Principles of Differential Geometry

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September 9, 2016

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Preface

The present text is a collection of notes about differential geometry prepared to some extent as part of tutorials about topics and applications related to tensor calculus. They can be regarded as continuation to the previous notes on tensor calculus [9, 10] as they are based on the materials and conventions given in those documents. They can be used as a reference for a first course on the subject or as part of a course on tensor calculus.

We generally follow the same notations and conventions employed in [9, 10], however the following points should be observed:

• Following the convention of several authors, when discussing issues related to 2D and 3D manifolds the Greek indices range over 1,2 while the Latin indices range over 1,2,3. Therefore, the use of Greek and Latin indices indicates the type of the intended manifold unless it is stated otherwise.

• The indexed u with Greek letters are used for the surface curvilinear coordinates while the indexed x with Latin letters are used largely for the space Cartesian coordinates but sometimes they are used for the space curvilinear coordinates. Comments are added when necessary to clarify the situation.

• For curvilinear coordinates of surfaces we use both u, v and u^1, u^2 as each has advantages in certain contexts; in particular the latter is necessary for expressing the equations of differential geometry in tensor forms.

• Unless stated otherwise, "surface" and "space" in the present notes mean 2D and 3D manifolds respectively.

• The indexed $\mathbf{E}$ are largely used for the surface, rather than space, basis vectors where they are labeled with Greek indices, e.g. $\mathbf{E}_\alpha$ and $\mathbf{E}^\beta$. However, in a few cases indexed $\mathbf{E}$ are also used for space basis vectors in which case they are distinguished by using Latin indices, e.g. $\mathbf{E}_i$ and $\mathbf{E}^j$. When the basis vectors are numbered rather than indexed, the distinction should be obvious from the context if it is not stated explicitly.

• The Christoffel symbols may be based on the space metric or the surface metric, hence when a number of Christoffel symbols in a certain context or equation are based on more than one metric, the type of indices (Greek or Latin) can be used as an indicator to the underlying metric where the Greek indices represent surface (e.g. $[\alpha\beta,\gamma]$ and $\Gamma^{\gamma}_{\alpha\beta}$) while the Latin indices represent space (e.g. $[ij,k]$ and Γ^{k}_{ij}). Nevertheless, comments are generally added to remove any ambiguity. In particular, when the Christoffel symbols are numbered (e.g. Γ^{1}_{22}) comments will be added to clarify the situation.

• The present notes are largely based on curves and surfaces embedded in a 3D flat space coordinated by a rectangular Cartesian system (x, y, z).

• Some of the definitions provided in the present text which are related to concepts from other subjects of mathematics such as calculus and topology are elementary because of the limits on the text size, moreover the notes are not prepared for these subjects. The purpose of these definitions is to provide a basic understanding of the related ideas in general. The readers are advised to refer to textbooks on those subjects for more technical and detailed definitions.

• For brevity, convenience and clean notation in certain contexts, we use overdot (e.g. $\dot{\mathbf{r}}$) to indicate derivative with respect to a general parameter t while we use prime (e.g. $\mathbf{r}'$) to indicate derivative with respect to a natural parameter s representing arc length.

• The materials of differential geometry are strongly interlinked and hence any elementary text about the subject, like the present one, will face the problem of arranging the materials in a natural order to ensure gradual development of concepts. In this text we largely followed such a scheme; however this is not always possible and hence in some cases references are provided for materials in later parts of the text for concepts needed in earlier parts.

• To facilitate linking related concepts and theorems, and hence ensuring a better understanding of the provided materials, we use hyperlinks (which are colored blue) extensively in the text. The reader, therefore, is advised to use these links.

• Twisted curves can reside in a 2D manifold (surface) or in a higher dimensionality manifold (usually 3D space). Hence we use "surface curves" and "space curves" to refer to the type of the manifold of residence. However, in most cases a single curve can be viewed as a resident of more than one manifold and hence it is a surface and space curve at the same time. For example, a curve embedded in a surface which in its turn is embedded in a 3D space is a surface curve and a space curve at the same time. Consequently, in the present text these terms should be interpreted flexibly. Many statements formulated in terms of a particular type of manifolds can be correctly and easily extended to another type with minimal adjustments of dimensionality and symbolism. Moreover, "space" in many statements should be understood in its general meaning as a manifold embracing the curve not as opposite to "surface" and hence it can include a 2D space, i.e. surface.

• We deliberately use a variety of notations for the same concepts (e.g. the above-mentioned u^1, u^2 and u, v) for convenience and to familiarize the reader with different notations all of which are in common use in the literature of differential geometry and tensor calculus. Having proficiency in these subjects requires familiarity with these various, and sometimes conflicting, notations.

• Of particular importance is an issue related to the previous point that is the use of different symbols for the coefficients of the first and second fundamental forms E, F, G, e, f, g and the coefficients of the surface covariant metric and curvature tensors $a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}$ despite the equivalence of these coefficients, i.e. $(E, F, G, e, f, g) = (a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22})$ and hence all these different formulations can be replaced by just one. However, we keep both notations as they are both in common use in the literature of differential geometry and tensor calculus; moreover in many situations the use of one of these notations or the other is advantageous depending on the context.

Contents

1 Preliminaries

1.1 Differential Geometry

• Differential geometry is a branch of mathematics that largely employs methods and techniques of other branches of mathematics such as differential and integral calculus, topology and tensor analysis to investigate geometric issues related to abstract objects, mainly space curves and surfaces, and their properties where these investigations are mostly focused on these properties at small scales. The investigations also include characterizing categories of these objects. There is also a close link between differential geometry and the disciplines of differential topology and differential equations.

• Differential geometry may be contrasted with "Algebraic geometry" which is another branch of geometry that uses algebraic tools to investigate geometric issues mainly of global nature.

• The investigation of the properties of curves and surfaces in differential geometry are closely linked. For instance, investigating the characteristics of space curves is largely exploited in the investigation of surfaces since common properties of surfaces are defined and quantified in terms of the properties of curves embedded in the surface. For example, several aspects of the surface curvature at a point are defined and quantified in terms of the parameters of the surface curves passing through that point.

1.2 Categories of Curve and Surface Properties

• The properties of curves and surfaces may be categorized into two main groups: local and global where these properties describe the geometry of the curves and surfaces *in the small* and *in the large* respectively. The local properties correspond to the characteristics of the object in the immediate neighborhood of a point on the object such as the curvature of a curve or surface at that point, while the global properties correspond to the characteristics of the object on a large scale and over extended parts of the object such as the number of stationary points of a curve or a surface or being a one-side surface like Mobius strip which is locally a double-sided surface. As indicated earlier, differential geometry of space curves and surfaces is mainly concerned with the local properties. The investigation of global properties normally involve topological treatments.[1]

• Another classification of the properties of curves and surfaces, based on their relation to the embedding external space in which they reside, may be made where the properties are divided into intrinsic and extrinsic. The first category corresponds to those properties which are independent in their existence and definition from the ambient space which embraces the object such as the distance along a given curve or the Gaussian curvature of a surface at a given point (see § 4.4.1), while the second category is related to those properties which depend in their existence and definition on the external embedding space such as having a normal vector at a point on the curve or surface. The idea of intrinsic and extrinsic properties may be illustrated by an inhabitant of a surface with a 2D perception (hereafter this creature will be called "2D inhabitant") where he can detect and measure intrinsic properties but not extrinsic properties as the former do not require appealing to an external embedding 3D space in which the surface is immersed while the latter do. Hence, in simple terms all the properties that can be detected and measured by a 2D inhabitant are intrinsic to the surface while the other properties are extrinsic. A 1D inhabitant of a curve may also be used, to a lesser extent, analogously to distinguish between intrinsic and extrinsic properties of space curves (refer for example to § 2.1).

• More technically, the intrinsic properties are defined and expressed in terms of the metric tensor (formulated in differential geometry as the first fundamental form; see § 3.3) while the extrinsic properties are expressed in terms of the surface curvature tensor (formulated

[1]There is a branch of differential geometry dedicated to the investigation of global (or *in the large*) properties. The focus of the present text is largely differential geometry *in the small* although a number of global differential geometric issues are investigated casually.

in differential geometry as the second fundamental form; see § 3.4).

• The "intrinsic geometry" of the surface comprises the collection of all the intrinsic properties of the surface.

• When two surfaces can have a coordinates system on each such that the first fundamental forms of the two surfaces are identical at each pair of corresponding points on the two surfaces then the two surfaces have identical intrinsic geometry. Such surfaces are isometric and can be mapped on each other by a transformation that preserves the line lengths, the angles and the surface areas.

1.3 Functions

• The domain of a functional mapping: $F : \mathbb{R}^m \to \mathbb{R}^n$ is the largest set of $\mathbb{R}^m$ on which the mapping is defined.

• A bicontinuous function or mapping is a continuous function with a continuous inverse.

• A scalar function is of class C^n if the function and all of its first n (but not $n+1$) partial derivatives do exist and are continuous. A vector function (e.g. a position vector representing a space curve or a surface) is of class C^n if one of its components is of this class while all the other components are of this class or higher. A curve or a surface is of class C^n if it is mathematically represented by a function of this class.

• In gross terms, a smooth curve or surface means that the functional relation that represents the object is sufficiently differentiable for the intended objective, being of class C^n at least where n is the minimum requirement for the differentiability index to satisfy the required conditions.

• A deleted neighborhood of a point P on a 1D interval on the real line is defined as the set of all points $x \in \mathbb{R}$ in the interval such that $0 < |x - x_P| < \epsilon$ where x_P is the coordinate of P on the real line and ϵ is a positive real number. Hence, the deleted neighborhood includes all the points in the open interval $(x_P - \epsilon, x_P + \epsilon)$ excluding x_P itself. For a space curve (which is not straight in general) represented by $\mathbf{r} = \mathbf{r}(t)$, where $\mathbf{r}$ is the spatial representation of the curve and t is a general parameter in the curve representation, the definition applies to the neighborhood of t_P where t_P is the value of t corresponding to the point P on the curve.

• A deleted neighborhood of a point P on a 2D flat surface is defined as the set of all points $(x, y) \in \mathbb{R}^2$ on the surface such that $0 < \sqrt{(x - x_P)^2 + (y - y_P)^2} < \epsilon$ where (x_P, y_P) are the coordinates of P on the plane and ϵ is a positive real number. Hence, the deleted neighborhood includes all the points inside a circle of radius ϵ and center (x_P, y_P) excluding the center itself. For a space surface (which is not flat in general) represented by $\mathbf{r} = \mathbf{r}(u, v)$, where $\mathbf{r}$ is the spatial representation of the surface and u, v are the surface coordinates, the definition applies to the neighborhood of (u_P, v_P) where (u_P, v_P) are the coordinates on the 2D uv plane corresponding to the point P on the surface.

• A quadratic expression $Q(x, y) = a_1x^2 + 2a_2xy + a_3y^2$ of real coefficients a_1, a_2, a_3 and real variables x, y is described as "positive definite" if it possesses positive values (> 0) for all pairs $(x, y) \neq (0, 0)$. The sufficient and necessary condition for Q to be positive definite is that $a_1 > 0$ and $(a_1a_3 - a_2a_2) > 0$.[2]

1.4 Coordinate Transformations

• An orthogonal coordinate transformation is a combination of translation, rotation and reflection of axes. The Jacobian of orthogonal transformations is unity, that is $J = \pm 1$. The orthogonal transformation is described as positive *iff* $J = +1$ and negative *iff* $J = -1$. Positive orthogonal transformations consist solely of translation and rotation (possibly trivial ones as in the case of an identity transformation) while negative orthogonal transformations include reflection, by applying an odd number of axes reversal, as well. Positive transformations can be decomposed into an infinite number of continuously varying

[2]The conditions $a_1 > 0$ and $(a_1a_3 - a_2a_2) > 0$ necessitate $a_3 > 0$ since these coefficients are real.

infinitesimal positive transformations each one of which assimilates an identity transformation. Such a decomposition is not possible in the case of negative orthogonal transformations because the shift from the identity transformation to reflection is impossible by a continuous process.

• Coordinate curves, which are also called parametric curves or parametric lines, on a surface are curves along which only one coordinate variable (u or v) varies while the other coordinate variable (v or u) remains constant.

• An invariant property of a curve or a surface is a property which is independent of allowable coordinate transformations and parameterizations.

• A regular representation of class C^m ($m > 0$) of a surface patch S in a 3D Euclidean space is a functional mapping of an open set Ω in the uv plane onto S that satisfies the following conditions:

(A) The functional mapping relation is of class C^m over the entire Ω.[3]

(B) The Jacobian matrix[4] for the transformation between the representations of the surface in 3D and 2D spaces is of rank 2 for all the points in Ω.

• Having a Jacobian matrix of rank 2 for the transformation is equivalent to the condition that $\mathbf{E}_1 \times \mathbf{E}_2 \neq \mathbf{0}$ where $\mathbf{E}_1 = \partial_u \mathbf{r}$ and $\mathbf{E}_2 = \partial_v \mathbf{r}$ are the surface basis vectors, which are the tangents to the u and v coordinate curves respectively, and $\mathbf{r} = \mathbf{r}(u, v)$ is the 3D spatial representation of the curves.[5]

• Having a Jacobian matrix of rank 2 is also equivalent to having a well-defined tangent

[3] It is noteworthy that the condition of being of class C^m in this context means that m is the minimum requirement for differentiability and hence the condition is satisfied by any function of class C^n where $n \geq m$.

[4] For a functional mapping of the form $\mathbf{S}(u, v) = (S_1(u, v), S_2(u, v), S_3(u, v))$, this Jacobian matrix is given by:

$$\begin{bmatrix} \partial_u S_1 & \partial_v S_1 \\ \partial_u S_2 & \partial_v S_2 \\ \partial_u S_3 & \partial_v S_3 \end{bmatrix} \tag{1}$$

[5] "Rank" here, and in similar contexts, refers to its meaning in linear algebra and should not be confused with the rank of tensor.

plane to the surface at the related point.

• A point on a surface which is not regular is called singular. Singularity occurs either because of a geometric reason, which is the case for instance for the apex of a cone, or because of the particular parametric representation of the surface. While the first type of singularity is inherent and hence it cannot be removed, the second type can be removed by changing the representation.

• Corresponding points on two curves refer to two points, one on each curve, with a common value of a common parameter of the two curves. When the two curves have two different parameterizations then a one-to-one correspondence between the two parameters should be established and the corresponding points then refer to two points with corresponding values of the two parameters. Corresponding points on two surfaces are defined in a similar manner taking into account that surfaces are parameterized by multiple values depending on the dimensionality of the reference space (e.g. two when using surface curvilinear coordinates u^1, u^2 and three when using 3D space coordinates x^1, x^2, x^3).

• In many cases of theoretical and practical situations, a mixed tensor A^i_α, which is contravariant with respect to transformations in space coordinates x^i and covariant with respect to transformations in surface coordinates u^α, is defined. Following a coordinate transformation in which both the space and surface coordinates change, the tensor A^i_α will be given in the new (barred) system by:

$$\bar{A}^i_\alpha = A^j_\beta \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \tag{2}$$

More generally, tensors with space and surface contravariant indices and space and surface covariant indices (e.g. $A^{i\alpha}_{j\beta}$) can also be defined similarly. The extension of the above transformation rule to include such tensors can be conducted trivially by following the obvious pattern seen in the last equation.

1.5 Intrinsic Distance

• The intrinsic distance between two points on a surface is the greatest lower bound (or infimum) of the lengths of all regular arcs connecting the two points on the surface.

• The intrinsic distance is an intrinsic property of the surface.

• The intrinsic distance d between two points is invariant under a local isometric mapping, that is $d(f(P_1), f(P_2)) = d(P_1, P_2)$ where f is an isometric mapping from a surface S_1 to a surface S_2 (see § 1.13), P_1 and P_2 are the two points on S_1 and $f(P_1)$ and $f(P_2)$ are their images on S_2.[6]

• The following conditions apply to the intrinsic distance d between points P_1, P_2 and P_3:

(A) Symmetry: $d(P_1, P_2) = d(P_2, P_1)$.

(B) Triangle inequality: $d(P_1, P_3) \leq d(P_1, P_2) + d(P_2, P_3)$.

(C) Positive definiteness: $d(P_1, P_2) > 0$ with $d(P_1, P_2) = 0$ *iff* P_1 and P_2 are the same point.

• An arc C connecting two points, P_1 and P_2, on a surface is described as an arc of minimum length[7] between P_1 and P_2 if the length of C is equal to the intrinsic distance between P_1 and P_2.

• The existence and uniqueness of an arc of minimum length between two specific points on a surface is not guaranteed, i.e. it may not exit and if it does it may not be unique (refer to § 5.7). Yes, for certain types of surface such an arc does exist and it is unique. For example, on a plane in a Euclidean space there exists an arc of minimum length between any two points on the plane and it is unique; this arc is the straight line segment connecting the two points.

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[6] In fact this may be taken as the definition of isometric mapping, i.e. it is the mapping that preserves intrinsic distance.

[7] Such an arc may not be unique (see next point).

1.6 Basis Vectors

• The set of basis vectors in a given manifold plays a pivotal role in the theoretical construction of the geometry of the manifold, and this applies to the basis vectors in differential geometry where these vectors are used in the definition and construction of essential concepts and objects such as the surface metric tensor. They are also employed to serve as moving coordinate frames for their underlying constructions (see § 2 and § 3).

• The differential geometry of curves and surfaces employs two main sets of basis vectors:[8]

(A) One set is constructed on space curves and consists of the three unit vectors: the tangent $\mathbf{T}$, the normal $\mathbf{N}$ and the binormal $\mathbf{B}$ to the curve.

(B) Another set is constructed on surfaces and consists of two linearly independent vectors tangent to the coordinate curves of the surface, $\mathbf{E}_1 = \frac{\partial \mathbf{r}}{\partial u^1}$ and $\mathbf{E}_2 = \frac{\partial \mathbf{r}}{\partial u^2}$, plus the normal to the surface $\mathbf{n}$, where $\mathbf{r}(u^1, u^2)$ is the spatial representation of a surface coordinate curve, and u^1, u^2 are the surface curvilinear coordinates[9] as will be explained in detail later on (refer to § **3**).

• Each one of the above basis sets is defined on each point of the curve or the surface and hence in general the vectors in each one of these basis sets vary from one point to another, i.e. they are position dependent.[10]

• In tensor notation, the surface basis vectors, $\mathbf{E}_1$ and $\mathbf{E}_2$, are given by $\frac{\partial x^i}{\partial u^\alpha}$ ($i = 1, 2, 3$ and $\alpha = 1, 2$) which is usually abbreviated as x^i_α. These vectors can be seen as contravariant space vectors or covariant surface vectors (see § 3.1 for further details).

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[8]Other sets of basis vectors are also defined and employed in differential geometry, see e.g. § 4.1 and § 4.4.

[9]The surface curvilinear coordinates are also called the Gaussian coordinates.

[10]The non-unit vectors (i.e. $\mathbf{E}_1$ and $\mathbf{E}_2$) vary in magnitude and direction while the unit vectors (which are the rest) vary in direction.

1.7 Flat and Curved Spaces

• A manifold, such as a 2D surface or a 3D space, is called "flat" if it is possible to find a coordinate system for the manifold with a diagonal metric tensor whose all diagonal elements are ± 1; the space is called "curved" otherwise. More formally, an nD space is described as flat *iff* it is possible to find a coordinate system for which the line element ds is given by:

$$(ds)^2 = \epsilon_1 (dx^1)^2 + \epsilon_2 (dx^2)^2 + \ldots \epsilon_n (dx^n)^2 = \sum_{i=1}^{n} \epsilon_i (dx^i)^2 \tag{3}$$

where the indexed ϵ are ± 1. Examples of flat space are the 3D Euclidean space which can be coordinated by a rectangular Cartesian system whose metric tensor is diagonal with all the diagonal elements being $+1$, and the 4D Minkowski space-time manifold whose metric is diagonal with elements of ± 1. An example of curved space is the 2D surface of a sphere or an ellipsoid.

• For the space to be flat, the condition given by Eq. 3 should apply all over the space and not just at certain points or regions.

• As discussed in [10], a necessary and sufficient condition for an nD space to be globally flat is that the Riemann-Christoffel curvature tensor of the space vanishes identically.

• Due to the connection between the Gaussian curvature and the Riemann-Christoffel curvature tensor which implies vanishing each one of these if the other does (see Eq. 17), we see that having an identically vanishing Gaussian curvature (see § 4.4.1) is another sufficient and necessary condition for a 2D space to be flat.[11]

• Curved spaces may have constant non-vanishing curvature all over the space, or have variable curvature and hence the curvature is position dependent. As an example of a

[11] The Gaussian curvature in differential geometry is defined for 2D spaces although the concept may be extended to higher dimensionality manifolds.

space of constant curvature is the surface of a sphere of radius R whose curvature[12] is $\frac{1}{R^2}$ at each point of the surface. Ellipsoid, paraboloid and torus are simple examples of surfaces with variable curvature.[13]

• A surface with positive/negative Gaussian curvature (see § 4.4.1) at each point is described as a surface of positive/negative curvature.[14] Ellipsoids, elliptic paraboloids and hyperboloids of two sheets are examples of surfaces of positive curvature while hyperbolic paraboloids and hyperboloids of one sheet are examples of surfaces of negative curvature (see § 6.2).

• Schur theorem related to nD spaces ($n > 2$) of constant curvature states that: if the Riemann-Christoffel curvature tensor at each point of a space is a function of the coordinates only, then the curvature is constant all over the space.[15]

• All 1D spaces are Euclidean and hence they cannot be curved. So twisted curves are curved only when viewed externally from the embedding space which they reside in (e.g. the 2D space of a surface curve or the 3D space of a space curve).

• The geometry of curved spaces is usually described as the Riemannian geometry. One approach for investigating the Riemannian geometry of a curved manifold is to embed the manifold in a Euclidean space of higher dimensionality. This approach is largely followed in the present notes where the geometry of curved 2D spaces (twisted surfaces) is investigated by immersing the surfaces in a 3D Euclidean space and examining their properties as viewed from this external enveloping 3D space. Such an external view is necessary for examining the extrinsic geometry of the surface but not its intrinsic geometry.

• The geometric description and quantification of flat spaces are simpler than those of

[12]This is the Gaussian curvature (refer to § 4.4.1).

[13]There are various characterizations and quantifications for the curvature and hence in the present context "curvature" may be a generic term. For 2D surfaces, curvature usually refers to the Gaussian curvature (see § 4.4.1) which is strongly linked to the Riemann curvature.

[14]Surfaces with constant non-zero Gaussian curvature K may be described as spherical if $K > 0$ and pseudo-spherical if $K < 0$.

[15]Schur theorem may also be stated as: the Riemannian curvature is constant over an isotropic region of an nD ($n > 2$) Riemannian space.

curved spaces, and hence in general the differential geometry of flat spaces is wealthier, more motivating and less challenging than that of curved spaces.[16]

1.8 Homogeneous Coordinate Systems

• When all the diagonal elements of a diagonal metric tensor of a flat space are +1, the coordinate system is described as homogeneous. In this case the line element of Eq. 3 becomes:

$$(ds)^2 = dx^i dx^i \tag{4}$$

An example of homogeneous coordinate systems is the rectangular Cartesian system (x, y, z) of a 3D Euclidean space.

• A homogeneous coordinate system can be transformed to another homogeneous coordinate system only by linear transformations.

• Any coordinate system obtained from a homogeneous coordinate system by an orthogonal transformation is homogeneous.

• As a consequence of the last points, infinitely many homogeneous coordinate systems can be constructed in any flat space.

• A coordinate system of a flat space can always be homogenized by allowing the coordinates to be imaginary. This is done by redefining the coordinates as:

$$\underline{x}^i = \sqrt{\epsilon_i} x^i \tag{5}$$

where the new coordinates $\underline{x}^i$ are imaginary when $\epsilon_i = -1$. Consequently, the line element will be given by:

$$(ds)^2 = d\underline{x}^i d\underline{x}^i \tag{6}$$

[16] As there is a subjective element in this type of statements, it may not apply to everyone.

which is of the same form as Eq. 4. An example of a homogeneous coordinate system with some real and some imaginary coordinates is the coordinate system of a Minkowski 4D space-time of special relativity.

1.9 Geodesic Coordinates

• It is always possible to introduce coordinates at particular points in a multi-dimensional manifold so that the Christoffel symbols vanish at these points. These coordinates are called geodesic coordinates.[17]

• Geodesic coordinates are employed as local coordinate systems for certain advantages. In geodesic coordinates the Christoffel symbols are made to vanish at certain allocated points described as the poles. These systems are called geodesic for these particular points and also described as locally Cartesian coordinates.

• The main reason for the use of geodesic coordinates is that the covariant and absolute derivatives in such systems become respectively partial and total derivatives at the poles since the Christoffel symbol terms in the covariant and absolute derivative expressions vanish at these points. Any tensor property can then be easily proved in the geodesic system at the pole and consequently generalized to other systems due to the invariance of the zero tensor under permissible coordinate transformations. If the allocated pole is a general point in the space, the property is then established over the whole space.

• In any Riemannian space it is possible to find a coordinate system for which the coordinates are geodesic at every point of a given analytic curve.

• There is an infinite number of ways by which geodesic coordinates can be defined over a coordinate patch.

[17]Some authors define geodesic coordinates on a coordinate patch of a surface as a coordinate system whose u and v coordinate curve families are orthogonal with one of these families (u or v) being a family of geodesic curves (refer to § 5.7). So, "geodesic coordinates" seems to have multiple usage.

1.10 Christoffel Symbols for Curves and Surfaces

• For 1D spaces, the Christoffel symbols are not defined.

• The Christoffel symbols of the first kind for a 2D surface are given by:

$$\begin{aligned}
[11,1] &= \frac{\partial_u a_{11}}{2} = \frac{E_u}{2} \\
[11,2] &= \partial_u a_{12} - \frac{\partial_v a_{11}}{2} = F_u - \frac{E_v}{2} \\
[12,1] &= \frac{\partial_v a_{11}}{2} = \frac{E_v}{2} = [21,1] \\
[12,2] &= \frac{\partial_u a_{22}}{2} = \frac{G_u}{2} = [21,2] \\
[22,1] &= \partial_v a_{12} - \frac{\partial_u a_{22}}{2} = F_v - \frac{G_u}{2} \\
[22,2] &= \frac{\partial_v a_{22}}{2} = \frac{G_v}{2}
\end{aligned} \tag{7}$$

where the indexed a are the elements of the surface covariant metric tensor (refer to 3.1) and E, F, G are the coefficients of the first fundamental form (refer to 3.3).[18] The subscripts u and v which suffix the coefficients stand for partial derivatives with respect to these variables (i.e. $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$). In orthogonal coordinate systems $F = a_{12} = a_{21} = 0$ and hence these formulae will be simplified accordingly by dropping any term involving these coefficients.

• The Christoffel symbols of the first kind are linked to the surface covariant basis vectors by the following relation:

$$[\alpha\beta, \gamma] = \frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot \mathbf{E}_\gamma \qquad (\alpha, \beta, \gamma = 1, 2) \tag{8}$$

which may be written as:

$$[\alpha\beta, \gamma] = \mathbf{r}_{\alpha\beta} \cdot \mathbf{r}_\gamma \qquad (\alpha, \beta, \gamma = 1, 2) \tag{9}$$

[18] As we will see in the forthcoming sections, $E = a_{11}$, $F = a_{12} = a_{21}$ and $G = a_{22}$.

where the subscripts represent partial derivatives with respect to these coordinate indices. The last equation provides an easier form to remember these formulae.

• Applying the index raising operator to Eq. 8, we obtain a similar expression for the Christoffel symbols of the second kind, that is:

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{\partial \mathbf{E}_{\alpha}}{\partial u^{\beta}} \cdot \mathbf{E}^{\gamma} \qquad (\alpha, \beta, \gamma = 1, 2) \tag{10}$$

where $\mathbf{E}^{\gamma}$ is the contravariant form of the surface basis vectors.

• The Christoffel symbols of the second kind[19] for a 2D surface are given by:

$$\begin{aligned}
\Gamma^{1}_{11} &= \frac{a_{22}\partial_u a_{11} - 2a_{12}\partial_u a_{12} + a_{12}\partial_v a_{11}}{2a} = \frac{GE_u - 2FF_u + FE_v}{2a} \\
\Gamma^{2}_{11} &= \frac{2a_{11}\partial_u a_{12} - a_{11}\partial_v a_{11} - a_{12}\partial_u a_{11}}{2a} = \frac{2EF_u - EE_v - FE_u}{2a} \\
\Gamma^{1}_{12} &= \frac{a_{22}\partial_v a_{11} - a_{12}\partial_u a_{22}}{2a} = \frac{GE_v - FG_u}{2a} = \Gamma^{1}_{21} \\
\Gamma^{2}_{12} &= \frac{a_{11}\partial_u a_{22} - a_{12}\partial_v a_{11}}{2a} = \frac{EG_u - FE_v}{2a} = \Gamma^{2}_{21} \\
\Gamma^{1}_{22} &= \frac{2a_{22}\partial_v a_{12} - a_{22}\partial_u a_{22} - a_{12}\partial_v a_{22}}{2a} = \frac{2GF_v - GG_u - FG_v}{2a} \\
\Gamma^{2}_{22} &= \frac{a_{11}\partial_v a_{22} - 2a_{12}\partial_v a_{12} + a_{12}\partial_u a_{22}}{2a} = \frac{EG_v - 2FF_v + FG_u}{2a}
\end{aligned} \tag{11}$$

where a is the determinant of the surface covariant metric tensor, and the other symbols are as explained in the previous points. The formulae will also be simplified in orthogonal coordinate systems where $F = a_{12} = a_{21} = 0$ by dropping the vanishing terms.

[19]They are also called affine connections.

• The Christoffel symbols of the second kind for a 2D surface may also be given by:

$$\begin{aligned}
\Gamma^1_{11} &= -\frac{(\mathbf{E}_2 \times \partial_1 \mathbf{E}_1) \cdot \mathbf{n}}{\sqrt{a}} \\
\Gamma^2_{11} &= +\frac{(\mathbf{E}_1 \times \partial_1 \mathbf{E}_1) \cdot \mathbf{n}}{\sqrt{a}} \\
\Gamma^1_{12} &= -\frac{(\mathbf{E}_2 \times \partial_2 \mathbf{E}_1) \cdot \mathbf{n}}{\sqrt{a}} = \Gamma^1_{21} \\
\Gamma^2_{12} &= +\frac{(\mathbf{E}_1 \times \partial_2 \mathbf{E}_1) \cdot \mathbf{n}}{\sqrt{a}} = \Gamma^2_{21} \\
\Gamma^1_{22} &= -\frac{(\mathbf{E}_2 \times \partial_2 \mathbf{E}_2) \cdot \mathbf{n}}{\sqrt{a}} \\
\Gamma^2_{22} &= +\frac{(\mathbf{E}_1 \times \partial_2 \mathbf{E}_2) \cdot \mathbf{n}}{\sqrt{a}}
\end{aligned} \tag{12}$$

where the indexed $\mathbf{E}$ are the surface covariant basis vectors, $\mathbf{n}$ is the unit vector normal to the surface and a is the determinant of the surface covariant metric tensor.

• Since the Christoffel symbols of both kinds are dependent on the metric only, as can be seen from the previous points, they represent intrinsic properties of the surface geometry and hence they are part of its intrinsic geometry.

1.11 Riemann-Christoffel and Ricci Curvature Tensors

• The Riemann-Christoffel curvature tensor is an absolute rank-4 tensor that characterizes important properties of spaces, including 2D surfaces, and hence it plays an important role in differential geometry. The tensor is used, for instance, to test for the space flatness.

• There are two kinds of Riemann-Christoffel curvature tensor: first and second. The Riemann-Christoffel curvature tensor of the first kind is a type $(0,4)$ tensor while the Riemann-Christoffel curvature tensor of the second kind is a type $(1,3)$ tensor. Shifting from one kind to the other is achieved by using the index-shifting operator.

• The first and second kinds of the Riemann-Christoffel curvature tensor are given respectively by:[20]

$$\begin{aligned} R_{ijkl} &= \partial_k [jl, i] - \partial_l [jk, i] + [il, r] \Gamma^r_{jk} - [ik, r] \Gamma^r_{jl} \\ R^i_{jkl} &= \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^r_{jl} \Gamma^i_{rk} - \Gamma^r_{jk} \Gamma^i_{rl} \end{aligned} \tag{13}$$

• The Riemann-Christoffel curvature tensor vanishes identically *iff* the space is globally flat; otherwise the space is curved.

• A surface is isometric to the Euclidean plane *iff* the Riemann-Christoffel curvature tensor is zero at each point on the surface.[21]

• From Eq. 13, it can be seen that the Riemann-Christoffel curvature tensor depends exclusively on the Christoffel symbols of the first and second kind which are both dependent on the metric (or the first fundamental form, see § 3.3) only. Hence, the Riemann-Christoffel curvature, as represented by the Riemann-Christoffel curvature tensor, is an intrinsic property of the manifold.

• Since the Riemann-Christoffel curvature tensor depends on the metric which, in general curvilinear coordinates, is a function of position, the Riemann-Christoffel curvature tensor follows this dependency on position.

• The Ricci curvature tensor of the first kind, which is a rank-2 symmetric tensor, is obtained by contracting the contravariant index with the last covariant index of the Riemann-Christoffel curvature tensor of the second kind, that is:

$$R_{ij} = R^a_{ija} = \partial_j \Gamma^a_{ia} - \partial_a \Gamma^a_{ij} + \Gamma^a_{bj} \Gamma^b_{ia} - \Gamma^a_{ba} \Gamma^b_{ij} \tag{14}$$

The Ricci tensor of the second kind is obtained by raising the first index of the Ricci tensor of the first kind using the index-raising operator.

[20] In these and the following equations in this subsection, the Latin indices do not necessarily range over $1, 2, 3$ as these equations are valid for general nD manifolds ($n \geq 2$) including surfaces and spaces with $n > 3$.

[21] The same statement also applies if the Gaussian curvature of the surface vanishes identically due to the link between the Riemann and Gaussian curvatures (refer to § 4.4.1 and see Eq. 17)

• The Ricci scalar, which is also called the curvature scalar and the curvature invariant, is the result of contracting the indices of the Ricci curvature tensor of the second kind, that is:

$$R = R^i_i \tag{15}$$

• The Riemann-Christoffel curvature tensor vanishes identically for 1D manifolds as represented by space and surface curves.

• As discussed in [10], the 2D Riemann-Christoffel curvature tensor has only one degree of freedom and hence it possesses a single independent non-vanishing component which is represented by R_{1212}. Hence for a 2D Riemannian space we have:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} \tag{16}$$

while all the other components of the tensor are identically zero. This can be expressed in a single equation as:

$$R_{\alpha\beta\gamma\delta} = R_{1212}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \frac{R_{1212}}{a}\underline{\epsilon}_{\alpha\beta}\underline{\epsilon}_{\gamma\delta} = K\underline{\epsilon}_{\alpha\beta}\underline{\epsilon}_{\gamma\delta} \tag{17}$$

where K is the Gaussian curvature (see § 4.4.1).

• The non-vanishing component of the 2D Riemann-Christoffel curvature tensor, R_{1212}, is given by:[22]

$$R_{1212} = \frac{1}{2}\left(2\partial_{12}a_{12} - \partial_{22}a_{11} - \partial_{11}a_{22}\right) + a_{\alpha\beta}\left(\Gamma^{\alpha}_{12}\Gamma^{\beta}_{12} - \Gamma^{\alpha}_{11}\Gamma^{\beta}_{22}\right) \tag{18}$$

where the indexed a are the coefficients of the surface covariant metric tensor, the Christoffel symbols are based on the surface metric, and $\alpha, \beta = 1, 2$.

• For 2D spaces, the Riemann-Christoffel curvature tensor is related to the Ricci tensor

[22] Here, $\partial_{\alpha\beta} \equiv \frac{\partial^2}{\partial u^\alpha \partial u^\beta}$.

by the following relations:

$$\frac{R_{1212}}{a} = -\frac{R_{11}}{a_{11}} = -\frac{R_{12}}{a_{12}} = -\frac{R_{21}}{a_{21}} = -\frac{R_{22}}{a_{22}} \tag{19}$$

where a is the determinant of the 2D covariant metric tensor (see § 3.1) and the indexed a are its elements. Since $K = \frac{R_{1212}}{a}$, the above relations also link the Gaussian curvature to the Ricci tensor.

• More details about the Riemann-Christoffel curvature tensor, Ricci curvature tensor and Ricci scalar can be found in [10].

1.12 Curves

• In simple terms, a space curve is a set of connected points[23] in the space such that any totally-connected subset of it can be twisted into a straight line segment without affecting the neighborhood of any point. More technically, a curve is defined as a differentiable parameterized mapping between an interval of the real line and a connected subset of the space, that is $C(t) : I \to \mathbb{R}^3$ where C is a space curve defined on the interval $I \subseteq \mathbb{R}$ and parameterized by the variable $t \in I$. Hence different parameterizations of the same "geometric curve" will lead to different "mapping curves". The image of the mapping in $\mathbb{R}^3$ is known as the trace of the curve; hence different mapping curves can have identical traces. The curve may also be defined as a topological image of a real interval and may be linked to the concept of Jordan arc.[24]

• Space curves can be defined symbolically in different ways; the most common of these is parametrically where the three space coordinates are given as functions of a real valued parameter, e.g. $x^i = x^i(t)$ where $t \in \mathbb{R}$ is the curve parameter and $i = 1, 2, 3$. The

[23] The points are usually assumed to be totally connected so that any point on the curve can be reached from any other point by passing through other curve points or at least they are piecewise connected. We also consider mostly open curves with simple connectivity and hence the curve does not intersect itself.

[24] Jordan arcs or Jordan curves are injective mappings with no self intersection.

parameter t may represent time or arc length or even an arbitrarily defined quantity. Similarly, surface curves are defined parametrically where the two surface coordinates are given as functions of a real valued parameter, e.g. $u^\alpha = u^\alpha(t)$ with $\alpha = 1, 2$.[25]

• Parameterized curves are oriented objects as they can be traversed in one direction or the other depending on the sense of increase of their parameter.

• The condition for a space curve $C(t) : I \to \mathbb{R}^3$, where $t \in I$ is the curve parameter and $I \subseteq \mathbb{R}$ is an interval over which the curve is defined, to be parameterized by arc length is that: for all t we have $|\frac{d\mathbf{r}}{dt}| = 1$ where $\mathbf{r}(t)$ is the position vector representing the curve in the 3D ambient space.

• As a consequence of the last point, parameterization by arc length is equivalent to traversing the curve with unity speed.

• The parameter symbol which is used normally for parameterization by arc length is s, while t is used to represent a general parameter which could be arc length or something else. This notation is followed in the present text.

• For curves parameterized by arc length, the length of a segment between two points on the curve corresponding to s_1 and s_2 is given by the simple formula: $L = \left|\int_{s_1}^{s_2} dt\right| = |s_2 - s_1|$.

• Parameterization by arc length s may be called natural representation or natural parameterization of the curve and hence s is called natural parameter.

• Natural parameterization is not unique; however any other natural parameterization $\check{s}$ is related to a given natural parameterization s by the relation $\check{s} = \pm s + c$ where c is a real constant and hence the above-stated condition $|\frac{d\mathbf{r}}{dt}| = 1$ remains valid.[26] This may be stated in a different way by saying that natural parameterization with arc length s is unique apart from the possibility of having a different sense of orientation and an additive

[25] In the present notes, many statements like this one about space curves apply as well to surface curves and vice versa. To be concise and avoid repetition and unnecessary complication of text we usually talk about one type only since the extension to the other type should be obvious with consideration of the difference in dimensionality and symbols.

[26] In this formula, t is a generic symbol and hence it stands for s.

constant to s.

• Natural parameterization may also be used for parameterization which is proportional to s and hence the transformation relation between two natural parameterizations becomes $\check{s} = \pm ms + c$ where m is another real constant. The two parameterizations then differ, apart from the sense of orientation and the constant shift, by the length scale which can be chosen arbitrarily.

• In a general nD space, the tangent vector to a space curve, represented parametrically by the spatial representation $\mathbf{r}(t)$ where t is a general parameter, is given by $\frac{d\mathbf{r}}{dt}$.

• The tangent line to a sufficiently smooth curve at one of its non-singular points P is a straight line passing through P but not through any point in a deleted neighborhood of P. More technically, a tangent line to a curve C at a point P is a straight line passing through P and having the same orientation as the derivative $\frac{d\mathbf{r}}{dt}$ where $\mathbf{r}(t)$ is the spatial representation of C. A vector tangent to a space curve at P is a vector oriented in either directions of the tangent line at P and hence it is a non-trivial scalar multiple of $\frac{d\mathbf{r}}{dt}$.

• The tangent line to a curve at a given point P on the curve may also be defined as the limit of a secant line passing through P and another neighboring point on the curve where the other point converges, while staying on the curve, to the tangent point. All these different definitions are equivalent as they represent the same entity.

• A vector tangent to a surface curve, represented parametrically by: $C(u(t), v(t))$ where u and v are the surface curvilinear coordinates and t is a general parameter, is given by:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} \tag{20}$$

where $\mathbf{r}(u(t), v(t))$ is the spatial representation of C and all these quantities are defined and evaluated at a particular point on the curve. The last equation, in tensor notation, becomes:

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial u^\alpha}\frac{du^\alpha}{dt} = x^i_\alpha \frac{du^\alpha}{dt} \tag{21}$$

where $i = 1, 2, 3$, $\alpha = 1, 2$ and $(u^1, u^2) \equiv (u, v)$.

• A space curve $C(t) : I \to \mathbb{R}^3$, where $I \subseteq R$ and $t \in I$ is a parameter, is "regular at point t_0" *iff* $\dot{C}(t_0)$ exists and $\dot{C}(t_0) \neq 0$ where the overdot stands for differentiation with respect to the general parameter t. The curve is "regular" *iff* it is regular at each interior point in I.

• On a regular parameterized curve there is a neighborhood to each point in its domain in which the curve is injective.

• On transforming a surface S by a differentiable regular mapping f of class C^n to a surface $\bar{S}$, a regular curve C of class C^n on S will be mapped on a regular curve $\bar{C}$ of class C^n on $\bar{S}$ by the same functional mapping relation, that is $\bar{\mathbf{r}}(t) = f(\mathbf{r}(t))$ where the barred and unbarred $\mathbf{r}(t)$ are the spatial parametric representations of the two curves on the barred and unbarred surfaces.

• A non-trivial vector $\mathbf{v}$ is said to be tangent to a regular surface S at a given point P on S if there is a regular curve C on S passing through P such that $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$ where $\mathbf{r}(t)$ is the spatial representation of C and $\frac{d\mathbf{r}(t)}{dt}$ is evaluated at P (also see 1.13 for further details).[27]

• A periodic curve C is a curve that can be represented parametrically by a continuous function of the form $\mathbf{r}(t + T) = \mathbf{r}(t)$ where $\mathbf{r}$ is the spatial representation of C, t is a real parameter and T is a real constant called the function period. Circles and ellipses are prominent examples of periodic curves where they can be represented respectively by $\mathbf{r}(t) = (a \cos t, a \sin t)$ and $\mathbf{r}(t) = (a \cos t, b \sin t)$ where a and b are real constants, $t \in \mathbb{R}$ and $\mathbf{r}(t + 2\pi) = \mathbf{r}(t)$. Hence, circles and ellipses are periodic curves with a period of 2π.

• A closed curve is a periodic curve defined over a minimum of one period.[28]

• Closed curves may be regarded as topological images of circles.

• A curve is described as plane curve if it can be embedded entirely in a plane with no

[27] In fact any vector $\mathbf{v} = c\frac{d\mathbf{r}(t)}{dt}$, where $c \neq 0$ is a real number, is a tangent although it may not be *the* tangent.

[28] Periodicity is not a necessary requirement for the definition of closed curves as the curves can be defined over a single period without being considered as such.

distortion.

• Orthogonal trajectories of a given family of curves is a family of curves that intersect the given family perpendicularly at their intersection points.

• Any curve can be mapped isometrically to a straight line segment where both are naturally parameterized by arc length.[29]

1.13 Surfaces

• A 2D surface embedded in a 3D space may be defined loosely as a set of connected points in the space such that the immediate neighborhood of each point on the surface can be deformed continuously to form a flat disk. Technically, a surface in a 3D manifold is a mapping from a subset of coordinate plane to a 3D space, that is $S : \Omega \to \mathbb{R}^3$, where Ω is a subset of $\mathbb{R}^2$ plane and S is a sufficiently smooth injective function. Other conditions may also be imposed to ensure the existence of a tangent plane and a normal at each point of the surface. In particular, the condition $\partial_u \mathbf{r} \times \partial_v \mathbf{r} \neq \mathbf{0}$ at all points on the surface is usually imposed to ensure regularity. Like space and surface curves, the image of the mapping in $\mathbb{R}^3$ is known as the trace of the surface.[30]

• A 2D surface embedded in a 3D space can be defined explicitly: $z = f(x, y)$, or implicitly: $F(x, y, z) = 0$, or parametrically: $x(u^1, u^2)$, $y(u^1, u^2)$, $z(u^1, u^2)$ where u^1 and u^2 are independent parameters described as the curvilinear coordinates of the surface. By substitution, elimination and algebraic manipulation these forms can be transformed interchangeably.

• A coordinate patch of a surface is an injective, bicontinuous, regular, parametric representation of a part of the surface. In more technical terms, a coordinate patch of class C^n

[29] From this statement plus the fact that isometric transformation is an equivalence relation (see § 6.5), it can be concluded that any two curves of equal length can be mapped isometrically to each other.

[30] For convenience, in the present text we use curve and surface for trace as well as for mapping; the meaning should be obvious from the context. Also, the trace of a curve or a surface should not be confused with the trace of a matrix which is the sum of its diagonal elements.

($n > 0$) on a surface S is a functional mapping of an open set Ω in the uv plane onto S that satisfies the following conditions:

(A) The functional mapping relation is of class C^n over Ω.

(B) The mapping is one-to-one and bicontinuous over Ω.

(C) $\mathbf{E}_1 \times \mathbf{E}_2 \neq \mathbf{0}$ at any point in Ω.

• As indicated previously, a vector $\frac{d\mathbf{r}(t)}{dt}$, where $\mathbf{r}$ is a t-parameterized position vector with $t \in I \subseteq \mathbb{R}$, is described as a tangent vector to the surface S at point P on the surface if there is a regular curve embedded in S and passing through P such that $\frac{d\mathbf{r}(t)}{dt}$ (or a non-trivial scalar multiple of $\frac{d\mathbf{r}(t)}{dt}$) is a tangent to the curve at P. The set of all tangent vectors to the surface S at point P forms a tangent plane to S at P. This set is called the tangent space of S at P and it is notated with $T_P S$.

• As we will see (also refer to § 1.6), the tangent space of a regular surface at a given point on the surface is the span of the two linearly independent basis vectors defined as $\mathbf{E}_1 = \frac{\partial \mathbf{r}}{\partial u^1}$ and $\mathbf{E}_2 = \frac{\partial \mathbf{r}}{\partial u^2}$ where $\mathbf{r}(u^1, u^2)$ is the spatial representation of the coordinate curves in a 3D coordinate system and u^1 and u^2 are the curvilinear coordinates of the surface. The tangent space therefore is the plane passing through P and is perpendicular to the vector $\mathbf{E}_1 \times \mathbf{E}_2$.

• As indicated previously, every vector tangent to a regular surface S at a given point P on S can be expressed as a linear combination of the surface basis vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ at P. The reverse is also true, that is every linear combination of $\mathbf{E}_1$ and $\mathbf{E}_2$ at P is a tangent vector to a regular curve embedded in S and passing through P and hence is a tangent to S at P.

• The tangent space at a specific point P of a surface is a property of the surface at P and hence it is independent of the patch that contains P.

• For any non-trivial vector $\mathbf{v}$ which is parallel to the tangent plane of a simple and smooth surface S at a given point P on S, there is a curve in S passing through P and represented parametrically by $\mathbf{r}(t)$ such that $\mathbf{v} = c\frac{d\mathbf{r}}{dt}$ where $c \neq 0$ is a real constant.

• A non-trivial vector is parallel to the tangent plane of a surface S at a given point P *iff* it is tangent to S at P.

• From the previous points, we see that the tangent plane of a surface at a given point P is given by:

$$\mathbf{r} = \mathbf{r}_P + p\mathbf{E}_1 + q\mathbf{E}_2 \tag{22}$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the tangent plane, $\mathbf{r}_P$ is the position vector of the point P, $p, q \in (-\infty, \infty)$ are real variables, and $\mathbf{E}_1$ and $\mathbf{E}_1$ are the surface basis vectors at P.

• The straight line passing through a given point P on a surface S in the direction of the normal vector $\mathbf{n}$ $\left(= \frac{\mathbf{E}_1 \times \mathbf{E}_2}{|\mathbf{E}_1 \times \mathbf{E}_2|}\right)$ of S at P is called the normal line to S at P.[31] The equation of this normal line is given by:

$$\mathbf{r} = \mathbf{r}_P + \lambda\mathbf{n} \tag{23}$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the normal line, $\mathbf{r}_P$ is the position vector of the point P, $\lambda \in (-\infty, \infty)$ is a real variable, and $\mathbf{n}$ is the unit normal vector of S at P.

• A regular curve of class C^n on a sufficiently smooth surface is an image of a unique regular plane curve of class C^n in the parameter plane Ω.[32]

• A surface is regular at a given point P *iff* $\mathbf{E}_1 \times \mathbf{E}_2 \neq \mathbf{0}$ at P where $\mathbf{E}_1 = \partial_u \mathbf{r}$ and $\mathbf{E}_2 = \partial_v \mathbf{r}$ are the tangent vectors to the surface coordinate curves. A surface is regular *iff* $\mathbf{E}_1 \times \mathbf{E}_2 \neq \mathbf{0}$ at any point on the surface.

• A surface of revolution is an axially-symmetric surface generated by a plane curve C

[31] This should not be confused with the normal line of a space curve C at point P which is normally called the principal normal line (see § 2). Anyway, the two should be distinguished easily by noticing their affiliation to a surface or a curve.

[32] Here, we are considering each connected part of the curve embedded in a coordinate patch if there is no single patch that contains the entire curve.

revolving around a straight line L contained in the plane of the curve but not intersecting the curve. The curve C is called the profile of the surface and the line L is called the axis of revolution which is also the axis of symmetry of the surface.

• Meridians of a surface of revolution are plane curves on the surface formed by the intersection of a plane containing the axis of revolution with the surface. Parallels of a surface of revolution are circles generated by intersecting the surface by planes perpendicular to the axis of revolution. Meridians and parallels intersect at right angles.[33]

• A "Monge patch" is a coordinate patch in a 3D space defined by a function in one of the following forms:

$$\begin{aligned} \mathbf{r}(u,v) &= (f(u,v),u,v) \\ \mathbf{r}(u,v) &= (u,f(u,v),v) \\ \mathbf{r}(u,v) &= (u,v,f(u,v)) \end{aligned} \tag{24}$$

where f is a differentiable function of the surface coordinates (u,v). When f is of class C^n then the coordinate patch is of this class.

• A "simply connected" region on a surface means that a closed curve contained in the region can be shrunk down continuously onto any point in the region without leaving the region. In simple terms, it means that the region contains no holes or gaps that separate its parts.

• In simple terms, a simple surface is a continuously deformed plane by compression, stretching and bending. Examples of simple surfaces are cylinders, cones and paraboloids.

• A connected surface S is a simple surface which cannot be entirely represented by the union of two disjoint open point sets in $\mathbb{R}^3$ where these sets have non-empty intersection with S. Hence, for any two arbitrary points, P_1 and P_2, on S there is a regular arc which

[33]For spheres, these curves are called meridians of longitude and parallels of latitude.

is totally-embedded in S with P_1 and P_2 being its end points. Examples of connected surfaces are planes, ellipsoids and cylinders.

• A closed surface is a simple surface with no open edges; examples of closed surfaces are spheres and ellipsoids.

• A bounded surface is a surface that can be contained entirely in a sphere of a finite radius.

• A compact surface is a simple surface which is bounded and closed like a torus or a Klein bottle.

• If f is a differentiable regular mapping from a surface S to a surface $\bar{S}$, then if S is compact then $\bar{S}$ is compact.

• If S_1 and S_2 are two simple surfaces where S_1 is connected and S_2 is closed and contained in S_1, then the two surfaces are equal as point sets. As a result, a simple closed surface cannot be a proper subset of a simple connected surface.

• An orientable surface is a simple surface over which a continuously-varying normal vector can be defined. Hence, spheres, cylinders and tori are orientable surfaces while the Mobius strip is a non-orientable surface since a normal vector moved continuously around the strip from a given point will return to the point in the opposite direction. An orientable surface which is connected can be oriented in only one of two possible ways.

• An oriented surface is an orientable surface over which the direction of the normal vector is determined.

• An elementary surface is a simple surface which possesses a single coordinate patch basis, and hence it is an orientable surface which can be mapped bicontinuously to an open set in the plane. Examples of elementary surfaces are planes, cones and elliptic paraboloids.

• A surface that can be flattened into a plane by unfolding without local distortion by compression or stretching is called developable surface.[34] A characteristic feature of a

[34]It is called developable because it can be developed into a plane by rolling the surface out on a plane without compression or stretching.

developable surface is that, like the plane, its Gaussian curvature (see § 4.4.1) is zero at every point on the surface.

• A topological property of a surface is a property which is invariant with respect to injective bicontinuous mappings. An example of a topological property is compactness.

• A differentiable regular mapping from a surface S to a surface $\bar{S}$ is called conformal if it preserves angles between oriented intersecting curves on the surface. The mapping is described as direct if it preserves the sense of the angles and inverse if it reverses it.

• Technically, the mapping is conformal if there is a function $\lambda(u, v) > 0$ that applies to all patches on the surface such that $a_{\alpha\beta} = \lambda \bar{a}_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ where the unbarred and barred indexed a are the coefficients of the surface covariant metric tensor in S and $\bar{S}$ respectively.[35] An example of conformal mapping is the stereographic projection from the Riemann sphere to a plane.

• An isometry or isometric mapping is a one-to-one mapping from a surface S to a surface $\bar{S}$ that preserves distances, hence any regular arc in S is mapped onto an arc in $\bar{S}$ with equal length. The two surfaces S and $\bar{S}$ are described as isometric surfaces. An example of isometric mapping is the deformation of a rectangular plane sheet into a cylinder with no local distortion by compression or stretching and hence the two surfaces are isometric since all distances are preserved.

• Isometry is a symmetric relation and hence the inverse of an isometric mapping is an isometric mapping, that is if f is an isometry from S to $\bar{S}$, then f^{-1} is an isometry from $\bar{S}$ to S.

• An injective mapping from a surface S onto a surface $\bar{S}$ is an isometry *iff* the coefficients of the first fundamental form (see § 3.3) for any patch on S are identical to the coefficients

[35] This may be stated by saying that the first fundamental forms of the two surfaces are proportional but proportionality should not be understood to mean that λ is constant.

of the first fundamental form of its image on $\bar{S}$, that is

$$E = \bar{E} \qquad F = \bar{F} \qquad G = \bar{G} \tag{25}$$

where the unbarred and barred E, F, G are the coefficients of the first fundamental form in the two surfaces.[36]

• The mapping that preserves distances but it is not injective is described as local isometry. The last point also applies to local isometry.

• Since intrinsic properties are dependent only on the coefficients of the first fundamental form of the surface, an intrinsic property of the surface is invariant with respect to isometric mappings.

• As a consequence of the equality of corresponding lengths of two isometric surfaces, the corresponding angles are also equal. However, the reverse is not valid, that is a mapping that attains the equality of corresponding angles (refer to conformal mapping above) does not necessarily ensue the equality of corresponding lengths.

• Similarly, isometric mapping preserves areas of mapped surfaces since it preserves lengths and angles.

• Isometric mapping is more restrictive than conformal mapping. In fact conformal mapping can be set up between any two surfaces and in many different ways but this is not always possible for isometric mapping.

• A surface generated by the collection of all the tangent lines to a given space curve is called the "tangent surface" of the curve while the tangent lines are called the generators of the surface.[37] The tangent surface of a curve may be demonstrated visually by a taut

[36]As we will see, the coefficients of the first fundamental form are the same as the coefficients of the surface covariant metric tensor, that is:

$$a_{11} = E \qquad a_{12} = a_{21} = F \qquad a_{22} = G \tag{26}$$

[37]These tangent lines are also called the rulings of the surface.

flexible string connected to the curve where it scans the surface while being directed tangentially at each point of the curve at its base.[38]

• A "branch" of the tangent surface of a curve C at a given point P on the curve refers to the tangent line of C at P.

• If C_e is a space curve with a tangent surface S_T and C_i is a curve embedded in S_T and it is orthogonal to all the tangent lines of C_e at their intersection points, then C_i is called an involute of C_e while C_e is called an evolute of C_i.

[38] The "tangent surface" of a *curve* should not be confused with the aforementioned "tangent plane" of a *surface* at a given point. Also the taut string visualization should extend to both directions to give the full extent of the tangent surface.

2 Curves in Space

• "Space" in this title is general and hence it includes surface, as explained in the Preface.

• Let have a space curve of class C^2 in a 3D Riemannian manifold with a given metric g_{ij} $(i, j = 1, 2, 3)$. The curve is parameterized by s which is the distance along the curve starting from an arbitrarily-chosen initial point on the curve, P_0.[39] The curve can therefore be represented by:

$$x^i = x^i(s) \qquad\qquad (i = 1, 2, 3) \tag{27}$$

where the indexed x represent the space coordinates.[40]

• Three mutually perpendicular vectors each of unit length can be defined at each point P with non-zero curvature of the above-described space curve: tangent $\mathbf{T}$, normal $\mathbf{N}$ and binormal $\mathbf{B}$ (see Figure 1). These vectors can serve as a moving coordinate system for the space.

• The unit vector tangent to the curve at point P on the curve is given by:[41]

$$[\mathbf{T}]^i = T^i = \frac{dx^i}{ds} \tag{28}$$

• For a t-parameterized curve, where t is not necessarily the arc length, the tangent vector is given by:

$$\mathbf{T} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \tag{29}$$

where the overdot represents differentiation with respect to t.

---

[39] We choose to parameterize the curve by s to have simpler formulae. Other formulae based on a more general parameterization will also be given. We also use a mix of tensor and symbolic notations as each has certain advantages and to familiarize the reader with both notations as different authors use different notations.

[40] Here, we have $\mathbf{r}(s) = x^i(s)\mathbf{e}_i$ where $\mathbf{r}$ is the spatial representation of the space curve and $\mathbf{e}_i$ are the space basis vectors.

[41] For simplicity, we employ Cartesian coordinates and hence ordinary derivatives (i.e. $\frac{d}{ds}$ and $\frac{d}{dt}$) are used in this and the following formulae. For general curvilinear coordinates, these ordinary derivatives should be replaced by absolute derivatives (i.e. $\frac{\delta}{\delta s}$ and $\frac{\delta}{\delta t}$) along the curves.

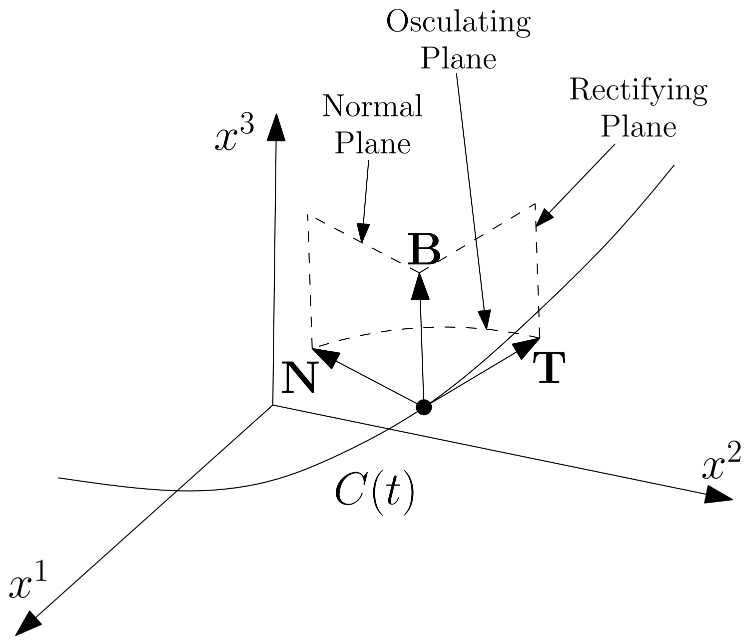

Figure 1: The vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ and their associated planes.

• The unit vector normal[42] to the tangent T^i, and hence to the curve, at the point P is given by:

$$[\mathbf{N}]^i = N^i = \frac{\frac{dT^i}{ds}}{\left|\frac{dT^i}{ds}\right|} = \frac{1}{\kappa}\frac{dT^i}{ds} \tag{30}$$

where κ is a scalar called the "curvature" of the curve at the point P and is defined, according to the normalization condition, by:[43]

$$\kappa = \sqrt{\frac{dT^i}{ds}\frac{dT^i}{ds}} \tag{31}$$

• For a t-parameterized curve, where t is not necessarily the arc length, the principal normal vector is given by:

$$\mathbf{N} = \frac{\dot{\mathbf{r}}(t) \times (\ddot{\mathbf{r}}(t) \times \dot{\mathbf{r}}(t))}{|\dot{\mathbf{r}}(t)|\,|\ddot{\mathbf{r}}(t) \times \dot{\mathbf{r}}(t)|} \tag{32}$$

[42] This is also called the unit principal normal vector. This vector is defined only on points of the curve where the curvature $\kappa \neq 0$.

[43] For general curvilinear coordinates, the formula becomes: $\kappa = \sqrt{g_{ij}\frac{\delta T^i}{\delta s}\frac{\delta T^j}{\delta s}}$ where g_{ij} is the space covariant metric tensor.

where the overdot represents differentiation with respect to t.

• The binormal unit vector is defined as:

$$[\mathbf{B}]^i = B^i = \frac{1}{\tau}\left(\kappa T^i + \frac{dN^i}{ds}\right) \tag{33}$$

which is a linear combination of two vectors both of which are perpendicular to N^i and hence it is perpendicular to N^i. In the last equation, the normalization scalar factor τ is the "torsion" whose sign is chosen to make T^i, N^i and B^i a right handed triad vectors satisfying the condition:[44]

$$\underline{\epsilon}_{ijk} T^i N^j B^k = 1 \tag{34}$$

• For a t-parameterized curve, where t is not necessarily the arc length, the binormal vector is given by:

$$\mathbf{B} = \frac{\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|} \tag{35}$$

where the overdot represents differentiation with respect to t.

• Apart from making $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ a right handed system, there is a geometric significance for the sign of the torsion as it affects the orientation of the space curve.

• At any point on the space curve, the triad T^i, N^i and B^i represents a mutually perpendicular right handed system fulfilling the condition:

$$B^i = [\mathbf{T} \times \mathbf{N}]^i = \underline{\epsilon}^{ijk} T_j N_k \tag{36}$$

• Since the triad $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ are mutually perpendicular, they satisfy the condition:

$$\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0 \tag{37}$$

[44] Some authors reverse the sign in the definition of τ and this reversal affects the signs in the forthcoming Frenet-Serret formulae (see § 2.3). The convention that we follow in these notes has certain advantages.

Also, because they are unit vectors we have:

$$\mathbf{T}\cdot\mathbf{T}=\mathbf{N}\cdot\mathbf{N}=\mathbf{B}\cdot\mathbf{B}=1 \tag{38}$$

• The tangent line of a curve C at a given point P on the curve is a straight line passing through P and is parallel to the tangent vector, $\mathbf{T}$, of C at P. The principal normal line of a curve C at a given point P on the curve is a straight line passing through P and is parallel to the principal normal vector, $\mathbf{N}$, of C at P. The binormal line of a curve C at a given point P on the curve is a straight line passing through P and is parallel to the binormal vector, $\mathbf{B}$, of C at P.

• Following the last point, the equations of the three lines can be given by the following generic form:

$$\mathbf{r}=\mathbf{r}_P+k\mathbf{V}_P \qquad (-\infty<k<\infty) \tag{39}$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the line, $\mathbf{r}_P$ is the position vector of the point P, k is a real variable and the vector $\mathbf{V}_P$ is the vector corresponding to the particular line, that is $\mathbf{V}_P\equiv\mathbf{T}$ for the tangent line, $\mathbf{V}_P\equiv\mathbf{N}$ for the normal line, and $\mathbf{V}_P\equiv\mathbf{B}$ for the binormal line.

• At any regular point P on the space curve, the triad T^i, N^i and B^i define three mutually-perpendicular planes where each one of these planes passes through the point P and is formed by a linear combination of two of these vectors in turn. These planes are: the "osculating plane" which is the span of T^i and N^i, the "rectifying plane" which is the span of T^i and B^i, and the "normal plane" which is the span of N^i and B^i and is orthogonal to the curve at P (see Figure 1).

• Following the last point, the equations of the three planes can be given by the following generic form:

$$(\mathbf{r}-\mathbf{r}_P)\cdot\mathbf{V}_P=0 \tag{40}$$

where $\mathbf{r}$ is the position vector of an arbitrary point on the plane, $\mathbf{r}_P$ is the position vector of the point P, and where for each plane the vector $\mathbf{V}_P$ is the perpendicular vector to the plane at P, that is $\mathbf{V}_P \equiv \mathbf{B}$ for the osculating plane, $\mathbf{V}_P \equiv \mathbf{N}$ for the rectifying plane, and $\mathbf{V}_P \equiv \mathbf{T}$ for the normal plane.

• Following the style of the definition of the tangent line as a limit of a secant line (see § 1.12), the osculating plane may be defined as the limiting position of a plane passing through P and two other points on the curve one on each side of P as the two points converge simultaneously along the curve to P.

• The positive sense of a parameterized curve, which corresponds to the direction in which the parameter increases and hence defines the orientation of the curve, can be determined in two opposite ways. While the sense of the tangent $\mathbf{T}$ and the binormal $\mathbf{B}$ is dependent on the curve orientation and hence they are in opposite directions in these two ways, the principal normal $\mathbf{N}$ is the same as it remains parallel to the normal plane in the direction in which the curve is turning.

2.1 Curvature and Torsion of Space Curves

• The curvature and torsion of space curves may also be called the first and second curvatures respectively, and hence a twisted curve with non-vanishing curvature and non-vanishing torsion is described as double-curvature curve. The expression $\sqrt{(ds_{\mathbf{T}})^2 + (ds_{\mathbf{B}})^2}$, where $ds_{\mathbf{T}}$ and $ds_{\mathbf{B}}$ are respectively the line element components in the tangent and binormal directions, may be described as the total or the third curvature of the curve.[45]

• The equation of Lancret states that:

$$(ds_{\mathbf{N}})^2 = (ds_{\mathbf{T}})^2 + (ds_{\mathbf{B}})^2 \tag{41}$$

[45] The "total curvature" is also used for surfaces (see § 4.4 and 4.4.4) but the meaning is different.

where $ds_{\mathbf{N}}$ is the line element component in the normal direction.

• According to the fundamental theorem of space curves in differential geometry, a space curve is completely determined by its curvature and torsion. More technically, given a real interval $I \subseteq \mathbb{R}$ and two differentiable real functions: $\kappa(s) > 0$ and $\tau(s)$ where $s \in I$, there is a uniquely defined parameterized regular space curve $C(s)$: $I \to \mathbb{R}^3$ of class C^2 with $\kappa(s)$ and $\tau(s)$ being the curvature and torsion of C respectively and s is its arc length. Hence, any other curve meeting these conditions will be different from C only by a rigid motion transformation (translation and rotation) which determines its position and orientation in space.[46]

• On the other hand, any curve with the properties given in the last point possesses uniquely defined $\kappa(s)$ and $\tau(s)$.

• From the previous points, the fundamental theorem of space curves provides the existence and uniqueness conditions for curves.

• The equations: $\kappa = \kappa(s)$ and $\tau = \tau(s)$, where s is the arc length, are called the intrinsic or natural equations of the curve.

• The curvature and torsion are invariants of the space curve and hence they do not depend on the employed coordinate system or the type of parameterization.

• While the curvature is always non-negative ($\kappa \geq 0$), as it represents the magnitude of a vector according to the above-stated definition (see e.g. Eqs. 30 and 31), the torsion can be negative as well as zero or positive.[47]

• The following are some examples of the curvature and torsion of a number of commonly-occurring simple curves:

(A) Straight line: $\kappa = 0$ and $\tau = 0$.

---

[46] In rigid motion transformations, which may also be called Euclidean motion, the distance between any two points on the image is the same as the distance between the corresponding points on the inverse image. Hence, rigid motion transformation is a form of isometric mapping.

[47] Some authors define the curvature vector (see § 4.1) and the principal normal vector of the curve in such a way that it is possible for the curvature to be negative.

(B) Circle of radius R: $\kappa = \frac{1}{R}$ and $\tau = 0$.[48]

(C) Helix parameterized by $\mathbf{r}(t) = (a\cos(t), a\sin(t), bt)$: $\kappa = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$.[49]

In these three examples, the curvature and torsion are constants along the whole curve. However, in general the curvature and torsion of space curves are position dependent and hence they vary from point to point.

• Following the example of 2D surfaces, a 1D inhabitant of a space curve can detect all the properties related to the arc length. Hence, the curvature and torsion, κ and τ, of the curve are extrinsic properties for such a 1D inhabitant. This fact may be expressed by saying that curves are intrinsically Euclidean, and hence their Riemann-Christoffel curvature tensor vanishes identically and they naturally admit 1D Cartesian systems represented by their natural parameterization of arc length.[50] Another demonstration of their intrinsic 1D nature is represented by the Frenet-Serret formulae (see § 2.3).

• Some authors resemble the role of κ and τ in curve theory to the surface curvature tensor $b_{\alpha\beta}$ in surface theory and describe κ and τ as the curve theoretic analogues of the $b_{\alpha\beta}$ in surface theory. In another context, κ and τ may be contrasted with the first and second fundamental forms of surfaces in their roles in defining the curve and surface in the fundamental theorems of these structures.

• From the first and the last of the Frenet-Serret formulae (Eq. 53), we have:

$$|\kappa\tau| = |\mathbf{T}' \cdot \mathbf{B}'| \tag{42}$$

where the prime stands for derivative with respect to a natural parameter s of the curve.

[48] Hence, the radius of curvature (see § 2.1.1) of a circle is its own radius

[49] It can be shown that a space curve of class C^3 with non-vanishing curvature is a helix *iff* the ratio of its torsion to curvature is constant. Geometrically, the helix is characterized by having a tangent vector that forms a constant angle with a specified direction which is the direction defined by its axis of rotation (the circle can be regarded as a degenerate helix).

[50] This should be obvious when we consider that any curve can be mapped isometrically to a straight line where both are naturally parameterized by arc length.

2.1.1 Curvature

• The curvature κ of a space curve is a measure of how much the curve bends as it progresses in the tangent direction at a particular point. The curvature represents the magnitude of the rate of change of the direction of the tangent vector with respect to the arc length and hence it is a measure for the departure of the curve from the orientation of the straight line passing through that point and oriented in the tangent direction. Consequently, the curvature vanishes identically for straight lines.[51]

• From the first of the Frenet-Serret formulae (see Eq. 53) and the fact that:

$$(\mathbf{N}\cdot\mathbf{T})' = (0)' = 0 \qquad \Rightarrow \qquad \mathbf{N}\cdot\mathbf{T}' = -\mathbf{N}'\cdot\mathbf{T} \tag{43}$$

the curvature κ can be expressed as:

$$\kappa = \mathbf{N}\cdot\mathbf{T}' = -\mathbf{N}'\cdot\mathbf{T} \tag{44}$$

where the prime represents differentiation with respect to the arc length s of the curve.

• The "radius of curvature", which is the radius of the osculating circle (see § 2.4), is defined at each point of the space curve for which $\kappa \neq 0$ as the reciprocal of the curvature, i.e. $R_\kappa = \frac{1}{\kappa}$.[52]

• There may be an advantage in using the concept of "curvature" as the principal concept instead of "radius of curvature", that is the curvature is defined at all regular points while the radius of curvature is defined only at the regular points with non-vanishing curvature.

• As indicated earlier, if C is a space curve of class C^2 which is defined on a real interval

[51] Having an identically vanishing curvature is a necessary and sufficient condition for a curve of class C^2 to be straight line.

[52] A different way for introducing these concepts, which is followed by some authors, is to define first the radius of curvature as the reciprocal of the magnitude of the acceleration vector, that is $R_\kappa = \frac{1}{|\mathbf{r}''(s)|}$ where $\mathbf{r}$ is the spatial representation of an s-parameterized curve; the curvature is then defined as the reciprocal of the radius of curvature. Hence, the radius of curvature may be described as the reciprocal of the norm of the acceleration vector where acceleration means the second derivative of the curve.

$I \subseteq \mathbb{R}$ and is parameterized by arc length $s \in I$, that is $C(s) : I \to \mathbb{R}^3$, then the curvature of C at a given point P on the curve is defined by: $\kappa = |\mathbf{r}''|$ where $\mathbf{r}(s)$ is the spatial representation of the curve, the double prime represents the second derivative with respect to s, and $\mathbf{r}''$ is evaluated at P.[53]

• For a space curve represented parametrically by $\mathbf{r}(t)$, where t is a general parameter not necessarily a natural parameter, we have:

$$\kappa = \frac{\left|\dot{\mathbf{T}}\right|}{|\dot{\mathbf{r}}|} = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{3/2}} \tag{45}$$

where all the quantities, which are functions of t, are evaluated at a given point corresponding to a given value of t, and the overdot represents derivative with respect to t.

• All surface curves passing through a point P on a surface S and have the same osculating plane at P have identical curvature κ at P if the osculating plane is not tangent to S at P.

2.1.2 Torsion

• The torsion τ represents the rate of change of the osculating plane, and hence it quantifies the twisting, in magnitude and sense, of the space curve out of the plane of curvature and its deviation from being a plane curve. The torsion therefore vanishes identically for plane curves.[54]

• If C is a space curve of class C^2 which is defined on a real interval $I \subseteq \mathbb{R}$ and it is parameterized by arc length $s \in I$, that is $C(s) : I \to \mathbb{R}^3$, then the torsion of C at a given

[53] There should be no confusion in the present and forthcoming points between the bare C symbol and the superscripted C symbol as the bare C symbolizes a curve while the superscripted C stands for the differentiability condition as explained earlier.

[54] Having an identically vanishing torsion is a necessary and sufficient condition for a curve of class C^2 to be plane curve.

point P on the curve is given by:[55]

$$\tau = \mathbf{N}' \cdot \mathbf{B} \tag{46}$$

where $\mathbf{N}'$ and $\mathbf{B}$ are evaluated at P and the prime represents differentiation with respect to s.

• For a space curve represented parametrically by $\mathbf{r}(t)$, where t is a general parameter not necessarily a natural parameter, we have:

$$\tau = \frac{\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}})}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})} = \frac{\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2} \tag{47}$$

where all the quantities, which are functions of t, are evaluated at a given point P corresponding to a given value of t, and the overdot represents derivative with respect to t. The curve should have non-vanishing curvature κ at P.

• For general curvilinear coordinates, the torsion of an s-parameterized curve is given in tensor notation by:

$$\tau = \epsilon^{ijk} T_i N_j \frac{\delta N_k}{\delta s} \tag{48}$$

• For rectangular Cartesian coordinates, the torsion of a t-parameterized curve is given in tensor notation by:[56]

$$\tau = \frac{\epsilon_{ijk} \dot{x}_i \ddot{x}_j \dddot{x}_k}{\kappa^2} \tag{49}$$

where κ is the curvature of the curve as defined previously.

• The "radius of torsion" is defined at each point of a space curve for which $\tau \neq 0$ as the absolute value of the reciprocal of the torsion, i.e. $R_\tau = \left|\frac{1}{\tau}\right|$.[57]

• The value of torsion is invariant under permissible coordinate transformations. It is also

[55] This can be obtained from the second of the Frenet-Serret formulae (see Eq. 53) by the dot product of both sides with $\mathbf{B}$.

[56] This is based on the formula in the previous point.

[57] Some authors do not take the absolute value and hence the radius of torsion can be negative.

independent of the nature of its parameterization and the orientation of the curve which is determined by the sense of increase of its parameter.

2.2 Geodesic Torsion

• Geodesic torsion, which is an attribute of a curve embedded in a surface, is also known as the relative torsion.

• The geodesic torsion of a surface curve C at a given point P is the torsion of the geodesic curve (see § 5.7) passing through P in the tangent direction of C at P.[58]

• The geodesic torsion τ_g of a surface curve represented spatially by $\mathbf{r} = \mathbf{r}(s)$ is given by the following scalar triple product:

$$\tau_g = \mathbf{n} \cdot (\mathbf{n}' \times \mathbf{r}') \tag{50}$$

where $\mathbf{n}$ is the unit normal vector to the surface, the primes represent differentiation with respect to a natural parameter s, and all these quantities are to be evaluated at a given point on the curve corresponding to the value of τ_g.

• The geodesic torsion of a curve C at a non-umbilical (see § 4.4.6) point P is given in terms of the principal curvatures (see § 4.4) by:

$$\tau_g = (\kappa_1 - \kappa_2) \sin\theta \cos\theta \tag{51}$$

where θ is the angle between the tangent vector $\mathbf{T}$ to the curve C at P and the first principal direction $\mathbf{d}_1$ (see Darboux frame in § 4.4).

• The geodesic torsion of a surface curve C parameterized by arc length s at a given point

[58] As we will see in § 5.7, in the neighborhood of a point P on a smooth surface and for any specified direction there is one and only one geodesic curve passing through P in that direction.

P is given by:[59]

$$\tau_g = \tau - \frac{d\phi}{ds} \tag{52}$$

where τ is the torsion of C at P and ϕ is the angle between the unit vector $\mathbf{n}$ normal to the surface and the principal normal $\mathbf{N}$ of C at P (i.e. $\phi = \arccos(\mathbf{n} \cdot \mathbf{N})$). This is known as the Bonnet formula. This formula demonstrates that when $\mathbf{n}$ and $\mathbf{N}$ are collinear (i.e. $\phi = 0$), the geodesic torsion and the torsion are equal (i.e. $\tau_g = \tau$). In this case, the geodesic curvature will vanish and the curve becomes a geodesic.[60]

• On a line of curvature (see § 5.8), the geodesic torsion vanishes identically.

• The geodesic torsion of a surface curve C at a given point P is zero *iff* C is a tangent to a line of curvature at P.

• The geodesic torsions of two orthogonal surface curves are equal in magnitude and opposite in sign.

2.3 Relationship between Curve Basis Vectors and their Derivatives

• The three basis vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ (see § 2) are connected to their derivatives by the Frenet-Serret formulae[61] which are given in rectangular Cartesian coordinates by:

$$\begin{aligned} \frac{dT^i}{ds} &= \kappa N^i \\ \frac{dN^i}{ds} &= \tau B^i - \kappa T^i \\ \frac{dB^i}{ds} &= -\tau N^i \end{aligned} \tag{53}$$

• The Frenet-Serret formulae can be cast in the following matrix form using symbolic

[59] The curve should not be asymptotic (see § 5.9).

[60] When $\mathbf{n}$ and $\mathbf{N}$ are collinear, the geodesic component of the curvature vector will vanish (see § 5.7).

[61] These are also called Frenet formulae. The sign of the terms involving τ depends on the convention about the torsion and hence these equations differ between different authors.

notation:[62]

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{54}$$

where all the quantities in this equation are functions of arc length s and the prime represents derivative with respect to s (i.e. $\frac{d}{ds}$).

• The Frenet-Serret formulae can also be given in the following form:

$$\begin{aligned} \mathbf{T}' &= \mathbf{d} \times \mathbf{T} \\ \mathbf{N}' &= \mathbf{d} \times \mathbf{N} \\ \mathbf{B}' &= \mathbf{d} \times \mathbf{B} \end{aligned} \tag{55}$$

where $\mathbf{d}$ is the "Darboux vector" which is given by:

$$\mathbf{d} = \tau \mathbf{T} + \kappa \mathbf{B} \tag{56}$$

• The three equations in the last point may be abbreviated in a single equation as:

$$(\mathbf{T}', \mathbf{N}', \mathbf{B}') = \mathbf{d} \times (\mathbf{T}, \mathbf{N}, \mathbf{B}) \tag{57}$$

• In general curvilinear coordinates, the Frenet-Serret formulae are given in terms of the absolute derivatives of the three vectors, that is:

$$\begin{aligned} \frac{\delta T^i}{\delta s} &= \frac{dT^i}{ds} + \Gamma^i_{jk} T^j \frac{dx^k}{ds} = \kappa N^i \\ \frac{\delta N^i}{\delta s} &= \frac{dN^i}{ds} + \Gamma^i_{jk} N^j \frac{dx^k}{ds} = \tau B^i - \kappa T^i \\ \frac{\delta B^i}{\delta s} &= \frac{dB^i}{ds} + \Gamma^i_{jk} B^j \frac{dx^k}{ds} = -\tau N^i \end{aligned} \tag{58}$$

[62] As seen, the coefficient matrix is anti-symmetric.

where the indexed x represent general spatial coordinates and s is a natural parameter; the other symbols are as given earlier.

• The triad **T**, **N** and **B** is called the Frenet frame which forms a set of orthonormal basis vectors for $\mathbb{R}^3$. This frame serves as a moving orthogonal coordinate system on the points of the curve.[63]

• According to the fundamental theorem of space curves, which is outlined previously in § 2.1, a curve does exist and it is unique *iff* its curvature and torsion as functions of arc length are given. Now, it is natural to expect that such a solution can be obtained from the system of differential equations given by Frenet-Serret formulae. However, such a solution cannot be obtained by direct integration of these equations. More elaborate methods (e.g. methods based on the Riccati equation for reducing a system of simultaneous differential equations to a first order differential equation) may be used to obtain the solution. Nevertheless, a solution can be obtained by direct integration of the Frenet-Serret formulae for plane curves where the torsion vanishes identically. A solution by direct integration of the Frenet-Serret formulae can also be obtained in simple cases such as when the curvature and torsion are constants.

2.4 Osculating Circle and Sphere

• At any point P with non-zero curvature of a smooth space curve C an "osculating circle"[64] can be defined, where this circle is characterized by:

(A) It is tangent to C at P (i.e. the circle and the curve have a common tangent vector at P).

(B) It lies in the osculating plane.

(C) Its radius is $\frac{1}{\kappa}$ where κ is the curvature of C at P.

---

[63] This triad may also be called the Frenet trihedron or the moving trihedron of the curve. This frame can suffer from problems or become undefined at inflection points where $\frac{d\mathbf{T}}{ds} = \mathbf{0}$.

[64] It may also be called the circle of curvature or the kissing circle.

(D) Its center $\mathbf{r}_C$ is at $\mathbf{r}_C = \mathbf{r}_P + \frac{\mathbf{N}}{\kappa}$ where $\mathbf{r}_P$ is the position vector of P and $\mathbf{N}$ is the principal normal of C at P.[65]

• The center of curvature of a curve at a point on the curve is defined as the center of the osculating circle at that point, as given in the last point.

• For all points on a circle, the center of curvature is the center of the circle, so the circle is its own osculating circle.

• The osculating circle provides a good approximation to the curve in the neighborhood of its regular points.

• Following the manner of defining the tangent line to a curve as a limit of the secant line (see § 1.12), the osculating circle to a curve at a point P may be defined as the limit of the circle passing through P and two other points on the curve one on each side of P as these two points converge to P while staying on the curve.

• The "osculating sphere" of a curve C at a point P may be defined as the limit of a sphere passing through P and three neighboring points on the curve as these points converge to P. The position of the center C_S of the osculating sphere at P, which is called the center of spherical curvature of C at P, is given by:

$$\mathbf{r}_S = \mathbf{r}_P + \frac{1}{\kappa}\mathbf{N} - \frac{\kappa'}{\tau\kappa^2}\mathbf{B} = \mathbf{r}_P + R_\kappa\mathbf{N} + R_\tau R'_\kappa\mathbf{B} \tag{59}$$

where $\mathbf{r}_S$ and $\mathbf{r}_P$ are the position vectors of C_S and P, $\mathbf{B}$ and $\mathbf{N}$ are the binormal and principal normal vectors, κ and τ are the curvature and torsion, R_κ and R_τ are the radii of curvature and torsion, and the prime represents derivative with respect to a natural parameter of C. All these quantities belong to C at P which should have non-vanishing curvature and torsion ($\kappa, \tau \neq 0$).

[65] In some cases the center may be defined geometrically but not analytically when the second derivative of the curve is not defined at the given point.

• From Eq. 59, the radius of the osculating sphere is given by:

$$|\mathbf{r}_S - \mathbf{r}_P| = \sqrt{R_\kappa^2 + (R_\tau R'_\kappa)^2} \tag{60}$$

2.5 Parallelism and Parallel Propagation

• In flat spaces parallelism is an absolute property as it is defined without reference to an external object. However, in Riemannian spaces the idea of parallelism is defined in comparison to a prescribed curve and hence it is different from the idea of parallelism in the Euclidean sense.

• A vector field A^α is described as being parallel along the surface curve $u^\beta = u^\beta(t)$ *iff* its absolute derivative (see § 7) along the curve vanishes, that is:

$$\frac{\delta A^\alpha}{\delta t} \equiv A^\alpha_{;\beta}\frac{du^\beta}{dt} \equiv \frac{dA^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}A^\gamma\frac{du^\beta}{dt} = 0 \tag{61}$$

This means that the sufficient and necessary condition for a vector field to be parallel along a surface curve is that the covariant derivative of the field is normal to the surface (see § 7).

• All the vectors of a field of parallel vectors have the same constant magnitude.

• A field of absolutely parallel unit vectors on a surface do exist *iff* there is an isometric correspondence between the plane and the surface.

• When two surfaces are tangent to each other along a given curve C, then a vector field which is parallel along C with respect to one of these surfaces will also be parallel along C with respect to the other surface.

• As a consequence of the definition of parallelism in Riemannian space we have:

(A) A surface vector field parallelly propagated along a given curve between two points P_1 and P_2 on the curve does not necessarily coincide with another vector field parallelly propagated along another curve connecting P_1 and P_2.

(B) Parallel propagation is path dependent, that is: given two points P_1 and P_2, the vector obtained at P_2 by parallel propagation of a vector from P_1 along a given curve C connecting P_1 to P_2 depends on the curve C.

(C) Starting from a given point P on a closed surface curve C enclosing a simply connected region on the surface, parallel propagation of a vector field around C starting from P does not necessarily result in the same vector field when arriving at P.[66]

• When two non-trivial vectors experience parallel propagation along a particular curve the angle between them stays constant.

[66] The angle between the initial and final vectors is a measure of the Gaussian curvature on the surface.

3 Surfaces in Space

• Here, we examine surfaces of class C^2 embedded in a 3D Euclidean space using a Cartesian coordinate system (x, y, z) for the most parts. Some notes are based on a more general Riemannian space with a curvilinear coordinate system.

• Assuming a parametric representation for the surface, where each one of the space coordinates (x, y, z) on the surface is a real differentiable function of the two surface curvilinear coordinates (u, v), the position vector of a point P on the surface as a function of the surface curvilinear coordinates is given by:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{e}_1 + y(u, v)\mathbf{e}_2 + z(u, v)\mathbf{e}_3 \tag{62}$$

where the indexed $\mathbf{e}$ are the Cartesian orthonormal basis vectors in the three directions. It is also assumed that $\partial_u \mathbf{r}$ and $\partial_v \mathbf{r}$ are linearly independent and hence they are not parallel or anti-parallel, that is:[67]

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0} \tag{63}$$

• To express the position vector of P in tensor notation, we re-label the space and surface coordinates as:

$$(x, y, z) \equiv (x^1, x^2, x^3) \qquad \& \qquad (u, v) \equiv (u^1, u^2) \tag{64}$$

and hence the position vector becomes:

$$\mathbf{r}(u^1, u^2) = x^i(u^1, u^2)\mathbf{e}_i \tag{65}$$

[67] This is a sufficient and necessary condition for the surface to be "regular" at the given point. The point is also described as "regular"; otherwise it is "singular" if the condition is violated. The surface is regular on Ω, a closed subset of $\mathbb{R}^2$, if it is regular at each interior point of Ω. The regularity condition guarantees that the surface mapping is one-to-one and possesses continuous inverse.

• To define a surface grid serving as a curvilinear positioning system for the surface, one of the coordinate variables is held fixed in turn while the other is varied. Hence, each one of the following two surface functions:

$$\mathbf{r}(u^1, c_2) \qquad \& \qquad \mathbf{r}(c_1, u^2) \tag{66}$$

where c_1 and c_2 are given real constants, defines a coordinate curve for the surface. These two curves meet at the common surface point (c_1, c_2). The grid is then generated by varying c_1 and c_2 uniformly to obtain coordinate curves at regular intervals.

• The surface coordinate curves of the above grid are orthogonal *iff* the surface metric tensor (see § 3.1) is diagonal everywhere on the surface.

• Corresponding to each one of the surface coordinate curves in the above order, a tangent vector to the curve at a given point on the curve is defined by:[68]

$$\mathbf{E}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\mathbf{e}_i = x^i_\alpha \mathbf{e}_i \tag{67}$$

where the derivatives are evaluated at that point, and $\alpha = 1, 2$ and $i = 1, 2, 3$. These tangent vectors serve as a set of basis vectors for the surface, and for each given point on the surface they generate, by their linear combination, any vector in the surface at that point.[69] They also define, by their linear combination, a plane tangent to the surface at that point.[70] A normal vector to the surface at that point is then defined as the cross product of these tangent basis vectors: $\mathbf{E}_1 \times \mathbf{E}_2$. This normal vector can be scaled by its

[68]That is $\mathbf{E}_1$ is tangent to the $\mathbf{r}(u^1, c_2)$ curve and $\mathbf{E}_2$ is tangent to the $\mathbf{r}(c_1, u^2)$ curve.

[69]This should be understood in an infinitesimal sense or, equivalently, as a vector lying in the tangent plane of the surface at the given point.

[70]The plane generated by the linear combination of $\mathbf{E}_1$ and $\mathbf{E}_2$ is the tangent space, $T_P S$, to the surface at point P as described earlier, and hence $\mathbf{E}_1(u^1_P, u^2_P)$ and $\mathbf{E}_2(u^1_P, u^2_P)$ form a basis for this space where the subscript P is a reference to the point P.

magnitude to produce a unit vector normal to the surface at that point:

$$\mathbf{n} = \frac{\mathbf{E}_1 \times \mathbf{E}_2}{|\mathbf{E}_1 \times \mathbf{E}_2|} = \frac{\mathbf{E}_1 \times \mathbf{E}_2}{\sqrt{a}} \tag{68}$$

where the vectors are labeled so that $\mathbf{E}_1, \mathbf{E}_2$ and $\mathbf{n}$ form a right-handed system, and a is the determinant of the surface covariant metric tensor (see § 3.1).

• The surface basis vectors, $\mathbf{E}_\alpha$, are symbolized in full tensor notation by:

$$[\mathbf{E}_\alpha]^i \equiv E^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} = x^i_\alpha \qquad (i = 1, 2, 3 \text{ and } \alpha = 1, 2) \tag{69}$$

and hence they can be regarded as 3D contravariant space vectors or as 2D covariant surface vectors (refer to § 3.1 for further details).

• It can be shown that the covariant form of the unit vector $\mathbf{n}$ normal to the surface is given by:

$$n_i = \frac{1}{2}\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}_{ijk} x^j_\alpha x^k_\beta \tag{70}$$

where $x^j_\alpha = \frac{\partial x^j}{\partial u^\alpha}$ and similarly for x^k_β. The implication of this equation, which defines $\mathbf{n}$ in terms of the surface basis vectors x^j_α and x^k_β, is that $\mathbf{n}$ is a space vector which is independent of the choice of the surface coordinates u^1, u^2 in support of the geometric intuition.

• Since $\mathbf{n}$ is normal to the surface, we have:

$$g_{ij} n^i x^j_\alpha = 0 \tag{71}$$

which is the statement, in tensor notation, that $\mathbf{n}$ is orthogonal to every vector in the tangent space of the surface at the given point.

• Although $\mathbf{E}_1$ and $\mathbf{E}_2$ are linearly independent they are not necessarily orthogonal or of unit length. However, they can be orthonormalized as follow:

$$\underline{\mathbf{E}}_1 = \frac{\mathbf{E}_1}{|\mathbf{E}_1|} = \frac{\mathbf{E}_1}{\sqrt{a_{11}}} \qquad \underline{\mathbf{E}}_2 = \frac{a_{11}\mathbf{E}_2 - a_{12}\mathbf{E}_1}{\sqrt{a_{11}a}} \tag{72}$$

where a is the determinant of the surface covariant metric tensor (see § 3.1), the indexed a are the coefficients of this tensor, and the underlined vectors are orthonormal basis vectors, that is:

$$\underline{\mathbf{E}}_1 \cdot \underline{\mathbf{E}}_1 = 1 \qquad \underline{\mathbf{E}}_2 \cdot \underline{\mathbf{E}}_2 = 1 \qquad \underline{\mathbf{E}}_1 \cdot \underline{\mathbf{E}}_2 = 0 \tag{73}$$

• The transformation rules from one curvilinear coordinate system of the surface to another coordinate system, notated with unbarred (u^1, u^2) and barred $(\bar{u}^1, \bar{u}^2)$ symbols respectively, where

$$\begin{aligned} u^1 &= u^1(\bar{u}^1, \bar{u}^2) \qquad \& \qquad u^2 = u^2(\bar{u}^1, \bar{u}^2) \\ \bar{u}^1 &= \bar{u}^1(u^1, u^2) \qquad \& \qquad \bar{u}^2 = \bar{u}^2(u^1, u^2) \end{aligned} \tag{74}$$

are similar to the general rules outlined in [9, 10] for the transformation between coordinate systems in a general nD space.

• Following a transformation from the unbarred surface coordinate system to the barred surface coordinate system, the surface becomes a function of the barred coordinates, and a new set of basis vectors for the surface, which are the tangents to the coordinate curves of the barred system, are defined by the following equations:

$$\begin{aligned} \bar{\mathbf{E}}_1 &= \frac{\partial \mathbf{r}}{\partial \bar{u}^1} = \frac{\partial \mathbf{r}}{\partial u^1}\frac{\partial u^1}{\partial \bar{u}^1} + \frac{\partial \mathbf{r}}{\partial u^2}\frac{\partial u^2}{\partial \bar{u}^1} = \mathbf{E}_1\frac{\partial u^1}{\partial \bar{u}^1} + \mathbf{E}_2\frac{\partial u^2}{\partial \bar{u}^1} \\ \bar{\mathbf{E}}_2 &= \frac{\partial \mathbf{r}}{\partial \bar{u}^2} = \frac{\partial \mathbf{r}}{\partial u^1}\frac{\partial u^1}{\partial \bar{u}^2} + \frac{\partial \mathbf{r}}{\partial u^2}\frac{\partial u^2}{\partial \bar{u}^2} = \mathbf{E}_1\frac{\partial u^1}{\partial \bar{u}^2} + \mathbf{E}_2\frac{\partial u^2}{\partial \bar{u}^2} \end{aligned} \tag{75}$$

These equations, which correlate the surface basis vectors in the barred and unbarred surface curvilinear coordinate systems, can be compactly presented in tensor notation as:

$$\frac{\partial x^i}{\partial \bar{u}^\alpha} = \frac{\partial x^i}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^\alpha} \qquad (i = 1, 2, 3 \text{ and } \alpha, \beta = 1, 2) \tag{76}$$

• A set of contravariant basis vectors for the surface may also be defined as the gradient of the surface curvilinear coordinates, that is:

$$\mathbf{E}^{\alpha} = \nabla u^{\alpha} \tag{77}$$

In tensor notation, this basis is given by:

$$[\mathbf{E}^{\alpha}]_i \equiv E_i^{\alpha} = \frac{\partial u^{\alpha}}{\partial x^i} = x_i^{\alpha} \qquad (i = 1, 2, 3 \text{ and } \alpha = 1, 2) \tag{78}$$

Hence, they can be regarded as 2D contravariant surface vectors or as 3D covariant space vectors.

• The contravariant and covariant forms of the surface basis vectors, $\mathbf{E}^{\alpha}$ and $\mathbf{E}_{\alpha}$, are obtained from each other by the index-shifting operator for the surface, that is:

$$\mathbf{E}_{\alpha} = a_{\alpha\beta}\mathbf{E}^{\beta} \qquad \mathbf{E}^{\alpha} = a^{\alpha\beta}\mathbf{E}_{\beta} \qquad (\alpha, \beta = 1, 2) \tag{79}$$

where the indexed a are the covariant and contravariant forms of the surface metric tensor (see § 3.1).

• The contravariant and covariant forms of the surface basis vectors, $\mathbf{E}^{\alpha}$ and $\mathbf{E}_{\alpha}$, are reciprocal systems and hence they satisfy the following relations:

$$\mathbf{E}_{\alpha} \cdot \mathbf{E}^{\beta} = \delta_{\alpha}^{\beta} \equiv a_{\alpha}^{\beta} \qquad \mathbf{E}^{\alpha} \cdot \mathbf{E}_{\beta} = \delta_{\beta}^{\alpha} \equiv a_{\beta}^{\alpha} \qquad (\alpha, \beta = 1, 2) \tag{80}$$

3.1 Surface Metric Tensor

• The surface metric tensor[71] is an absolute, rank-2, 2×2 symmetric tensor.[72]

• Following the example of the metric in general nD spaces, as explained in [9, 10], the

[71] In differential geometry, the surface metric tensor $a_{\alpha\beta}$ may be called the first groundform.

[72] The coefficients of the metric tensor are real numbers.

surface metric tensor of a 2D surface embedded in a 3D Euclidean flat space with metric $g_{ij} = \delta_{ij}$ is given in its covariant form by:

$$a_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} \tag{81}$$

where the indexed x and u are the space Cartesian coordinates and the surface curvilinear coordinates respectively, and $i = 1, 2, 3$ and $\alpha, \beta = 1, 2$.

• The surface and space metric tensors in a general Riemannian space with general metric g_{ij} are related by:

$$a_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} = g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta} = g_{ij}x^i_\alpha x^j_\beta \tag{82}$$

where $a_{\alpha\beta}$ and g_{ij} are respectively the surface and space covariant metric tensors, the indexed x and u are the curvilinear coordinates of the space and surface respectively, and $i, j = 1, 2, 3$ and $\alpha, \beta = 1, 2$. It is noteworthy that Eq. 81 is a special instance of Eq. 82 for the case of a flat space with a Cartesian system where the space metric is the unity tensor.

• Eq. 82 is the fundamental relation that provides the crucial link between the surface and its enveloping space. As indicated before, the partial derivatives in this relations, $\frac{\partial x^i}{\partial u^\alpha}$ and $\frac{\partial x^j}{\partial u^\beta}$, may be considered as contravariant rank-1 3D space tensors or as covariant rank-1 2D surface tensors. A tensor like $\frac{\partial x^i}{\partial u^\alpha}$ is usually labeled as x^i_α where it represents two surface vectors which are contravariantly-transformed with respect to the three space coordinates x^i:

$$x^i_1 = \left(\frac{\partial x^1}{\partial u^1}, \frac{\partial x^2}{\partial u^1}, \frac{\partial x^3}{\partial u^1}\right) \qquad\qquad x^i_2 = \left(\frac{\partial x^1}{\partial u^2}, \frac{\partial x^2}{\partial u^2}, \frac{\partial x^3}{\partial u^2}\right) \tag{83}$$

or three space vectors which are covariantly-transformed with respect to the two surface coordinates u^α:

$$x^1_\alpha = \left(\frac{\partial x^1}{\partial u^1}, \frac{\partial x^1}{\partial u^2}\right) \qquad x^2_\alpha = \left(\frac{\partial x^2}{\partial u^1}, \frac{\partial x^2}{\partial u^2}\right) \qquad x^3_\alpha = \left(\frac{\partial x^3}{\partial u^1}, \frac{\partial x^3}{\partial u^2}\right) \tag{84}$$

• Any surface vector A^α $(\alpha = 1, 2)$, defined as a linear combination of the surface basis vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, can also be considered as a space vector A^i $(i = 1, 2, 3)$ where the two representations are linked through the relation:

$$A^i = \frac{\partial x^i}{\partial u^\alpha} A^\alpha = x^i_\alpha A^\alpha \qquad (i = 1, 2, 3 \text{ and } \alpha = 1, 2) \tag{85}$$

Now, since we have (see Eqs. 82 and 85):

$$a_{\alpha\beta} A^\alpha A^\beta = g_{ij} x^i_\alpha x^j_\beta A^\alpha A^\beta = g_{ij} x^i_\alpha A^\alpha x^j_\beta A^\beta = g_{ij} A^i A^j \tag{86}$$

then the two representations are equivalent, that is they define a vector of the same magnitude and direction.

• The surface basis vectors in their covariant and contravariant forms, x^i_α and x^α_i, and the unit vector normal to the surface n_i are linked by the following relation:

$$x^i_\alpha = \underline{\epsilon}^{ijk} \underline{\epsilon}_{\alpha\beta} x^\beta_j n_k \tag{87}$$

This equation means that the given product (which looks like a vector cross product) of the surface contravariant basis vector x^β_j and the unit normal vector n_k produces a surface covariant basis vector x^i_α and hence it is perpendicular to both.[73]

• The contravariant form of the surface metric tensor is defined as the inverse of the surface

[73] Being a surface basis vector implies orthogonality to the unit normal vector, while being a covariant surface basis vector implies orthogonality to the contravariant surface basis vector.

covariant metric tensor[74], that is:

$$a^{\alpha\gamma}\, a_{\gamma\beta} = \delta^{\alpha}_{\ \beta} \qquad\qquad a_{\alpha\gamma}\, a^{\gamma\beta} = \delta_{\alpha}^{\ \beta} \tag{88}$$

• Similar to the metric tensor in general nD spaces, the covariant and contravariant forms of the surface metric tensor, $a_{\alpha\beta}$ and $a^{\alpha\beta}$, are used for lowering and raising indices and related tensor operations.

• The covariant form of the surface metric tensor $a_{\alpha\beta}$ is given by:

$$[a_{\alpha\beta}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{89}$$

where E, F, G are the coefficients of the first fundamental form (refer to § 3.3).

• The contravariant form of the surface metric tensor $a^{\alpha\beta}$ is the inverse of its covariant form and hence it is given by:

$$[a^{\alpha\beta}] = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \tag{90}$$

where the symbols are as defined previously.

• The mixed form of the surface metric tensor a^{α}_{β} is the identity tensor, that is:

$$[a^{\alpha}_{\beta}] = [\delta^{\alpha}_{\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{91}$$

• Similar to the space metric tensor (refer to [9, 10]), the surface metric tensor transforms

[74]Since the first fundamental form is positive definite (see § 3.3), and hence $a > 0$, the existence of an inverse is guaranteed.

between the barred and unbarred surface coordinate systems as:

$$\bar{a}_{\alpha\beta} = a_{\gamma\delta} \frac{\partial u^{\gamma}}{\partial \bar{u}^{\alpha}} \frac{\partial u^{\delta}}{\partial \bar{u}^{\beta}} \tag{92}$$

where the indexed $\bar{a}$ and a are the surface covariant metric tensors in the barred and unbarred systems respectively. The contravariant and mixed forms of the surface metric tensor also follow similar rules to their counterparts of the space metric tensor as outlined in [9, 10].

• Similar to the determinants of the space metric (refer to [10]), the determinants of the surface metric in the barred and unbarred coordinate systems are linked through the Jacobian of transformation by the following relation:

$$\bar{a} = J^2 a \tag{93}$$

where $\bar{a}$ and a are respectively the determinants of the covariant form of the surface metric tensor in the barred and unbarred systems respectively and $J \left(= \left|\frac{\partial u}{\partial \bar{u}}\right|\right)$ is the Jacobian of the transformation between the two surface systems. This relation can be obtained directly by taking the determinant of the two sides of Eq. 92.

• The Christoffel symbols of the first kind $[\alpha\beta, \gamma]$ are linked to the basis vectors and their partial derivatives by the following relation:

$$[\alpha\beta, \gamma] = \frac{\partial \mathbf{E}_{\alpha}}{\partial u^{\beta}} \cdot \mathbf{E}_{\gamma} \tag{94}$$

• The relation between the partial derivative of the surface metric tensor and the Christoffel symbols of the first kind is given by:

$$\frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}} = \frac{\partial \left(\mathbf{E}_{\alpha} \cdot \mathbf{E}_{\beta}\right)}{\partial u^{\gamma}} = \frac{\partial \mathbf{E}_{\alpha}}{\partial u^{\gamma}} \cdot \mathbf{E}_{\beta} + \mathbf{E}_{\alpha} \cdot \frac{\partial \mathbf{E}_{\beta}}{\partial u^{\gamma}} = [\alpha\gamma, \beta] + [\beta\gamma, \alpha] \tag{95}$$

• Similar relations between the surface metric tensor and the Christoffel symbols of the second kind can be obtained from the equations in the previous point by using the index-raising operator for the surface.

• Scaling a surface up or down by a constant factor $c > 0$, which is equivalent to scaling all the distances on the surface by that factor, can be done by multiplying the surface metric tensor by c^2.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the surface covariant metric tensor $\mathbf{a}$ is given by:

$$\mathbf{a} \equiv \mathbf{I}_S = \begin{bmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{bmatrix} \tag{96}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates, and $\mathbf{I}_S$ is the tensor of the first fundamental form of the surface (see § 3.3).

• If $C : I \to S$ is a regular curve on a surface S defined on the interval $I \subseteq \mathbb{R}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel vector fields over C, then the dot product associated with the metric tensor $\mathbf{v}_1 \cdot \mathbf{v}_2$, the norm of the vector fields $|\mathbf{v}_1|$ and $|\mathbf{v}_2|$, and the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ are constants.

• In the following subsections, we investigate arc length, area and angle between two vectors on a surface. All these entities depend in their definition and quantification on the metric tensor. We will see that their geometric and tensor formulation is identical to that given in [9, 10] for a general nD space with the use of the surface metric tensor and the surface representation of the involved vectors.

3.1.1 Arc Length

• Following the example of the length of an element of arc of a curve embedded in a general nD space, the length of an element of arc of a curve on a 2D surface is given in its general form by:

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta = \mathbf{E}_\alpha \cdot \mathbf{E}_\beta \, du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta \tag{97}$$

where $a_{\alpha\beta}$ is the covariant type of the surface metric tensor, $\mathbf{r}$ is the spatial representation of the curve and $\alpha, \beta = 1, 2$.

• From the above formula we have the following identity which is valid at each point of a surface curve:

$$a_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 1 \tag{98}$$

• Based on the above formula (Eq. 97), the length of a segment of a t-parameterized curve between a starting point corresponding to $t = t_1$ and a terminal point corresponding to $t = t_2$ is given by:

$$\begin{aligned} L &= \int_I ds \\ &= \int_{t_1}^{t_2} \frac{ds}{dt} dt \\ &= \int_{t_1}^{t_2} \sqrt{a_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt}} \, dt \\ &= \int_{t_1}^{t_2} \sqrt{a_{11} \left(\frac{\partial u^1}{\partial t} \right)^2 + 2a_{12} \frac{\partial u^1}{\partial t} \frac{\partial u^2}{\partial t} + a_{22} \left(\frac{\partial u^2}{\partial t} \right)^2} \, dt \\ &= \int_{t_1}^{t_2} \sqrt{E \left(\frac{\partial u^1}{\partial t} \right)^2 + 2F \frac{\partial u^1}{\partial t} \frac{\partial u^2}{\partial t} + G \left(\frac{\partial u^2}{\partial t} \right)^2} \, dt \end{aligned} \tag{99}$$

where $I \subset \mathbb{R}$ is an interval on the real line and E, F, G are the coefficients of the first fundamental form.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the length of an element of arc of a curve is given by:

$$ds = \sqrt{(1 + f_u^2) \, du du + 2 f_u f_v du dv + (1 + f_v^2) \, dv dv} \tag{100}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

• The length of a space curve is an intrinsic property since it depends on the metric tensor only.

• The length of a space curve is invariant with respect to the type of parameterization.

3.1.2 Surface Area

• The area of an infinitesimal element of a surface in the neighborhood of a point P on the surface is given by:[75]

$$\begin{aligned} d\sigma &= |d\mathbf{r}_1 \times d\mathbf{r}_2| \\ &= |\mathbf{E}_1 \times \mathbf{E}_2|\, du^1 du^2 \\ &= \sqrt{|\mathbf{E}_1|^2 |\mathbf{E}_2|^2 - (\mathbf{E}_1 \cdot \mathbf{E}_2)^2}\, du^1 du^2 \\ &= \sqrt{a_{11}a_{22} - (a_{12})^2}\, du^1 du^2 \\ &= \sqrt{EG - F^2}\, du^1 du^2 \\ &= \sqrt{a}\, du^1 du^2 \end{aligned} \tag{101}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the surface covariant basis vectors, E, F, G are the coefficients of the first fundamental form, a is the determinant of the surface covariant metric tensor and the indexed a are its elements.[76] All the quantities in these expressions belong to the point P.

• The area of a surface patch $S : \Omega \to \mathbb{R}^3$, where Ω is a proper subset of the $\mathbb{R}^2$ plane, is given by:[77]

$$\sigma = \int_\Omega d\sigma = \iint_\Omega \sqrt{a_{11}a_{22} - (a_{12})^2}\, du^1 du^2 = \iint_\Omega \sqrt{EG - F^2}\, du^1 du^2 = \iint_\Omega \sqrt{a}\, du^1 du^2 \tag{102}$$

[75] Here, we assume $du^1 du^2$ is positive.

[76] $a = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - (a_{12})^2$.

[77] S should be injective, sufficiently differentiable, and regular on the interior of Ω.

• The formulae for the area are reminder of the volume formulae (see [10]), so the area can be regraded as a *volume* in a 2D space.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the surface area is given by:

$$\sigma = \iint_{\Omega} \sqrt{1 + f_u^2 + f_v^2}\, du dv \tag{103}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

3.1.3 Angle Between Two Surface Curves

• The angle between two sufficiently smooth surface curves intersecting at a given point on the surface is defined as the angle between their tangent vectors at that point. As there are two opposite directions for each curve, corresponding to the two senses of traversing the curve, there are two main angles θ_1 and θ_2 such that $\theta_1 + \theta_2 = \pi$. The principal angle between the two curves is usually taken as the smaller of the two angles and hence the directions are determined accordingly.[78]

• The angle between two surface curves passing through a given point P on the surface with tangent vectors $\mathbf{A}$ and $\mathbf{B}$ at P is given by:

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{a_{\alpha\beta} A^{\alpha} B^{\beta}}{\sqrt{a_{\gamma\delta} A^{\gamma} A^{\delta}}\sqrt{a_{\epsilon\zeta} B^{\epsilon} B^{\zeta}}} \tag{104}$$

where the indexed a are the elements of the surface covariant metric tensor and the Greek indices run over $1, 2$.

• If $\mathbf{A}$ and $\mathbf{B}$ are two unit surface vectors with surface representations A^{δ} and B^{δ} $(\delta = 1, 2)$ and space representations A^k and B^k $(k = 1, 2, 3)$ then the angle θ between the two vectors

[78] In fact there are still two possibilities for the directions but this has no significance as far as the angle between the two curves is concerned.

is given by (see Eqs. 82 and 85):

$$\cos\theta = a_{\alpha\beta}A^{\alpha}B^{\beta} = g_{ij}x^{i}_{\alpha}x^{j}_{\beta}A^{\alpha}B^{\beta} = g_{ij}A^{i}B^{j} \qquad (\alpha,\beta = 1,2 \text{ and } i,j = 1,2,3) \tag{105}$$

and hence the surface and space representations of the two vectors define the same angle.

• The vectors **A** and **B** in the previous points are orthogonal *iff* $a_{\alpha\beta}A^{\alpha}B^{\beta} = 0$ at P.

• The coordinate curves at a given point P on a surface are orthogonal *iff* $a_{12} \equiv F = 0$ at P.

• The corresponding angles of two isometric surfaces, like the corresponding lengths, are equal. However, the reverse is not true in general, that is the equality of angles on two surfaces related by a given mapping, as in the conformal mapping, does not lead to the equality of the corresponding lengths on the two mapped surfaces.

• The sine of the angle θ ($\leq \pi$) between two surface unit vectors, **A** and **B**, is given by:

$$\sin\theta = \underline{\epsilon}_{\alpha\beta}A^{\alpha}B^{\beta} \tag{106}$$

which is numerically equal to the area of the parallelogram with sides **A** and **B**. Consequently, the sufficient and necessary condition for **A** and **B** to be orthogonal is that:

$$\left|\underline{\epsilon}_{\alpha\beta}A^{\alpha}B^{\beta}\right| = 1 \tag{107}$$

3.2 Surface Curvature Tensor

• The surface curvature tensor[79], $b_{\alpha\beta}$, is an absolute, rank-2, 2×2 symmetric tensor.[80]

[79]The surface curvature tensor $b_{\alpha\beta}$ may be called the second groundform.
[80]The coefficients of the curvature tensor are real numbers.

• The elements of the surface covariant curvature tensor are given by:

$$b_{\alpha\beta} = -\frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{n}}{\partial u^\beta} = -\mathbf{E}_\alpha \cdot \frac{\partial \mathbf{n}}{\partial u^\beta} = -\frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{n}}{\partial u^\beta} + \frac{\partial \mathbf{r}}{\partial u^\beta} \cdot \frac{\partial \mathbf{n}}{\partial u^\alpha}\right) \tag{108}$$

and also by:[81]

$$b_{\alpha\beta} = \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \cdot \mathbf{n} = \frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot \mathbf{n} = \frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot \left(\frac{\mathbf{E}_1 \times \mathbf{E}_2}{\sqrt{a}}\right) = \frac{\frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot (\mathbf{E}_1 \times \mathbf{E}_2)}{\sqrt{a}} \tag{109}$$

• Considering Eq. 109, the symmetry of the surface curvature tensor (i.e. $b_{12} = b_{21}$) follows from the fact that:

$$\frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} = \frac{\partial \mathbf{r}}{\partial u^\beta \partial u^\alpha} = \frac{\partial \mathbf{r}}{\partial u^\alpha \partial u^\beta} = \frac{\partial \mathbf{E}_\beta}{\partial u^\alpha} \tag{110}$$

• In full tensor notation, the surface covariant curvature tensor is given by:

$$b_{\alpha\beta} = \frac{1}{2}\underline{\epsilon}^{\gamma\delta}\underline{\epsilon}_{ijk} x^i_{\alpha;\beta} x^j_\gamma x^k_\delta = \frac{1}{\sqrt{a}}\underline{\epsilon}_{ijk} x^i_{\alpha;\beta} x^j_1 x^k_2 \tag{111}$$

where a is the determinant of the surface covariant metric tensor. This formula will simplify to the following when the space coordinates are rectangular Cartesian:

$$b_{\alpha\beta} = \frac{1}{\sqrt{a}}\epsilon_{ijk}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} x^j_1 x^k_2 \tag{112}$$

• The surface curvature tensor obeys the same transformation rules as the surface metric tensor. Hence, we have:

$$\bar{b} = J^2 b \tag{113}$$

where $\bar{b}$ and b are respectively the determinants of the surface covariant curvature tensor

[81] The equality $\frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot \mathbf{n} = -\mathbf{E}_\alpha \cdot \frac{\partial \mathbf{n}}{\partial u^\beta}$ is based on the equality $\frac{\partial (\mathbf{E}_\alpha \cdot \mathbf{n})}{\partial u^\beta} = \frac{\partial (0)}{\partial u^\beta} = 0$ and the product rule for differentiation.

in the barred and unbarred systems respectively and J $\left(= \left|\frac{\partial u}{\partial \bar{u}}\right|\right)$ is the Jacobian of the transformation between the two surface curvilinear coordinate systems.

• The surface covariant curvature tensor $b_{\alpha\beta}$ in matrix form is given by:

$$\mathbf{b} \equiv [b_{\alpha\beta}] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} \tag{114}$$

where e, f, g are the coefficients of the second fundamental form of the surface (refer to § 3.4).

• The mixed form of the surface curvature tensor $b^{\alpha}{}_{\beta}$ is given by:[82]

$$\left[b^{\alpha}{}_{\beta}\right] = [a^{\alpha\gamma} b_{\gamma\beta}] = \frac{1}{a} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \frac{1}{a} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix} \tag{115}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms and $a = EG - F^2$ is the determinant of the surface covariant metric tensor. As seen, the coefficients of b^{α}_{β} depend on the coefficients of both the first and second fundamental forms.

•The contravariant form of the surface curvature tensor is given by:

$$\begin{aligned} \left[b^{\alpha\beta}\right] &= \left[a^{\alpha\gamma} b_{\gamma}{}^{\beta}\right] \\ &= \left[b^{\alpha}{}_{\gamma} a^{\gamma\beta}\right] \\ &= \frac{1}{a^2} \begin{bmatrix} eG^2 - 2fFG + gF^2 & fEG - eFG - gEF + fF^2 \\ fEG - eFG - gEF + fF^2 & gE^2 - 2fEF + eF^2 \end{bmatrix} \end{aligned} \tag{116}$$

where the symbols are as explained in the previous point. Like the covariant form, the contravariant form is a symmetric tensor.

[82]The mixed form $b_{\alpha}{}^{\beta} = b_{\alpha\gamma} a^{\gamma\beta}$ is the transpose of the given form.

• As we will see, the trace of the surface mixed curvature tensor b^{α}_{β} is twice the mean curvature H (see § 4.4.2), while its determinant is the Gaussian curvature K (see § 4.4.1), that is:[83]

$$H = \frac{\text{tr}(b^{\alpha}_{\beta})}{2} \qquad\qquad K = \det(b^{\alpha}_{\beta}) \tag{117}$$

• The surface covariant curvature tensor of a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$ is given by:

$$\mathbf{b} \equiv [b_{\alpha\beta}] = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix} \tag{118}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates. Since $f_{uv} = f_{vu}$, the tensor is symmetric as it should be.

• The Riemann-Christoffel curvature tensor of the second kind and the curvature tensor of a surface are linked by the following relation:

$$R^{\delta}{}_{\alpha\beta\gamma} = b_{\alpha\gamma} b^{\delta}_{\ \beta} - b_{\alpha\beta} b^{\delta}_{\ \gamma} \tag{119}$$

This relation may also be given in terms of the Riemann-Christoffel curvature tensor of the first kind using the index-lowering operator for the surface:

$$a_{\alpha\omega} R^{\omega}{}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} = b_{\alpha\gamma} b_{\beta\delta} - b_{\alpha\delta} b_{\beta\gamma} \tag{120}$$

• From Eqs. 13 and 119, the surface curvature tensor and the surface Christoffel symbols of the second kind and their derivatives are related by:

$$b_{\alpha\gamma} b^{\delta}_{\beta} - b_{\alpha\beta} b^{\delta}_{\gamma} = \frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} + \Gamma^{\omega}_{\alpha\gamma} \Gamma^{\delta}_{\omega\beta} - \Gamma^{\omega}_{\alpha\beta} \Gamma^{\delta}_{\omega\gamma} \tag{121}$$

[83] Since the trace and the determinant of a tensor are its main two invariants under permissible transformations, then H and K are invariant, as will be established in the forthcoming notes.

• Considering the fact that the 2D Riemann-Christoffel curvature tensor has only one degree of freedom and hence it possesses a single independent non-vanishing component which is represented by R_{1212}, we see that Eq. 120 has only one independent component which is given by:

$$R_{1212} = b_{11}b_{22} - b_{12}b_{21} = b \tag{122}$$

where b is the determinant of the surface covariant curvature tensor. This equation shows that each one of the following provisions: $R_{\alpha\beta\gamma\delta} = 0$ and $b_{\alpha\beta} = 0$ if satisfied identically is a sufficient and necessary condition for having a flat 2D space, i.e. a plane surface. Hence, for a plane surface, all the coefficients of the Riemann-Christoffel curvature tensor and the coefficients of the surface curvature tensor vanish identically throughout the surface.

• From Eqs. 13 and 120 we see that the Riemann-Christoffel curvature tensor can be expressed in terms of the coefficients of the surface curvature tensor as well as in terms of the coefficients of the surface metric tensor where the two sets of coefficients are linked through Eq. 121.[84]

• The sign of the surface curvature tensor $\mathbf{b}$ (i.e. the sign of its coefficients) is dependent on the choice of the direction of the unit vector $\mathbf{n}$ normal to the surface.

3.3 First Fundamental Form

• As indicated previously, the first fundamental form[85], which is based on the metric, encompasses all the intrinsic information about the surface that a 2D inhabitant of the surface can obtain from measurements performed on the surface without appealing to an external dimension.

• The first fundamental form of the length of an element of arc of a curve on a surface is

[84] This does not mean that Riemann curvature is an extrinsic property but it means that some intrinsic properties can also be defined in terms of extrinsic parameters.

[85] In older books, this is also called the first fundamental quadratic form.

a quadratic expression given by:

$$\begin{aligned} I_S &= (ds)^2 \\ &= d\mathbf{r} \cdot d\mathbf{r} \\ &= a_{\alpha\beta} du^\alpha du^\beta \\ &= E(du^1)^2 + 2F\, du^1 du^2 + G(du^2)^2 \end{aligned} \tag{123}$$

where E, F and G, which in general are continuous variable functions of the surface coordinates (u, v), are given by:

$$\begin{aligned} E &= a_{11} = \mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^1} = g_{ij} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^1} \\ F &= a_{12} = \mathbf{E}_1 \cdot \mathbf{E}_2 = \frac{\partial \mathbf{r}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} = g_{ij} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} = \mathbf{E}_2 \cdot \mathbf{E}_1 = a_{21} \\ G &= a_{22} = \mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{\partial \mathbf{r}}{\partial u^2} \cdot \frac{\partial \mathbf{r}}{\partial u^2} = g_{ij} \frac{\partial x^i}{\partial u^2} \frac{\partial x^j}{\partial u^2} \end{aligned} \tag{124}$$

where the indexed a are the elements of the surface covariant metric tensor, the indexed x are the curvilinear coordinates of the enveloping space and g_{ij} is its covariant metric tensor.

• For a flat space with a Cartesian coordinate system x^i, the space metric is $g_{ij} = \delta_{ij}$ and hence the above equations become:

$$\begin{aligned} E &= \frac{\partial x^i}{\partial u^1} \frac{\partial x^i}{\partial u^1} \\ F &= \frac{\partial x^i}{\partial u^1} \frac{\partial x^i}{\partial u^2} \\ G &= \frac{\partial x^i}{\partial u^2} \frac{\partial x^i}{\partial u^2} \end{aligned} \tag{125}$$

• The first fundamental form can be cast in the following matrix form:

$$
\begin{aligned}
I_S &= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \cdot \mathbf{E}_1 & \mathbf{E}_1 \cdot \mathbf{E}_2 \\ \mathbf{E}_2 \cdot \mathbf{E}_1 & \mathbf{E}_2 \cdot \mathbf{E}_2 \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \mathbf{v}\, \mathbf{I}_S\, \mathbf{v}^T
\end{aligned}
\tag{126}
$$

where $\mathbf{v}$ is a direction vector, $\mathbf{v}^T$ is its transpose, and $\mathbf{I}_S$ is the first fundamental form tensor which is the surface covariant metric tensor. Hence, the matrix associated with the first fundamental form is the covariant metric tensor of the surface.

• The first fundamental form is not a unique characteristic of the surface and hence two geometrically different surfaces as seen from the enveloping space, such as plane and cylinder, can have the same first fundamental form.[86]

• The first fundamental form is positive definite at regular points of 2D surfaces, hence its coefficients are subject to the conditions $E > 0$ and $\det(\mathbf{I}_S) = EG - F^2 > 0$.[87] However, this condition may be amended to allow for metrics with imaginary coordinates as it is the case in the coordinate systems of relativistic mechanics.

• As indicated previously, the first fundamental form encompasses the intrinsic properties of the surface geometry. Hence, as seen in § 3.1, the first fundamental form is used to define

[86] Such surfaces are different extrinsically as seen from the embedding space although they are identical intrinsically as viewed internally from the surface by a 2D inhabitant.

[87] As indicated earlier, the conditions $E > 0$ and $EG - F^2 > 0$ imply $G > 0$.

and quantify things like arc length, area and angle between curves on a surface based on its qualification as a metric. For example, Eq. 99 shows that the length of a curve segment on a surface is obtained by integrating the square root of the first fundamental form of the surface along the segment.

• If a surface S_1 can be mapped isometrically (see § 1.13) onto another surface S_2 then the two surfaces have identical first fundamental form coefficients, that is $E_1 = E_2$, $F_1 = F_2$ and $G_1 = G_2$ where the subscripts are labels for the two surfaces.

• From a previous point (see Eq. 96), for a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the first fundamental form is given by:

$$I_S = \left(1 + f_u^2\right) dudu + 2f_u f_v dudv + \left(1 + f_v^2\right) dvdv \tag{127}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

3.4 Second Fundamental Form

• The mathematical entity that characterizes the extrinsic geometry of a surface is the normal vector to the surface. This entity can only be observed externally from outside the surface by an observer in a reference frame in the space that envelopes the surface. Hence, the normal vector and all its subsidiaries are strange to a 2D inhabitant to the surface who can only access the intrinsic attributes of the surface as represented by and contained in the first fundamental form.

• As a consequence of the last point, the variation of the normal vector as it moves around the surface can be used as an indicator to characterize the surface shape from an external point of view and that is how this indicator is exploited in the second fundamental form to represent the extrinsic geometry of the surface as will be seen from the forthcoming formulations.

• The following quadratic expression is called the second fundamental form[88], of the surface:

$$\begin{aligned} II_S &= -d\mathbf{r}\cdot d\mathbf{n} \\ &= -\left(\frac{\partial \mathbf{r}}{\partial u^\alpha}du^\alpha\right)\cdot\left(\frac{\partial \mathbf{n}}{\partial u^\beta}du^\beta\right) \\ &= -\left(\frac{\partial \mathbf{r}}{\partial u^1}du^1+\frac{\partial \mathbf{r}}{\partial u^2}du^2\right)\cdot\left(\frac{\partial \mathbf{n}}{\partial u^1}du^1+\frac{\partial \mathbf{n}}{\partial u^2}du^2\right) \\ &= e(du^1)^2+2f\,du^1du^2+g(du^2)^2 \end{aligned} \tag{128}$$

where[89]

$$\begin{aligned} e &= -\frac{\partial \mathbf{r}}{\partial u^1}\cdot\frac{\partial \mathbf{n}}{\partial u^1} = -\mathbf{E}_1\cdot\frac{\partial \mathbf{n}}{\partial u^1} \\ f &= -\frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial u^1}\cdot\frac{\partial \mathbf{n}}{\partial u^2}+\frac{\partial \mathbf{r}}{\partial u^2}\cdot\frac{\partial \mathbf{n}}{\partial u^1}\right) = -\frac{1}{2}\left(\mathbf{E}_1\cdot\frac{\partial \mathbf{n}}{\partial u^2}+\mathbf{E}_2\cdot\frac{\partial \mathbf{n}}{\partial u^1}\right) \\ g &= -\frac{\partial \mathbf{r}}{\partial u^2}\cdot\frac{\partial \mathbf{n}}{\partial u^2} = -\mathbf{E}_2\cdot\frac{\partial \mathbf{n}}{\partial u^2} \end{aligned} \tag{129}$$

In the above equations, $\mathbf{r}(u^1,u^2)$ is the spatial representation of the surface, $\mathbf{n}(u^1,u^2)$ is the unit vector normal to the surface and $\alpha,\beta=1,2$.

[88] In older books, this is also called the second fundamental quadratic form.

[89] Some authors use L, M, N instead of e, f, g. However, the use of e, f, g is advantageous since they correspond to the coefficients of the first fundamental form E, F, G nicely making the formulae involving the first and second fundamental forms more symmetric and memorable. On the other hand, the use of L, M, N is also advantageous when reading formulae containing the coefficients of both fundamental forms; moreover, it is less susceptible to errors when writing or typing these formulae. Another point is that the coefficient g should not be confused with the determinant of the space covariant metric tensor which is used in the previous notes [10] but we do not use it in the present text.

• The second fundamental form is also given by:

$$
\begin{aligned}
II_S &= d^2\mathbf{r} \cdot \mathbf{n} \\
&= \left(\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} du^\alpha du^\beta \right) \cdot \mathbf{n} \\
&= \left(\frac{\partial \mathbf{E}_1}{\partial u^1} (du^1)^2 + 2 \frac{\partial \mathbf{E}_1}{\partial u^2} du^1 du^2 + \frac{\partial \mathbf{E}_2}{\partial u^2} (du^2)^2 \right) \cdot \mathbf{n} \\
&= \frac{\partial \mathbf{E}_1}{\partial u^1} \cdot \mathbf{n} \, (du^1)^2 + 2 \frac{\partial \mathbf{E}_1}{\partial u^2} \cdot \mathbf{n} \, du^1 du^2 + \frac{\partial \mathbf{E}_2}{\partial u^2} \cdot \mathbf{n} \, (du^2)^2
\end{aligned}
\tag{130}
$$

where $d^2\mathbf{r}$ is the second order differential of the position vector $\mathbf{r}$ of an arbitrary point on the surface in the direction (du^1, du^2).

• From the last point, the coefficients of the second fundamental form can also be given by the following alternative expressions:

$$
\begin{aligned}
e &= \mathbf{n} \cdot \frac{\partial \mathbf{E}_1}{\partial u^1} = -\frac{\partial \mathbf{n}}{\partial u^1} \cdot \mathbf{E}_1 \\
f &= \mathbf{n} \cdot \frac{\partial \mathbf{E}_1}{\partial u^2} = \mathbf{n} \cdot \frac{\partial \mathbf{E}_2}{\partial u^1} = -\frac{\partial \mathbf{n}}{\partial u^2} \cdot \mathbf{E}_1 = -\frac{\partial \mathbf{n}}{\partial u^1} \cdot \mathbf{E}_2 \\
g &= \mathbf{n} \cdot \frac{\partial \mathbf{E}_2}{\partial u^2} = -\frac{\partial \mathbf{n}}{\partial u^2} \cdot \mathbf{E}_2
\end{aligned}
\tag{131}
$$

and also by:

$$
\begin{aligned}
e &= \frac{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot \frac{\partial \mathbf{E}_1}{\partial u^1}}{\sqrt{a}} \\
f &= \frac{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot \frac{\partial \mathbf{E}_1}{\partial u^2}}{\sqrt{a}} \\
g &= \frac{(\mathbf{E}_1 \times \mathbf{E}_2) \cdot \frac{\partial \mathbf{E}_2}{\partial u^2}}{\sqrt{a}}
\end{aligned}
\tag{132}
$$

where $a = a_{11}a_{22} - a_{12}a_{21} = EG - F^2$ is the determinant of the surface covariant metric tensor.

• The second fundamental form can be cast in the following matrix form:

$$\begin{aligned}
II_S &= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} -\mathbf{E}_1 \\ -\mathbf{E}_2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \mathbf{n}}{\partial u^1} & \frac{\partial \mathbf{n}}{\partial u^2} \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} -\mathbf{E}_1 \cdot \frac{\partial \mathbf{n}}{\partial u^1} & -\mathbf{E}_1 \cdot \frac{\partial \mathbf{n}}{\partial u^2} \\ -\mathbf{E}_2 \cdot \frac{\partial \mathbf{n}}{\partial u^1} & -\mathbf{E}_2 \cdot \frac{\partial \mathbf{n}}{\partial u^2} \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{E_1}}{\partial u^1} \cdot \mathbf{n} & \frac{\partial \mathbf{E_1}}{\partial u^2} \cdot \mathbf{n} \\ \frac{\partial \mathbf{E_2}}{\partial u^1} \cdot \mathbf{n} & \frac{\partial \mathbf{E_2}}{\partial u^2} \cdot \mathbf{n} \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \qquad (133) \\
&= \begin{bmatrix} du^1 & du^2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix} \\
&= \mathbf{v}\, \mathbf{II}_S\, \mathbf{v}^T
\end{aligned}$$

where $\mathbf{v}$ is a direction vector, $\mathbf{v}^T$ is its transpose, and $\mathbf{II}_S$ is the second fundamental form tensor.

• Like the coefficients of the first fundamental form, the coefficients of the second fundamental form are, in general, continuous variable functions of the surface coordinates (u, v).

• The coefficients of the second fundamental form tensor satisfy the following relations:

$$e = b_{11} \qquad f = b_{12} = b_{21} \qquad g = b_{22} \qquad (134)$$

Hence, the second fundamental form tensor $\mathbf{II}_S$ is the same as the surface covariant curvature tensor $\mathbf{b}$, that is:

$$\mathbf{II}_S = \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \mathbf{b} \qquad (135)$$

• From the previous points, we see that the second fundamental form can also be given by:

$$II_S = b_{\alpha\beta} du^\alpha du^\beta \tag{136}$$

where the indexed b are the elements of the surface covariant curvature tensor and $\alpha, \beta = 1, 2$.

• The second fundamental form of the surface can be expressed in terms of the first fundamental form I_S and the normal curvature κ_n (see § 4.2) of the surface at a given point and in a given direction as:

$$II_S = \kappa_n I_S = \kappa_n (ds)^2 \tag{137}$$

• From a previous point (see Eq. 118), for a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the second fundamental form is given by:

$$II_S = \frac{f_{uu} dudu + 2 f_{uv} dudv + f_{vv} dvdv}{\sqrt{1 + f_u^2 + f_v^2}} \tag{138}$$

where the subscripts u and v stand for partial derivatives with respect to these surface curvilinear coordinates.

• The second fundamental form is invariant under permissible coordinate transformations that maintain the sense of the normal vector to the surface, $\mathbf{n}$. The second fundamental form changes its sign if the sense of $\mathbf{n}$ is reversed.

• As indicated earlier, while the first fundamental form encompasses the intrinsic geometry of the surface, the second fundamental form encompasses its extrinsic geometry.

• As seen, the first fundamental form is associated with the surface covariant metric tensor, while the second fundamental form is associated with the surface covariant curvature tensor.

• While the first fundamental form is positive definite, as stated previously, the second fundamental form is not necessarily positive or definite.

• Unlike space curves which are completely defined by specified curvature and torsion, κ and τ, providing arbitrary first and second fundamental forms is not a sufficient condition for the existence of a surface with these forms, because the first and second fundamental forms do not provide full identification for the surface. In a more technical terms, defining six functions E, F, G, e, f and g of class C^3 on a subset of $\mathbb{R}^2$ where these functions satisfy the conditions for the coefficients of the first and second fundamental forms (in particular $E, G > 0$ and $EG - F^2 > 0$) does not guarantee the existence of a surface over the given subset with a first fundamental form $E\left(du^1\right)^2 + 2F du^1 du^2 + G\left(du^2\right)^2$ and a second fundamental form $e\left(du^1\right)^2 + 2f du^1 du^2 + g\left(du^2\right)^2$. Further compatibility conditions relating the first and second fundamental forms are required to fully identify the surface and secure its existence.[90]

• Following the last point, the required compatibility conditions for the existence of a surface with predefined first and second fundamental forms are given by the Codazzi-Mainardi equations (Eq. 161) plus the following equation:

$$\begin{aligned} eg - f^2 \quad = \quad & F\left[\frac{\partial \Gamma^2_{22}}{\partial u} - \frac{\partial \Gamma^2_{12}}{\partial v} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}\right] + \\ & E\left[\frac{\partial \Gamma^1_{22}}{\partial u} - \frac{\partial \Gamma^1_{12}}{\partial v} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}\right] \end{aligned} \tag{139}$$

• From the last two points the fundamental theorem of surfaces in differential geometry, which is the equivalent of the fundamental theorem of curves (see § 2.1), emerges. The

[90] This may be linked to the fact that the curve conditions are established based on the existence theorem for ordinary differential equations where these equations generally have a solution, while the surface conditions should be established based on the existence theorem for partial differential equations which have solutions only when they meet additional integrability conditions. The details should be sought in more extensive books of differential geometry.

theorem states that: given six sufficiently-smooth functions E, F, G, e, f and g on a subset of $\mathbb{R}^2$ satisfying the following conditions:

(A) $E, G > 0$ and $EG - F^2 > 0$, and

(B) E, F, G, e, f and g satisfy Eqs. 139 and 161,

then there is a unique surface with E, F, G as its first fundamental form coefficients and e, f, g as its second fundamental form coefficients. Hence, if two surfaces meet all these conditions, then they are identical within a rigid motion transformation in space.

• As seen, the fundamental theorem of surfaces, like the fundamental theorem of curves, provides the existence and uniqueness conditions for surfaces.

• On the other hand, according to the theorem of Bonnet, if two surfaces of class C^3, $S_1 : \Omega \to \mathbb{R}^3$ and $S_2 : \Omega \to \mathbb{R}^3$, are defined over a connected set $\Omega \subseteq \mathbb{R}^2$ with identical first and second fundamental forms, then the two surfaces can be mapped on each other by a purely rigid motion transformation.

• Two surfaces having identical first fundamental forms but different second fundamental forms may be described as applicable. An example of applicable surfaces are plane and cylinder.

3.4.1 Dupin Indicatrix

• Dupin indicatrix at a given point P on a sufficiently smooth surface is a function of the coefficients of the second fundamental form at the point and hence it is a function of the surface coordinates at P.

• Dupin indicatrix is an indicator of the departure of the surface from the tangent plane in the close proximity of the point of tangency. Accordingly, the second fundamental form is used in Dupin indicatrix to measure this departure.

• In quantitative terms, Dupin indicatrix is the family of conic sections given by the following quadratic equation:

$$ex^2 + 2fxy + gy^2 = \pm 1 \tag{140}$$

where e, f, g are the coefficients of the second fundamental form at P.

• As a consequence of the previous points, Dupin indicatrix can be used to classify the surface points with respect to the local shape of the surface as flat, elliptic, parabolic or hyperbolic; more details about this are given in § 4.4.5.

3.5 Third Fundamental Form

• The third fundamental form of a space surface is defined by:

$$III_S = d\mathbf{n} \cdot d\mathbf{n} = c_{\alpha\beta} du^\alpha du^\beta \tag{141}$$

where $\mathbf{n}$ is the unit vector normal to the surface at a given point P, $c_{\alpha\beta}$ are the coefficients of the third fundamental form at P and $\alpha, \beta = 1, 2$.

• The coefficients of the third fundamental form are given by:[91]

$$c_{\alpha\beta} = g_{ij} n^i{}_{;\alpha} n^j{}_{;\beta} \tag{142}$$

where g_{ij} is the space covariant metric tensor and the indexed n is the unit vector normal to the surface.

• The coefficients of the third fundamental form are also given by:

$$c_{\alpha\beta} = a^{\gamma\delta} b_{\alpha\gamma} b_{\beta\delta} \tag{143}$$

[91] These coefficients are real numbers.

where $a^{\gamma\delta}$ is the surface contravariant metric tensor and the indexed b are the coefficients of the surface covariant curvature tensor.

• The first, second and third fundamental forms are linked, through the Gaussian curvature K and the mean curvature H (see § 4.4.1 and 4.4.2), by the following relation:

$$KI_S - 2HII_S + III_S = 0 \tag{144}$$

• The coefficients of the first, second and third fundamental forms are correlated, through the mean curvature H and the Gaussian curvature K, by the following relation:

$$Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} = 0 \tag{145}$$

By multiplying both sides with $a^{\alpha\beta}$ and contracting we obtain:

$$Ka^{\alpha}_{\alpha} - 2Hb^{\alpha}_{\alpha} + c^{\alpha}_{\alpha} = 0 \tag{146}$$

that is:[92]

$$\text{tr}\left(c_{\alpha\beta}\right) = 4H^2 - 2K \tag{147}$$

3.6 Relationship between Surface Basis Vectors and their Derivatives

• The focus of this subsection is the equations of Gauss and Weingarten which, for surfaces, are the analogue of the equations of Frenet-Serret for curves. While the Frenet-Serret formulae express the derivatives of $\mathbf{T}, \mathbf{N}, \mathbf{B}$ as combinations of these vectors using κ and τ as coefficients, the equations of Gauss and Weingarten express the derivatives of $\mathbf{E}_1, \mathbf{E}_2, \mathbf{n}$ as combinations of these vectors with coefficients based on the first and second fundamental

[92] We have: $a^{\alpha}_{\alpha} = \delta^{\alpha}_{\alpha} = \delta^1_1 + \delta^2_2 = 2$ and $H = \frac{\text{tr}(b_{\alpha\beta})}{2}$ and hence the formula is justified.

forms.

• As shown earlier (see § 2), three unit vectors can be constructed on each point at which the curvature does not vanish of a class C^2 space curve: the tangent $\mathbf{T}$, the normal $\mathbf{N}$ and the binormal $\mathbf{B}$. These mutually orthogonal vectors (i.e. $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$) can serve as a set of basis vectors. Hence, the derivatives of these vectors with respect to the distance traversed along the curve, s, can be expressed as combinations of this set as demonstrated by the Frenet-Serret formulae (refer to § 2.3).

• Similarly, the surface vectors: $\mathbf{E}_1 = \frac{\partial \mathbf{r}}{\partial u^1}$, $\mathbf{E}_2 = \frac{\partial \mathbf{r}}{\partial u^2}$ and the unit vector normal to the surface, $\mathbf{n}$, at each regular point on a class C^2 surface also form a basis set and hence their partial derivatives with respect to the surface curvilinear coordinates, u^1 and u^2, can be expressed as combinations of this set. The equations of Gauss and Weingarten demonstrate this fact.

• The equations of Gauss express the partial derivatives of the surface vectors, $\mathbf{E}_1$ and $\mathbf{E}_2$, with respect to the surface curvilinear coordinates as combinations of the surface basis set, that is:

$$\begin{aligned}\frac{\partial \mathbf{E}_1}{\partial u^1} &= \Gamma^1_{11}\mathbf{E}_1 + \Gamma^2_{11}\mathbf{E}_2 + e\mathbf{n} \\ \frac{\partial \mathbf{E}_1}{\partial u^2} &= \Gamma^1_{12}\mathbf{E}_1 + \Gamma^2_{12}\mathbf{E}_2 + f\mathbf{n} = \frac{\partial \mathbf{E}_2}{\partial u^1} \\ \frac{\partial \mathbf{E}_2}{\partial u^2} &= \Gamma^1_{22}\mathbf{E}_1 + \Gamma^2_{22}\mathbf{E}_2 + g\mathbf{n}\end{aligned} \tag{148}$$

where e, f, g are the coefficients of the second fundamental form. These equations can be expressed compactly, with partial use of tensor notation, as:

$$\frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{E}_\gamma + b_{\alpha\beta}\mathbf{n} \qquad (\alpha, \beta = 1, 2) \tag{149}$$

where the Christoffel symbol $\Gamma^\gamma_{\alpha\beta}$ is based on the surface metric, as given by Eq. 11, and

$b_{\alpha\beta}$ is the surface covariant curvature tensor. The last equation can be expressed in full tensor notation as:

$$x^i{}_{\alpha,\beta} = \Gamma^{\gamma}_{\alpha\beta} x^i_{\gamma} + b_{\alpha\beta} n^i \tag{150}$$

• Likewise, the equations of Weingarten express the partial derivatives of the unit vector normal to the surface, **n**, with respect to the surface curvilinear coordinates as combinations of the surface vectors, $\mathbf{E}_1$ and $\mathbf{E}_2$, that is:

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial u^1} &= \frac{fF - eG}{a} \mathbf{E}_1 + \frac{eF - fE}{a} \mathbf{E}_2 \\ \frac{\partial \mathbf{n}}{\partial u^2} &= \frac{gF - fG}{a} \mathbf{E}_1 + \frac{fF - gE}{a} \mathbf{E}_2 \end{aligned} \tag{151}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms and $a = EG - F^2$ is the determinant of the surface covariant metric tensor. These equations can be expressed compactly, with partial use of tensor notation, as:

$$\frac{\partial \mathbf{n}}{\partial u^{\alpha}} = -b_{\alpha}^{\ \beta} \mathbf{E}_{\beta} \tag{152}$$

where $b_{\alpha}^{\ \beta}$ $(= b_{\alpha\gamma} a^{\gamma\beta})$ is the mixed type of the surface curvature tensor, $b_{\alpha\gamma}$ is the surface covariant curvature tensor and $a^{\gamma\beta}$ is the surface contravariant metric tensor. They can also be expressed with full use of tensor notation employing curvilinear space coordinates as:[93]

$$n^i{}_{,\alpha} = -b_{\alpha\gamma} a^{\gamma\beta} x^i_{\beta} = -b_{\alpha}^{\ \beta} x^i_{\beta} \tag{153}$$

[93] The vector $n^i{}_{,\alpha}$ is orthogonal to n^i and hence it is parallel to the tangent space of the surface, so it can be expressed as a linear combination of the surface basis vectors x^i_{β}, that is $n^i{}_{,\alpha} = d^{\beta}_{\alpha} x^i_{\beta}$ for a certain set of coefficients $d^{\beta}_{\alpha} = -b_{\alpha\gamma} a^{\gamma\beta}$, as given above.

• Weingarten equations can be expressed in matrix form as:

$$\begin{bmatrix} \frac{\partial \mathbf{n}}{\partial u^1} \\ \frac{\partial \mathbf{n}}{\partial u^2} \end{bmatrix} = -\mathbf{II}_S \mathbf{I}_S^{-1} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} \tag{154}$$

where $\mathbf{II}_S$ is the surface covariant curvature tensor and $\mathbf{I}_S^{-1}$ is the surface contravariant metric tensor.

• The partial derivatives of the unit vector normal to the surface, $\mathbf{n}$, with respect to the surface curvilinear coordinates (u^1, u^2) are linked to the Gaussian curvature K (see § 4.4.1) and the surface basis vectors, $\mathbf{E}_1$ and $\mathbf{E}_2$, by the following relation:

$$\frac{\partial \mathbf{n}}{\partial u^1} \times \frac{\partial \mathbf{n}}{\partial u^2} = \frac{eg - f^2}{EG - F^2} (\mathbf{E}_1 \times \mathbf{E}_2) = K (\mathbf{E}_1 \times \mathbf{E}_2) \tag{155}$$

• The partial derivatives of the unit vector normal to the surface, $\mathbf{n}$, with respect to the surface curvilinear coordinates (u^1, u^2) are linked to the Gaussian and mean curvatures and the coefficients of the first and second fundamental forms by the following relations:

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial u^1} \cdot \frac{\partial \mathbf{n}}{\partial u^1} &= 2eH - EK \\ \frac{\partial \mathbf{n}}{\partial u^1} \cdot \frac{\partial \mathbf{n}}{\partial u^2} &= 2fH - FK \\ \frac{\partial \mathbf{n}}{\partial u^2} \cdot \frac{\partial \mathbf{n}}{\partial u^2} &= 2gH - GK \end{aligned} \tag{156}$$

where K is the Gaussian curvature, H is the mean curvature, and E, F, G, e, f, g are the coefficients of the first and second fundamental forms.

• The above equations of Weingarten (Eq. 151) can be solved for the surface basis vectors, $\mathbf{E}_1$ and $\mathbf{E}_2$, and hence these vectors can be expressed as combinations of the partial derivatives of the normal vector, $\mathbf{n}$, that is:

$$\begin{aligned}\mathbf{E}_1 &= \frac{fF - gE}{b}\frac{\partial \mathbf{n}}{\partial u^1} + \frac{fE - eF}{b}\frac{\partial \mathbf{n}}{\partial u^2} \\ \mathbf{E}_2 &= \frac{fG - gF}{b}\frac{\partial \mathbf{n}}{\partial u^1} + \frac{fF - eG}{b}\frac{\partial \mathbf{n}}{\partial u^2}\end{aligned} \tag{157}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms and $b = eg - f^2$ is the determinant of the surface covariant curvature tensor.

• From Eq. 149, it can be seen that the coefficients of the surface covariant curvature tensor, $b_{\alpha\beta}$, are the projections of the partial derivative of the surface basis vectors, $\frac{\partial \mathbf{E}_\alpha}{\partial u^\beta}$, in the direction of the unit vector normal to the surface, $\mathbf{n}$, that is:

$$b_{\alpha\beta} = \frac{\partial \mathbf{E}_\alpha}{\partial u^\beta} \cdot \mathbf{n} \tag{158}$$

• The main conclusion from the above equations of Gauss and Weingarten is that the partial derivatives of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{n}$ can be represented as combinations of these vectors with coefficients provided by the elements of the first and second fundamental forms and their partial derivatives.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the Gauss equations are given by:

$$\begin{aligned}\frac{\partial \mathbf{E}_1}{\partial u} &= \frac{1}{1 + f_u^2 + f_v^2}\left(f_u f_{uu}\mathbf{E}_1 + f_v f_{uu}\mathbf{E}_2 + f_{uu}\sqrt{1 + f_u^2 + f_v^2}\mathbf{n}\right) \\ \frac{\partial \mathbf{E}_1}{\partial v} &= \frac{1}{1 + f_u^2 + f_v^2}\left(f_u f_{uv}\mathbf{E}_1 + f_v f_{uv}\mathbf{E}_2 + f_{uv}\sqrt{1 + f_u^2 + f_v^2}\mathbf{n}\right) = \frac{\partial \mathbf{E}_2}{\partial u} \\ \frac{\partial \mathbf{E}_2}{\partial v} &= \frac{1}{1 + f_u^2 + f_v^2}\left(f_u f_{vv}\mathbf{E}_1 + f_v f_{vv}\mathbf{E}_2 + f_{vv}\sqrt{1 + f_u^2 + f_v^2}\mathbf{n}\right)\end{aligned} \tag{159}$$

where the subscripts u and v represent partial derivatives with respect to the surface curvilinear coordinates u and v.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the Weingarten equations are given by:

$$\begin{aligned}\frac{\partial \mathbf{n}}{\partial u} &= \frac{\left(f_u f_v f_{uv} - f_{uu} f_v^2 - f_{uu}\right) \mathbf{E}_1 + \left(f_u f_v f_{uu} - f_u^2 f_{uv} - f_{uv}\right) \mathbf{E}_2}{\sqrt{\left(1 + f_u^2 + f_v^2\right)^3}} \\ \frac{\partial \mathbf{n}}{\partial v} &= \frac{\left(f_u f_v f_{vv} - f_{uv} f_v^2 - f_{uv}\right) \mathbf{E}_1 + \left(f_u f_v f_{uv} - f_u^2 f_{vv} - f_{vv}\right) \mathbf{E}_2}{\sqrt{\left(1 + f_u^2 + f_v^2\right)^3}}\end{aligned} \tag{160}$$

where the subscripts u and v are as explained in the previous point.

3.6.1 Codazzi-Mainardi Equations

• From the aforementioned equations of Gauss and Weingarten, supported by further compatibility conditions, the following equations, called Codazzi or Codazzi-Mainardi equations, can be derived:

$$\begin{aligned}\frac{\partial b_{12}}{\partial u^1} - \frac{\partial b_{11}}{\partial u^2} &= b_{22}\Gamma^2_{11} - b_{12}\left(\Gamma^2_{12} - \Gamma^1_{11}\right) - b_{11}\Gamma^1_{12} \\ \frac{\partial b_{22}}{\partial u^1} - \frac{\partial b_{21}}{\partial u^2} &= b_{22}\Gamma^2_{12} - b_{12}\left(\Gamma^2_{22} - \Gamma^1_{12}\right) - b_{11}\Gamma^1_{22}\end{aligned} \tag{161}$$

where the Christoffel symbols are based on the surface metric. These equations can be expressed compactly in tensor notation as:

$$\frac{\partial b_{\alpha\beta}}{\partial u^\delta} - \frac{\partial b_{\alpha\delta}}{\partial u^\beta} = b_{\gamma\beta}\Gamma^\gamma_{\alpha\delta} - b_{\gamma\delta}\Gamma^\gamma_{\alpha\beta} \tag{162}$$

• If we arrange the terms of the Codazzi-Mainardi equation (Eq. 162) and subtract the term $\Gamma^\gamma_{\delta\beta} b_{\alpha\gamma}$ from both sides we obtain:

$$\frac{\partial b_{\alpha\beta}}{\partial u^\delta} - b_{\gamma\beta}\Gamma^\gamma_{\alpha\delta} - \Gamma^\gamma_{\delta\beta} b_{\alpha\gamma} = \frac{\partial b_{\alpha\delta}}{\partial u^\beta} - b_{\gamma\delta}\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\delta\beta} b_{\alpha\gamma} \tag{163}$$

which can be expressed compactly, using the covariant derivative notation, as:

$$b_{\alpha\beta;\gamma} = b_{\alpha\gamma;\beta} \tag{164}$$

• The Codazzi-Mainardi equations in the form given by Eq. 164 reveals that there are only two independent components for these equations because, adding to the fact that all the indices range over 1 and 2, the covariant derivative according to Eq. 164 is symmetric in its last two indices (i.e. β and γ), and the covariant curvature tensor is symmetric in its two indices (i.e. $b_{\alpha\beta} = b_{\beta\alpha}$).[94] These two independent components are given by:

$$b_{\alpha\alpha;\beta} = b_{\alpha\beta;\alpha} \tag{165}$$

where $\alpha \neq \beta$ and there is no summation over α. Writing these equations in full, using the covariant derivative expression[95] and noting that one term of the covariant derivative expression is the same on both sides and hence it drops away, we have:

$$\frac{\partial b_{\alpha\alpha}}{\partial u^\beta} - \Gamma^\delta_{\alpha\beta} b_{\alpha\delta} = \frac{\partial b_{\alpha\beta}}{\partial u^\alpha} - \Gamma^\delta_{\alpha\alpha} b_{\delta\beta} \qquad (\alpha \neq \beta, \text{ no sum on } \alpha) \tag{166}$$

• There is also another more general equation called the Gauss-Codazzi equation which is given by:

$$R^\delta{}_{\gamma\alpha\beta} x^i_\delta = x^i_\delta b^\delta_\alpha b_{\beta\gamma} - x^i_\delta b^\delta_\beta b_{\alpha\gamma} + n^i b_{\alpha\gamma;\beta} - n^i b_{\beta\gamma;\alpha} \tag{167}$$

The tangential component of this equation is the *Theorema Egregium*[96] while its normal component is the Codazzi equation.

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[94] As a consequence of these two symmetries, the covariant derivative of the surface covariant curvature tensor, $b_{\alpha\beta;\gamma}$, is fully symmetric in all of its indices.

[95] That is $b_{\alpha\beta;\gamma} = \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \Gamma^\delta_{\alpha\gamma} b_{\delta\beta} - \Gamma^\delta_{\beta\gamma} b_{\alpha\delta}$.

[96] In the form given by Eqs. 119 and 120. Refer to § 4.4.3 about the essence of *Theorema Egregium* as an expression of the fact that certain types of curvature are intrinsic properties of the surface and hence they can be evaluated from purely intrinsic parameters obtained from the first fundamental form.

3.7 Gauss Mapping

• The Gauss mapping or sphere mapping is a correlation between the points of a surface and the unit sphere where each point on the surface is projected onto its unit normal as a point on the unit sphere centered at the origin of coordinates.[97]

• In technical terms, let S be a surface embedded in an $\mathbb{R}^3$ space and S_1 represents the origin-centered unit sphere in this space, then Gauss mapping is given by:

$$\left\{N : S \rightarrow S_1,\ N(P) = \check{P}\right\} \tag{168}$$

where the point $P(x, y, z)$ on the trace of S is mapped by N onto the point $\check{P}(\check{x}, \check{y}, \check{z})$ on the trace of the unit sphere with $\check{x}, \check{y}, \check{z}$ being the coordinates of the origin-based position vector of the normal vector to the surface, $\mathbf{n}$, at P.

• The image $\bar{\mathfrak{S}}$ on the unit sphere of a Gauss mapping of a region $\mathfrak{S}$ on S is called the spherical image of $\mathfrak{S}$.

• To have a single-valued sphere mapping, the functional relation representing the surface S should be one-to-one.

• The limit of the ratio of the area of a region $\bar{\mathcal{R}}$ on the spherical image to the area of a region $\mathcal{R}$ on the surface in the neighborhood of a given point P equals the absolute value of the Gaussian curvature $|K|$ at P as $\mathcal{R}$ shrinks to the point P, that is:

$$\lim_{\mathcal{R}\rightarrow P} \frac{\sigma(\bar{\mathcal{R}})}{\sigma(\mathcal{R})} = |K_P| \tag{169}$$

where σ stands for area, and K_P is the Gaussian curvature at P. The tendency of $\mathcal{R}$ to P should be understood in the given sense.

• At a given point P on a surface, where the Gaussian curvature is non-zero, there exists

[97] This sort of mapping for surfaces is similar to the spherical indicatrix mapping (see § 5.5) for space curves.

a neighborhood $\mathcal{N}$ of P where an injective mapping can be established between $\mathcal{N}$ and its spherical image $\bar{\mathcal{N}}$.

• A conformal correspondence can be established between a surface and its spherical image *iff* the surface is a sphere or a minimal surface (see § 6.7).

3.8 Global Surface Theorems

• In this subsection we state a few global theorems related to surfaces.

• Planes are the only connected surfaces of class C^2 whose all points are flat.

• Spheres are the only connected closed surfaces of class C^3 whose all points are spherical umbilical.

• Spheres are the only connected compact surfaces of class C^3 with constant Gaussian curvature.

• Spheres are the only connected compact surfaces with constant mean curvature and positive Gaussian curvature.

• Also refer to § 4.4.4 for the global form of the Gauss-Bonnet theorem.

4 Curvature

• "Curvature" is a property of both curves and surfaces at a given point which is determined by the shape of the curve or surface at that point.[98] In this section we investigate this property in its general sense and examine the main parameters used to describe and quantify it.

4.1 Curvature Vector

• At a given point P on a surface S, a plane containing the unit vector $\mathbf{n}$ normal to the surface at P intersects the surface in a surface curve C having a tangent vector $\mathbf{t}$ at P. The curve C is called the normal section of S at P in the direction of $\mathbf{t}$. This curve can be parameterized by s, representing the distance traversed along the curve starting from an arbitrary given point P_0, and hence it is defined by the position vector $\mathbf{r} = \mathbf{r}(s)$. At point P, the vector $\mathbf{T} = \frac{d\mathbf{r}}{ds}$ is a unit vector tangent to C at P in the direction of increasing s and contained in the plane tangent to the surface at P as defined previously in § 2.[99]

• The curvature vector of C at P, which is orthogonal to $\mathbf{T}$, is defined by:[100]

$$\mathbf{K} = \frac{d\mathbf{T}}{ds} \tag{170}$$

• The curvature, κ, of C at P (which is defined previously in § 2) is the magnitude of the curvature vector, that is: $\kappa = |\mathbf{K}|$, and the radius of curvature when $\kappa \neq 0$ is its reciprocal, i.e. $R_\kappa = \frac{1}{\kappa}$.[101] The curvature vector can then be expressed as:

[98] There are also global characteristics of curvature like total curvature K_t (see § 4.4.4) but they are based in general on the local characterization of curvature at individual points.

[99] The vector $\mathbf{T}$ can be parallel or anti-parallel to $\mathbf{t}$. We chose to introduce $\mathbf{t}$ and define it in this way for generality since the sense of increasing s and hence the curve orientation can be in one direction or the other.

[100] $\mathbf{K}$ here is the uppercase Greek letter kappa.

[101] This is the radius of the osculating circle of C at P.

$$\mathbf{K} = |\mathbf{K}| \frac{\mathbf{K}}{|\mathbf{K}|} = \kappa \mathbf{N} \tag{171}$$

where $\mathbf{N}$ is the unit vector normal to the curve C at P as defined previously in § 2.

• The surface curvature vector is independent of the orientation and parameterization of the surface and the curve. However, in general it is a function of the position on the surface and the direction and hence it depends on the point and direction.[102]

• A point on the curve at which $\mathbf{K} = \mathbf{0}$, and hence $\kappa = 0$, is called inflection point. At such points, the radius of curvature is infinite and the normal vector $\mathbf{N}$ and the osculating circle are not defined.[103]

• On introducing a new unit vector, which is orthogonal simultaneously to $\mathbf{n}$ and $\mathbf{T}$ and defined by the following cross product:

$$\mathbf{u} = \mathbf{n} \times \mathbf{T} \tag{172}$$

the curvature vector, which lies in the plane spanned by $\mathbf{n}$ and $\mathbf{u}$, can then be resolved in the $\mathbf{n}$ and $\mathbf{u}$ directions as:

$$\mathbf{K} = \mathbf{K}_n + \mathbf{K}_g = \kappa_n \mathbf{n} + \kappa_g \mathbf{u} \tag{173}$$

where the subscripts n and g are labels and not indices, and κ_n and κ_g are respectively the "normal curvature" and the "geodesic curvature".

[102] Direction here refers to the tangential direction as represented by a vector lying in the tangent plane of the surface at a given point on the surface.

[103] As we assume that the curve is of class C^2, the curvature vector varies smoothly and hence at points of inflection on such a curve $\mathbf{N}$ may be defined in such a way to ensure continuity when this is possible, which is not always the case even for some C^∞ curves unless the curve is analytic in which case the curve can have a continuous normal vector in the neighborhood of inflection points. However, this does not apply to straight lines where the curvature is identically zero and hence $\mathbf{N}$ is not defined naturally on any point on the curve although it can be defined as a constant vector in an arbitrary direction over the whole line although the above relations between the curve basis vectors will not hold.

• The normal and geodesic curvatures are given respectively by:

$$\begin{aligned} \kappa_n &= \mathbf{n} \cdot \mathbf{K} = -\mathbf{T} \cdot \frac{d\mathbf{n}}{ds} = -\frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{n}}{ds} \\ \kappa_g &= \mathbf{u} \cdot \mathbf{K} = \mathbf{u} \cdot \frac{d\mathbf{T}}{ds} = (\mathbf{n} \times \mathbf{T}) \cdot \frac{d\mathbf{T}}{ds} \end{aligned} \tag{174}$$

• The vector $\mathbf{u}$, which is a unit vector normal to the curve C, is called the geodesic normal vector. This vector is the orthonormal projection of $\mathbf{K}$ onto the tangent space and hence it is contained in the tangent space of the surface at point P.

• While the normal curvature κ_n is an extrinsic property, since it depends on the first and second fundamental form coefficients, the geodesic curvature κ_g in an intrinsic property as it depends only on the first fundamental form coefficients and their derivatives.

• The triad $(\mathbf{n}, \mathbf{T}, \mathbf{u})$ is another roaming frame in use in differential geometry in addition to the Frenet curve-based frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ and the surface-based frame $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{n})$.

• Let C be a curve on a sufficiently smooth surface S. If ϕ is the angle between the principal normal vector N^i of C at a given point P and the unit vector n_i normal to the surface at P then we have:

$$\cos\phi = n_i N^i \tag{175}$$

The normal and geodesic curvatures, κ_n and κ_g, of C at P are then given by:

$$\kappa_n = \kappa \cos\phi \qquad\qquad \kappa_g = \kappa \sin\phi \tag{176}$$

where κ is the curvature of C at P as defined previously. According to the theorem of Meusnier (see § 4.2.1), all the curves on S that pass through P with the same tangent direction at P have the same normal curvature.[104]

[104] The theorem of Meusnier may be stated in this context as: the curvature of any surface curve at a given point P on the curve equals the curvature of the normal section which is tangent to the curve at P divided by the cosine of the angle between the principal normal to the curve at P and the normal to the surface at P.

4.2 Normal Curvature

• Using the first and second fundamental forms, given by Eqs. 123 and 128, the normal curvature in the $\frac{du}{dv}$ direction can be expressed as the following quotient of the second fundamental form involving the coefficients of the surface covariant curvature tensor to the first fundamental form involving the coefficients of the surface covariant metric tensor:[105]

$$\kappa_n = \frac{II_S}{I_S} = \frac{e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\gamma\delta}du^\gamma du^\delta} \tag{177}$$

where the overdot means differentiation with respect to a general parameter t of the curve. This equation can be obtained as follow:

$$\begin{aligned}\kappa_n &= -\mathbf{T} \cdot \frac{d\mathbf{n}}{ds} \\ &= -\left(\frac{\partial \mathbf{r}}{\partial u^\alpha}\frac{du^\alpha}{ds}\right) \cdot \left(\frac{\partial \mathbf{n}}{\partial u^\beta}\frac{du^\beta}{ds}\right) \\ &= -\left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{n}}{\partial u^\beta}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds} \\ &= \frac{-\left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{n}}{\partial u^\beta}\right) du^\alpha du^\beta}{ds\, ds} \\ &= \frac{b_{\alpha\beta}du^\alpha du^\beta}{(ds)^2} \\ &= \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\gamma\delta}du^\gamma du^\delta}\end{aligned} \tag{178}$$

• From the previous point, it can be seen that the normal component of the curvature vector is also given by:

$$\mathbf{K}_n = \left[e\left(\frac{du^1}{ds}\right)^2 + 2f\frac{du^1}{ds}\frac{du^2}{ds} + g\left(\frac{du^2}{ds}\right)^2\right]\mathbf{n} \tag{179}$$

• From Eq. 177, it can be seen that the sign of the normal curvature κ_n (i.e. being greater

[105] The last part of the above equation should be interpreted as the sum of the terms in the numerator divided by the sum of the terms in the denominator.

than, less than or equal to zero) is determined solely by the sign of the second fundamental form since the first fundamental form is positive definite. Now, since the sign of the second fundamental form is dependent on its determinant b, this sign is determined by b.

• As given earlier, all surface curves passing through a point P on a surface and have the same tangent line at P have identical normal curvature at P. Hence, the normal curvature is a property of the surface at a given point and in a given direction.

• The normal curvature κ_n of a normal section C of a surface at a point P is equal in magnitude to the curvature κ of C at P, i.e. $|\kappa_n| = \kappa$. This can be explained by the fact that the normal vector $\mathbf{n}$ to the surface at P is collinear with the principal normal vector $\mathbf{N}$ of C at P so there is only a normal component to the curvature vector with no geodesic component.

• The normal curvature κ_n of a surface at a given point and in a given direction is an invariant property apart from its sign which is dependent on the choice of the direction of the unit vector $\mathbf{n}$ normal to the surface.

• At every point on a sphere and in any direction, the normal curvature is a constant given by: $|\kappa_n| = \frac{1}{R}$ where R is the sphere radius.

• At any point P on a sphere, any surface curve C passing through P in any direction is a normal section *iff* C is a great circle.[106] All these great circles have constant curvature κ and normal curvature κ_n which are equal in magnitude to $\frac{1}{R}$ where R is the sphere radius.

• At flat points on a surface $\kappa_n = 0$ in all directions. At elliptic points $\kappa_n \neq 0$ and have the same sign in all directions. At parabolic points κ_n have the same sign in all directions except in the direction for which the second fundamental form vanishes where $\kappa_n = 0$. At hyperbolic points κ_n is negative, positive or zero depending on the direction (for the definition of flat, elliptic, parabolic and hyperbolic points see § 4.4.5).

• In any two orthogonal directions at a given point P on a sufficiently smooth surface, the

[106]The great circles of a sphere are the plane sections formed by the intersection of the sphere with the planes passing through the center of the sphere.

sum of the normal curvatures corresponding to these directions at P is constant.

• At any point P of a sufficiently smooth surface S, there exists a paraboloid tangent at its vertex to the tangent plane of S at P such that the normal curvature of the paraboloid in a given direction at P is equal to the normal curvature of S at P in that direction.

• The normal curvature κ_n is an extrinsic property, since it depends on the second fundamental form coefficients.[107]

• The normal curvatures of surface curves at a given point P in the directions of the u and v coordinate curves are given respectively by $\frac{b_{11}}{a_{11}}$ and $\frac{b_{22}}{a_{22}}$ where these are evaluated at P.

• As we will see (refer to § 4.4), at each non-umbilical point P of a sufficiently smooth surface there are two perpendicular directions along which the normal curvature of the surface at P takes its maximum and minimum values of all the normal curvature values at P (for the definition of umbilical point see § 4.4.6).

• The necessary and sufficient condition for a given point P on a sufficiently smooth surface S to be umbilical point is that the coefficients of the first and second fundamental forms of the surface are proportional, that is:

$$\frac{e}{E} = \frac{f}{F} = \frac{g}{G} \, (= \kappa_n) \tag{180}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms at P, and κ_n is the normal curvature of S at P in any direction.[108]

4.2.1 Meusnier Theorem

• According to the theorem of Meusnier, all the surface curves passing through a point P on a surface and have the same tangent direction[109] at P have identical normal curvatures

[107] That is it cannot be expressed purely in terms of the first fundamental form coefficients.
[108] This can be seen from Eq. 177 where κ_n in this case becomes independent from the direction.
[109] It should be non-asymptotic direction (see § 5.9).

which is the normal curvature κ_n of the normal section at P in the given direction. Moreover, the osculating circles of these curves lie on a sphere S_s with radius $\frac{1}{\kappa}$ and with center at $\mathbf{r}_C = \mathbf{r}_P + \frac{\mathbf{N}}{\kappa}$ where κ (which is equal to κ_n) is the curvature of the normal section at P, $\mathbf{N}$ is the principal normal vector of the normal section at P and $\mathbf{r}_P$ is the position vector of P.

• Following the last point, we have:

(A) The center of the sphere S_s is the center of curvature of the normal section at P in the given direction (see § 2.4).

(B) These curves are characterized by being tangent to the normal section at P in the given direction and by being plane sections of the surfaces with shared tangent direction at P.

(C) The osculating circles of these curves are the intersection of the sphere S_s with the osculating planes of these curves.

(D) The sphere S_s is tangent to the tangent plane of the surface at P.

• The theorem of Meusnier may also be stated as follow: the center of curvature of a surface curve at a given point P on the curve is obtained by orthogonal projection of the center of curvature of the normal section, which is tangent to the curve at P, on the osculating plane of the curve.

4.3 Geodesic Curvature

• As described earlier, the curvature vector $\mathbf{K}$ of a surface curve lies in a plane perpendicular to $\mathbf{T}$ and it can be resolved into a normal component $\mathbf{K}_n = \kappa_n \mathbf{n}$ and a geodesic component $\mathbf{K}_g = \kappa_g \mathbf{u}$ where the normal and geodesic curvatures, κ_n and κ_g, are given by Eq. 174.

• The geodesic component $\mathbf{K}_g$ of the curvature vector $\mathbf{K}$ of a surface curve at a given point P is the projection of $\mathbf{K}$ onto the tangent space $T_P S$ of the surface at P.

• The geodesic component of the curvature vector of a surface curve is given by:

$$\mathbf{K}_g = \kappa_g \mathbf{u} = \left(\frac{d^2u^1}{ds^2} + \Gamma^1_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right) \mathbf{E}_1 + \left(\frac{d^2u^2}{ds^2} + \Gamma^2_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right) \mathbf{E}_2 \tag{181}$$

where the Christoffel symbols are derived from the surface metric.

• While the curvature vector $\mathbf{K}$ is an extrinsic property of the surface, the geodesic curvature vector $\mathbf{K}_g$ is an intrinsic property. This can be seen, for example, from Eq. 181 where all the elements in the equation (Christoffel symbols and surface basis vectors) are solely dependent on the first fundamental form.

• The geodesic component of the curvature vector is also given by:

$$\mathbf{K}_g = \left[\mathbf{n} \times \left(\frac{\partial \mathbf{E}_1}{\partial u^1} \left(\frac{du^1}{ds} \right)^2 + 2 \frac{\partial \mathbf{E}_1}{\partial u^2} \frac{du^1}{ds} \frac{du^2}{ds} + \frac{\partial \mathbf{E}_2}{\partial u^2} \left(\frac{du^2}{ds} \right)^2 \right) \right] \times \mathbf{n} + \mathbf{E}_1 \frac{d^2u^1}{ds^2} + \mathbf{E}_2 \frac{d^2u^2}{ds^2} \tag{182}$$

• It can be shown that κ_g can also be given by:

$$\begin{aligned} \kappa_g =& \sqrt{a} \Bigg[\Gamma^2_{11} \left(\frac{du^1}{ds} \right)^3 + \left(2\Gamma^2_{12} - \Gamma^1_{11} \right) \left(\frac{du^1}{ds} \right)^2 \frac{du^2}{ds} + \\ & \left(\Gamma^2_{22} - 2\Gamma^1_{12} \right) \frac{du^1}{ds} \left(\frac{du^2}{ds} \right)^2 - \Gamma^1_{22} \left(\frac{du^2}{ds} \right)^3 + \frac{du^1}{ds} \frac{d^2u^2}{ds^2} - \frac{d^2u^1}{ds^2} \frac{du^2}{ds} \Bigg] \end{aligned} \tag{183}$$

where the Christoffel symbols are derived from the surface metric and $a = EG - F^2$ is the determinant of the surface covariant metric tensor.

• On the u^1 coordinate curves (i.e. in the orientation of $\mathbf{E}_1$), $\frac{du^1}{ds} = \frac{1}{\sqrt{E}}$ and $\frac{du^2}{ds} = 0$; hence Eq. 183 will simplify to:

$$(\kappa_g)_{\mathbf{E}_1} = \sqrt{a} \, \Gamma^2_{11} \left(\frac{du^1}{ds} \right)^3 = \frac{\sqrt{a}}{E^{3/2}} \Gamma^2_{11} \tag{184}$$

The last formula will be simplified further if the surface coordinate curves u^1, u^2 are orthogonal, since in this case $F = 0$ and $\Gamma^2_{11} = -\frac{E_v}{2G}$ (see Eq. 11), and the formula will become:

$$(\kappa_g)_{\mathbf{E}_1} = -\frac{E_v}{2E\sqrt{G}} \tag{185}$$

• On the u^2 coordinate curves (i.e. in the orientation of $\mathbf{E}_2$), $\frac{du^1}{ds} = 0$ and $\frac{du^2}{ds} = \frac{1}{\sqrt{G}}$; hence Eq. 183 will simplify to:

$$(\kappa_g)_{\mathbf{E}_2} = -\sqrt{a}\,\Gamma^1_{22}\left(\frac{du^2}{ds}\right)^3 = -\frac{\sqrt{a}}{G^{3/2}}\,\Gamma^1_{22} \tag{186}$$

The last formula will be simplified further if the coordinate curves u^1, u^2 are orthogonal, since in this case $F = 0$ and $\Gamma^1_{22} = -\frac{G_u}{2E}$ (see Eq. 11), and the formula will become:

$$(\kappa_g)_{\mathbf{E}_2} = \frac{G_u}{2G\sqrt{E}} \tag{187}$$

• Geodesic curvature can take any real value: positive, negative or zero. Since $\mathbf{u} = \mathbf{n} \times \mathbf{T}$, the sense of the geodesic curvature vector depends on the orientation of the surface and the orientation of the curve.

• Geodesic curvature can also be calculated extrinsically by:

$$\kappa_g = \frac{\ddot{\mathbf{r}} \cdot (\mathbf{n} \times \dot{\mathbf{r}})}{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \tag{188}$$

where the overdot means differentiation with respect to a general parameter t for the curve.[110]

• For any surface curve, the curvature κ and the geodesic curvature κ_g of the curve at a given point P on the curve are related by:

$$\kappa_g = \kappa \sin\theta \tag{189}$$

[110] As indicated previously, some intrinsic properties can also be defined in terms of extrinsic parameters.

where θ is the angle between the principal normal $\mathbf{N}$ to the curve at P and the unit normal $\mathbf{n}$ to the surface.

• On a surface patch of class C^2 with orthogonal coordinate curves, an s-parameterized curve C of class C^2 given by $\mathbf{r}(u(s), v(s))$ has a geodesic curvature given by:[111]

$$\kappa_g = \frac{d\phi}{ds} + \kappa_u \cos\phi + \kappa_v \sin\phi \tag{190}$$

where κ_u and κ_v are the geodesic curvatures of the u and v coordinate curves, ϕ is the angle such that $\mathbf{T} = \frac{\mathbf{E}_1}{|\mathbf{E}_1|}\cos\phi + \frac{\mathbf{E}_2}{|\mathbf{E}_2|}\sin\phi$ where $\mathbf{T}$ is the tangent unit vector of C, and $\mathbf{E}_1$ and $\mathbf{E}_2$ are the surface basis vectors in the u and v directions. All the given quantities are evaluated at a given point on the patch.

4.4 Principal Curvatures and Directions

• On rotating the plane containing $\mathbf{n}$, i.e. the unit vector normal to the surface at a given point P on the surface, around $\mathbf{n}$ the normal section and hence its curvature κ at P will vary in general.[112] The normal curvature κ_n (which is equal to the curvature κ for a normal section) of the surface at P in a given direction λ is given by:

$$\begin{aligned} \kappa_n &= \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} \\ &= \frac{f + g\lambda}{F + G\lambda} \\ &= \frac{e + f\lambda}{E + F\lambda} \end{aligned} \tag{191}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms and $\lambda = \frac{du^2}{du^1}$.[113] The two principal curvatures of the surface at P, κ_1 and κ_2, which represent

[111] This is known as Liouville formula.

[112] The plane containing $\mathbf{n}$ is orthogonal to the tangent plane to the surface at P.

[113] The directions represented by $\frac{du^2}{du^1}$ are the directions of the tangents to the normal sections at P.

the maximum and minimum values of the normal curvature κ_n of the surface at P as given by the last equation, correspond to the two λ roots of the following quadratic equation:

$$(gF - fG)\lambda^2 + (gE - eG)\lambda + (fE - eF) = 0 \qquad (gF - fG) \neq 0 \tag{192}$$

The last equation is obtained by equating the derivative of κ_n (as given by Eq. 191) with respect to λ to zero to obtain the extremum values.

• Eq. 192 possesses two roots, λ_1 and λ_2, which are linked by the following relations:

$$\lambda_1 + \lambda_2 = \frac{gE - eG}{gF - fG} \qquad \lambda_1\lambda_2 = \frac{fE - eF}{gF - fG} \qquad (gF - fG \neq 0) \tag{193}$$

These roots represent the two directions corresponding to the two principal curvatures, κ_1 and κ_2, of the surface.

• On an oriented and sufficiently smooth surface, the principal curvatures are continuous functions of the surface coordinates.

• The following two vectors on the surface, which are defined in terms of the two roots of λ as given above, represent the directions of the principal curvatures:

$$\begin{aligned} \left(\frac{d\mathbf{r}}{du^1}\right)_1 &= \frac{\partial \mathbf{r}}{\partial u^1} + \lambda_1 \frac{\partial \mathbf{r}}{\partial u^2} = \mathbf{E}_1 + \lambda_1 \mathbf{E}_2 \\ \left(\frac{d\mathbf{r}}{du^1}\right)_2 &= \frac{\partial \mathbf{r}}{\partial u^1} + \lambda_2 \frac{\partial \mathbf{r}}{\partial u^2} = \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 \end{aligned} \tag{194}$$

These directions, which are called the principal directions[114] of the surface at point P, are orthogonal at non-umbilical points where $\kappa_1 \neq \kappa_2$.[115]

• On each non-umbilical (including non-flat) point P of a smooth surface S an orthonormal moving "Darboux frame" can be defined. This frame consists of the vector triad $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{n})$

[114] They may also be called curvature directions.

[115] At umbilical points (see § 4.4.6) the normal curvature is the same in all directions and hence there are no principal directions to be orthogonal (or every direction is a principal direction and hence there is no sensible meaning for being orthogonal).

where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the unit vectors corresponding to the principal directions at P, and $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ is the unit vector normal to the surface at P.[116]

• The centers of curvature of the normal sections corresponding to the two principal curvatures[117] at a given point P on a surface S are given in tensor notation by (see § 2.4):

$$\begin{aligned} x_1^i &= x_P^i + \frac{N^i}{\kappa_1} \\ x_2^i &= x_P^i + \frac{N^i}{\kappa_2} \end{aligned} \tag{195}$$

where x_1^i and x_2^i are the spatial coordinates of the first and second center of curvature, x_P^i are the spatial coordinates of P, N^i is the principal normal vector of the normal sections at P, κ_1 and κ_2 are the principal curvatures of S at P, and $i = 1, 2, 3$.[118]

• The centers of curvature of the normal sections of the principal curvatures at P are described as the principal centers of curvature of the surface at P.

• According to one of the Euler theorems, the normal curvature at a given point P on a surface of class C^2 in a given direction can be expressed as a combination of the principal curvatures, κ_1 and κ_2, at P as:

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \tag{196}$$

where θ is the angle between the principal direction of κ_1 at P and the given direction.[119]

• A number of invariant parameters of the surface at a given point P on the surface are

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[116] This is another moving frame in use in differential geometry to be added to the three previously-described frames: the $(\mathbf{T},\mathbf{N},\mathbf{B})$ frame, the $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{n})$ frame and the $(\mathbf{n}, \mathbf{T}, \mathbf{u})$ frame. The first of these frames is associated with curves while the last three are associated with surfaces.

[117] These two normal sections, which are the normal sections in the principal directions, may be called the principal normal sections.

[118] The principal normal vector $\mathbf{N}$ of a normal section at a point P on the surface is collinear with the unit vector $\mathbf{n}$ normal to the surface at P and hence the principal normal vector is the same for all the normal sections at P.

[119] Since the principal directions at non-umbilical points are orthogonal, θ could represent the angle with the other principal direction but with relabeling of the two κ.

Table 1: The limiting conditions on the principal curvatures, κ_1 and κ_2, for a number of surfaces of simple geometric shapes alongside the corresponding mean curvature H and Gaussian curvature K. Apart from the plane, the unit vector normal to the surface, $\mathbf{n}$, is assumed to be in the outside direction.

	κ_1	κ_2	H	K
Plane	0	0	0	0
Cylinder	$\kappa_1 = 0$	$\kappa_2 < 0$	$H < 0$	0
Sphere	$\kappa_1 = \kappa_2 < 0$	$\kappa_2 = \kappa_1 < 0$	$H < 0$	$K > 0$
Ellipsoid	$\kappa_1 < 0$	$\kappa_2 < 0$	$H < 0$	$K > 0$
Hyperboloid of one sheet (see § 6.2)	$\kappa_1 > 0$	$\kappa_2 < 0$	—	$K < 0$

defined in terms of the principal curvatures; these include:

(A) The principal radii of curvature: $R_1 = \left|\frac{1}{\kappa_1}\right|$ and $R_2 = \left|\frac{1}{\kappa_2}\right|$.

(B) The Gaussian curvature:[120] $K = \kappa_1 \kappa_2$.

(C) The mean curvature: $H = \frac{\kappa_1+\kappa_2}{2}$.[121]

• Table 1 shows the restricting conditions on the principal curvatures, κ_1 and κ_2, for a number of common surfaces with simple geometric shapes and the effect on the Gaussian curvature K and the mean curvature H.

• Let P be a point on a sufficiently smooth surface S embedded in a 3D space coordinated by a rectangular Cartesian system (x, y, z) with P being above the origin, the tangent plane of $S(x, y)$ at P being parallel to the xy plane, and the principal directions being along the x and y coordinate lines. The equation of S in the neighborhood of P can then be expressed, up and including the quadratic terms, in the following form:

$$S(x, y) \simeq S(0, 0) + \frac{\kappa_1 x^2}{2} + \frac{\kappa_2 y^2}{2} \tag{197}$$

where κ_1 and κ_2 are the principal curvatures of S at P. This means that in the immediate

[120] Some authors use "total curvature" for the "Gaussian curvature" and hence they are synonym, while others use total curvature for the area integral $\int K d\sigma$ as, for example, in the Gauss-Bonnet theorem (refer to § 4.4.4). In the present text we follow the latter convention and we use total curvature strictly for the integral. Hence, we label the Gaussian curvature with K and the total curvature with K_t.

[121] Some authors define H as the sum of κ_1 and κ_2, that is: $H = \kappa_1 + \kappa_2$. Each convention has its merit. In the present notes we define H as the average, not the sum, of the two principal curvatures.

neighborhood of P, S resembles a quadratic surface of the given form.[122]

• The necessary and sufficient condition for a number $\kappa \in \mathbb{R}$ to be a principal curvature of a surface S at a given point P and in a given direction $\frac{du}{dv}$, where $(du)^2 + (dv)^2 \neq 0$, is that the following equations are satisfied:[123]

$$\begin{aligned}(e-\kappa E)du+(f-\kappa F)dv&=0\\(f-\kappa F)du+(g-\kappa G)dv&=0\end{aligned} \tag{198}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms at P.

• The above equations can be cast in a matrix form as:

$$\begin{bmatrix} e-\kappa E & f-\kappa F \\ f-\kappa F & g-\kappa G \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{199}$$

This system of homogeneous linear equations has a non-trivial solution, du and dv, *iff* the determinant of the coefficient matrix is zero, that is:

$$\begin{vmatrix} e-\kappa E & f-\kappa F \\ f-\kappa F & g-\kappa G \end{vmatrix} = \left(EG-F^2\right)\kappa^2-(gE-2fF+eG)\,\kappa+\left(eg-f^2\right)=0 \tag{200}$$

The above quadratic equation in κ has a non-negative discriminant and hence it possesses either two distinct real roots or a repeated real root. In the former case there are two distinct principal curvatures at P corresponding to two orthogonal principal directions, while in the latter case the point is umbilical and hence there is no specific principal direction as each direction can be a principal direction. So in brief, a given real number κ

[122] This includes umbilical points where the principal directions can be arbitrarily chosen as the directions of the x and y coordinate lines.

[123] For simplicity in notation, we use κ in the following equations to represent principal curvature. This use should not be confused with the curve curvature which is also symbolized by κ. However, for this case κ is equal to the principal curvature since the latter is the curvature of a normal section and hence the use of κ is justified.

is a principal curvature of S at P *iff* it is a solution of Eq. 200.

• From Eq. 200, the principal curvatures of a surface at a given point P are the solutions of the above quadratic equation and hence they are given by:

$$\kappa_{1,2} = \frac{gE - 2fF + eG \pm \sqrt{(gE - 2fF + eG)^2 - 4(EG - F^2)(eg - f^2)}}{2(EG - F^2)} \tag{201}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms at P.

• On dividing Eq. 200 by $a = EG - F^2$ we obtain:

$$\kappa^2 - 2H\kappa + K = 0 \tag{202}$$

where H and K are the mean and Gaussian curvatures (see Eqs. 208 and 235). Hence, Eq. 201 can be expressed compactly as:[124]

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{203}$$

• The principal directions are invariant with respect to permissible coordinate transformations and parameterizations.

• When the u and v coordinate curves of a surface at a point P are aligned along the principal directions at P, the principal curvatures at P will be given by:

$$\kappa_1 = \frac{b_{11}}{a_{11}} = \frac{e}{E} \qquad\qquad \kappa_2 = \frac{b_{22}}{a_{22}} = \frac{g}{G} \tag{204}$$

where E, G, e, g are the coefficients of the first and second fundamental forms at P, and the indexed a and b are the coefficients of the surface covariant metric and covariant curvature tensors.

• For a non-umbilical point P on a sufficiently smooth surface S, the direction $\frac{du}{dv}$ is a

[124] This formula can be obtained directly from Eq. 202 using the quadratic formula.

principal direction of S at P *iff* the following condition is true:[125]

$$(fE - eF)\,dudu + (gE - eG)\,dudv + (gF - fG)\,dvdv = 0 \tag{205}$$

The last equation can be factored into two linear equations each of the form $Adu + Bdv = 0$ (with A and B being real parameters) where these equations represent the two orthogonal principal directions.

• At a given point P on a sufficiently smooth surface S, a direction $\frac{du}{dv}$ is a principal direction *iff* for a real number κ the following relation holds true:

$$d\mathbf{n} = -\kappa d\mathbf{r} \tag{206}$$

where $d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u}du + \frac{\partial \mathbf{n}}{\partial v}dv$ and $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv$. If these conditions are satisfied, then κ is the principal curvature of S at P corresponding to the principal direction $\frac{du}{dv}$. Eq. 206 is called the Rodrigues curvature formula.

• From the Rodrigues curvature formula (Eq. 206) we have:

$$\frac{\partial \mathbf{n}}{\partial u} = -\kappa \mathbf{E}_1 \qquad\qquad \frac{\partial \mathbf{n}}{\partial v} = -\kappa \mathbf{E}_2 \tag{207}$$

• At any point on a plane surface, all the directions are principal directions.[126]

• At any point on a sphere, all the directions are principal directions (or there is no principal direction).

• The principal curvatures, κ_1 and κ_2, are the eigenvalues of the mixed type surface curvature tensor b^{α}_{β}.

[125] This equation is based on Eq. 192.

[126] Or, alternatively, there is no principal direction depending on allowing more than two principal directions or not.

4.4.1 Gaussian Curvature

• Gaussian curvature represents a generalization of curve curvature to surfaces since it is the product of two curvatures of curves belonging to the surface and hence in a sense it is a "2D curvature".

• The Gaussian curvature K of a surface[127] at a given point P on the surface is given by:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{b}{a} = \frac{R_{1212}}{a} \tag{208}$$

where E, F, G and e, f, g are the coefficients of the first and second fundamental forms at P, a and b are the determinants of the surface covariant metric and curvature tensors, and R_{1212} is the component of the 2D covariant Riemann-Christoffel curvature tensor.[128]

• The above formulae (Eq. 208) are based on the fact that the Gaussian curvature K is the determinant of the mixed curvature tensor b^{α}_{β} of the surface, that is:[129]

$$K = \det(b^{\alpha}_{\beta}) = \det(a^{\alpha\gamma} b_{\gamma\beta}) = \det(a^{\alpha\gamma})\det(b_{\gamma\beta}) = \frac{\det(b_{\gamma\beta})}{\det(a_{\alpha\gamma})} = \frac{b}{a} \tag{209}$$

• From Eq. 208, we see that the sign of K (i.e. $K > 0$, $K < 0$ or $K = 0$) is the sign of b since a is positive definite.

• Since both R_{1212} (see Eq. 13) and a depend exclusively on the surface metric tensor, Eq. 208 reveals that K depends only on the first fundamental form coefficients and hence it is an intrinsic property of the surface (refer to § 4.4.3). The dependence of K on the second fundamental form coefficients in Eq. 209 does not affect its qualification as an intrinsic

[127] Gaussian curvature may also be called the Riemannian curvature of the surface.

[128] As discussed in [10], the 2D Riemann-Christoffel curvature tensor has only one independent non-vanishing component which is represented by R_{1212}. From Eq. 120 we get $R_{1212} = b_{11}b_{22} - b_{12}b_{21} = eg - f^2 = b$ where the indexed b are the coefficients of the surface covariant curvature tensor and b is its determinant.

[129] Being the determinant of a tensor establishes the status of K as an invariant under permissible transformations. Also, the chain of formulae in Eq. 209 may be taken in the opposite direction starting primarily from $K = \frac{b}{a}$ as a definition or as a derived result from other arguments, and hence the statement $K = \det(b^{\alpha}_{\beta})$ will be obtained as a secondary result, as done by some authors.

property since this dependency is not indispensable as K can be expressed in terms of the first fundamental form coefficients exclusively.

• The Gaussian curvature is an invariant with respect to permissible coordinate transformations in 2D manifolds and hence:

$$K = \frac{R_{1212}}{a} = \frac{\bar{R}_{1212}}{\bar{a}} \tag{210}$$

where the barred and unbarred symbols represent the quantities in the barred and unbarred surface coordinate systems.

• The Gaussian curvature is also invariant with respect to the type of representation and parameterization of the surface. In particular, the Gaussian curvature is independent of the orientation of the surface which is based on the choice of the direction of the normal vector $\mathbf{n}$ to the surface.

• From Table 1 we see that the Gaussian curvature of planes and cylinders are both identically zero. At the root of this is the fact that the Gaussian curvature is an intrinsic property and the cylinder is a developable surface obtained by wrapping a plane with no localized distortion by stretching or compression. Hence, the planes and cylinders possess identical first fundamental forms, as indicated previously in § 3.3, and consequently identical Gaussian curvature (also see § 4.4.3).

• Since the magnitude of the normal curvature of a sphere of radius R is $|\kappa_n| = \frac{1}{R}$ at any point on its surface and for any normal section in any direction, its Gaussian curvature is a constant given by $K = \frac{1}{R^2}$.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the Gaussian curvature is given by:

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{\left(1 + f_u^2 + f_v^2\right)^2} \tag{211}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

• The Gaussian curvature of a surface of revolution generated by revolving a plane curve of class C^2 having the form $y = f(x)$ around the x-axis is given by:

$$K = -\frac{f_{xx}}{f\left(1 + f_x^2\right)^2} \tag{212}$$

where the subscript x represents the partial derivative of f with respect to this variable.

• At any point on a sufficiently smooth surface the Gaussian curvature satisfies the following relation:

$$\partial_u \mathbf{n} \times \partial_v \mathbf{n} = K\left(\mathbf{E}_1 \times \mathbf{E}_2\right) \tag{213}$$

On dot producting both sides by $\mathbf{n}$ we obtain:

$$\mathbf{n} \cdot \left(\partial_u \mathbf{n} \times \partial_v \mathbf{n}\right) = K\,\mathbf{n} \cdot \left(\mathbf{E}_1 \times \mathbf{E}_2\right) = K\sqrt{a} \tag{214}$$

Hence:

$$K = \frac{\mathbf{n} \cdot \left(\partial_u \mathbf{n} \times \partial_v \mathbf{n}\right)}{\sqrt{a}} \tag{215}$$

• On a 2D surface, the Gaussian curvature K is related to the Ricci curvature scalar R (see § 1.11) by the following relation:

$$K = \frac{R}{2} \tag{216}$$

• There are surfaces with constant zero Gaussian curvature $K = 0$ (e.g. planes, cylinders and cones excluding the apex), surfaces with constant positive Gaussian curvature $K > 0$ (e.g. spheres with $K = \frac{1}{R^2}$ where R is the sphere radius), and surfaces with constant negative Gaussian curvature $K < 0$ (e.g. Beltrami pseudo-spheres with $K = -\frac{1}{\rho^2}$ where ρ is the pseudo-radius of the pseudo-sphere).[130] However, in general the Gaussian curvature is a variable function in sign and magnitude of the surface coordinates and a single surface can have Gaussian curvature of different magnitudes and signs.

• On scaling a surface up or down by a constant factor $c > 0$ (refer to § 3.1), the Gaussian curvature K will scale by a factor of $\frac{1}{c^2}$.

• The Gaussian curvature is invariant with respect to all isometric transformations, and hence two isometric surfaces have identical Gaussian curvature at each pair of their corresponding points. However, two mapped surfaces with equal Gaussian curvature at their corresponding points are not necessarily isometric. Yes in the case of two sufficiently smooth surfaces with equal constant Gaussian curvature the two surfaces have local isometry i.e. they are isometric in the immediate neighborhood of their corresponding points.

• In 3D manifolds, there is no compact surface of class C^2 with non-positive Gaussian curvature, i.e. $K \leq 0$.

• The sphere is the only connected, compact and sufficiently smooth surface with constant Gaussian curvature.

• According to the Hilbert lemma, if P is a point on a sufficiently smooth surface S with κ_1 and κ_2 being the principal curvatures of S at P such that: $\kappa_1 > \kappa_2$, κ_1 is a local maximum, and κ_2 is a local minimum, then the Gaussian curvature of S at P is non-positive, that is $K \leq 0$.

• At a given point P on a spherically-mapped (see § 3.7) and sufficiently smooth surface

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[130] The tractrix is a plane curve in the xy plane starting from point $(\rho, 0)$ on the x-axis ($\rho > 0$) with the property that the length of the line segment of its tangent between the tangency point and the point of intersection with the y-axis is equal ρ. Hence, the tractrix is a solution of the following differential equation:

$$\frac{dy}{dx} = -\frac{\sqrt{\rho^2 - x^2}}{x} \tag{217}$$

with the condition $y(\rho) = 0$. The real parameter ρ is called pseudo-radius. The Beltrami pseudo-sphere is a surface of revolution generated by revolving a tractrix around its asymptote which is the y-axis in the above formulation.

S, the ratio of the area of the spherical image $\bar{\mathfrak{S}}$ of a region surrounding P to the area of the mapped region $\mathfrak{S}$ on S converges to the value of the Gaussian curvature at P as $\mathfrak{S}$ shrinks to P.

• From the Gauss-Bonnet theorem (see § 4.4.4), it can be shown that a surface will have identically-vanishing Gaussian curvature if at any point P on the surface there are two families of geodesic curves (see § 5.7) in the neighborhood of P intersecting at a constant angle.

• From Eqs. 18 and 208, the Gaussian curvature K of a sufficiently smooth surface represented by $\mathbf{r} = \mathbf{r}(u, v) = \mathbf{r}(u^1, u^2)$ can also be given by:

$$K = \frac{1}{a}\left[F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} + a_{\alpha\beta}\left(\Gamma^{\alpha}_{12}\Gamma^{\beta}_{12} - \Gamma^{\alpha}_{11}\Gamma^{\beta}_{22}\right)\right] \qquad (\alpha,\beta{=}1,2) \tag{218}$$

where E, F, G are the coefficients of the first fundamental form, the subscripts u and v stand for partial derivatives with respect to these surface coordinates, a is the determinant of the surface covariant metric tensor and the indexed a are its coefficients. The Christoffel symbols are based on the surface metric.

• The Gaussian curvature of a sufficiently smooth surface is also given by:

$$K = \frac{1}{2\sqrt{a}}\left[\partial_u\left(\frac{FE_v}{E\sqrt{a}} - \frac{G_u}{\sqrt{a}}\right) + \partial_v\left(\frac{2F_u}{\sqrt{a}} - \frac{E_v}{\sqrt{a}} - \frac{FE_u}{E\sqrt{a}}\right)\right] \tag{219}$$

where E, F, G are the coefficients of the first fundamental form, $a = EG - F^2$ is the determinant of the surface covariant metric tensor, and the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

• The Gaussian curvature K of a surface of class C^3 represented by $\mathbf{r}(u, v)$ with orthogonal surface coordinate curves is given by:

$$K = -\frac{1}{2\sqrt{EG}}\left[\partial_u\left(\frac{G_u}{\sqrt{EG}}\right) + \partial_v\left(\frac{E_v}{\sqrt{EG}}\right)\right] \tag{220}$$

where E, G are the coefficients of the first fundamental form, and the subscripts u and v stand for partial derivatives with respect to these surface coordinates. This formula is obtained from the previous formula by setting $F = 0$.

• The last formula will simplify to:

$$K = -\frac{\partial_{uu}\sqrt{G}}{\sqrt{G}} \tag{221}$$

when the surface $\mathbf{r}(u, v)$ is represented by geodesic coordinates (see § 1.9) with the u coordinate curves being geodesics and u is a natural parameter.[131]

• The sign of the Gaussian curvature is independent of the choice of the direction of the unit vector $\mathbf{n}$ normal to the surface. This is because a change in the direction ensuing a change in the sign of the principal curvatures will change the sign of both of these curvatures and hence their product will not be affected.

• The Gaussian curvature K can be expressed in terms of the mean curvature H as:

$$K = \kappa_1\kappa_2 = (H + C)(H - C) = H^2 - C^2 \tag{222}$$

where

$$C = \frac{\sqrt{\left\{e(EG - 2F^2) + 2fEF - gE^2\right\}^2 + 4\left(fE - eF\right)^2}}{2E\left(EG - F^2\right)} \tag{223}$$

and E, F, G, e, f, g are the coefficients of the first and second fundamental forms.

• The Gaussian curvature of a surface S at a given point P on the surface is positive if all the surface points in a deleted neighborhood of P on S are on the same side of the plane tangent to S at P. The Gaussian curvature is negative if for all deleted neighborhoods of P on S some points are on one side and some are on the other. The Gaussian curvature

[131] Here, geodesic coordinates means a coordinate system on a coordinate patch of a surface whose u and v coordinate curve families are orthogonal with one of these families (u or v) being a family of geodesic curves (refer to § 1.9 for more details).

is zero if, in a deleted neighborhood, either all the points lie in the tangent plane or all the points are on one side except some which lie on a curve in the tangent plane. Hence,

(A) A sphere has a positive Gaussian curvature at all points.

(B) A hyperbolic paraboloid has a negative Gaussian curvature at all points.

(C) A plane has a zero Gaussian curvature at all points.

(D) A cylinder has a zero Gaussian curvature at all points.

(E) A torus has points with positive Gaussian curvature (outer half), points with zero Gaussian curvature (top and bottom circles) and points with negative Gaussian curvature (inner half).

• The Gaussian curvature of a developable surface is identically zero. Hence, beside the plane, there are other surfaces with constant zero Gaussian curvature. Examples are: cones, cylinders and tangent surfaces of curves (refer to § 6.6).

• Examples of the Gaussian curvature, K, for a number of simple surfaces:

(A) Plane: $K = 0$.

(B) Sphere of radius R: $K = \frac{1}{R^2}$.

(C) Torus parameterized by $x = (R + r\sin\phi)\cos\theta$, $y = (R + r\sin\phi)\sin\theta$ and $z = r\cos\phi$: $K = \frac{\sin\phi}{r(R+r\sin\phi)}$.[132]

• The total curvature K_t is the area integral of the Gaussian curvature K over a surface S or a patch of a surface, that is:

$$K_t = \iint_S K d\sigma \tag{224}$$

where $d\sigma$ symbolizes infinitesimal area element on the surface.

[132] In these parameterizations, R is the torus radius (i.e. the distance between the center of symmetry of the torus and the center of the tube), r is the radius of the generating circle ($r < R$), $\phi \in [0, 2\pi)$ is the angle of variation of r, and $\theta \in [0, 2\pi)$ is the angle of variation of R.

• From Eqs. 101 and 155, the total curvature K_t may be given by:

$$K_t \equiv \iint_S K d\sigma = \iint_S K \left|\mathbf{E}_1 \times \mathbf{E}_2\right| dudv = \iint_S \operatorname{sgn}(K) \left|\partial_u \mathbf{n} \times \partial_v \mathbf{n}\right| dudv \tag{225}$$

where $\operatorname{sgn}(K)$ is the sign function of K as a function of the surface coordinates, u and v.

• The Riemann-Christoffel curvature tensor is related to the Gaussian curvature through the alternating absolute tensor of the surface by the following relation:

$$R_{\alpha\beta\gamma\delta} = K \underline{\epsilon}_{\alpha\beta} \underline{\epsilon}_{\gamma\delta} \tag{226}$$

On multiplying both sides of the last equation by $\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta}$ we get:

$$\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta} R_{\alpha\beta\gamma\delta} = K \underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta} \underline{\epsilon}_{\alpha\beta} \underline{\epsilon}_{\gamma\delta} \tag{227}$$

Now, since $\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}_{\alpha\beta} = \underline{\epsilon}^{\gamma\delta}\underline{\epsilon}_{\gamma\delta} = 2$, the last equation becomes:[133]

$$K = \frac{1}{4}\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{1}{4}\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta}\left(b_{\alpha\gamma}b_{\beta\delta} - b_{\alpha\delta}b_{\beta\gamma}\right) \tag{228}$$

• The Riemann-Christoffel curvature tensor is also linked to the Gaussian curvature through the surface metric tensor by the following relation:

$$R_{\alpha\beta\gamma\delta} = K\left(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma}\right) \tag{229}$$

• The Gaussian curvature K may also be given by:

$$K = \frac{1}{2}\underline{\epsilon}^{\alpha\beta}\underline{\epsilon}^{\gamma\delta} b_{\gamma\alpha}b_{\delta\beta} \tag{230}$$

[133]Hence, K is a rank-0 tensor or scalar.

where the indexed $\underline{\epsilon}$ are the 2D absolute alternating tensor, the indexed b are the components of the surface covariant curvature tensor and all the indices range over 1 and 2.

• From Eqs. 120 and 226, the Gaussian curvature and the surface curvature tensor are related by:

$$K \underline{\epsilon}_{\alpha\beta} \underline{\epsilon}_{\gamma\delta} = b_{\alpha\gamma} b_{\beta\delta} - b_{\alpha\delta} b_{\beta\gamma} \tag{231}$$

• Another formula for the Gaussian curvature (in terms of the surface basis vectors, their derivatives and the coefficients of the first fundamental form) can be obtained from the formula $K = \frac{b}{a}$, that is:

$$\begin{aligned} b &= eg - f^2 \\ &= \frac{(\partial_u \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)(\partial_v \mathbf{E}_2 \cdot \mathbf{E}_1 \times \mathbf{E}_2) - (\partial_v \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)^2}{|\mathbf{E}_1 \times \mathbf{E}_2|^2} \\ &= \frac{(\partial_u \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)(\partial_v \mathbf{E}_2 \cdot \mathbf{E}_1 \times \mathbf{E}_2) - (\partial_v \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)^2}{a} \\ &= \frac{(\partial_u \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)(\partial_v \mathbf{E}_2 \cdot \mathbf{E}_1 \times \mathbf{E}_2) - (\partial_v \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)^2}{EG - F^2} \end{aligned} \tag{232}$$

Hence:

$$\begin{aligned} K &= \frac{b}{a} \\ &= \frac{(\partial_u \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)(\partial_v \mathbf{E}_2 \cdot \mathbf{E}_1 \times \mathbf{E}_2) - (\partial_v \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)^2}{a^2} \\ &= \frac{(\partial_u \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)(\partial_v \mathbf{E}_2 \cdot \mathbf{E}_1 \times \mathbf{E}_2) - (\partial_v \mathbf{E}_1 \cdot \mathbf{E}_1 \times \mathbf{E}_2)^2}{(EG - F^2)^2} \end{aligned} \tag{233}$$

This formula is another confirmation that the Gaussian curvature is an intrinsic property of the surface.

4.4.2 Mean Curvature

• As given earlier, the mean curvature H is the average[134] of the two principal curvatures, that is:

$$H = \frac{\kappa_1 + \kappa_2}{2} \tag{234}$$

• The mean curvature H is also given by the following formula:

$$H = \frac{eG - 2fF + gE}{2\left(EG - F^2\right)} = \frac{\operatorname{tr}\left(b_\alpha^\beta\right)}{2} = \frac{b_\alpha^\alpha}{2} \qquad (\alpha, \beta = 1, 2) \tag{235}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental forms, the indexed b are the coefficients of the surface mixed curvature tensor, and tr stands for the trace of matrix.[135]

• The sign of the mean curvature is dependent on the choice of the direction of the unit vector $\mathbf{n}$ normal to the surface.

• Like the Gaussian curvature, the mean curvature is invariant under permissible coordinate transformations and representations.

• The mean curvature of a surface at a given point P is a measure of the rate of change of area of the surface elements in the neighborhood of P.

• Examples of the mean curvature, H, for a number of simple surfaces:

(A) Plane: $H = 0$.

(B) Sphere of radius R: $|H| = \frac{1}{R}$.

(C) Torus parameterized by $x = (R + r\sin\phi)\cos\theta$, $y = (R + r\sin\phi)\sin\theta$ and $z = r\cos\phi$: $H = \frac{R+2r\sin\phi}{2r(R+r\sin\phi)}$.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the mean curvature is given by:

[134] Or the sum depending on the authors although it will not be a mean anymore.

[135] Being half the trace of a tensor establishes the status of H as an invariant under permissible transformations.

$$H = \frac{\left(1 + f_v^2\right) f_{uu} - 2 f_u f_v f_{uv} + \left(1 + f_u^2\right) f_{vv}}{2\left(1 + f_u^2 + f_v^2\right)^{3/2}} \tag{236}$$

where the subscripts u and v stand for partial derivatives with respect to these surface coordinates.

• The mean curvature may be considered as the 2D equivalent of the geodesic curvature in 1D.[136] Accordingly, the 2D minimal surfaces (see § 6.7) correspond to the 1D geodesic curves (see § 5.7).

4.4.3 Theorema Egregium

• The essence of Gauss *Theorema Egregium* or *Remarkable Theorem* is that the Gaussian curvature K of a surface is an intrinsic property of the surface and hence it can be expressed as a function of the coefficients of the first fundamental form and their partial derivatives only with no involvement of the coefficients of the second fundamental form. This can be guessed for example from the last part[137] of Eq. 208.[138]

• The essence of *Theorema Egregium*, as a statement of the fact that certain types of curvature are intrinsic to the surface, is contained in several forms and equations; some of which are indicated in these notes when they occur. For example, Eq. 120 which links the surface curvature tensor to the Riemann-Christoffel curvature tensor (which is an intrinsic

[136] The equivalence should be obvious since the mean curvature is a measure for minimizing the surface area while the geodesic curvature is a measure for minimizing the curve length.

[137] In fact even the first part can be used in this argument since b can be expressed purely in terms of E, F, G and their derivatives as:

$$\begin{aligned} b &= eg - f^2 = F\left(\frac{\partial \Gamma_{22}^2}{\partial u} - \frac{\partial \Gamma_{12}^2}{\partial v} + \Gamma_{22}^1 \Gamma_{11}^2 - \Gamma_{12}^1 \Gamma_{12}^2\right) + \\ & \quad E\left(\frac{\partial \Gamma_{22}^1}{\partial u} - \frac{\partial \Gamma_{12}^1}{\partial v} + \Gamma_{22}^1 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{12}^1 - \Gamma_{12}^1 \Gamma_{12}^1 - \Gamma_{12}^2 \Gamma_{22}^1\right) \end{aligned} \tag{237}$$

[138] In several places of these notes we see that some quantities can be expressed once in terms of the coefficients of the first fundamental form exclusively and once in terms of expressions involving the coefficients of the second fundamental form as well. In this regard, a quantity is classified as intrinsic if it can be expressed as a function of the first fundamental form only even if it can also be expressed in terms involving the second fundamental form.

property of the surface and is related to the Gaussian curvature by Eqs. 17 and 226 for instance) can be regarded as a statement of *Theorema Egregium* since it expresses a form of surface curvature in terms of a combination of purely intrinsic surface parameters.

• An example may be given to demonstrate the significance of *Theorema Egregium* that is, if a piece of plane is rolled into a cylinder of radius R, then κ_1, κ_2, H will change from $0, 0, 0$ to $\frac{1}{R}, 0, \frac{1}{2R}$.[139] However, as a consequence of *Theorema Egregium,* K will not change since K is dependent exclusively on the first fundamental form which is the same for planes and cylinders as indicated previously in § 3.3.

• According to *Theorema Egregium,* the Gaussian curvature of a sufficiently smooth surface of class C^3 at a given point P can be represented by the following function of the coefficients of the first fundamental form and their partial derivatives at P:

$$\begin{aligned} K &= \frac{1}{(EG-F^2)^2} \left\{ \begin{vmatrix} C & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right\} \\ &= \frac{1}{(EG-F^2)^2} \left\{ \begin{vmatrix} C & [22,1] & [22,2] \\ [11,1] & a_{11} & a_{12} \\ [11,2] & a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} 0 & [21,1] & [21,2] \\ [21,1] & a_{11} & a_{12} \\ [21,2] & a_{21} & a_{22} \end{vmatrix} \right\} \end{aligned} \tag{238}$$

where $C = \frac{1}{2}\left(-E_{vv} + 2F_{uv} - G_{uu}\right)$ and the subscripts u and v stand for the partial derivatives with respect to these surface coordinates.

4.4.4 Gauss-Bonnet Theorem

• This theorem ties the geometry of surfaces to their topology. There are several variants of this theorem; some of which are local while others are global. Due to the importance and subtlety of this theorem we give different variants of the theorem and several examples

[139] For the cylinder, we are assuming a normal unit vector in the inner direction.

for both plane and curved surfaces.

• According to the Gauss-Bonnet theorem, if $\mathfrak{S}$ is a simply connected region on a surface of class C^3 where $\mathfrak{S}$ is bordered by a finite number of piecewise regular curves C_i that meet in corners[140] then we have:[141]

$$\sum_{s} \int_{C_i} \kappa_g + \sum_{c} \phi_j + \iint_{\mathfrak{S}} K d\sigma = 2\pi \tag{239}$$

where the first sum is over the sides while the second sum is over the corners, κ_g is the geodesic curvature of the curves C_i as a function of their coordinates, ϕ_j are the exterior angles of the corners and K is the Gaussian curvature of $\mathfrak{S}$ as a function of the coordinates over $\mathfrak{S}$.[142]

• As indicated previously, the term $\iint_{\mathfrak{S}} K d\sigma$, which is the area integral of the Gaussian curvature over the region $\mathfrak{S}$ of the above-described surface, is called the total curvature K_t of $\mathfrak{S}$.

• Some examples for the application of the Gauss-Bonnet theorem are given below:

(A) A disc in a plane with radius R where Eq. 239 becomes $\frac{1}{R} 2\pi R + 0 + 0 = 2\pi + 0 + 0 = 2\pi$ which is an identity.

(B) A semi-circular disc in a plane with radius R where Eq. 239 becomes $\left(\frac{1}{R}\pi R + 0 \times 2R\right) + 2\left(\frac{\pi}{2}\right) + 0 = \pi + \pi + 0 = 2\pi$ which is an identity again.

(C) A spherical triangle on a sphere of radius R whose sides are two half meridians connecting a pole to the equator and one quarter of an equatorial parallel and all of its three corners are right angles ($= \frac{\pi}{2}$) where Eq. 239 becomes $\left[0\left(3 \times \frac{\pi R}{2}\right)\right] + 3\left(\frac{\pi}{2}\right) + \frac{1}{R^2}\frac{4\pi R^2}{8} =$

[140] These corners can be defined as the points of discontinuity of the tangents of the boundary curves. The angles of these corners are therefore defined as the angles between the tangent vectors at the points of discontinuity when traversing the boundary curves in a predefined sense. Hence, these angles are exterior to the curves and region. Sometimes, artificial "corners" at regular points are introduced, for convenience to establish an argument, in which case the exterior angle is zero.

[141] The geodesic and Gaussian curvatures in this theorem should be continuous and finite over their domain.

[142] This form of the Gauss-Bonnet theorem may be labeled as a local variant of the theorem although its locality may not be obvious. However, these labels are not of crucial importance.

$0 + \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi$.

(D) The upper half of a sphere (or hemisphere in general) of radius R where Eq. 239 becomes $0\,(2\pi R) + 0 + \frac{1}{R^2} 2\pi R^2 = 0 + 0 + 2\pi = 2\pi$.

• The fact that the sum of the interior angles of a planar triangle is π can be regarded as an instance of the Gauss-Bonnet theorem since for the planar triangle Eq. 239 becomes:

$$\begin{aligned} 0 + \sum_{i=1}^{3} (\pi - \theta_i) + 0 &= \\ 3\pi - \sum_{i=1}^{3} \theta_i &= 2\pi \end{aligned}$$

where θ_i are the interior angles of the triangle and hence $\sum_{i=1}^{3} \theta_i = \pi$. By a similar argument, we can obtain the sum of the interior angles of a planar polygon of n sides using the Gauss-Bonnet theorem, that is:

$$\begin{aligned} 0 + \sum_{i=1}^{n} (\pi - \theta_i) + 0 &= \\ n\pi - \sum_{i=1}^{n} \theta_i &= 2\pi \end{aligned}$$

where θ_i are the interior angles of the n-polygon and hence $\sum_{i=1}^{n} \theta_i = (n-2)\,\pi$.

• The fact that the perimeter of a planar circle of radius R is $2\pi R$ can be regarded as another instance of the Gauss-Bonnet theorem since for such a planar circle Eq. 239 becomes:

$$\frac{1}{R} L + 0 + 0 = 2\pi \tag{240}$$

where L is the length of the circle perimeter and hence $L = 2\pi R$.

• As a result of the Gauss-Bonnet theorem, the sum θ_s of the interior angles of a geodesic triangle[143] on a surface with constant (i.e. have the same sign) Gaussian curvature is:

[143] A geodesic triangle is a triangle with geodesic sides. Also, "triangle" here is more general than a

(A) $\theta_s < \pi$ *iff* $K < 0$.

(B) $\theta_s = \pi$ *iff* $K = 0$.

(C) $\theta_s > \pi$ *iff* $K > 0$.

This, in the light of Eq. 239, shows that the total curvature provides the excess over π for the sum when $K > 0$ on the surface and the deficit when $K < 0$. The vanishing total curvature in the case of $K = 0$ is the intermediate case where the total curvature has no contribution to the sum.

• As a consequence of the last point, two geodesic curves on a simply connected patch of a surface with negative Gaussian curvature cannot intersect at two points because on introducing an artificial vertex at a regular point on one curve we will have a new corner with zero exterior angle and hence π interior angle. We will then have a geodesic triangle with $\theta_s > \pi$ on a surface over which $K < 0$, in violation of the above-stated condition.

• By a similar argument to the above, the area of a geodesic polygon[144] on a surface with constant non-zero Gaussian curvature is determined by the polygon interior angles.

• Because the geodesic curvature is an intrinsic property, as discussed in § 4.3, the Gauss-Bonnet theorem is another indication to the fact that the Gaussian curvature is an intrinsic property and hence it is another demonstration of *Theorema Egregium* (see § 4.4.3).

• The Euler characteristic, or the Euler-Poincare characteristic, is a topological parameter of surfaces which, for polyhedral surfaces, is given by $\chi = \mathcal{V} + \mathcal{F} - \mathcal{E}$ where $\mathcal{V}, \mathcal{F}, \mathcal{E}$ are the numbers of vertices, faces and edges of the polyhedron. The Euler characteristic is also defined for more general surfaces.

• The Euler characteristic of a compact orientable non-polyhedral surface, like sphere and torus, can be obtained by a polygonal decomposition by dividing the entire surface into a finite number of non-overlapping curvilinear polygons which share at most edges or

three-side planar polygon with three straight segments as it can be on a curved surface with curved non-planar sides.

[144] A geodesic polygon is a polygon with geodesic sides. Again, "polygon" here is general and hence it includes that on curved surface with non-planar curved sides.

vertices.

• The Gauss-Bonnet theorem has also a global variant which links the Euler characteristic χ, which is a topological invariant of the surface, to the Gaussian curvature K, which is a geometric invariant of the surface. This global form of the Gauss-Bonnet theorem states that: on a compact orientable surface S of class C^3 these two invariants are linked through the following equation:

$$\iint_S K d\sigma = 2\pi\chi \tag{241}$$

Now, since χ is a topological invariant of the surface, Eq. 241 reveals that the total curvature is also a topological invariant of the surface.

• The global Gauss-Bonnet theorem can be used to determine the total curvature K_t of a surface. For example, the Euler characteristic of a sphere is 2 and hence from Eq. 241 its total curvature is $K_t = 4\pi$ with no need for evaluation of the area integral. Similarly, the Euler characteristic of a torus is 0 and hence it can be concluded immediately that its total curvature is $K_t = 0$ with no need for evaluating the integral.

• The Euler characteristics of the sphere and torus in the above examples can be obtained easily by a polygonal decomposition, as described above. For example, the Euler characteristic of the sphere can be calculated by dividing the surface of the sphere to 4 curved polygonal faces with 4 vertices and 6 edges and hence the Euler characteristic is $\chi = 4 + 4 - 6 = 2$, as given in the last point.

• The global Gauss-Bonnet theorem can be used in the opposite direction, that is it may be used for determining the Euler characteristic of a surface knowing its Gaussian, and hence total, curvature although this may be practically of little use. For instance, the Gaussian curvature of a sphere of radius R is $\frac{1}{R^2}$ at every point on the sphere and hence its total curvature is $K\sigma = \frac{1}{R^2} 4\pi R^2 = 4\pi$, therefore from Eq. 241 its Euler characteristic is $\chi = \frac{4\pi}{2\pi} = 2$.

• The Gauss-Bonnet theorem can also be used to find the total curvature of a surface which is topologically equivalent (technically described as homeomorphic) to another surface with known total curvature. For example, the ellipsoid is homeomorphic to the sphere and hence they have the same Euler characteristic, therefore they have the same total curvature, according to Eq. 241, which is 4π as known from the aforementioned sphere example.

• The total curvature K_t of a surface with a complex shape can be obtained from the Gauss-Bonnet theorem by reducing the surface to a topologically-equivalent simpler surface whose total curvature is known or can be computed more easily. For example, for an orientable surface of genus[145] $\mathfrak{g}$ the Euler characteristic is $\chi = 2(1 - \mathfrak{g})$ and hence its total curvature is given by $K_t = 2\pi\chi = 4\pi(1 - \mathfrak{g})$. So, for a compact orientable complexly-shaped surface which is topologically-equivalent to a sphere with 2 handles the total curvature is $K_t = -4\pi$. Similarly, the genus of a torus is $\mathfrak{g} = 1$ and hence its total curvature is $K_t = 4\pi(1 - 1) = 0$, as found earlier by another method.

• The obvious implication of the global variant of the Gauss-Bonnet theorem is that the total curvature of a closed surface is dependent on its genus and not on its shape and hence it is a topological parameter of the surface as indicated previously.

4.4.5 Local Shape of Surface

• Using the principal curvatures, κ_1 and κ_2, a point P on a surface is classified according to the shape of the surface in the close proximity of P as:

(A) Flat when $\kappa_1 = \kappa_2 = 0$, and hence $K = H = 0$.

(B) Elliptic when either $\kappa_1 > 0$ and $\kappa_2 > 0$ or $\kappa_1 < 0$ and $\kappa_2 < 0$, and hence $K > 0$.

(C) Hyperbolic when $\kappa_1 > 0$ and $\kappa_2 < 0$, and hence $K < 0$.

(D) Parabolic when either $\kappa_1 = 0$ and $\kappa_2 \neq 0$ or $\kappa_2 = 0$ and $\kappa_1 \neq 0$, and hence $K = 0$ and $H \neq 0$.

[145] In simple terms, the genus (in topology) of a surface is the number of handles or topological holes on the surface.

These constraints on K and H are sufficient and necessary conditions for determining the type of the surface point as described above.

• The following are some examples for the above classification:

(A) The points of a plane are flat.

(B) The points of an ellipsoid are elliptic.

(C) The points of a catenoid are hyperbolic.

(D) The points of a cone (excluding the apex) and the points of a cylinder are parabolic.

• Surfaces normally contain points of different shapes. For example, the torus has elliptic points on its outside rim, parabolic points on its top and bottom parallels[146], and hyperbolic points on its inside rim.[147] However, there are some types of surfaces whose all points are of the same shape; e.g. all points of planes are flat, all points of spheres are elliptic, all points of catenoids are hyperbolic, and all points of cylinders are parabolic.

• The above classification can also be based on the coefficients of the second fundamental form of the surface where:[148]

(A) $eg - f^2 = 0$ and $e = f = g = 0$ for flat points.

(B) $eg - f^2 = 0$ and $e^2 + f^2 + g^2 \neq 0$ for parabolic points.

(C) $eg - f^2 > 0$ for elliptic points.

(D) $eg - f^2 < 0$ for hyperbolic points.

The reader is referred to § 4.2 for the significance of the sign of b.

• The above classification of the shape of a surface in the immediate neighborhood of a point on a surface (i.e. being flat, elliptic, hyperbolic or parabolic) is an invariant property with respect to permissible coordinate transformations. This can be concluded from the dependence of the classification on the sign of b as explained above, plus Eq. 113 where the Jacobian (which is real) is squared and hence the sign of b and $\bar{b}$ is the same. The

[146] These parallels correspond to the two circles contacting its two tangent planes at the top and bottom which are perpendicular to its axis of symmetry.

[147] In fact it is elliptic on the outside half and hyperbolic on the inside half.

[148] We have $eg - f^2 = b$ where b is the determinant of the surface covariant curvature tensor.

classification is also independent of the representation and parameterization of the surface since these types are real geometric properties of the surface in their local definitions.

• The invariance of the shape type of the surface points, as explained in the previous bullet point, holds true even for the transformations that reverse the direction of the vector $\mathbf{n}$ normal to the surface, because the classification depends on the Gaussian curvature which is invariant even under this type of transformations (refer to § 4.4.1). As for the distinction between the flat and parabolic points which involves H as well, the distinction is not affected since it depends on the magnitude of H (i.e. being zero or not) and not on its sign and the magnitude is not affected by such transformations.

• In the immediate neighborhood of an elliptic point P of a surface S, the surface lies completely on one side of the tangent plane to S at P, while at a hyperbolic point the tangent plane cuts through S and hence some parts of S are on one side while other parts are on the other side. In the neighborhood of a parabolic point, the surface lies entirely on one side except for some points on a curve which lies in the tangent plane itself. As for planar points, the neighborhood of the point lies in the tangent plane.

• The surface points can also be classified according to the geometric shape of Dupin indicatrix (refer to § 3.4.1) as follow:

(A) If $eg - f^2 > 0$ then κ_1 and κ_2 have the same sign; hence the point is elliptic and the Dupin indicatrix is an ellipse or circle.

(B) If $eg - f^2 = 0$ and $e^2 + f^2 + g^2 > 0$, then either $\kappa_1 = 0$ and $\kappa_2 \neq 0$ or $\kappa_2 = 0$ and $\kappa_1 \neq 0$; hence the point is parabolic and the Dupin indicatrix becomes two parallel lines.[149]

(C) If $eg - f^2 < 0$ then κ_1 and κ_2 have opposite signs; hence the point is hyperbolic and the Dupin indicatrix becomes two conjugate hyperbolas.[150]

(D) If $e = f = g = 0$, and hence $eg - f^2 = 0$, then the point is flat and the Dupin indicatrix

[149] These lines are characterized by having identically vanishing normal curvature.

[150] The normal curvature is positive along one of these hyperbolas and negative along the other, while along the common asymptotes of these hyperbolas the normal curvature is zero.

is not defined.[151]

Consequently, because of these correlations between the type of point and its Dupin indicatrix, the Dupin indicatrix can be used to classify the point as flat, parabolic, elliptic, or hyperbolic.

• In the immediate neighborhood of a point on a surface, the surface may be approximated by:

(A) An elliptic paraboloid for an elliptic point.

(B) A hyperbolic paraboloid for a hyperbolic point.

(C) A parabolic cylinder for a parabolic point.

(D) A plane for a flat point.

• In the neighborhood of a parabolic point P on a surface S, the tangent plane of S at P meets S in a single line passing through P.

• In the neighborhood of a hyperbolic point P on a surface S, the tangent plane meets S in two lines intersecting at P where these two lines divide S alternatively into regions above the tangent plane and regions below the tangent plane.

• The following function:

$$\frac{II_S}{2} = \frac{e\,dudu + 2f\,dudv + g\,dvdv}{2} \tag{242}$$

evaluated at a given point P of a class C^2 surface is called the osculating paraboloid of P. This function is used to determine the shape of the surface at P.

4.4.6 Umbilical Point

• A point on a surface is called "umbilical" or "umbilic" or "navel" if all the normal sections of the surface at the point have the same curvature κ.[152]

[151] Hence, having undefined Dupin indicatrix is a characteristic for planar points.

[152] For normal sections, the normal curvature is equal to the curvature.

• The following are some examples of umbilical points on common surfaces:

(A) All points of planes are umbilical.[153]

(A) All points of spheres are umbilical.[154]

(B) The vertex of a paraboloid of revolution is an umbilical point.

(C) The two vertices of an ellipsoid of revolution are umbilical points.

• If all points of a surface of class C^3 are umbilical then the surface must be a sphere (the plane is a special case of sphere as it can be considered a sphere with infinite radius).

• The sufficient and necessary condition for a point to be umbilical is that the curvature tensor $b_{\alpha\beta}$ is proportional to the metric tensor $a_{\alpha\beta}$, that is:

$$b_{\alpha\beta} = \lambda(u^1, u^2) a_{\alpha\beta} \qquad \alpha, \beta = 1, 2 \tag{243}$$

where λ is independent of the direction of the tangent to the normal section at the umbilical point. As a result, the determinants of the two tensors, a and b, satisfy the relation:

$$b = \lambda^2 a \tag{244}$$

• Since the first fundamental form is positive definite, $a > 0$. Therefore, if at the umbilical point $\lambda = 0$ then $b = 0$ according to Eq. 244 and the point is a flat or parabolic umbilic; otherwise $b > 0$ (since λ is real) and the point is an elliptic umbilic.[155]

• On a plane surface all points are planar umbilic, while on a sphere all points are elliptic umbilic.[156]

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[153] Some authors impose the condition $K > 0$ at umbilical points and hence the points of plane are not umbilical according to these authors.

[154] Hence, umbilical points may be called spherical points.

[155] By definition, a hyperbolic point cannot be umbilical point.

[156] They may also be called spherical umbilic.

• At umbilical points $\kappa_1 = \kappa_2$ and hence we have:

$$K = \kappa_1 \kappa_2 = \kappa_1 \kappa_1 = (\kappa_1)^2 = \left(\frac{2\kappa_1}{2}\right)^2 = \left(\frac{\kappa_1 + \kappa_2}{2}\right)^2 = H^2 \tag{245}$$

where K and H are the Gaussian and mean curvatures at the point. This can also be obtained from Eq. 202 where the discriminant of this quadratic equation is zero, i.e. $4H^2 - 4K = 0$, since at an umbilical point the two roots are equal and hence $H^2 = K$.[157]

• The relation between K and H at umbilical points, as expressed in Eq. 245, may be stated by some authors in the following disguised form:

$$\left(a^{\alpha\beta} b_{\alpha\beta}\right)^2 = \frac{4}{a}\left(b_{11} b_{22} - b_{12}^2\right) \qquad (\alpha, \beta = 1, 2) \tag{246}$$

[157] The provision $H^2 = K$ is a sufficient and necessary condition for the point to be umbilic.

5 Special Types of Curve

• There are many classifications of space curves depending on their properties and relations; a few of these categories are given in the following subsections.

5.1 Straight Lines

• A necessary and sufficient condition for a curve of class C^2 to be a straight line is that its curvature is zero at every point on the curve.

• A criterion for a curve to be a straight line is that all the tangents of the curve are parallel.[158]

• Another criterion for a curve $C(t) : I \to \mathbb{R}^3$ where $t \in I \subseteq \mathbb{R}$ to be a straight line is that for all the points t in the domain of the curve, $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ are linearly dependent where the overdot represents derivative with respect to the general parameter t of the curve.

• A straight line lying in a surface has the same tangent plane at each of its points, and hence the line is contained in this tangent plane.

5.2 Plane Curves

• A curve is described as plane curve if the whole curve can be contained in a plane with no distortion.

• A necessary and sufficient condition for a curve parameterized by a general parameter t to be a plane curve is that the relation $\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}}) = 0$ holds identically where the overdots represent differentiation with respect to t.

• Plane curves are characterized by having identically vanishing torsion.[159]

• For plane curves, the osculating plane at each regular point contains the entire curve.

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[158] "Parallel" here is in its absolute Euclidean sense.

[159] An identically vanishing torsion is a necessary and sufficient condition for a regular curve of class C^2 to be plane curve.

Therefore, the plane curve may be characterized by having a common intersection point for all of its osculating planes.[160]

• Two curves are plane curves if they have the same binormal lines at each pair of corresponding points on the two curves.

• The locus of the centers of curvature of a curve C is an evolute of C *iff* C is a plane curve.

• On a smooth surface, a geodesic curve (see § 5.7) which is also a line of curvature (see § 5.8) is a plane curve.

5.3 Involutes and Evolutes

• If C_e is a space curve with a tangent surface S_T (see § 6.6) and C_i is a curve embedded in S_T and it is orthogonal to all the tangent lines of C_e at their intersection points, then C_i is called an involute of C_e while C_e is called an evolute of C_i. Hence, the involute is an orthogonal trajectory of the generators of the tangent surface of its evolute.

• Accordingly, the equation of an involute C_i to a curve C_e is given by:

$$\mathbf{r}_i = \mathbf{r}_e + (c - s)\,\mathbf{T}_e \tag{247}$$

where $\mathbf{r}_i$ is an arbitrary point on the involute, $\mathbf{r}_e$ is the point on the curve C_e corresponding to $\mathbf{r}_i$, c is a given constant, s is a natural parameter of C_e and $\mathbf{T}_e$ is the unit vector tangent to C_e at $\mathbf{r}_e$.

• A visual demonstration of how to generate an involute C_i of a curve C_e, when $(c - s)$ in Eq. 247 is positive, may be given by unrolling a taut string wrapped around C_e where the string is kept in the tangent direction as it is unrolled. A fixed point P on the string, where the distance between P and the point of contact of the string with C_e represents a

[160] It also implies that the curve have the same osculating plane at all of its points.

natural parameter of C_e, then traces an involute of C_e.

• A curve has infinitely many involutes corresponding to different values of c and/or s in Eq. 247.

• For any tangent of a given curve, the length of the line segment confined between two given involutes is constant which is the difference between the two c in Eq. 247 of the two involutes.

• If C_e is an evolute of C_i, then for a given point P_e on C_e and the corresponding point P_i on C_i the principal normal line of C_e at P_e is parallel to the tangent line of C_i at P_i.

• A curve C_i is a plane curve *iff* the locus of the centers of curvature of C_i is an evolute of C_i.

• The involutes of a circle are congruent.

• The evolutes of plane curves are helices.

5.4 Bertrand Curves

• Bertrand curves are two associated space curves with common principal normal lines at their continuously-varying one-to-one corresponding points.[161]

• Associated Bertrand curves are characterized by the following properties:

(A) The product of the torsions of their corresponding points is constant; i.e. $\tau_1(s_o)\tau_2(s_o) =$ constant where τ_1 and τ_2 are the torsions of the two curves and s_o is a given value of their common parameter.

(B) The distance between their corresponding points is constant.

(C) The angle between their corresponding tangent lines is constant.

• For a plane curve C_1, there is always a curve C_2 such that C_1 and C_2 are associated Bertrand curves.[162]

[161] Some authors characterize Bertrand curves by having a common principal normal vector.

[162] This may also be stated as: a plane curve C_1 has always a Bertrand curve associate C_2 (i.e. C_1 and C_2 are associated Bertrand curves).

• If C_1 is a curve with non-vanishing torsion such that C_1 has more than one Bertrand curve associate, then C_1 is a circular helix. The reverse is also true.

• If C_1 is a curve with non-vanishing torsion then a necessary and sufficient condition for C_1 to be a Bertrand curve (i.e. it possesses an associate curve C_2 such that C_1 and C_2 are Bertrand curves) is that there are two constants c_1 and c_2 such that:

$$\kappa = c_1\tau + c_2 \tag{248}$$

where κ and τ are the curvature and torsion of the curve C_1.

• If a plane curve C has two involutes C_1 and C_2, then C_1 and C_2 are Bertrand curves.

5.5 Spherical Indicatrix

• A spherical indicatrix of a continuously-varying unit vector is a continuous curve $\bar{C}$ on the origin-based unit sphere generated by mapping the unit vector (e.g. $\mathbf{T}$ or $\mathbf{N}$ or $\mathbf{B}$) of a particular space curve C on an equal unit vector represented by a point on the origin-based unit sphere. Hence, we have $\bar{C_\mathbf{T}}$, $\bar{C_\mathbf{N}}$ and $\bar{C_\mathbf{B}}$ as the spherical indicatrices of C corresponding respectively to the tangent, principal normal and binormal vectors of C.

• When C is a naturally parameterized curve then $\bar{\mathbf{r}}(t) = \mathbf{T}(t)$ where $\bar{\mathbf{r}}$ is the origin-based position vector of $\bar{C_\mathbf{T}}$.

• A necessary and sufficient condition for $\bar{\mathbf{r}}(t)$ to be a natural parameterization of $\bar{C_\mathbf{T}}$ is that $\kappa(t) = 1$ identically.

• The tangent to the curve $\bar{C_\mathbf{T}}$ of a curve C is parallel to the normal vector $\mathbf{N}$ of C at the corresponding points of the two curves.

• The curvature of the curve $\bar{C_\mathbf{T}}$ of a curve C is related to the curvature and torsion of C by:

$$\kappa_\mathbf{T}^2 = \frac{\kappa^2 + \tau^2}{\kappa^2} \tag{249}$$

where $\kappa_{\mathbf{T}}$ is the curvature of $\bar{C_{\mathbf{T}}}$ while κ and τ are the curvature and torsion of C respectively.

• The torsion of the curve $\bar{C_{\mathbf{T}}}$ of a curve C is given by:

$$\tau_{\mathbf{T}} = \frac{\kappa' \tau - \kappa \tau'}{\kappa \left(\kappa^2 + \tau^2\right)} \tag{250}$$

where $\tau_{\mathbf{T}}$ is the torsion of $\bar{C_{\mathbf{T}}}$, κ and τ are the curvature and torsion of C respectively, and the prime stands for the derivative with respect to the natural parameter s.

• The curvature of the curve $\bar{C_{\mathbf{B}}}$ of a curve C is given by:

$$\kappa_{\mathbf{B}} = \frac{\kappa^2 + \tau^2}{\kappa^2} \tag{251}$$

where $\kappa_{\mathbf{B}}$ is the curvature of $\bar{C_{\mathbf{B}}}$ while κ and τ are the curvature and torsion of C respectively.

• The torsion of the curve $\bar{C_{\mathbf{B}}}$ of a curve C is given by:

$$\tau_{\mathbf{B}} = \frac{\kappa' \tau - \kappa \tau'}{\tau \left(\kappa^2 + \tau^2\right)} \tag{252}$$

where $\tau_{\mathbf{B}}$ is the torsion of $\bar{C_{\mathbf{B}}}$, κ and τ are the curvature and torsion of C respectively, and the prime stands for the derivative with respect to the natural parameter s.

• The necessary and sufficient condition for the curve $\bar{C_{\mathbf{T}}}$ of a curve C to be a circle is that C is a helix.

• The tangent to the curve $\bar{C_{\mathbf{T}}}$ of a curve C is parallel to the tangent to the $\bar{C_{\mathbf{B}}}$ of C at the corresponding points of the two curves.

5.6 Spherical Curves

• Spherical curve is a curve that lies completely on the surface of a sphere. Spherical indicatrices are common examples of spherical curves (see § 5.5).

• Circles are the only spherical curves with constant curvature.

• At all points of a spherical curve, the normal plane passes through the center of the embedding sphere. Conversely, if all the normal planes of a curve meet in a common point, then the curve is spherical with the common point being the center of the sphere that envelopes the curve.

• The sufficient and necessary condition that should be satisfied by a spherical curve is given by:

$$\frac{R_\kappa}{R_\tau} + \frac{d}{ds}\left(R_\tau \frac{dR_\kappa}{ds}\right) = 0 \tag{253}$$

where R_κ and R_τ are the radii of curvature and torsion and s is a natural parameter of the curve.

• The center of curvature of a twisted spherical curve C at a given point P on C is the perpendicular projection of the center of the enveloping sphere on the osculating plane of C at P.

5.7 Geodesic Curves

• The characteristic feature of a geodesic curve is that it has vanishing geodesic curvature κ_g at every point on the curve. This is a necessary and sufficient condition for a curve to be geodesic.

• In more technical terms, let $S : \Omega \to \mathbb{R}^3$ be a surface defined on a set $\Omega \subseteq \mathbb{R}^2$ and let $C(t) : I \to \mathbb{R}^3$, where $I \subseteq \mathbb{R}$, be a regular curve on S, then C is a geodesic curve *iff* $\kappa_g(t) = 0$ on all points $t \in I$ in its domain.

• The path of the shortest distance connecting two points in a Riemannian space is a geodesic. The length of arc, as given by Eq. 97, is used in the definition of geodesic in this sense. The geodesic is a straight line in a Euclidean space, but it is a generalized curved path in a general Riemannian space.

• Examples of geodesic curves on simple surfaces are arcs of great circles on spheres and arcs of helices on cylinders.[163] The meridians of surfaces of revolution are also geodesics. The arcs of parallel circles on a surface of revolution corresponding to stationary points on the generating curve of the surface are also geodesic curves.

• All straight lines on any surface are geodesic curves.[164]

• If a geodesic surface curve is not a straight line then its principal normal vector **N** is collinear with the normal vector **n** to the surface at each point on the curve with non-vanishing curvature.[165] The opposite is also true.

• Being an arc of a great circle on a sphere is a sufficient and necessary condition for being geodesic.

• Intrinsically, the geodesic curves are straight lines in the sense that a 2D inhabitant will see them straight since he cannot measure their curvature. Any deviation from such "straight lines" within the surface is therefore a geodesic curvature and can be detected intrinsically by a 2D inhabitant.

• Although a geodesic curve is usually the curve of the shortest distance between two points on the surface it is not necessarily so. For instance, the largest of the two arcs forming a great circle on a sphere is a geodesic curve but it is not the curve of the shortest distance on the sphere between its two end points; in fact it is the curve of the longest distance.[166]

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[163] The generating straight lines and the circles on cylinders are also geodesics (they may be considered as degenerate helices).

[164] For plane surfaces in particular, being a straight line on a plane is a sufficient and necessary condition for being geodesic.

[165] Collinearity is equivalent to the condition that **n** lies in the osculating plane of the curve at the given point.

[166] A constraint my be imposed to make the geodesic minimal by stating that geodesics minimize distance locally but not necessarily globally where an infinitesimal element of arc length is considered in this

• If on a surface S there is exactly one geodesic curve connecting two given points, P_1 and P_2, then the length of the geodesic segment between P_1 and P_2 is the shortest distance on S between these points.

• Being a shortest path is a sufficient but not necessary condition for being a geodesic, that is all shortest paths connecting two given points are geodesics but not all geodesics are shortest paths, as explained in the previous points.

• A sufficient and necessary condition for a curve to be a geodesic is that the first variation[167] of its length is zero.[168]

• It can be shown that a geodesic curve satisfies the Euler-Lagrange variational principle which is a necessary and sufficient condition for extremizing the arc length.[169]

• A physical interpretation may also be given to the geodesic curve that a free particle restricted to move on the surface will follow a geodesic path.

• The geodesic, even in its restricted sense as the curve of the shortest distance, is not necessarily unique; for example all semicircular meridians of longitude connecting the two poles (or any antipodal points) of a sphere are geodesics and there is an infinite number of them.

• In fact even the existence, let alone uniqueness, of a geodesic connecting two points on a surface is not guaranteed. An example is the xy plane excluding the origin with two points on a straight line lying in the plane and passing through the origin where there is a

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constraint.

[167]The first variation of a functional $F(x)$ may be defined by the Gateaux derivative of the functional as:

$$\delta F(x,h) = \lim_{\alpha \to 0} \frac{F(x+\alpha h) - F(x)}{\alpha} \tag{254}$$

where x and h are variable functions and α is a scalar parameter (refer to the Calculus of Variations for details).

[168]In fact this may be taken as the basis for the definition of geodesic as the curve connecting two fixed points, P_1 and P_2, whose length possesses a stationary value with regard to small variations, that is: $\delta \int_{P_1}^{P_2} ds = 0$.

[169]This principle in its generic, simple and most common form is given by: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$. The reader is advised to consult textbooks on the Calculus of Variations for details.

lower limit for the length of any curve connecting the two points. This limit is the straight segment connecting the two points but this segment cannot be a geodesic since it includes the origin which is not on the surface. Any curve C (other than the straight segment) connecting the two points cannot be a geodesic since there is always another curve on the plane connecting the two points which is shorter than C.

• In the neighborhood of a given point P on a surface and for any specified direction, there is exactly one geodesic curve that passes through P in that direction. More technically, for any specific point P on a surface S of class C^3, and for any tangent vector $\mathbf{v}$ in the tangent space of S at P, there exists a geodesic curve on the surface in the direction of $\mathbf{v}$ in the immediate neighborhood of P.[170]

• The obvious examples of the last point is the plane, where a straight line passes through any point and in any direction, and the sphere where a great circle passes through any point and in any direction. A less obvious example is the cylinder where a helix (including the straight line generators and the circles which can be regarded as degenerate forms of helix) passes through any point and in any direction.

• Similarly, there is exactly one geodesic passing through two sufficiently-close points on a smooth surface.

• Geodesics in curved spaces are the equivalent of straight lines in flat spaces.

• For planes (or in fact any Euclidean nD manifold) there is a unique geodesic passing between any two points (close or not) which is the straight line segment connecting the two points.

• The necessary and sufficient conditions that should be satisfied by a naturally-parameterized curve on a surface, both of class C^2, to be a geodesic curve are given by the following second

[170]This is based on the existence of a unique solution to the geodesic differential equations (Eq. 255) for given initial values for a point on the curve and its derivative (which represents its tangent direction) at that point.

order non-linear differential equations:[171]

$$\begin{aligned} \frac{d^2u^1}{ds^2} + \Gamma^1_{11}\left(\frac{du^1}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^1_{22}\left(\frac{du^2}{ds}\right)^2 &= 0 \\ \frac{d^2u^2}{ds^2} + \Gamma^2_{11}\left(\frac{du^1}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^2_{22}\left(\frac{du^2}{ds}\right)^2 &= 0 \end{aligned} \tag{255}$$

where s is the arc length, and the Christoffel symbols are derived from the surface metric. The last equations can be merged in a single equation using tensor notation:

$$\frac{\delta}{\delta s}\left(\frac{du^\alpha}{ds}\right) \equiv \frac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0 \tag{256}$$

where $\alpha, \beta, \gamma = 1, 2$. These equations, which can be obtained from Eq. 181 by setting the two components of the geodesic curvature to zero, have no closed form solutions in general. Similar equations are used to find the geodesic curves between two given points in general nD spaces.

• From Eq. 256, it can be seen that being a geodesic is an intrinsic property since it depends exclusively on the Christoffel symbols which depend only on the coefficients of the first fundamental form and their partial derivatives.

• From Eq. 256, it can be seen that for planes (or indeed for any Euclidean nD manifold) the geodesic is a straight line since in this case the Christoffel symbols vanish identically and Eq. 256 is reduced to $\frac{d^2u^\alpha}{ds^2} = 0$ which has a straight line solution.

• From Eq. 184, we see that the u^1 coordinate curves on a sufficiently smooth surface are geodesics *iff* $\Gamma^2_{11} = 0$. Similarly, from Eq. 186, we see that the u^2 coordinate curves are geodesics *iff* $\Gamma^1_{22} = 0$.

• We also see from Eqs. 185 and 187 that for orthogonal coordinate systems the coordinate curves are geodesics *iff* E is independent of v and G is independent of u.

• For a Monge patch of the form $\mathbf{r}(u, v) = (u, v, f(u, v))$, the geodesic differential equations

[171] Due to their non-linearity, closed form explicit solutions of these equations are rare.

are given by:

$$\begin{aligned} \left(1+f_u^2+f_v^2\right)u''+f_u f_{uu}(u')^2+2f_u f_{uv}u'v'+f_u f_{vv}(v')^2 &= 0 \\ \left(1+f_u^2+f_v^2\right)v''+f_v f_{uu}(u')^2+2f_v f_{uv}u'v'+f_v f_{vv}(v')^2 &= 0 \end{aligned} \tag{257}$$

where the subscripts u and v represent partial derivatives with respect to the surface curvilinear coordinates u and v, and the prime represents derivatives with respect to a natural parameter.

• Each one of the following provisions is a necessary and sufficient condition for a curve C on a surface S to be a geodesic curve:

(A) The geodesic component of the curvature vector is zero at each point on the curve, that is $\mathbf{K}_g = \mathbf{0}$ identically.

(B) The osculating plane of the curve at each point of the curve is orthogonal to the tangent plane of S at that point.[172]

(C) The normal vector $\mathbf{n}$ to the surface at any point on the curve lies in the osculating plane.[173]

(D) The principal normal vector $\mathbf{N}$ of C is normal to the surface at each point on C.

(E) The curvature vector $\mathbf{K}$ is normal to the tangent plane at each point on the curve.

• Being a geodesic is independent of the choice of the coordinate system and hence it is invariant under permissible transformations. It is also independent of the type of representation and parameterization and hence it is invariant in this sense.[174]

• Geodesics can be open or closed curves and may be self-intersecting.[175]

[172] On geodesic curves $\kappa_g = 0$ and hence $\mathbf{n}$ and $\mathbf{N}$ are in the same orientation (parallel or anti-parallel) on all points along the curve (refer to Eqs. 171 and 173) and hence $\mathbf{n}$ lies in the osculating plane.

[173] This is because for a geodesic curve, $\mathbf{K}_g$ vanishes identically and hence $\mathbf{K} = \kappa\mathbf{N} = \kappa_n\mathbf{n} = \mathbf{K}_n$.

[174] This may be regarded as another reason for non-uniqueness of geodesic because even if the trace of the curve is unique, its parameterization is not unique in general due to different mappings.

[175] From the previous points, we see that "geodesic curves" have two common uses: (a) curves with identically vanishing geodesic curvature and (b) curves of shortest distance between two points, where (b) is a subset of (a). Here, "geodesics" is used mainly in the first sense.

• As a result of the Gauss-Bonnet theorem (see § 4.4.4), a surface with negative Gaussian curvature cannot have a geodesic that intersects itself.[176] Also on such a surface, two geodesics cannot intersect at more than one point if the geodesics enclose a simply-connected region.[177]

• The lines of curvature (see § 5.8) are geodesics.

• On a patch on a surface of class C^2 with orthogonal coordinate curves and with first fundamental form coefficients being dependent on only the u-coordinate variable[178], the following statements apply:

(A) The u-coordinate curves with constant v are geodesics at a point *iff* $\partial_u G = 0$.

(B) The v-coordinate curves with constant u are geodesics.

(C) A curve C represented by $\mathbf{r} = \mathbf{r}(u, v(u))$ is a geodesic *iff*:

$$v = \pm \int_C \frac{\alpha\sqrt{E}}{\sqrt{G(G - \alpha^2)}} du \tag{258}$$

where α is a constant.

• A vector attained by parallel propagation (see § 2.5) of a tangent vector to a geodesic curve stays always tangent to the geodesic curve.

• A sufficient and necessary condition for a surface curve to be geodesic is being a tangent to a parallel vector field.

• As a consequence of the previous points, a vector field attained by parallel propagation along a geodesic makes a constant angle with the geodesic.

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[176]Because the first term on the LHS of Eq. 239 will vanish and the third term will be negative and hence the sum of the angles (which is the angle of intersection) in the second term will not satisfy the relation unless it is greater than 2π which is not possible since $K < 0$ over the surface (see § 4.4.4).

[177]Similarly, on introducing an artificial vertex at a regular point on one of these curves we will have a new corner with π interior angle and hence the sum of the geodesic triangle will exceed π. Also, on introducing an artificial vertex at a regular point on each one of these curves we will have a curvilinear quadrilateral whose internal angles sum is greater than 2π on a surface with $K < 0$ which is not possible (see § 4.4.4).

[178]That is: $E = E(u)$, $F = 0$ and $G = G(u)$. The case of dependence on only the v-coordinate variable can be obtained by re-labeling the coordinate variables and coefficients.

• Geodesic in curved spaces is a generalization of straight lines in flat spaces.[179]

• Geodesic curves on a developable surface become straight lines when the surface is developed into a plane by unrolling.

• Isometric surfaces possess identical geodesic equations.

• On a surface with orthogonal coordinate curves, the curves of constant u^α are geodesics *iff* $a_{\beta\beta}$ $(\beta \neq \alpha)$ is a function of u^β only.

5.8 Line of Curvature

• A "line of curvature" is a curve C on a surface S defined on an interval $I \subseteq \mathbb{R}$ as $C : I \to S$ with the condition that the tangent of C at each point on C is collinear with one of the principal directions (see § 4.4) of the surface at that point.

• Since the definition of the line of curvature is based on the existence of distinct principal directions, the line of curvature should not include umbilical (including flat) points (see § 4.4.6) due to the absence of distinct principal directions at these points.

• Referring to Eq. 51, on a line of curvature either $\sin\theta = 0$ or $\cos\theta = 0$ and hence the lines of curvature are characterized by having identically vanishing geodesic torsion (i.e. $\tau_g = 0$).

• The lines of intersection of each pair of a triply orthogonal system are lines of curvature.[180]

• Examples of lines of curvature are meridians and parallels of surfaces of revolution of class C^2.

• At a non-umbilical point P on a sufficiently smooth surface S, the u and v coordinate curves are aligned with the principal directions *iff* $f = F = 0$ at P. Hence, the coordinate

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[179] Hence, geodesics may be described as the straightest curves in the space.

[180] Three families of surfaces in a subset V of a 3D space form a triply orthogonal system if at each point P of V there is a single surface of each family passing through P such that each pair of these surfaces intersect orthogonally at their curve of intersection.

curves on S, excluding the umbilical points, are lines of curvature *iff* $f = F = 0$ over the entire surface.

• When the u and v coordinate curves of a surface patch are lines of curvature, the principal curvatures over the entire patch will be given by:

$$\kappa_1 = \frac{e}{E} \qquad \kappa_2 = \frac{g}{G} \tag{259}$$

where E, G, e, g are the coefficients of the first and second fundamental forms of the patch. These coefficients are functions of position in general.

• On a surface of class C^3, there are two perpendicular families of lines of curvature in the neighborhood of any non-umbilical point.

• If the curve of intersection of two surfaces is a line of curvature for one surface then it is a line of curvature for the other surface when the two surfaces are intersecting each other at a constant angle.

• The lines of curvature satisfy the following relation:

$$(a_{12}b_{11} - a_{11}b_{12})\,du^1du^1 + (a_{22}b_{11} - a_{11}b_{22})\,du^1du^2 + (a_{22}b_{12} - a_{12}b_{22})\,du^2du^2 = 0 \tag{260}$$

where the indexed a and b are the coefficients of the surface covariant metric and curvature tensors respectively.

• The condition that should be satisfied by a line of curvature on a surface may be given in tensor notation by:

$$\underline{\epsilon}^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta}du^{\alpha}du^{\beta} = 0 \tag{261}$$

• The generators of developable surfaces are lines of curvature.

• The lines of curvature form a real orthogonal grid over the surface. On surfaces with constant Gaussian curvature, the lines of curvature form an isometric conjugate grid.

• For a developable surface, the lines of curvature consist of its generators and their orthogonal trajectories.

• On a smooth surface, excluding planes and spheres, if the lines of curvature are selected as the net of coordinate curves then $a_{12} = b_{12} = 0$ over the entire surface.

• On a sufficiently smooth surface, any geodesic which is a plane curve is a line of curvature.

• A curve is a line of curvature *iff* the tangent to the curve and the tangent to its spherical image (see § 3.7) at corresponding points are parallel.

• The lines of curvature on a surface, which is not a sphere or minimal surface, are represented by an orthogonal net on its spherical image.

• In the neighborhood of a non-umbilical point on a sufficiently smooth surface there are two orthogonal families of lines of curvature. Hence, at each point P on such a surface a coordinate patch including P can be introduced in the neighborhood of P where the coordinate curves at P are aligned with the principal directions.

• On a surface patch where the Gaussian curvature does not vanish, the angles between the asymptotic lines (see § 5.9) are bisected by the lines of curvature.

5.9 Asymptotic Lines

• An asymptotic direction of a surface at a point P is a direction for which the normal curvature vanishes, i.e. $\kappa_n = 0$.[181] Hence, at an asymptotic point we have:

$$\mathbf{K} = \mathbf{K}_g = \kappa_g \mathbf{u} \tag{262}$$

• As a consequence of Eq. 177, κ_n is zero for directions for which the second fundamental form is zero. Hence the necessary and sufficient condition for the asymptotic directions is

[181] Asymptotic directions are defined only at points for which $K \le 0$.

that:

$$b_{\alpha\beta}du^\alpha du^\beta = b_{11}(du^1)^2 + 2b_{12}\,du^1du^2 + b_{22}(du^2)^2 = 0 \tag{263}$$

• The number of asymptotic directions at elliptic, parabolic and hyperbolic points is 0, 1 and 2 respectively, while at flat points all directions are asymptotic. The two asymptotic directions of a hyperbolic point separate the directions of positive normal curvature from the directions of negative normal curvature. The sign of the normal curvature at elliptic and parabolic points is the same in all directions.[182]

• A t-parameterized surface curve $C(t) : I \to S$, where $I \subseteq \mathbb{R}$ is an open interval over which $K < 0$ and S represents the surface, is described as an asymptotic line if at each point $t \in I$ the vector $\mathbf{T}$, which is the tangent to the curve, is collinear with one of the asymptotic directions at that point on the curve.

• From the previous points, it can be seen that the asymptotic lines are characterized by the following:

(A) The normal component of the curvature vector is zero at each point on the curve, that is $\mathbf{K}_n = \mathbf{0}$ identically.

(B) The tangent plane to the surface at each point of the curve coincides with the osculating plane of the curve at that point.

• The differential equations representing asymptotic lines can be obtained from the condition that the normal curvature vanishes identically over the line, that is (see Eq. 177):

$$e\left(\frac{du^1}{ds}\right)^2 + 2f\frac{du^1}{ds}\frac{du^2}{ds} + g\left(\frac{du^2}{ds}\right)^2 = 0 \tag{264}$$

• From Eq. 264, it can be seen that the necessary and sufficient condition for the u^1 and u^2 coordinate curves to become asymptotic lines is that $e = g = 0$ on every point on the

[182] The directions here are those of the tangents to the normal sections at the given point (excluding any asymptotic direction).

curve.[183]

• On a sufficiently smooth surface of class C^3, there are two distinct families of asymptotic lines in the neighborhood of a hyperbolic point.

• According to Eq. 174 $\kappa_n = \mathbf{n} \cdot \mathbf{K}$ where $\mathbf{n}$ and $\mathbf{K}$ are respectively the normal vector to the surface and the curve curvature vector. Hence, a curve on a sufficiently smooth surface is an asymptotic line *iff* $\mathbf{n} \cdot \mathbf{K} = 0$ identically. This condition is realized if at each point on the curve either $\mathbf{K} = \mathbf{0}$ or $\mathbf{K}$ and $\mathbf{n}$ are orthogonal vectors.[184] In the former case the point is an inflection point while in the latter case the osculating plane is tangent to the surface at the point. Therefore, all points on an asymptotic line should be one of these types or the other. The reverse is also true, i.e. a curve whose all points are one of these types or the other is an asymptotic line.

• As a results of the previous point, any straight line on a surface is an asymptotic line since the curve curvature vector $\mathbf{K}$ vanishes identically on such a line.

• According to the theorem of Beltrami-Enneper, along an asymptotic non-straight line on a sufficiently smooth surface the square of the torsion τ is equal to the negative of the Gaussian curvature K, that is:

$$\tau^2 = -K \tag{265}$$

where τ and K are evaluated at each individual point along the curve.[185]

• The torsions of two asymptotic lines passing through a given point on a surface are equal in magnitude and opposite in sign.

• As we will see (refer to § 5.10), asymptotic directions are self-conjugate.[186]

• From the definition of the asymptotic direction plus the Euler equation (Eq. 196), we

[183] This is because on a coordinate curve either $du^1 = 0$ and $du^2 \neq 0$ or $du^1 \neq 0$ and $du^2 = 0$ and hence the middle term is vanishing anyway.

[184] The vector $\mathbf{n}$ cannot vanish on a regular point.

[185] Since asymptotic directions are defined only at points for which $K \le 0$, the square of the torsion in the above equation is equal to the absolute value of the Gaussian curvature at the point, that is: $\tau^2 = |K|$.

[186] Some authors take self-conjugation as the defining characteristic for being asymptotic.

see that the angle θ which an asymptotic direction makes with the principal direction of κ_1 at a given point P on a sufficiently smooth surface S is given by:

$$\tan^2\theta = -\frac{\kappa_1}{\kappa_2} \tag{266}$$

where κ_1 and κ_2 are the principal curvatures of S at P.

• On a sufficiently smooth surface with orthogonal families of asymptotic lines the mean curvature H is zero.

• The principal directions at a given point on a smooth surface bisect the asymptotic directions at the point.

• Eq. 263 is quadratic and hence it possesses two solutions which are real and distinct, or real and coincident, or conjugate imaginary depending on its discriminant Δ which is opposite in sign to the determinant b of the surface covariant curvature tensor.[187] Hence, the asymptotic directions at a given point on a surface can be classified according to the determinant b at the point as:

(A) Real and distinct for $\Delta > 0$ and hence $b < 0$.

(B) Real and coincident for $\Delta = 0$ and hence $b = 0$.

(C) Conjugate imaginary for $\Delta < 0$ and hence $b > 0$.

• From Eq. 208, the sign of the Gaussian curvature K is the same as the sign of b, since $a > 0$. Hence, the classification in the previous point can also be based on K as stated for b in the previous point.

• A straight line contained in a surface is an asymptotic line. This is due to the fact that such a line is wholly contained in a plane, which is the tangent space of each of its points, and hence the normal curvature vanishes identically along the line.

• As a consequence of having identically vanishing normal curvature, the osculating planes

[187]The discriminant is: $\Delta = 4(b_{12})^2 - 4b_{11}b_{22}$ while the determinant is: $b = b_{11}b_{22} - (b_{12})^2$, and hence $\Delta = -4b$.

at each point of a curved asymptotic line on a surface are tangent to the surface.

• On a sufficiently smooth surface, if a geodesic curve C is a line of curvature then C is a plane curve.

5.10 Conjugate Directions

• A direction $\frac{\delta u}{\delta v}$ at a point on a sufficiently smooth surface is described as conjugate to the direction $\frac{du}{dv}$ if the following relation holds:

$$d\mathbf{r} \cdot \delta\mathbf{n} = 0 \tag{267}$$

where $d\mathbf{r} = \mathbf{E}_1 du + \mathbf{E}_2 dv$ and $\delta\mathbf{n} = \partial_u \mathbf{n} \delta u + \partial_v \mathbf{n} \delta v$. Due to the symmetry, $\frac{du}{dv}$ is also conjugate to $\frac{\delta u}{\delta v}$, and hence the two directions are described as conjugate directions.

• Two families of curves on a sufficiently smooth surface are described as conjugate families if the directions of their tangents at each point on the curves are conjugate directions.

• The u and v coordinate curves on a smooth surface are conjugate families of curves *iff* f, which is the coefficient of the second fundamental form, vanishes identically.

• At a hyperbolic or elliptic point on a sufficiently smooth surface, each direction has a unique conjugate direction.

• An asymptotic direction is self-conjugate direction.

6 Special Types of Surface

• There are many classifications of surfaces in 3D spaces depending on their properties and relations; a few of these classifications are given in the following subsections.

6.1 Plane Surfaces

• Planes are simple, ruled, connected, elementary surfaces.

• All the coefficients of the surface curvature tensor vanish identically throughout plane surfaces.

• The Riemann-Christoffel curvature tensor vanishes identically over plane surfaces.

• The Gaussian curvature K and the mean curvature H vanish identically over planes.

• Planes are minimal surfaces.

• All points on planes are flat umbilical.

• At any point on a plane surface, $\kappa_1 = \kappa_2 = 0$ and hence all the directions are principal directions (or there is no principal direction).

• At any point on a plane surface, all the directions are asymptotic.

• A sufficient and necessary condition for a surface to be isometric with the plane is having an identically vanishing Riemann-Christoffel curvature tensor. The same applies for identically vanishing Gaussian curvature.

6.2 Quadratic Surfaces

• Quadratic surfaces are defined by the following quadratic equation:[188]

$$A_{ij}x^i x^j + B_i x^i + C = 0 \qquad (i, j = 1, 2, 3) \tag{268}$$

[188] We assume a Euclidean 3D space with a rectangular Cartesian coordinate system.

where the coefficients A_{ij} and B_i are real-valued tensors of rank-2 and rank-1 respectively and C is a real scalar.

• There are six main non-degenerate types of quadratic surfaces: ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic paraboloid, hyperbolic paraboloid, and quadric cone. By rigid motion transformations, consisting of translation and rotation of coordinates whose purpose is to put the center of symmetry of these surfaces at the origin and orient their axes with the coordinate lines, these types can be given in the following canonical forms:[189]

(A) Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{269}$$

(B) Hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{270}$$

(C) Hyperboloid of two sheets:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{271}$$

(D) Elliptic paraboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0 \tag{272}$$

(E) Hyperbolic paraboloid:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0 \tag{273}$$

(F) Quadric cone:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{274}$$

[189] In the following equations, a, b, c are real parameters, and for convenience we use x, y, z for x^1, x^2, x^3 respectively. Also for the first three of these surfaces the origin of coordinates is not a valid point.

6.3 Ruled Surfaces

• A "ruled surface", or "scroll", is a surface generated by a continuous translational-rotational motion of a straight line in space. Hence, at each point of the surface there is a straight line passing through the point and lying entirely in the surface. Planes, cones, cylinders and Mobius strips are common examples of ruled surface. The different perspectives of the generating line along its movement are described as the rulings of the surface.

• A ruled surface that can be generated by two different families of lines is called doubly-ruled surface. Examples of doubly-ruled surface are hyperbolic paraboloids and hyperboloids of one sheet.

• At any point of a regular ruled surface, the Gaussian curvature is non-positive ($K \leq 0$).

• The tangent surface (see § 1.13) of a smooth curve is a ruled surface generated by the tangent line of the curve.

• The tangent plane is constant along a branch, represented by the tangent line at a given point, of the tangent surface of a curve. Examples are the tangent planes of cylinders and cones along their generators.

• If P is a point on a curve C where C has a tangent surface S, then the tangent plane to S along the ruling that passes through P coincides with the osculating plane of C at P. Hence, the tangent surface may be described as the envelope of the osculating planes of the curve.

• The tangent plane to a cylinder or a cone is constant along their generators.

6.4 Developable Surfaces

• As defined previously (see § 1.13), a surface that can be flattened into a plane without local distortion is called developable surface.

• A developable surface can also be defined as a surface that is isometric to the Euclidean plane.

• In 3D manifolds, all developable surfaces are ruled surfaces but not all ruled surfaces are developable surfaces. A ruled surface is developable if the tangent plane is constant along every ruling of the surface as it is the case with cones and cylinders.

• The neighborhood of each point on a sufficiently smooth surface with no flat points is developable *iff* the Gaussian curvature vanishes identically on the surface.

• The generators of a developable surface and their orthogonal trajectories are their lines of curvature.

• A developable surface, excluding cylinder and cone, is a tangent surface of a curve.

• If a developable surface S, excluding cylinder and cone, is rolled out on a plane then all the points of an orthogonal trajectory of the tangent planes of S will map on a single point on the plane.

• The collection of normal lines to a surface S along a given curve C on S make a developable surface *iff* C is a line of curvature (see § 5.8).

• Intrinsically, any developable surface is equivalent[190] to a plane and hence any two developable surfaces are equivalent to each other.

• The generators of developable surfaces are lines of curvature (see § 5.8).

6.5 Isometric Surfaces

• An isometry is an injective mapping from a surface S to a surface $\bar{S}$ which preserves distances.

• As a consequence of preserving the lengths in isometric mappings, the angles and areas are also preserved.

• Examples of isometric surfaces are cylinder and cone which are both isometric to plane.

• Two isometric surfaces, such as a cylinder and a cone or each one of these and a plane,

[190] Equivalent means having the same metric characteristics.

appear identical to a 2D inhabitant. Any difference between the two can only be perceived by an external observer residing in a reference frame in the enveloping space.

• Isometry is an equivalence relation and hence it is reflective, symmetric and transitive, that is for three surfaces S_1, S_2 and S_3 we have:[191]

(A) $S_1 \sim S_1$.

(B) $S_1 \sim S_2 \qquad \Longleftrightarrow \qquad S_2 \sim S_1$.

(C) $S_1 \sim S_2$ and $S_2 \sim S_3 \qquad \Longrightarrow \qquad S_1 \sim S_3$.

• Two isometric surfaces possess identical first fundamental forms and hence any difference between them, as viewed extrinsically from the embedding space, is based on the difference between their second fundamental forms.

• If two surfaces have constant equal Gaussian curvature then they are isometric. The mapping relation between the two surfaces then include three constants corresponding to the three independent coefficients of the first fundamental form.

• A surface of revolution is isometric to itself in infinitely-many ways, each of which corresponds to a rotation of the surface through a given angle around its axis of symmetry. Hence, a surface S_1 which is isometric to a surface of revolution S_2 is equivalent to S_2.

6.6 Tangent Surfaces

• As stated previously, the tangent surface of a space curve is a surface generated by the assembly of all the tangent lines to the curve. The tangent lines of the curve are called the generators of the tangent surface.

• Accordingly, the equation of a tangent surface S_T to a curve C is given by:

$$\mathbf{r}_T = \mathbf{r}_i + k\mathbf{T}_i \qquad (-\infty < k < \infty) \tag{275}$$

[191] The symbol $\sim$ represents an isometric relation.

where $\mathbf{r}_T$ is an arbitrary point on the tangent surface, $\mathbf{r}_i$ is a given point on the curve C, k is a real variable, and $\mathbf{T}_i$ is the unit vector tangent to C at $\mathbf{r}_i$. The tangent surface is generated by varying i along C.

• The tangent surface of a curve is made of two parts: one part corresponding to $k > 0$ and the other corresponding to $k < 0$ where the curve is a border line between these two parts. The two parts of the tangent surface are tangent to each other along the curve which forms a sharp edge between the two.[192]

• The tangent plane is constant along a branch[193] of the tangent surface of a curve. This tangent plane is the osculating plane of the curve at the point of contact of the branch with the curve.

• According to the definition of involute, all the involutes of a curve C_e are wholly embedded in the tangent surface of C_e.

• The normal to the tangent surface of a space curve C at a point of a given ruling $\mathfrak{R}$ is parallel to the binormal line of C at the point of contact of C with $\mathfrak{R}$.

6.7 Minimal Surfaces

• A "minimal surface" is a surface whose area is minimum compared to the area of any other surface sharing the same boundary. Hence, the minimal surface is an extremal with regard to the integral of area over its domain.

• A common physical example of a minimal surface is a soap film formed between two coaxial rings where it takes the minimal surface shape of a catenoid due to the surface tension. This problem, and its alike of investigations related to the physical realization of minimal surfaces, may be described as the Plateau problem.

• Since the mean curvature of a surface at a point P is a measure of the rate of change of

[192] The curve is called the edge of regression of the surface.
[193] A branch in this context is the tangent line of the curve. It is also called generator.

area of the surface elements in the neighborhood of P, a minimal surface is characterized by having an identically vanishing mean curvature and hence the principal curvatures at each point have the same magnitude and opposite signs.

• Examples of minimal surface shapes are planes, catenoids, helicoids and ennepers.[194]

• A minimal surface is characterized by having an orthogonal net of asymptotic lines and a conjugate net of minimal lines.[195]

[194]The catenoid is a surface of revolution generated by revolving a catenary around its directrix. The helicoid is a ruled surface (see § 6.3) with the property that for each point P on the surface there is a helix passing through P and contained entirely in the surface. The enneper is a self-intersecting surface which can be described parametrically in various ways; one of which is:

$$\begin{aligned} x &= -\frac{u^3}{3} + u + uv^2 \\ y &= -u^2 v - v + \frac{v^3}{3} \\ z &= u^2 - v^2 \end{aligned}$$

where $u, v \in (-\infty, \infty)$. It is noteworthy that the catenoid and helicoid are locally isometric.

[195]Minimal lines are curves of minimal length. Having an orthogonal net of asymptotic lines is a sufficient and necessary condition for having zero mean curvature.

7 Tensor Differentiation

• Tensor differentiation, whether covariant or absolute, over a 2D surface follows similar rules to those stated in [9, 10] for general nD curved spaces. Some of these rules are:

(A) The sum and product rules of differentiation apply to covariant and absolute differentiation as usual.

(B) The covariant and absolute derivatives of tensors are tensors.

(C) The covariant and absolute derivatives of scalars and invariant tensors of higher ranks are the same as the ordinary derivatives.

(D) The covariant and absolute derivative operators commute with contraction of indices.

(E) The covariant and absolute derivatives of the metric, Kronecker and alternating tensors (and their associated tensors) vanish identically in any coordinate system, that is:

$$\begin{aligned} a_{\alpha\beta|\gamma} &= 0 & a^{\alpha\beta}{}_{|\gamma} &= 0 \\ \delta^{\alpha}_{\beta|\gamma} &= 0 & \delta^{\alpha\delta}_{\beta\omega|\gamma} &= 0 \\ \underline{\epsilon}_{\alpha\beta|\gamma} &= 0 & \underline{\epsilon}^{\alpha\beta}{}_{|\gamma} &= 0 \end{aligned} \tag{276}$$

where the sign | represents covariant or absolute differentiation with respect to the surface coordinate u^γ. Hence, these tensors should be treated like constants in tensor differentiation.

• An exception of these rules is the covariant derivative of the space basis vectors in their covariant and contravariant forms which is identically zero, as stated previously in [10], that is:

$$\begin{aligned} \mathbf{E}_{i;j} &= \partial_j \mathbf{E}_i - \Gamma^k_{ij}\mathbf{E}_k = \Gamma^k_{ij}\mathbf{E}_k - \Gamma^k_{ij}\mathbf{E}_k = \mathbf{0} \\ \mathbf{E}^i_{;j} &= \partial_j \mathbf{E}^i + \Gamma^i_{kj}\mathbf{E}^k = -\Gamma^i_{kj}\mathbf{E}^k + \Gamma^i_{kj}\mathbf{E}^k = \mathbf{0} \end{aligned} \tag{277}$$

but this is not the case with the surface basis vectors in their covariant and contravariant forms, $\mathbf{E}_\alpha$ and $\mathbf{E}^\alpha$, whose covariant derivatives do not vanish identically. The reason is that, due to curvature, the partial derivatives of the surface basis vectors do not necessarily lie in the tangent plane and hence the following relations:

$$\begin{aligned} \partial_j \mathbf{E}_i &= +\Gamma^k_{ij} \mathbf{E}_k \\ \partial_j \mathbf{E}^i &= -\Gamma^i_{kj} \mathbf{E}^k \end{aligned} \tag{278}$$

which are valid in the enveloping space and are used in Eq. 277, are not valid on the surface anymore.

• At a point P on a sufficiently smooth surface with geodesic surface coordinates and Cartesian rectangular space coordinates, the covariant and absolute derivatives reduce respectively to the partial and total derivatives at P.

• The covariant derivative of the surface basis vectors is symmetric in its two indices, that is:

$$\begin{aligned} \mathbf{E}_{\alpha;\beta} &= \partial_\beta \mathbf{E}_\alpha - \Gamma^\gamma_{\alpha\beta} \mathbf{E}_\gamma \\ &= \partial_\alpha \mathbf{E}_\beta - \Gamma^\gamma_{\beta\alpha} \mathbf{E}_\gamma \\ &= \mathbf{E}_{\beta;\alpha} \end{aligned} \tag{279}$$

• The covariant derivative of the surface basis vectors, $\mathbf{E}_{\alpha;\beta}$, represents space vectors which are normal to the surface with no tangential component.

• The covariant derivative of a space tensor with respect to a surface coordinate u^α is formed by the inner product of the covariant derivative of the tensor with respect to the space coordinates x^l by the tensor x^l_α. For example, the covariant derivative of A^i with respect to u^α is given by:

$$A^i{}_{;\alpha} = A^i{}_{;k} x^k_\alpha \tag{280}$$

• The covariant derivative with respect to a surface coordinate u^β of a mixed tensor A^i_α, which is contravariant with respect to transformations in space coordinates x^i and covariant with respect to transformations in surface coordinates u^α, is given by:[196]

$$A^i_{\alpha;\beta} = \frac{\partial A^i_\alpha}{\partial u^\beta} + \Gamma^i_{jk} A^k_\alpha \frac{\partial x^j}{\partial u^\beta} - \Gamma^\gamma_{\alpha\beta} A^i_\gamma \tag{281}$$

where the Christoffel symbols with Latin and Greek indices are derived respectively from the space and surface metrics. This pattern can be easily generalized to a mixed tensor of type (m, n) $A^{i_1 \dots i_m}_{\alpha_1 \dots \alpha_n}$ which is contravariant in transformations of space coordinates x^i and covariant in transformations of surface coordinates u^α. For example, the covariant derivative of a tensor $T^{ij}_{\alpha\beta}$ with respect to u^γ is given by:

$$A^{ij}_{\alpha\beta;\gamma} = \frac{\partial A^{ij}_{\alpha\beta}}{\partial u^\gamma} + \Gamma^i_{mk} A^{mj}_{\alpha\beta} \frac{\partial x^k}{\partial u^\gamma} + \Gamma^j_{mk} A^{im}_{\alpha\beta} \frac{\partial x^k}{\partial u^\gamma} - \Gamma^\delta_{\alpha\gamma} A^{ij}_{\delta\beta} - \Gamma^\delta_{\beta\gamma} A^{ij}_{\alpha\delta} \tag{282}$$

• The above rules can be extended further to include tensors with space and surface contravariant indices and space and surface covariant indices. For Example, the covariant derivative of a tensor $A^{i\alpha}_{j\beta}$ with respect to a surface coordinate u^γ, where i and j are space indices and α and β are surface indices, is given by:

$$A^{i\alpha}_{j\beta;\gamma} = \frac{\partial A^{i\alpha}_{j\beta}}{\partial u^\gamma} + \Gamma^i_{mk} A^{m\alpha}_{j\beta} \frac{\partial x^k}{\partial u^\gamma} + \Gamma^\alpha_{\delta\gamma} A^{i\delta}_{j\beta} - \Gamma^m_{jk} A^{i\alpha}_{m\beta} \frac{\partial x^k}{\partial u^\gamma} - \Gamma^\delta_{\beta\gamma} A^{i\alpha}_{j\delta} \tag{283}$$

This example can be easily extended to the most general form of a tensor with any combination of covariant and contravariant space and surface indices.

• The covariant derivative of the surface basis vector $\mathbf{E}_\alpha$, which in tensor notation is denoted by x^i_α, is given by:

$$x^i_{\alpha;\beta} = \frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} + \Gamma^i_{jk} x^j_\alpha x^k_\beta - \Gamma^\delta_{\alpha\beta} x^i_\delta = x^i_{\beta;\alpha} \tag{284}$$

[196] An example of such a tensor is x^i_α which was discussed in § 3.1.

where curvilinear space coordinates are in use.

• The mixed second order covariant derivative of the surface basis vectors is given by:

$$x^i_{\alpha;\beta\gamma} = b_{\alpha\beta;\gamma} n^i + b_{\alpha\beta} n^i_{;\gamma} = b_{\alpha\beta;\gamma} n^i - b_{\alpha\beta} a^{\delta\omega} b_{\delta\gamma} x^i_{\omega} \tag{285}$$

where the covariant derivative of the surface covariant curvature tensor is given, as usual, by:

$$b_{\alpha\beta;\gamma} = \frac{\partial b_{\alpha\beta}}{\partial u^{\alpha}} - \Gamma^{\delta}_{\alpha\gamma} b_{\delta\beta} - \Gamma^{\delta}_{\beta\gamma} b_{\alpha\delta} \tag{286}$$

• The covariant differentiation operators in mixed derivatives are not commutative and hence for a surface covariant vector A^{γ} we have:

$$A^{\gamma}{}_{;\alpha\beta} - A^{\gamma}{}_{;\beta\alpha} = R^{\gamma}{}_{\delta\alpha\beta} A^{\delta} \tag{287}$$

where $R^{\gamma}{}_{\delta\alpha\beta}$ is the Riemann-Christoffel curvature tensor of the second kind for the surface.

• The mixed second order covariant derivatives of the surface basis vectors satisfy the following relation:[197]

$$x^i_{\alpha;\beta\gamma} - x^i_{\alpha;\gamma\beta} = R^{\delta}{}_{\alpha\beta\gamma} x^i_{\delta} \tag{288}$$

• As defined in [10], the absolute or intrinsic derivative of a tensor field along a t-parameterized curve in an nD space with respect to the parameter t is the inner product of the covariant derivative of the tensor and the tangent vector to the curve. This identically applies to the absolute derivative of curves contained in 2D surfaces.

• The absolute derivative of a tensor field along a t-parameterized curve on a surface with respect to the parameter t follows similar rules to those of a space curve in a general nD space, as outlined in the previous notes [10]. Hence, the absolute derivative of a differentiable surface vector field **A** in its covariant and contravariant forms with respect

[197] This is an instance of the relation: $A_{j;kl} - A_{j;lk} = R^i_{jkl} A_i$ which is given and explained in [10].

to the parameter t is given by:

$$\begin{aligned}\frac{\delta A_\alpha}{\delta t} &= \frac{dA_\alpha}{dt} - \Gamma^\gamma_{\alpha\beta} A_\gamma \frac{du^\beta}{dt} \\ \frac{\delta A^\alpha}{\delta t} &= \frac{dA^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} A^\gamma \frac{du^\beta}{dt}\end{aligned}$$

where the Christoffel symbols are derived from the surface metric. It should be remarked that if **A** is a space vector field defined along the above surface curve then the above formulae will take a similar form but with change from surface to space coordinates, and hence the curve is treated as a space curve, that is:

$$\begin{aligned}\frac{\delta A_i}{\delta t} &= \frac{dA_i}{dt} - \Gamma^j_{ik} A_j \frac{dx^k}{dt} \\ \frac{\delta A^i}{\delta t} &= \frac{dA^i}{dt} + \Gamma^i_{jk} A^j \frac{dx^k}{dt}\end{aligned}$$

where the Christoffel symbols are derived from the space metric.

• The absolute derivative of the tensor A^i_α, defined in the previous points, along a t-parameterized surface curve is given by:

$$\frac{\delta A^i_\alpha}{\delta t} = A^i_{\alpha;\beta} \frac{du^\beta}{dt} = \left(\frac{\partial A^i_\alpha}{\partial u^\beta} + \Gamma^i_{jk} A^k_\alpha \frac{\partial x^j}{\partial u^\beta} - \Gamma^\gamma_{\alpha\beta} A^i_\gamma \right) \frac{du^\beta}{dt} \tag{289}$$

• To extend the idea of geodesic coordinates to deal with mixed tensors of the type A^i_α, a rectangular Cartesian coordinate system over the space and a geodesic system on the surface can be introduced and hence at the poles the absolute and covariant derivatives become total and partial derivatives respectively.

• The covariant and absolute derivatives of space and surface metric, permutation and Kronecker tensors in their covariant, contravariant and mixed forms vanish identically and hence they behave as constants with respect to tensor differentiation when involved in inner or outer product operations with other tensors and commute with these operators.

• The *surface* covariant and absolute derivatives of *space* metric tensor, space Kronecker tensor, space alternating tensor and space basis vectors vanish identically, that is:

$$\begin{aligned} g_{ij|\gamma} &= 0 & g^{ij}{}_{|\gamma} &= 0 \\ \delta^{i}_{j|\gamma} &= 0 & \delta^{ij}_{kl|\gamma} &= 0 \\ \underline{\epsilon}_{ijk|\gamma} &= 0 & \underline{\epsilon}^{ijk}{}_{|\gamma} &= 0 \\ \mathbf{E}_{i|\gamma} &= \mathbf{0} & \mathbf{E}^{i}{}_{|\gamma} &= \mathbf{0} \end{aligned} \tag{290}$$

where the sign | represents covariant or absolute differentiation with respect to the surface coordinate u^{γ}. Hence, these space tensors are in lieu of constants with respect to surface tensor differentiation.

• The nabla ∇ based differential operations, such as gradient and divergence, apply to the surface as for any general curved space and hence the formulae given in [10] can be used with the substitution of the surface metric parameters. For example, the divergence of a surface vector field A^{α} is given by:

$$\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{a}} \partial_{\alpha} \left(\sqrt{a} A^{\alpha} \right) \tag{291}$$

and the Laplacian of a surface scalar field f is given by:

$$\nabla^{2} f = \frac{1}{\sqrt{a}} \partial_{\alpha} \left(\sqrt{a} a^{\alpha\beta} \partial_{\beta} f \right) \tag{292}$$

where the indexed a is the surface contravariant metric tensor, a is the determinant of the surface covariant metric tensor and $\alpha, \beta = 1, 2$.

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