-0), the vector $\mathbf{0}$ (rank-1), Kronecker delta δ_{ij} (rank-2) and Levi-Civita tensor ϵ_{ijk} (rank-3). Many tensors describing physical properties of materials, such as stress and magnetic susceptibility, are anisotropic.
- Direct and inner products of isotropic tensors are isotropic tensors.
- The zero tensor of any rank is isotropic; therefore if the components of a tensor vanish in a particular coordinate system they will vanish in all properly and improperly rotated coordinate systems.⁷ Consequently, if the components of two tensors are identical in a particular coordinate system they are identical in all transformed coordinate systems.
- As indicated, all rank-0 tensors (scalars) are isotropic. Also, the zero vector, $\mathbf{0}$, of any dimension is isotropic; in fact it is the only rank-1 isotropic tensor.

2.5.5 Symmetric and Anti-symmetric Tensors

- These types of tensor apply to high ranks only (rank ≥ 2). Moreover, these types are not exhaustive, even for tensors of ranks ≥ 2 , as there are high-rank tensors which are neither symmetric nor anti-symmetric.

⁷For improper rotation, this is more general than being isotropic.

- A rank-2 tensor A_{ij} is symmetric *iff* for all i and j

$$A_{ji} = A_{ij} \quad (34)$$

and anti-symmetric or skew-symmetric *iff*

$$A_{ji} = -A_{ij} \quad (35)$$

Similar conditions apply to contravariant type tensors (refer also to the following).

- A rank- n tensor $A_{i_1 \dots i_n}$ is symmetric in its two indices i_j and i_l *iff*

$$A_{i_1 \dots i_l \dots i_j \dots i_n} = A_{i_1 \dots i_j \dots i_l \dots i_n} \quad (36)$$

and anti-symmetric or skew-symmetric in its two indices i_j and i_l *iff*

$$A_{i_1 \dots i_l \dots i_j \dots i_n} = -A_{i_1 \dots i_j \dots i_l \dots i_n} \quad (37)$$

- Any rank-2 tensor A_{ij} can be synthesized from (or decomposed into) a symmetric part $A_{(ij)}$ (marked with round brackets enclosing the indices) and an anti-symmetric part $A_{[ij]}$ (marked with square brackets) where

$$A_{ij} = A_{(ij)} + A_{[ij]}, \quad A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) \quad \& \quad A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) \quad (38)$$

- A rank-3 tensor A_{ijk} can be symmetrized by

$$A_{(ijk)} = \frac{1}{3!}(A_{ijk} + A_{kij} + A_{jki} + A_{ikj} + A_{jik} + A_{kji}) \quad (39)$$

and anti-symmetrized by

$$A_{[ijk]} = \frac{1}{3!} (A_{ijk} + A_{kij} + A_{jki} - A_{ikj} - A_{jik} - A_{kji}) \quad (40)$$

- A rank- n tensor $A_{i_1 \dots i_n}$ can be symmetrized by

$$A_{(i_1 \dots i_n)} = \frac{1}{n!} (\text{sum of all even \& odd permutations of indices } i\text{'s}) \quad (41)$$

and anti-symmetrized by

$$A_{[i_1 \dots i_n]} = \frac{1}{n!} (\text{sum of all even permutations minus sum of all odd permutations}) \quad (42)$$

- For a symmetric tensor A_{ij} and an anti-symmetric tensor B^{ij} (or the other way around) we have

$$A_{ij}B^{ij} = 0 \quad (43)$$

- The indices whose exchange defines the symmetry and anti-symmetry relations should be of the same variance type, i.e. both upper or both lower.
- The symmetry and anti-symmetry characteristic of a tensor is invariant under coordinate transformation.
- A tensor of high rank (> 2) may be symmetrized or anti-symmetrized with respect to only some of its indices instead of all of its indices, e.g.

$$A_{(ij)k} = \frac{1}{2} (A_{ijk} + A_{jik}) \quad \& \quad A_{[ij]k} = \frac{1}{2} (A_{ijk} - A_{jik}) \quad (44)$$

- A tensor is totally symmetric *iff*

$$A_{i_1 \dots i_n} = A_{(i_1 \dots i_n)} \quad (45)$$

and totally anti-symmetric *iff*

$$A_{i_1 \dots i_n} = A_{[i_1 \dots i_n]} \quad (46)$$

- For a totally skew-symmetric tensor (i.e. anti-symmetric in all of its indices), nonzero entries can occur only when all the indices are different.

2.6 Tensor Operations

- There are many operations that can be performed on tensors to produce other tensors in general. Some examples of these operations are addition/subtraction, multiplication by a scalar (rank-0 tensor), multiplication of tensors (each of rank > 0), contraction and permutation. Some of these operations, such as addition and multiplication, involve more than one tensor while others are performed on a single tensor, such as contraction and permutation.
- In tensor algebra, division is allowed only for scalars, hence if the components of an indexed tensor should appear in a denominator, the tensor should be redefined to avoid this, e.g. $B_i = \frac{1}{A_i}$.

2.6.1 Addition and Subtraction

- Tensors of the same rank and type (covariant/contravariant/mixed and true/pseudo) can be added algebraically to produce a tensor of the same rank and type, e.g.

$$a = b + c \quad (47)$$

$$A_i = B_i - C_i \quad (48)$$

$$A_j^i = B_j^i + C_j^i \quad (49)$$

- The added/subtracted terms should have the same indicial structure with regard to their free indices, as explained in § 2.2, hence A_{jk}^i and B_{ik}^j cannot be added or subtracted although they are of the same rank and type, but A_{mjk}^{mi} and B_{jk}^i can be added and subtracted.
- Addition of tensors is associative and commutative:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad (50)$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (51)$$

2.6.2 Multiplication by Scalar

- A tensor can be multiplied by a scalar, which generally should not be zero, to produce a tensor of the same variance type and rank, e.g.

$$A_{ik}^j = aB_{ik}^j \quad (52)$$

where a is a non-zero scalar.

- As indicated above, multiplying a tensor by a scalar means multiplying each component of the tensor by that scalar.
- Multiplication by a scalar is commutative, and associative when more than two factors are involved.

2.6.3 Tensor Multiplication

- This may also be called outer or exterior or direct or dyadic multiplication, although some of these names may be reserved for operations on vectors.
- On multiplying each component of a tensor of rank r by each component of a tensor of rank k , both of dimension m , a tensor of rank $(r + k)$ with m^{r+k} components is obtained where the variance type of each index (covariant or contravariant) is preserved, e.g.

$$A_i B_j = C_{ij} \quad (53)$$

$$A^{ij} B_{kl} = C^{ij}_{kl} \quad (54)$$

- The outer product of a tensor of type (m, n) by a tensor of type (p, q) results in a tensor of type $(m + p, n + q)$.
- Direct multiplication of tensors may be marked by the symbol \otimes , mostly when using symbolic notation for tensors, e.g. $\mathbf{A} \otimes \mathbf{B}$. However, in the present text no symbol will be used for the operation of direct multiplication.
- Direct multiplication of tensors is not commutative.
- The outer product operation is distributive with respect to the algebraic sum of tensors:

$$\mathbf{A} (\mathbf{B} \pm \mathbf{C}) = \mathbf{A}\mathbf{B} \pm \mathbf{A}\mathbf{C} \quad \& \quad (\mathbf{B} \pm \mathbf{C}) \mathbf{A} = \mathbf{B}\mathbf{A} \pm \mathbf{C}\mathbf{A} \quad (55)$$

- Multiplication of a tensor by a scalar (refer to § 2.6.2) may be regarded as a special case of direct multiplication.
- The rank-2 tensor constructed as a result of the direct multiplication of two vectors is commonly called dyad.
- Tensors may be expressed as an outer product of vectors where the rank of the resultant

product is equal to the number of the vectors involved (e.g. 2 for dyads and 3 for triads).

- Not every tensor can be synthesized as a product of lower rank tensors.
- In the outer product, it is understood that all the indices of the involved tensors have the same range.
- The outer product of tensors yields a tensor.

2.6.4 Contraction

- Contraction of a tensor of rank > 1 is to make two free indices identical, by unifying their symbols, and perform summation over these repeated indices, e.g.

$$A_i^j \xrightarrow{\text{contraction}} A_i^i \quad (56)$$

$$A_{il}^{jk} \xrightarrow{\text{contraction on } jl} A_{im}^{mk} \quad (57)$$

- Contraction results in a reduction of the rank by 2 since it implies the annihilation of two free indices. Therefore, the contraction of a rank-2 tensor is a scalar, the contraction of a rank-3 tensor is a vector, the contraction of a rank-4 tensor is a rank-2 tensor, and so on.
- For general non-Cartesian coordinate systems, the pair of contracted indices should be different in their variance type, i.e. one upper and one lower. Hence, contraction of a mixed tensor of type (m, n) will, in general, produce a tensor of type $(m - 1, n - 1)$.
- A tensor of type (p, q) can have $p \times q$ possible contractions, i.e. one contraction for each pair of lower and upper indices.
- A common example of contraction is the dot product operation on vectors which can be regarded as a direct multiplication (refer to § 2.6.3) of the two vectors, which results in a rank-2 tensor, followed by a contraction.

- In matrix algebra, taking the trace (summing the diagonal elements) can also be considered as contraction of the matrix, which under certain conditions can represent a rank-2 tensor, and hence it yields the trace which is a scalar.
- Applying the index contraction operation on a tensor results into a tensor.
- Application of contraction of indices operation on a relative tensor (see § 2.5.3) produces a relative tensor of the same weight as the original tensor.

2.6.5 Inner Product

- On taking the outer product (refer to § 2.6.3) of two tensors of rank ≥ 1 followed by a contraction on two indices of the product, an inner product of the two tensors is formed. Hence if one of the original tensors is of rank- m and the other is of rank- n , the inner product will be of rank- $(m + n - 2)$.
- The inner product operation is usually symbolized by a single dot between the two tensors, e.g. $\mathbf{A} \cdot \mathbf{B}$, to indicate contraction following outer multiplication.
- In general, the inner product is not commutative. When one or both of the tensors involved in the inner product are of rank > 1 the order of the multiplicands does matter.
- The inner product operation is distributive with respect to the algebraic sum of tensors:

$$\mathbf{A} \cdot (\mathbf{B} \pm \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} \pm \mathbf{A} \cdot \mathbf{C} \quad \& \quad (\mathbf{B} \pm \mathbf{C}) \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{A} \pm \mathbf{C} \cdot \mathbf{A} \quad (58)$$

- As indicated before (see § 2.6.4), the dot product of two vectors is an example of the inner product of tensors, i.e. it is an inner product of two rank-1 tensors to produce a rank-0 tensor:

$$[\mathbf{ab}]_i^j = a_i b^j \quad \xrightarrow{\text{contraction}} \quad \mathbf{a} \cdot \mathbf{b} = a_i b^i \quad (59)$$

- Another common example (from linear algebra) of inner product is the multiplication of a matrix (representing a rank-2 tensor assuming certain conditions) by a vector (rank-1

tensor) to produce a vector, e.g.

$$[\mathbf{A}\mathbf{b}]_{ij}^k = A_{ij}b^k \quad \xrightarrow{\text{contraction on } jk} \quad [\mathbf{A} \cdot \mathbf{b}]_i = A_{ij}b^j \quad (60)$$

The multiplication of two $n \times n$ matrices is another example of inner product (see Eq. 119).

- For tensors whose outer product produces a tensor of rank > 2 , various contraction operations between different sets of indices can occur and hence more than one inner product, which are different in general, can be defined. Moreover, when the outer product produces a tensor of rank > 3 more than one contraction can take place simultaneously.
- There are more specialized types of inner product; some of which may be defined differently by different authors. For example, a double inner product of two rank-2 tensors, \mathbf{A} and \mathbf{B} , may be defined and denoted by double vertically- or horizontally-aligned dots (e.g. $\mathbf{A} : \mathbf{B}$ or $\mathbf{A} \cdot \cdot \mathbf{B}$) to indicate double contraction taking place between different pairs of indices. An instance of these types is the inner product with double contraction of two dyads which is commonly defined by⁸

$$\mathbf{ab} : \mathbf{cd} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \quad (61)$$

and the result is a scalar. The single dots in the right hand side of the last equation symbolize the conventional dot product of two vectors. Some authors may define a different type of double-contraction inner product of two dyads, symbolized by two horizontally-aligned dots, which may be called a “transposed contraction”, and is given by

$$\mathbf{ab} \cdot \cdot \mathbf{cd} = \mathbf{ab} : \mathbf{dc} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \quad (62)$$

⁸It is also defined differently by some authors.

where the result is also a scalar. However, different authors may have different conventions and hence one should be vigilant about such differences.

- For two rank-2 tensors, the aforementioned double-contraction inner products are similarly defined as in the case of two dyads:

$$\mathbf{A} : \mathbf{B} = A_{ij}B_{ij} \quad \& \quad \mathbf{A} \cdot \cdot \mathbf{B} = A_{ij}B_{ji} \quad (63)$$

- Inner products with higher multiplicity of contractions are similarly defined, and hence can be regarded as trivial extensions of the inner products with lower contraction multiplicities.
- The inner product of tensors produces a tensor because the inner product is an outer product operation followed by an index contraction operation and both of these operations on tensors produce tensors.

2.6.6 Permutation

- A tensor may be obtained by exchanging the indices of another tensor, e.g. transposition of rank-2 tensors.
- Tensor permutation applies only to tensors of rank ≥ 2 .
- The collection of tensors obtained by permuting the indices of a basic tensor may be called isomers.

2.7 Tensor Test: Quotient Rule

- Sometimes a tensor-like object may be suspected for being a tensor; in such cases a test based on the “quotient rule” can be used to clarify the situation. According to this rule, if the inner product of a suspected tensor with a known tensor is a tensor then the suspect is a tensor. In more formal terms, if it is not known if \mathbf{A} is a tensor but it is known that

B and **C** are tensors; moreover it is known that the following relation holds true in all rotated (properly-transformed) coordinate frames:

$$A_{pq\dots k\dots m}B_{ij\dots k\dots n} = C_{pq\dots mij\dots n} \quad (64)$$

then **A** is a tensor. Here, **A**, **B** and **C** are respectively of ranks m , n and $(m + n - 2)$, due to the contraction on k which can be any index of **A** and **B** independently.

- Testing for being a tensor can also be done by applying first principles through direct substitution in the transformation equations. However, using the quotient rule is generally more convenient and requires less work.
- The quotient rule may be considered as a replacement for the division operation which is not defined for tensors.

3 δ and ϵ Tensors

- These tensors are of particular importance in tensor calculus due to their distinctive properties and unique transformation attributes. They are numerical tensors with fixed components in all coordinate systems. The first is called Kronecker delta or unit tensor, while the second is called Levi-Civita⁹, permutation, anti-symmetric and alternating tensor.
- The δ and ϵ tensors are conserved under coordinate transformations and hence they are the same for all systems of coordinate.¹⁰

3.1 Kronecker δ

- This is a rank-2 symmetric tensor in all dimensions, i.e.

$$\delta_{ij} = \delta_{ji} \quad (i, j = 1, 2, \dots, n) \quad (65)$$

Similar identities apply to the contravariant and mixed types of this tensor.

- It is invariant in all coordinate systems, and hence it is an isotropic tensor.¹¹
- It is defined as:

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \quad (66)$$

⁹This name is usually used for the rank-3 tensor. Also some authors distinguish between the permutation tensor and Levi-Civita tensor even for rank-3. Moreover, some of the common labels and descriptions of ϵ are more specific to rank-3.

¹⁰For the permutation tensor, the statement applies to proper coordinate transformations.

¹¹In fact it is more general than isotropic as it is invariant even under improper coordinate transformations.

and hence it can be considered as the identity matrix, e.g. for 3D

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (67)$$

- Covariant, contravariant and mixed type of this tensor are the same, that is

$$\delta_j^i = \delta_i^j = \delta^{ij} = \delta_{ij} \quad (68)$$

3.2 Permutation ϵ

- This is an isotropic tensor. It has a rank equal to the number of dimensions; hence, a rank- n permutation tensor has n^n components.
- It is totally anti-symmetric in each pair of its indices, i.e. it changes sign on swapping any two of its indices, that is

$$\epsilon_{i_1 \dots i_k \dots i_l \dots i_n} = -\epsilon_{i_1 \dots i_l \dots i_k \dots i_n} \quad (69)$$

The reason is that any exchange of two indices requires an even/odd number of single-step shifts to the right of the first index plus an odd/even number of single-step shifts to the left of the second index, so the total number of shifts is odd and hence it is an odd permutation of the original arrangement.

- It is a pseudo tensor since it acquires a minus sign under improper orthogonal transformation of coordinates (inversion of axes with possible superposition of rotation).
- Definition of rank-2 ϵ (ϵ_{ij}):

$$\epsilon_{12} = 1, \quad \epsilon_{21} = -1 \quad \& \quad \epsilon_{11} = \epsilon_{22} = 0 \quad (70)$$

- Definition of rank-3 ϵ (ϵ_{ijk}):

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k \text{ is even permutation of } 1,2,3) \\ -1 & (i, j, k \text{ is odd permutation of } 1,2,3) \\ 0 & (\text{repeated index}) \end{cases} \quad (71)$$

- The definition of rank- n ϵ ($\epsilon_{i_1 i_2 \dots i_n}$) is similar to the definition of rank-3 ϵ considering index repetition and even or odd permutations of its indices (i_1, i_2, \dots, i_n) corresponding to $(1, 2, \dots, n)$, that is

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} 1 & [(i_1, i_2, \dots, i_n) \text{ is even permutation of } (1, 2, \dots, n)] \\ -1 & [(i_1, i_2, \dots, i_n) \text{ is odd permutation of } (1, 2, \dots, n)] \\ 0 & [\text{repeated index}] \end{cases} \quad (72)$$

- ϵ may be considered a contravariant relative tensor of weight +1 or a covariant relative tensor of weight -1. Hence, in 2D, 3D and n D spaces respectively we have:

$$\epsilon_{ij} = \epsilon^{ij} \quad (73)$$

$$\epsilon_{ijk} = \epsilon^{ijk} \quad (74)$$

$$\epsilon_{i_1 i_2 \dots i_n} = \epsilon^{i_1 i_2 \dots i_n} \quad (75)$$

3.3 Useful Identities Involving δ or/and ϵ

3.3.1 Identities Involving δ

- When an index of the Kronecker delta is involved in a contraction operation by repeating an index in another tensor in its own term, the effect of this is to replace the shared index

in the other tensor by the other index of the Kronecker delta, that is

$$\delta_{ij}A_j = A_i \quad (76)$$

In such cases the Kronecker delta is described as the substitution or index replacement operator. Hence,

$$\delta_{ij}\delta_{jk} = \delta_{ik} \quad (77)$$

Similarly,

$$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = n \quad (78)$$

where n is the space dimension.

- Because the coordinates are independent of each other:

$$\frac{\partial x_i}{\partial x_j} = \partial_j x_i = x_{i,j} = \delta_{ij} \quad (79)$$

Hence, in an n D space we have

$$\partial_i x_i = \delta_{ii} = n \quad (80)$$

- For orthonormal Cartesian systems:

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial x^j}{\partial x^i} = \delta_{ij} = \delta^{ij} \quad (81)$$

- For a set of orthonormal basis vectors in orthonormal Cartesian systems:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (82)$$

- The double inner product of two dyads formed by orthonormal basis vectors of an

orthonormal Cartesian system is given by:

$$\mathbf{e}_i \mathbf{e}_j : \mathbf{e}_k \mathbf{e}_l = \delta_{ik} \delta_{jl} \quad (83)$$

3.3.2 Identities Involving ϵ

- For rank-3 ϵ :

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \quad (\text{sense of cyclic order}) \quad (84)$$

These equations demonstrate the fact that rank-3 ϵ is totally anti-symmetric in all of its indices since a shift of any two indices reverses the sign. This also reflects the fact that the above tensor system has only one independent component.

- For rank-2 ϵ :

$$\epsilon_{ij} = (j - i) \quad (85)$$

- For rank-3 ϵ :

$$\epsilon_{ijk} = \frac{1}{2} (j - i) (k - i) (k - j) \quad (86)$$

- For rank-4 ϵ :

$$\epsilon_{ijkl} = \frac{1}{12} (j - i) (k - i) (l - i) (k - j) (l - j) (l - k) \quad (87)$$

- For rank- n ϵ :

$$\epsilon_{a_1 a_2 \dots a_n} = \prod_{i=1}^{n-1} \left[\frac{1}{i!} \prod_{j=i+1}^n (a_j - a_i) \right] = \frac{1}{S(n-1)} \prod_{1 \leq i < j \leq n} (a_j - a_i) \quad (88)$$

where $S(n-1)$ is the super-factorial function of $(n-1)$ which is defined as

$$S(k) = \prod_{i=1}^k i! = 1! \cdot 2! \cdot \dots \cdot k! \quad (89)$$

A simpler formula for rank- n ϵ can be obtained from the previous one by ignoring the magnitude of the multiplication factors and taking only their signs, that is

$$\epsilon_{a_1 a_2 \dots a_n} = \prod_{1 \leq i < j \leq n} \sigma(a_j - a_i) = \sigma \left(\prod_{1 \leq i < j \leq n} (a_j - a_i) \right) \quad (90)$$

where

$$\sigma(k) = \begin{cases} +1 & (k > 0) \\ -1 & (k < 0) \\ 0 & (k = 0) \end{cases} \quad (91)$$

- For rank- n ϵ :

$$\epsilon_{i_1 i_2 \dots i_n} \epsilon_{i_1 i_2 \dots i_n} = n! \quad (92)$$

because this is the sum of the squares of $\epsilon_{i_1 i_2 \dots i_n}$ over all the permutations of n different indices which is equal to $n!$ where the value of ϵ of each one of these permutations is either $+1$ or -1 and hence in both cases their square is 1.

- For a symmetric tensor A_{jk} :

$$\epsilon_{ijk} A_{jk} = 0 \quad (93)$$

because an exchange of the two indices of A_{jk} does not affect its value due to the symmetry whereas a similar exchange in these indices in ϵ_{ijk} results in a sign change; hence each term in the sum has its own negative and therefore the total sum will vanish.

-

$$\epsilon_{ijk} A_i A_j = \epsilon_{ijk} A_i A_k = \epsilon_{ijk} A_j A_k = 0 \quad (94)$$

because, due to the commutativity of multiplication, an exchange of the indices in A 's will not affect the value but a similar exchange in the corresponding indices of ϵ_{ijk} will cause a change in sign; hence each term in the sum has its own negative and therefore the total sum will be zero.

- For a set of orthonormal basis vectors in a 3D space with a right-handed orthonormal Cartesian coordinate system:

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k \quad (95)$$

$$\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \epsilon_{ijk} \quad (96)$$

3.3.3 Identities Involving δ and ϵ

-

$$\epsilon_{ijk} \delta_{1i} \delta_{2j} \delta_{3k} = \epsilon_{123} = 1 \quad (97)$$

- For rank-2 ϵ :

$$\epsilon_{ij} \epsilon_{kl} = \begin{vmatrix} \delta_{ik} & \delta_{il} \\ \delta_{jk} & \delta_{jl} \end{vmatrix} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \quad (98)$$

$$\epsilon_{il} \epsilon_{kl} = \delta_{ik} \quad (99)$$

$$\epsilon_{ij} \epsilon_{ij} = 2 \quad (100)$$

- For rank-3 ϵ :

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl} \quad (101)$$

$$\epsilon_{ijk}\epsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad (102)$$

The last identity is very useful in manipulating and simplifying tensor expressions and proving vector and tensor identities.

$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il} \quad (103)$$

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6 \quad (104)$$

since the rank and dimension of ϵ are the same, which is 3 in this case.

- For rank- n ϵ :

$$\epsilon_{i_1 i_2 \dots i_n} \epsilon_{j_1 j_2 \dots j_n} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{vmatrix} \quad (105)$$

- According to Eqs. [71](#) and [76](#):

$$\epsilon_{ijk}\delta_{ij} = \epsilon_{ijk}\delta_{ik} = \epsilon_{ijk}\delta_{jk} = 0 \quad (106)$$

3.4 Generalized Kronecker delta

- The generalized Kronecker delta is defined by:

$$\delta_{j_1 \dots j_n}^{i_1 \dots i_n} = \begin{cases} 1 & [(j_1 \dots j_n) \text{ is even permutation of } (i_1 \dots i_n)] \\ -1 & [(j_1 \dots j_n) \text{ is odd permutation of } (i_1 \dots i_n)] \\ 0 & [\text{repeated } j\text{'s}] \end{cases} \quad (107)$$

It can also be defined by the following $n \times n$ determinant:

$$\delta_{j_1 \dots j_n}^{i_1 \dots i_n} = \begin{vmatrix} \delta_{j_1}^{i_1} & \delta_{j_2}^{i_1} & \dots & \delta_{j_n}^{i_1} \\ \delta_{j_1}^{i_2} & \delta_{j_2}^{i_2} & \dots & \delta_{j_n}^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{j_1}^{i_n} & \delta_{j_2}^{i_n} & \dots & \delta_{j_n}^{i_n} \end{vmatrix} \quad (108)$$

where the δ_j^i entries in the determinant are the normal Kronecker delta as defined by Eq. 66.

- Accordingly, the relation between the rank- n ϵ and the generalized Kronecker delta in an n D space is given by:

$$\epsilon_{i_1 i_2 \dots i_n} = \delta_{i_1 i_2 \dots i_n}^{1 2 \dots n} \quad \& \quad \epsilon^{i_1 i_2 \dots i_n} = \delta_{1 2 \dots n}^{i_1 i_2 \dots i_n} \quad (109)$$

Hence, the permutation tensor ϵ may be considered as a special case of the generalized Kronecker delta. Consequently the permutation symbol can be written as an $n \times n$ determinant consisting of the normal Kronecker deltas.

- If we define

$$\delta_{lm}^{ij} = \delta_{lmk}^{ijk} \quad (110)$$

then Eq. 102 will take the following form:

$$\delta_{lm}^{ij} = \delta_l^i \delta_m^j - \delta_m^i \delta_l^j \quad (111)$$

Other identities involving δ and ϵ can also be formulated in terms of the generalized Kronecker delta.

- On comparing Eq. 105 with Eq. 108 we conclude

$$\delta_{j_1 \dots j_n}^{i_1 \dots i_n} = \epsilon^{i_1 \dots i_n} \epsilon_{j_1 \dots j_n} \quad (112)$$

4 Applications of Tensor Notation and Techniques

4.1 Common Definitions in Tensor Notation

- The trace of a matrix \mathbf{A} representing a rank-2 tensor is:

$$\text{tr}(\mathbf{A}) = A_{ii} \quad (113)$$

- For a 3×3 matrix representing a rank-2 tensor in a 3D space, the determinant is:

$$\det(\mathbf{A}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \quad (114)$$

where the last two equalities represent the expansion of the determinant by row and by column. Alternatively

$$\det(\mathbf{A}) = \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} A_{il} A_{jm} A_{kn} \quad (115)$$

- For an $n \times n$ matrix representing a rank-2 tensor in an n D space, the determinant is:

$$\det(\mathbf{A}) = \epsilon_{i_1 \dots i_n} A_{1i_1} \dots A_{ni_n} = \epsilon_{i_1 \dots i_n} A_{i_1 1} \dots A_{i_n n} = \frac{1}{n!} \epsilon_{i_1 \dots i_n} \epsilon_{j_1 \dots j_n} A_{i_1 j_1} \dots A_{i_n j_n} \quad (116)$$

- The inverse of a matrix \mathbf{A} representing a rank-2 tensor is:

$$[\mathbf{A}^{-1}]_{ij} = \frac{1}{\det(\mathbf{A})} \epsilon_{jmn} \epsilon_{ipq} A_{mp} A_{nq} \quad (117)$$

- The multiplication of a matrix \mathbf{A} by a vector \mathbf{b} as defined in linear algebra is:

$$[\mathbf{Ab}]_i = A_{ij} b_j \quad (118)$$

It should be noticed that here we are using matrix notation. The multiplication operation, according to the symbolic notation of tensors, should be denoted by a dot between the tensor and the vector, i.e. $\mathbf{A} \cdot \mathbf{b}$.¹²

- The multiplication of two $n \times n$ matrices \mathbf{A} and \mathbf{B} as defined in linear algebra is:

$$[\mathbf{AB}]_{ik} = A_{ij}B_{jk} \quad (119)$$

Again, here we are using matrix notation; otherwise a dot should be inserted between the two matrices.

- The dot product of two vectors is:

$$\mathbf{A} \cdot \mathbf{B} = \delta_{ij}A_iB_j = A_iB_i \quad (120)$$

The readers are referred to § 2.6.5 for a more general definition of this type of product that includes higher rank tensors.

- The cross product of two vectors is:

$$[\mathbf{A} \times \mathbf{B}]_i = \epsilon_{ijk}A_jB_k \quad (121)$$

- The scalar triple product of three vectors is:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \epsilon_{ijk}A_iB_jC_k \quad (122)$$

¹²The matrix multiplication in matrix notation is equivalent to a dot product operation in tensor notation.

- The vector triple product of three vectors is:

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i = \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m \quad (123)$$

4.2 Scalar Invariants of Tensors

- In the following we list and write in tensor notation a number of invariants of low rank tensors which have special importance due to their widespread applications in vector and tensor calculus. All These invariants are scalars.
- The value of a scalar (rank-0 tensor), which consists of a magnitude and a sign, is invariant under coordinate transformation.
- An invariant of a vector (rank-1 tensor) under coordinate transformations is its magnitude, i.e. length (the direction is also invariant but it is not scalar!).¹³
- The main three independent scalar invariants of a rank-2 tensor \mathbf{A} under change of basis are:

$$I = \text{tr}(\mathbf{A}) = A_{ii} \quad (124)$$

$$II = \text{tr}(\mathbf{A}^2) = A_{ij}A_{ji} \quad (125)$$

$$III = \text{tr}(\mathbf{A}^3) = A_{ij}A_{jk}A_{ki} \quad (126)$$

- Different forms of the three invariants of a rank-2 tensor \mathbf{A} , which are also widely used, are:

$$I_1 = I = A_{ii} \quad (127)$$

¹³In fact the magnitude alone is invariant under coordinate transformations even for pseudo vectors because it is a scalar.

$$I_2 = \frac{1}{2} (I^2 - II) = \frac{1}{2} (A_{ii}A_{jj} - A_{ij}A_{ji}) \quad (128)$$

$$I_3 = \det(\mathbf{A}) = \frac{1}{3!} (I^3 - 3I II + 2III) = \frac{1}{3!} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} A_{kr} \quad (129)$$

- The invariants I , II and III can similarly be defined in terms of the invariants I_1 , I_2 and I_3 as follow:

$$I = I_1 \quad (130)$$

$$II = I_1^2 - 2I_2 \quad (131)$$

$$III = I_1^3 - 3I_1 I_2 + 3I_3 \quad (132)$$

- Since the determinant of a matrix representing a rank-2 tensor is invariant, then if the determinant vanishes in one coordinate system it will vanish in all coordinate systems and vice versa. Consequently, if a rank-2 tensor is invertible in a particular coordinate system, it is invertible in all coordinate systems.

- Ten joint invariants between two rank-2 tensors, \mathbf{A} and \mathbf{B} , can be formed; these are: $\text{tr}(\mathbf{A})$, $\text{tr}(\mathbf{B})$, $\text{tr}(\mathbf{A}^2)$, $\text{tr}(\mathbf{B}^2)$, $\text{tr}(\mathbf{A}^3)$, $\text{tr}(\mathbf{B}^3)$, $\text{tr}(\mathbf{A} \cdot \mathbf{B})$, $\text{tr}(\mathbf{A}^2 \cdot \mathbf{B})$, $\text{tr}(\mathbf{A} \cdot \mathbf{B}^2)$ and $\text{tr}(\mathbf{A}^2 \cdot \mathbf{B}^2)$.

4.3 Common Differential Operations in Tensor Notation

- Here we present the most common differential operations as defined by tensor notation. These operations are mostly based on the various types of interaction between the vector differential operator nabla ∇ with tensors of different ranks as well as interaction with other types of operation like dot and cross products.

- ∇ is essentially a spatial partial differential operator defined in Cartesian coordinate systems by:

$$\nabla_i = \frac{\partial}{\partial x_i} \quad (133)$$

The definition of ∇ in some non-Cartesian systems will be given in § 4.3.2.

4.3.1 Cartesian System

- The gradient of a differentiable scalar function of position f is a vector given by:

$$[\nabla f]_i = \nabla_i f = \frac{\partial f}{\partial x_i} = \partial_i f = f_{,i} \quad (134)$$

- The gradient of a differentiable vector function of position \mathbf{A} (which is the outer product, as defined in § 2.6.3, between the ∇ operator and the vector) is a rank-2 tensor defined by:

$$[\nabla \mathbf{A}]_{ij} = \partial_i A_j \quad (135)$$

- The gradient operation is distributive but not commutative or associative:

$$\nabla(f + h) = \nabla f + \nabla h \quad (136)$$

$$\nabla f \neq f \nabla \quad (137)$$

$$(\nabla f) h \neq \nabla(fh) \quad (138)$$

where f and h are differentiable scalar functions of position.

- The divergence of a differentiable vector \mathbf{A} is a scalar given by:

$$\nabla \cdot \mathbf{A} = \delta_{ij} \frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} = \nabla_i A_i = \partial_i A_i = A_{i,i} \quad (139)$$

The divergence operation can also be viewed as taking the gradient of the vector followed by a contraction. Hence, the divergence of a vector is invariant because it is the trace of a rank-2 tensor.¹⁴

- The divergence of a differentiable rank-2 tensor \mathbf{A} is a vector defined in one of its forms by:

$$[\nabla \cdot \mathbf{A}]_i = \partial_j A_{ji} \quad (140)$$

and in another form by

$$[\nabla \cdot \mathbf{A}]_j = \partial_i A_{ji} \quad (141)$$

These two different forms can be given, respectively, in symbolic notation by:

$$\nabla \cdot \mathbf{A} \quad \& \quad \nabla \cdot \mathbf{A}^T \quad (142)$$

where \mathbf{A}^T is the transpose of \mathbf{A} . More generally, the divergence of a tensor of rank $n \geq 2$, which is a tensor of rank- $(n - 1)$, can be defined in several forms, which are different in general, depending on the combination of the contracted indices.

- The divergence operation is distributive but not commutative or associative:

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \quad (143)$$

$$\nabla \cdot \mathbf{A} \neq \mathbf{A} \cdot \nabla \quad (144)$$

¹⁴It may also be argued that the divergence of a vector is a scalar and hence it is invariant.

$$\nabla \cdot (f \mathbf{A}) \neq \nabla f \cdot \mathbf{A} \quad (145)$$

where \mathbf{A} and \mathbf{B} are differentiable tensor functions of position.

- The curl of a differentiable vector \mathbf{A} is a vector given by:

$$[\nabla \times \mathbf{A}]_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} = \epsilon_{ijk} \nabla_j A_k = \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} A_{k,j} \quad (146)$$

- The curl operation may be generalized to tensors of rank > 1 , and hence the curl of a differentiable rank-2 tensor \mathbf{A} can be defined as a rank-2 tensor given by:

$$[\nabla \times \mathbf{A}]_{ij} = \epsilon_{imn} \partial_m A_{nj} \quad (147)$$

- The curl operation is distributive but not commutative or associative:

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \quad (148)$$

$$\nabla \times \mathbf{A} \neq \mathbf{A} \times \nabla \quad (149)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \neq (\nabla \times \mathbf{A}) \times \mathbf{B} \quad (150)$$

- The Laplacian scalar operator, also called the harmonic operator, acting on a differentiable scalar f is given by:

$$\Delta f = \nabla^2 f = \delta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_i \partial x_i} = \nabla_{ii} f = \partial_{ii} f = f_{,ii} \quad (151)$$

- The Laplacian operator acting on a differentiable vector \mathbf{A} is defined for each component

of the vector similar to the definition of the Laplacian acting on a scalar, that is

$$[\nabla^2 \mathbf{A}]_i = \partial_{jj} A_i \quad (152)$$

- The following scalar differential operator is commonly used in science (e.g. in fluid dynamics):

$$\mathbf{A} \cdot \nabla = A_i \nabla_i = A_i \frac{\partial}{\partial x_i} = A_i \partial_i \quad (153)$$

where \mathbf{A} is a vector. As indicated earlier, the order of A_i and ∂_i should be respected.

- The following vector differential operator also has common applications in science:

$$[\mathbf{A} \times \nabla]_i = \epsilon_{ijk} A_j \partial_k \quad (154)$$

- The differentiation of a tensor increases its rank by one, by introducing an extra covariant index, unless it implies a contraction in which case it reduces the rank by one. Therefore the gradient of a scalar is a vector and the gradient of a vector is a rank-2 tensor ($\partial_i A_j$), while the divergence of a vector is a scalar and the divergence of a rank-2 tensor is a vector ($\partial_j A_{ji}$ or $\partial_i A_{ji}$). This may be justified by the fact that ∇ is a vector operator. On the other hand the Laplacian operator does not change the rank since it is a scalar operator; hence the Laplacian of a scalar is a scalar and the Laplacian of a vector is a vector.

4.3.2 Other Coordinate Systems

- For completeness, we define here some differential operations in the most commonly used non-Cartesian coordinate systems, namely cylindrical and spherical systems, as well as general orthogonal coordinate systems.
- We can use indexed generalized coordinates like q_1 , q_2 and q_3 for the cylindrical coordinates (ρ, ϕ, z) and the spherical coordinates (r, θ, ϕ) . However, for more clarity at this

level and to follow the more conventional practice, we use the coordinates of these systems as suffixes in place of the indices used in the tensor notation.¹⁵

- For the cylindrical system identified by the coordinates (ρ, ϕ, z) with an orthonormal basis vectors \mathbf{e}_ρ , \mathbf{e}_ϕ and \mathbf{e}_z :¹⁶

The ∇ operator is:

$$\nabla = \mathbf{e}_\rho \partial_\rho + \mathbf{e}_\phi \frac{1}{\rho} \partial_\phi + \mathbf{e}_z \partial_z \quad (155)$$

The Laplacian operator is:

$$\nabla^2 = \partial_{\rho\rho} + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial_{\phi\phi} + \partial_{zz} \quad (156)$$

The gradient of a differentiable scalar f is:

$$\nabla f = \mathbf{e}_\rho \frac{\partial f}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \mathbf{e}_z \frac{\partial f}{\partial z} \quad (157)$$

The divergence of a differentiable vector \mathbf{A} is:

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \left[\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial A_\phi}{\partial \phi} + \frac{\partial(\rho A_z)}{\partial z} \right] \quad (158)$$

The curl of a differentiable vector \mathbf{A} is:

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad (159)$$

For plane polar coordinate systems, these operators and operations can be obtained by dropping the z components or terms from the cylindrical form of the above operators and

¹⁵There is another reason that is these are physical components not covariant or contravariant.

¹⁶It should be obvious that since ρ , ϕ and z are specific coordinates and not variable indices, the summation convention does not apply.

operations.

- For the spherical system identified by the coordinates (r, θ, ϕ) with an orthonormal basis vectors \mathbf{e}_r , \mathbf{e}_θ and \mathbf{e}_ϕ :¹⁷

The ∇ operator is:

$$\nabla = \mathbf{e}_r \partial_r + \mathbf{e}_\theta \frac{1}{r} \partial_\theta + \mathbf{e}_\phi \frac{1}{r \sin \theta} \partial_\phi \quad (160)$$

The Laplacian operator is:

$$\nabla^2 = \partial_{rr} + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi\phi} \quad (161)$$

The gradient of a differentiable scalar f is:

$$\nabla f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \quad (162)$$

The divergence of a differentiable vector \mathbf{A} is:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial (r^2 A_r)}{\partial r} + r \frac{\partial (\sin \theta A_\theta)}{\partial \theta} + r \frac{\partial A_\phi}{\partial \phi} \right] \quad (163)$$

The curl of a differentiable vector \mathbf{A} is:

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix} \quad (164)$$

- For a general orthogonal system in a 3D space identified by the coordinates (u_1, u_2, u_3) with unit basis vectors \mathbf{u}_1 , \mathbf{u}_2 and \mathbf{u}_3 and scale factors h_1 , h_2 and h_3 where $h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$ and \mathbf{r} is the position vector:

¹⁷Again, the summation convention does not apply to r , θ and ϕ .

The ∇ operator is:

$$\nabla = \frac{\mathbf{u}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\mathbf{u}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\mathbf{u}_3}{h_3} \frac{\partial}{\partial u_3} \quad (165)$$

The Laplacian operator is:

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right] \quad (166)$$

The gradient of a differentiable scalar f is:

$$\nabla f = \frac{\mathbf{u}_1}{h_1} \frac{\partial f}{\partial u_1} + \frac{\mathbf{u}_2}{h_2} \frac{\partial f}{\partial u_2} + \frac{\mathbf{u}_3}{h_3} \frac{\partial f}{\partial u_3} \quad (167)$$

The divergence of a differentiable vector \mathbf{A} is:

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \quad (168)$$

The curl of a differentiable vector \mathbf{A} is:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{u}_1 & h_2 \mathbf{u}_2 & h_3 \mathbf{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (169)$$

4.4 Common Identities in Vector and Tensor Notation

- Here we present some of the widely used identities of vector calculus in the traditional vector notation and in its equivalent tensor notation. In the following bullet points, f and h are differentiable scalar fields; \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are differentiable vector fields; and $\mathbf{r} = x_i \mathbf{e}_i$ is the position vector.

•

$$\begin{aligned}
 \nabla \cdot \mathbf{r} &= n \\
 \Downarrow \\
 \partial_i x_i &= n
 \end{aligned} \tag{170}$$

where n is the space dimension.

•

$$\begin{aligned}
 \nabla \times \mathbf{r} &= \mathbf{0} \\
 \Downarrow \\
 \epsilon_{ijk} \partial_j x_k &= 0
 \end{aligned} \tag{171}$$

•

$$\begin{aligned}
 \nabla (\mathbf{a} \cdot \mathbf{r}) &= \mathbf{a} \\
 \Downarrow \\
 \partial_i (a_j x_j) &= a_i
 \end{aligned} \tag{172}$$

where \mathbf{a} is a constant vector.

•

$$\begin{aligned}
 \nabla \cdot (\nabla f) &= \nabla^2 f \\
 \Downarrow \\
 \partial_i (\partial_i f) &= \partial_{ii} f
 \end{aligned} \tag{173}$$

•

$$\begin{aligned}
 \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\
 \Downarrow & \\
 \epsilon_{ijk} \partial_i \partial_j A_k &= 0
 \end{aligned} \tag{174}$$

•

$$\begin{aligned}
 \nabla \times (\nabla f) &= \mathbf{0} \\
 \Downarrow & \\
 \epsilon_{ijk} \partial_j \partial_k f &= 0
 \end{aligned} \tag{175}$$

•

$$\begin{aligned}
 \nabla (fh) &= f \nabla h + h \nabla f \\
 \Downarrow & \\
 \partial_i (fh) &= f \partial_i h + h \partial_i f
 \end{aligned} \tag{176}$$

•

$$\begin{aligned}
 \nabla \cdot (f \mathbf{A}) &= f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f \\
 \Downarrow & \\
 \partial_i (f A_i) &= f \partial_i A_i + A_i \partial_i f
 \end{aligned} \tag{177}$$

•

$$\begin{aligned}
\nabla \times (f \mathbf{A}) &= f \nabla \times \mathbf{A} + \nabla f \times \mathbf{A} \\
&\Downarrow \\
\epsilon_{ijk} \partial_j (f A_k) &= f \epsilon_{ijk} \partial_j A_k + \epsilon_{ijk} (\partial_j f) A_k
\end{aligned} \tag{178}$$

•

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \\
&\Downarrow \qquad \qquad \Downarrow \\
\epsilon_{ijk} A_i B_j C_k &= \epsilon_{kij} C_k A_i B_j = \epsilon_{jki} B_j C_k A_i
\end{aligned} \tag{179}$$

•

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \\
&\Downarrow \\
\epsilon_{ijk} A_j \epsilon_{klm} B_l C_m &= B_i (A_m C_m) - C_i (A_l B_l)
\end{aligned} \tag{180}$$

•

$$\begin{aligned}
\mathbf{A} \times (\nabla \times \mathbf{B}) &= (\nabla \mathbf{B}) \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} \\
&\Downarrow \\
\epsilon_{ijk} \epsilon_{klm} A_j \partial_l B_m &= (\partial_i B_m) A_m - A_l (\partial_l B_i)
\end{aligned} \tag{181}$$

•

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\
&\Downarrow \\
\epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m &= \partial_i (\partial_m A_m) - \partial_{ll} A_i
\end{aligned} \tag{182}$$

•

$$\begin{aligned}
\nabla (\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} \\
&\Downarrow \\
\partial_i (A_m B_m) &= \epsilon_{ijk} A_j (\epsilon_{klm} \partial_l B_m) + \epsilon_{ijk} B_j (\epsilon_{klm} \partial_l A_m) + (A_l \partial_l) B_i + (B_l \partial_l) A_i
\end{aligned} \tag{183}$$

•

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\
&\Downarrow \\
\partial_i (\epsilon_{ijk} A_j B_k) &= B_k (\epsilon_{kij} \partial_i A_j) - A_j (\epsilon_{jik} \partial_i B_k)
\end{aligned} \tag{184}$$

•

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B} \\
&\Downarrow \\
\epsilon_{ijk} \epsilon_{klm} \partial_j (A_l B_m) &= (B_m \partial_m) A_i + (\partial_m B_m) A_i - (\partial_j A_j) B_i - (A_j \partial_j) B_i
\end{aligned} \tag{185}$$

•

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix} \\
&\Downarrow \\
\epsilon_{ijk} A_j B_k \epsilon_{ilm} C_l D_m &= (A_l C_l) (B_m D_m) - (A_m D_m) (B_l C_l)
\end{aligned} \tag{186}$$

•

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{C} - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{D} \\
&\Downarrow \\
\epsilon_{ijk} \epsilon_{jmn} A_m B_n \epsilon_{kpq} C_p D_q &= (\epsilon_{qmn} D_q A_m B_n) C_i - (\epsilon_{pmn} C_p A_m B_n) D_i
\end{aligned} \tag{187}$$

• In vector and tensor notations, the condition for a vector field \mathbf{A} to be solenoidal is:

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= 0 \\
&\Downarrow \\
\partial_i A_i &= 0
\end{aligned} \tag{188}$$

• In vector and tensor notations, the condition for a vector field \mathbf{A} to be irrotational is:

$$\begin{aligned}
\nabla \times \mathbf{A} &= \mathbf{0} \\
&\Downarrow \\
\epsilon_{ijk} \partial_j A_k &= 0
\end{aligned} \tag{189}$$

4.5 Integral Theorems in Tensor Notation

- The divergence theorem for a differentiable vector field \mathbf{A} in vector and tensor notation is:

$$\begin{aligned} \iiint_V \nabla \cdot \mathbf{A} d\tau &= \iint_S \mathbf{A} \cdot \mathbf{n} d\sigma \\ &\Downarrow \\ \int_V \partial_i A_i d\tau &= \int_S A_i n_i d\sigma \end{aligned} \quad (190)$$

where V is a bounded region in an n D space enclosed by a generalized surface S , $d\tau$ and $d\sigma$ are generalized volume and surface elements respectively, \mathbf{n} and n_i are unit normal to the surface and its i^{th} component respectively, and the index i ranges over $1, \dots, n$.

- The divergence theorem for a differentiable rank-2 tensor field \mathbf{A} in tensor notation for the first index is given by:

$$\int_V \partial_i A_{il} d\tau = \int_S A_{il} n_i d\sigma \quad (191)$$

- The divergence theorem for differentiable tensor fields of higher ranks \mathbf{A} in tensor notation for the index k is:

$$\int_V \partial_k A_{ij\dots k\dots m} d\tau = \int_S A_{ij\dots k\dots m} n_k d\sigma \quad (192)$$

- Stokes theorem for a differentiable vector field \mathbf{A} in vector and tensor notation is:

$$\begin{aligned} \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} d\sigma &= \int_C \mathbf{A} \cdot d\mathbf{r} \\ &\Downarrow \\ \int_S \epsilon_{ijk} \partial_j A_k n_i d\sigma &= \int_C A_i dx_i \end{aligned} \quad (193)$$

where C stands for the perimeter of the surface S and $d\mathbf{r}$ is the vector element tangent to

the perimeter.

- Stokes theorem for a differentiable rank-2 tensor field \mathbf{A} in tensor notation for the first index is:

$$\int_S \epsilon_{ijk} \partial_j A_{kl} n_i d\sigma = \int_C A_{il} dx_i \quad (194)$$

- Stokes theorem for differentiable tensor fields of higher ranks \mathbf{A} in tensor notation for the index k is:

$$\int_S \epsilon_{ijk} \partial_j A_{lm\dots k\dots n} n_i d\sigma = \int_C A_{lm\dots k\dots n} dx_k \quad (195)$$

4.6 Examples of Using Tensor Techniques to Prove Identities

- $\nabla \cdot \mathbf{r} = n$:

$$\begin{aligned} \nabla \cdot \mathbf{r} &= \partial_i x_i && \text{(Eq. 139)} \\ &= \delta_{ii} && \text{(Eq. 80)} \\ &= n && \text{(Eq. 80)} \end{aligned} \quad (196)$$

- $\nabla \times \mathbf{r} = \mathbf{0}$:

$$\begin{aligned} [\nabla \times \mathbf{r}]_i &= \epsilon_{ijk} \partial_j x_k && \text{(Eq. 146)} \\ &= \epsilon_{ijk} \delta_{kj} && \text{(Eq. 79)} \\ &= \epsilon_{ijj} && \text{(Eq. 76)} \\ &= 0 && \text{(Eq. 71)} \end{aligned} \quad (197)$$

Since i is a free index the identity is proved for all components.

- $\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$:

$$\begin{aligned}
 [\nabla(\mathbf{a} \cdot \mathbf{r})]_i &= \partial_i(a_j x_j) && \text{(Eqs. 134 \& 120)} \\
 &= a_j \partial_i x_j + x_j \partial_i a_j && \text{(product rule)} \\
 &= a_j \partial_i x_j && (a_j \text{ is constant}) \\
 &= a_j \delta_{ji} && \text{(Eq. 79)} \\
 &= a_i && \text{(Eq. 76)} \\
 &= [\mathbf{a}]_i && \text{(definition of index)}
 \end{aligned} \tag{198}$$

Since i is a free index the identity is proved for all components.

- $\nabla \cdot (\nabla f) = \nabla^2 f$:

$$\begin{aligned}
 \nabla \cdot (\nabla f) &= \partial_i [\nabla f]_i && \text{(Eq. 139)} \\
 &= \partial_i (\partial_i f) && \text{(Eq. 134)} \\
 &= \partial_i \partial_i f && \text{(rules of differentiation)} \\
 &= \partial_{ii} f && \text{(definition of 2nd derivative)} \\
 &= \nabla^2 f && \text{(Eq. 151)}
 \end{aligned} \tag{199}$$

- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \partial_i [\nabla \times \mathbf{A}]_i && \text{(Eq. 139)} \\
&= \partial_i (\epsilon_{ijk} \partial_j A_k) && \text{(Eq. 146)} \\
&= \epsilon_{ijk} \partial_i \partial_j A_k && (\partial \text{ not acting on } \epsilon) \\
&= \epsilon_{ijk} \partial_j \partial_i A_k && \text{(continuity condition)} \\
&= -\epsilon_{jik} \partial_j \partial_i A_k && \text{(Eq. 84)} \\
&= -\epsilon_{ijk} \partial_i \partial_j A_k && \text{(relabeling dummy indices } i \text{ and } j) \\
&= 0 && \text{(since } \epsilon_{ijk} \partial_i \partial_j A_k = -\epsilon_{jik} \partial_j \partial_i A_k)
\end{aligned} \tag{200}$$

This can also be concluded from line three by arguing that: since by the continuity condition ∂_i and ∂_j can change their order with no change in the value of the term while a corresponding change of the order of i and j in ϵ_{ijk} results in a sign change, we see that each term in the sum has its own negative and hence the terms add up to zero (see Eq. 94).

- $\nabla \times (\nabla f) = \mathbf{0}$:

$$\begin{aligned}
[\nabla \times (\nabla f)]_i &= \epsilon_{ijk} \partial_j [\nabla f]_k && \text{(Eq. 146)} \\
&= \epsilon_{ijk} \partial_j (\partial_k f) && \text{(Eq. 134)} \\
&= \epsilon_{ijk} \partial_j \partial_k f && \text{(rules of differentiation)} \\
&= \epsilon_{ijk} \partial_k \partial_j f && \text{(continuity condition)} \\
&= -\epsilon_{ikj} \partial_k \partial_j f && \text{(Eq. 84)} \\
&= -\epsilon_{ijk} \partial_j \partial_k f && \text{(relabeling dummy indices } j \text{ and } k) \\
&= 0 && \text{(since } \epsilon_{ijk} \partial_j \partial_k f = -\epsilon_{ikj} \partial_k \partial_j f)
\end{aligned} \tag{201}$$

This can also be concluded from line three by a similar argument to the one given in the

previous point. Because $[\nabla \times (\nabla f)]_i$ is an arbitrary component, then each component is zero.

- $\nabla (fh) = f\nabla h + h\nabla f$:

$$\begin{aligned}
 [\nabla (fh)]_i &= \partial_i (fh) && \text{(Eq. 134)} \\
 &= f\partial_i h + h\partial_i f && \text{(product rule)} \\
 &= [f\nabla h]_i + [h\nabla f]_i && \text{(Eq. 134)} \\
 &= [f\nabla h + h\nabla f]_i && \text{(Eq. 16)}
 \end{aligned} \tag{202}$$

Because i is a free index the identity is proved for all components.

- $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$:

$$\begin{aligned}
 \nabla \cdot (f\mathbf{A}) &= \partial_i [f\mathbf{A}]_i && \text{(Eq. 139)} \\
 &= \partial_i (fA_i) && \text{(definition of index)} \\
 &= f\partial_i A_i + A_i\partial_i f && \text{(product rule)} \\
 &= f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f && \text{(Eqs. 139 \& 153)}
 \end{aligned} \tag{203}$$

- $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$:

$$\begin{aligned}
 [\nabla \times (f\mathbf{A})]_i &= \epsilon_{ijk}\partial_j [f\mathbf{A}]_k && \text{(Eq. 146)} \\
 &= \epsilon_{ijk}\partial_j (fA_k) && \text{(definition of index)} \\
 &= f\epsilon_{ijk}\partial_j A_k + \epsilon_{ijk}(\partial_j f)A_k && \text{(product rule \& commutativity)} \\
 &= f\epsilon_{ijk}\partial_j A_k + \epsilon_{ijk}[\nabla f]_j A_k && \text{(Eq. 134)} \\
 &= [f\nabla \times \mathbf{A}]_i + [\nabla f \times \mathbf{A}]_i && \text{(Eqs. 146 \& 121)} \\
 &= [f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}]_i && \text{(Eq. 16)}
 \end{aligned} \tag{204}$$

Because i is a free index the identity is proved for all components.

• $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$:

$$\begin{aligned}
 \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \epsilon_{ijk} A_i B_j C_k && \text{(Eq. 122)} \\
 &= \epsilon_{kij} A_i B_j C_k && \text{(Eq. 84)} \\
 &= \epsilon_{kij} C_k A_i B_j && \text{(commutativity)} \\
 &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) && \text{(Eq. 122)} \\
 &= \epsilon_{jki} A_i B_j C_k && \text{(Eq. 84)} \\
 &= \epsilon_{jki} B_j C_k A_i && \text{(commutativity)} \\
 &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) && \text{(Eq. 122)}
 \end{aligned} \tag{205}$$

The negative permutations of these identities can be similarly obtained and proved by changing the order of the vectors in the cross products which results in a sign change.

• $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$:

$$\begin{aligned}
[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i &= \epsilon_{ijk} A_j [\mathbf{B} \times \mathbf{C}]_k && \text{(Eq. 121)} \\
&= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m && \text{(Eq. 121)} \\
&= \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m && \text{(commutativity)} \\
&= \epsilon_{ijk} \epsilon_{lmk} A_j B_l C_m && \text{(Eq. 84)} \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j B_l C_m && \text{(Eq. 102)} \\
&= \delta_{il} \delta_{jm} A_j B_l C_m - \delta_{im} \delta_{jl} A_j B_l C_m && \text{(distributivity)} \\
&= (\delta_{il} B_l) (\delta_{jm} A_j C_m) - (\delta_{im} C_m) (\delta_{jl} A_j B_l) && \text{(commutativity and grouping)} \\
&= B_i (A_m C_m) - C_i (A_l B_l) && \text{(Eq. 76)} \\
&= B_i (\mathbf{A} \cdot \mathbf{C}) - C_i (\mathbf{A} \cdot \mathbf{B}) && \text{(Eq. 120)} \\
&= [\mathbf{B}(\mathbf{A} \cdot \mathbf{C})]_i - [\mathbf{C}(\mathbf{A} \cdot \mathbf{B})]_i && \text{(definition of index)} \\
&= [\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})]_i && \text{(Eq. 16)}
\end{aligned} \tag{206}$$

Because i is a free index the identity is proved for all components. Other variants of this identity [e.g. $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$] can be obtained and proved similarly by changing the order of the factors in the external cross product with adding a minus sign.

• $\mathbf{A} \times (\nabla \times \mathbf{B}) = (\nabla \mathbf{B}) \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}$:

$$\begin{aligned}
[\mathbf{A} \times (\nabla \times \mathbf{B})]_i &= \epsilon_{ijk} A_j [\nabla \times \mathbf{B}]_k && \text{(Eq. 121)} \\
&= \epsilon_{ijk} A_j \epsilon_{klm} \partial_l B_m && \text{(Eq. 146)} \\
&= \epsilon_{ijk} \epsilon_{klm} A_j \partial_l B_m && \text{(commutativity)} \\
&= \epsilon_{ijk} \epsilon_{lmk} A_j \partial_l B_m && \text{(Eq. 84)} \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j \partial_l B_m && \text{(Eq. 102)} \\
&= \delta_{il} \delta_{jm} A_j \partial_l B_m - \delta_{im} \delta_{jl} A_j \partial_l B_m && \text{(distributivity)} \\
&= A_m \partial_i B_m - A_l \partial_l B_i && \text{(Eq. 76)} \\
&= (\partial_i B_m) A_m - A_l (\partial_l B_i) && \text{(commutativity \& grouping)} \\
&= [(\nabla \mathbf{B}) \cdot \mathbf{A}]_i - [\mathbf{A} \cdot \nabla \mathbf{B}]_i && \text{(Eq. 135 \& § 2.6.5)} \\
&= [(\nabla \mathbf{B}) \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B}]_i && \text{(Eq. 16)}
\end{aligned} \tag{207}$$

Because i is a free index the identity is proved for all components.

- $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$:

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{A})]_i &= \epsilon_{ijk} \partial_j [\nabla \times \mathbf{A}]_k && \text{(Eq. 146)} \\
&= \epsilon_{ijk} \partial_j (\epsilon_{klm} \partial_l A_m) && \text{(Eq. 146)} \\
&= \epsilon_{ijk} \epsilon_{klm} \partial_j (\partial_l A_m) && (\partial \text{ not acting on } \epsilon) \\
&= \epsilon_{ijk} \epsilon_{lmk} \partial_j \partial_l A_m && \text{(Eq. 84 \& definition of derivative)} \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l A_m && \text{(Eq. 102)} \\
&= \delta_{il} \delta_{jm} \partial_j \partial_l A_m - \delta_{im} \delta_{jl} \partial_j \partial_l A_m && \text{(distributivity)} \\
&= \partial_m \partial_i A_m - \partial_l \partial_l A_i && \text{(Eq. 76)} \\
&= \partial_i (\partial_m A_m) - \partial_{ll} A_i && (\partial \text{ shift, grouping \& Eq. 4)} \\
&= [\nabla (\nabla \cdot \mathbf{A})]_i - [\nabla^2 \mathbf{A}]_i && \text{(Eqs. 139, 134 \& 152)} \\
&= [\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}]_i && \text{(Eqs. 16)}
\end{aligned} \tag{208}$$

Because i is a free index the identity is proved for all components. This identity can also be considered as an instance of the identity before the last one, observing that in the second term on the right hand side the Laplacian should precede the vector, and hence no independent proof is required.

- $\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$:

We start from the right hand side and end with the left hand side

$$\begin{aligned}
& [\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}]_i = \\
& [\mathbf{A} \times (\nabla \times \mathbf{B})]_i + [\mathbf{B} \times (\nabla \times \mathbf{A})]_i + [(\mathbf{A} \cdot \nabla) \mathbf{B}]_i + [(\mathbf{B} \cdot \nabla) \mathbf{A}]_i = \quad (\text{Eq. 16}) \\
& \epsilon_{ijk} A_j [\nabla \times \mathbf{B}]_k + \epsilon_{ijk} B_j [\nabla \times \mathbf{A}]_k + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{Eqs. 121, 139 \& indexing}) \\
& \epsilon_{ijk} A_j (\epsilon_{klm} \partial_l B_m) + \epsilon_{ijk} B_j (\epsilon_{klm} \partial_l A_m) + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{Eq. 146}) \\
& \epsilon_{ijk} \epsilon_{klm} A_j \partial_l B_m + \epsilon_{ijk} \epsilon_{klm} B_j \partial_l A_m + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{commutativity}) \\
& \epsilon_{ijk} \epsilon_{lmk} A_j \partial_l B_m + \epsilon_{ijk} \epsilon_{lmk} B_j \partial_l A_m + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{Eq. 84}) \\
& (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j \partial_l B_m + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) B_j \partial_l A_m + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{Eq. 102}) \quad (209) \\
& (\delta_{il} \delta_{jm} A_j \partial_l B_m - \delta_{im} \delta_{jl} A_j \partial_l B_m) + (\delta_{il} \delta_{jm} B_j \partial_l A_m - \delta_{im} \delta_{jl} B_j \partial_l A_m) + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{distributivity}) \\
& \delta_{il} \delta_{jm} A_j \partial_l B_m - A_l \partial_l B_i + \delta_{il} \delta_{jm} B_j \partial_l A_m - B_l \partial_l A_i + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{Eq. 76}) \\
& \delta_{il} \delta_{jm} A_j \partial_l B_m - (A_l \partial_l) B_i + \delta_{il} \delta_{jm} B_j \partial_l A_m - (B_l \partial_l) A_i + (A_l \partial_l) B_i + (B_l \partial_l) A_i = \quad (\text{grouping}) \\
& \delta_{il} \delta_{jm} A_j \partial_l B_m + \delta_{il} \delta_{jm} B_j \partial_l A_m = \quad (\text{cancellation}) \\
& A_m \partial_i B_m + B_m \partial_i A_m = \quad (\text{Eq. 76}) \\
& \partial_i (A_m B_m) = \quad (\text{product rule}) \\
& = [\nabla (\mathbf{A} \cdot \mathbf{B})]_i \quad (\text{Eqs. 134 \& 139})
\end{aligned}$$

Because i is a free index the identity is proved for all components.

• $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$:

$$\begin{aligned}
 \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \partial_i [\mathbf{A} \times \mathbf{B}]_i && \text{(Eq. 139)} \\
 &= \partial_i (\epsilon_{ijk} A_j B_k) && \text{(Eq. 121)} \\
 &= \epsilon_{ijk} \partial_i (A_j B_k) && (\partial \text{ not acting on } \epsilon) \\
 &= \epsilon_{ijk} (B_k \partial_i A_j + A_j \partial_i B_k) && \text{(product rule)} \\
 &= \epsilon_{ijk} B_k \partial_i A_j + \epsilon_{ijk} A_j \partial_i B_k && \text{(distributivity)} \tag{210} \\
 &= \epsilon_{kij} B_k \partial_i A_j - \epsilon_{jik} A_j \partial_i B_k && \text{(Eq. 84)} \\
 &= B_k (\epsilon_{kij} \partial_i A_j) - A_j (\epsilon_{jik} \partial_i B_k) && \text{(commutativity \& grouping)} \\
 &= B_k [\nabla \times \mathbf{A}]_k - A_j [\nabla \times \mathbf{B}]_j && \text{(Eq. 146)} \\
 &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) && \text{(Eq. 120)}
 \end{aligned}$$

$$\bullet \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B}:$$

$$\begin{aligned} [\nabla \times (\mathbf{A} \times \mathbf{B})]_i &= \epsilon_{ijk} \partial_j [\mathbf{A} \times \mathbf{B}]_k && \text{(Eq. 146)} \\ &= \epsilon_{ijk} \partial_j (\epsilon_{klm} A_l B_m) && \text{(Eq. 121)} \\ &= \epsilon_{ijk} \epsilon_{klm} \partial_j (A_l B_m) && (\partial \text{ not acting on } \epsilon) \\ &= \epsilon_{ijk} \epsilon_{klm} (B_m \partial_j A_l + A_l \partial_j B_m) && \text{(product rule)} \\ &= \epsilon_{ijk} \epsilon_{lmk} (B_m \partial_j A_l + A_l \partial_j B_m) && \text{(Eq. 84)} \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) (B_m \partial_j A_l + A_l \partial_j B_m) && \text{(Eq. 102)} \\ &= \delta_{il} \delta_{jm} B_m \partial_j A_l + \delta_{il} \delta_{jm} A_l \partial_j B_m - \delta_{im} \delta_{jl} B_m \partial_j A_l - \delta_{im} \delta_{jl} A_l \partial_j B_m && \text{(distributivity)} \\ &= B_m \partial_m A_i + A_i \partial_m B_m - B_i \partial_j A_j - A_j \partial_j B_i && \text{(Eq. 76)} \\ &= (B_m \partial_m) A_i + (\partial_m B_m) A_i - (\partial_j A_j) B_i - (A_j \partial_j) B_i && \text{(grouping)} \\ &= [(\mathbf{B} \cdot \nabla) \mathbf{A}]_i + [(\nabla \cdot \mathbf{B}) \mathbf{A}]_i - [(\nabla \cdot \mathbf{A}) \mathbf{B}]_i - [(\mathbf{A} \cdot \nabla) \mathbf{B}]_i && \text{(Eqs. 153 \& 139)} \\ &= [(\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B}]_i && \text{(Eq. 16)} \\ & && (211) \end{aligned}$$

Because i is a free index the identity is proved for all components.

$$\bullet (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix};$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \times \mathbf{B}]_i [\mathbf{C} \times \mathbf{D}]_i \quad (\text{Eq. 120})$$

$$= \epsilon_{ijk} A_j B_k \epsilon_{ilm} C_l D_m \quad (\text{Eq. 121})$$

$$= \epsilon_{ijk} \epsilon_{ilm} A_j B_k C_l D_m \quad (\text{commutativity})$$

$$= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) A_j B_k C_l D_m \quad (\text{Eqs. 84 \& 102})$$

$$= \delta_{jl} \delta_{km} A_j B_k C_l D_m - \delta_{jm} \delta_{kl} A_j B_k C_l D_m \quad (\text{distributivity})$$

$$= (\delta_{jl} A_j C_l) (\delta_{km} B_k D_m) - (\delta_{jm} A_j D_m) (\delta_{kl} B_k C_l) \quad (\text{commutativity \& grouping})$$

$$= (A_l C_l) (B_m D_m) - (A_m D_m) (B_l C_l) \quad (\text{Eq. 76})$$

$$= (\mathbf{A} \cdot \mathbf{C}) (\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D}) (\mathbf{B} \cdot \mathbf{C}) \quad (\text{Eq. 120})$$

$$= \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix} \quad (\text{definition of determinant})$$

(212)

$$\bullet (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{C} - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{D}:$$

$$\begin{aligned} [(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})]_i &= \epsilon_{ijk} [\mathbf{A} \times \mathbf{B}]_j [\mathbf{C} \times \mathbf{D}]_k && \text{(Eq. 121)} \\ &= \epsilon_{ijk} \epsilon_{jmn} A_m B_n \epsilon_{kpq} C_p D_q && \text{(Eq. 121)} \\ &= \epsilon_{ijk} \epsilon_{kpq} \epsilon_{jmn} A_m B_n C_p D_q && \text{(commutativity)} \\ &= \epsilon_{ijk} \epsilon_{pqk} \epsilon_{jmn} A_m B_n C_p D_q && \text{(Eq. 84)} \\ &= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) \epsilon_{jmn} A_m B_n C_p D_q && \text{(Eq. 102)} \\ &= (\delta_{ip} \delta_{jq} \epsilon_{jmn} - \delta_{iq} \delta_{jp} \epsilon_{jmn}) A_m B_n C_p D_q && \text{(distributivity)} \\ &= (\delta_{ip} \epsilon_{qmn} - \delta_{iq} \epsilon_{pmn}) A_m B_n C_p D_q && \text{(Eq. 76)} \\ &= \delta_{ip} \epsilon_{qmn} A_m B_n C_p D_q - \delta_{iq} \epsilon_{pmn} A_m B_n C_p D_q && \text{(distributivity)} \\ &= \epsilon_{qmn} A_m B_n C_i D_q - \epsilon_{pmn} A_m B_n C_p D_i && \text{(Eq. 76)} \\ &= \epsilon_{qmn} D_q A_m B_n C_i - \epsilon_{pmn} C_p A_m B_n D_i && \text{(commutativity)} \\ &= (\epsilon_{qmn} D_q A_m B_n) C_i - (\epsilon_{pmn} C_p A_m B_n) D_i && \text{(grouping)} \\ &= [\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})] C_i - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})] D_i && \text{(Eq. 122)} \\ &= [[\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{C}]_i - [[\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{D}]_i && \text{(definition of index)} \\ &= [[\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{C} - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})] \mathbf{D}]_i && \text{(Eq. 16)} \end{aligned} \tag{213}$$

Because i is a free index the identity is proved for all components.

5 Metric Tensor

- This is a rank-2 tensor which may also be called the fundamental tensor.
- The main purpose of the metric tensor is to generalize the concept of distance to general curvilinear coordinate frames and maintain the invariance of distance in different coordinate systems.
- In orthonormal Cartesian coordinate systems the distance element squared, $(ds)^2$, between two infinitesimally neighboring points in space, one with coordinates x^i and the other with coordinates $x^i + dx^i$, is given by

$$(ds)^2 = dx^i dx^i = \delta_{ij} dx^i dx^j \quad (214)$$

This definition of distance is the key to introducing a rank-2 tensor, g_{ij} , called the metric tensor which, for a general coordinate system, is defined by

$$(ds)^2 = g_{ij} dx^i dx^j \quad (215)$$

The metric tensor has also a contravariant form, i.e. g^{ij} .

- The components of the metric tensor are given by:

$$g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j \quad \& \quad g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j \quad (216)$$

where the indexed \mathbf{E} are the covariant and contravariant basis vectors as defined in § 2.5.1.

- The mixed type metric tensor is given by:

$$g^i_j = \mathbf{E}^i \cdot \mathbf{E}_j = \delta^i_j \quad \& \quad g_i^j = \mathbf{E}_i \cdot \mathbf{E}^j = \delta_i^j \quad (217)$$

and hence it is the same as the unity tensor.

- For a coordinate system in which the metric tensor can be cast in a diagonal form where the diagonal elements are ± 1 the metric is called flat.
- For Cartesian coordinate systems, which are orthonormal flat-space systems, we have

$$g^{ij} = \delta^{ij} = g_{ij} = \delta_{ij} \quad (218)$$

- The metric tensor is symmetric, that is

$$g_{ij} = g_{ji} \quad \& \quad g^{ij} = g^{ji} \quad (219)$$

- The contravariant metric tensor is used for raising indices of covariant tensors and the covariant metric tensor is used for lowering indices of contravariant tensors, e.g.

$$A^i = g^{ij} A_j \quad A_i = g_{ij} A^j \quad (220)$$

where the metric tensor acts, like a Kronecker delta, as an index replacement operator. Hence, any tensor can be cast into a covariant or a contravariant form, as well as a mixed form. However, the order of the indices should be respected in this process, e.g.

$$A^i_j = g_{jk} A^{ik} \neq A_j^i = g_{jk} A^{ki} \quad (221)$$

Some authors insert dots (e.g. $A_j^{\cdot i}$) to remove any ambiguity about the order of the indices.

- The covariant and contravariant metric tensors are inverses of each other, that is

$$[g_{ij}] = [g^{ij}]^{-1} \quad \& \quad [g^{ij}] = [g_{ij}]^{-1} \quad (222)$$

Hence

$$g^{ik} g_{kj} = \delta^i_j \quad \& \quad g_{ik} g^{kj} = \delta_i^j \quad (223)$$

- It is common to reserve the “metric tensor” to the covariant form and call the contravariant form, which is its inverse, the “associate” or “conjugate” or “reciprocal” metric tensor.
- As a tensor, the metric has a significance regardless of any coordinate system although it requires a coordinate system to be represented in a specific form.
- For orthogonal coordinate systems the metric tensor is diagonal, i.e. $g_{ij} = g^{ij} = 0$ for $i \neq j$.
- For flat-space orthonormal Cartesian coordinate systems in a 3D space, the metric tensor is given by:

$$[g_{ij}] = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\delta^{ij}] = [g^{ij}] \quad (224)$$

- For cylindrical coordinate systems with coordinates (ρ, ϕ, z) , the metric tensor is given by:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \& \quad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (225)$$

- For spherical coordinate systems with coordinates (r, θ, ϕ) , the metric tensor is given by:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad \& \quad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix} \quad (226)$$

6 Covariant Differentiation

- The ordinary derivative of a tensor is not a tensor in general. The objective of covariant differentiation is to ensure the invariance of derivative (i.e. being a tensor) in general coordinate systems, and this results in applying more sophisticated rules using Christoffel symbols where different differentiation rules for covariant and contravariant indices apply. The resulting covariant derivative is a tensor which is one rank higher than the differentiated tensor.
- Christoffel symbol of the second kind is defined by:

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \frac{g^{kl}}{2} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \quad (227)$$

where the indexed g is the metric tensor in its contravariant and covariant forms with implied summation over l . It is noteworthy that Christoffel symbols are not tensors.

- The Christoffel symbols of the second kind are symmetric in their two lower indices:

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \left\{ \begin{matrix} k \\ ji \end{matrix} \right\} \quad (228)$$

- For Cartesian coordinate systems, the Christoffel symbols are zero for all the values of indices.
- For cylindrical coordinate systems (ρ, ϕ, z) , the Christoffel symbols are zero for all the values of indices except:

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} &= -\rho \\ \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} &= \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \frac{1}{\rho} \end{aligned} \quad (229)$$

where $(1, 2, 3)$ stand for (ρ, ϕ, z) .

- For spherical coordinate systems (r, θ, ϕ) , the Christoffel symbols are zero for all the values of indices except:

$$\begin{aligned}
 \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} &= -r & (230) \\
 \left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} &= -r \sin^2 \theta \\
 \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} &= \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \frac{1}{r} \\
 \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} &= -\sin \theta \cos \theta \\
 \left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\} &= \left\{ \begin{matrix} 3 \\ 31 \end{matrix} \right\} = \frac{1}{r} \\
 \left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\} &= \left\{ \begin{matrix} 3 \\ 32 \end{matrix} \right\} = \cot \theta
 \end{aligned}$$

where $(1, 2, 3)$ stand for (r, θ, ϕ) .

- For a differentiable scalar f the covariant derivative is the same as the normal partial derivative, that is:

$$f_{;i} = f_{,i} = \partial_i f \quad (231)$$

This is justified by the fact that the covariant derivative is different from the normal partial derivative because the basis vectors in general coordinate systems are dependent on their spatial position, and since a scalar is independent of the basis vectors the covariant and partial derivatives are identical.

- For a differentiable vector \mathbf{A} the covariant derivative is:

$$\begin{aligned}
 A_{j;i} &= \partial_i A_j - \left\{ \begin{matrix} k \\ ji \end{matrix} \right\} A_k && \text{(covariant)} \\
 A^j_{;i} &= \partial_i A^j + \left\{ \begin{matrix} j \\ ki \end{matrix} \right\} A^k && \text{(contravariant)}
 \end{aligned} \quad (232)$$

- For a differentiable rank-2 tensor \mathbf{A} the covariant derivative is:

$$\begin{aligned}
A_{jk;i} &= \partial_i A_{jk} - \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} A_{lk} - \left\{ \begin{matrix} l \\ ki \end{matrix} \right\} A_{jl} && \text{(covariant)} \\
A^{jk}_{;i} &= \partial_i A^{jk} + \left\{ \begin{matrix} j \\ li \end{matrix} \right\} A^{lk} + \left\{ \begin{matrix} k \\ li \end{matrix} \right\} A^{jl} && \text{(contravariant)} \\
A^k_{j;i} &= \partial_i A^k_j + \left\{ \begin{matrix} k \\ li \end{matrix} \right\} A^l_j - \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} A^k_l && \text{(mixed)}
\end{aligned} \tag{233}$$

- For a differentiable rank- n tensor \mathbf{A} the covariant derivative is:

$$\begin{aligned}
A^{ij\dots k}_{lm\dots p;q} &= \partial_q A^{ij\dots k}_{lm\dots p} + \left\{ \begin{matrix} i \\ aq \end{matrix} \right\} A^{aj\dots k}_{lm\dots p} + \left\{ \begin{matrix} j \\ aq \end{matrix} \right\} A^{ia\dots k}_{lm\dots p} + \dots + \left\{ \begin{matrix} k \\ aq \end{matrix} \right\} A^{ij\dots a}_{lm\dots p} \\
&\quad - \left\{ \begin{matrix} a \\ lq \end{matrix} \right\} A^{ij\dots k}_{am\dots p} - \left\{ \begin{matrix} a \\ mq \end{matrix} \right\} A^{ij\dots k}_{la\dots p} - \dots - \left\{ \begin{matrix} a \\ pq \end{matrix} \right\} A^{ij\dots k}_{lm\dots a}
\end{aligned} \tag{234}$$

• From the last three points a pattern for covariant differentiation emerges, that is it starts with a partial derivative term then for each tensor index an extra Christoffel symbol term is added, positive for superscripts and negative for subscripts, where the differentiation index is the second of the lower indices in the Christoffel symbol.

- Since the Christoffel symbols are identically zero in Cartesian coordinate systems, the covariant derivative is the same as the normal partial derivative for all tensor ranks.
- The covariant derivative of the metric tensor is zero in all coordinate systems.
- Several rules of normal differentiation similarly apply to covariant differentiation. For example, covariant differentiation is a linear operation with respect to algebraic sums of tensor terms:

$$\partial_{;i} (a\mathbf{A} \pm b\mathbf{B}) = a\partial_{;i}\mathbf{A} \pm b\partial_{;i}\mathbf{B} \tag{235}$$

where a and b are scalar constants and \mathbf{A} and \mathbf{B} are differentiable tensor fields. The product rule of normal differentiation also applies to covariant differentiation of tensor multiplication:

$$\partial_{;i} (\mathbf{AB}) = (\partial_{;i}\mathbf{A})\mathbf{B} + \mathbf{A}\partial_{;i}\mathbf{B} \tag{236}$$

This rule is also valid for the inner product of tensors because the inner product is an outer product operation followed by a contraction of indices, and covariant differentiation and contraction of indices commute.

- The covariant derivative operator can bypass the raising/lowering index operator:

$$A_i = g_{ij} A^j \quad \implies \quad \partial_{;m} A_i = g_{ij} \partial_{;m} A^j \quad (237)$$

and hence the metric behaves like a constant with respect to the covariant operator.

- A principal difference between normal partial differentiation and covariant differentiation is that for successive differential operations the partial derivative operators do commute with each other (assuming certain continuity conditions) but the covariant operators do not commute, that is

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{but} \quad \partial_{;i} \partial_{;j} \neq \partial_{;j} \partial_{;i} \quad (238)$$

- Higher order covariant derivatives are similarly defined as derivatives of derivatives; however the order of differentiation should be respected (refer to the previous point).

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Note: As well as the references cited above, I benefited during the writing of these notes from many sources such as tutorials, presentations, and articles which I found on the

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