MICRO AND MACRO-LEVELS OF NEURAL OBSERVATION DESCRIBE THE SAME BRAIN ACTIVITY

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ABSTRACT

Brain activity can be assessed either at anatomical/functional micro-, meso- and macro- spatiotemporal scales of observation, or at different intertwined levels with mutual interactions. Here we show, based on topological findings, that nervous activities occurring in micro-levels project to single activities at the macro-levels. This means that brain functions assessed at the higher scale of the whole brain necessarily display a counterpart in the lower ones, and vice versa. Furthermore, we point out how it is possible for different functional and anatomical levels to be stitched (sewn) together to become condensed brain activities, giving rise, for example, to ideas and concepts. Also, a topological approach makes it possible to assess brain functions in the general terms of particle trajectories taking place on donut-like manifolds. Indeed, every macro-scale activity, independent of the subtending theoretical model, can be described in terms of a multi-dimensional torus mapping to activities at lower levels. In physics, the term duality refers to a case where two seemingly different systems turn out to be equivalent, because they are mathematically distinguishable descriptions of the same phenomenon. Our framework permits a topological duality among different neuro-techniques, because it holds for all the types of spatio-temporal brain activities, independent of their inter- and intra-level relationships, strength, magnitude and boundaries.

INTRODUCTION

Many neuro-techniques have been developed throughout the years, in order to assess brain activity at different levels or dimensions of spatio-temporal observation. The rather general term "brain activity" stands here for a large repertoire of brain functions and mental faculties, such as attention and perception, emotions and cognition, memory and learning, higher cognitive processes (decision making, goal-directed choice, and so on) (Gazzaniga, 2009), mind wandering (Andrews-Hanna et al., 2014) and so on. Every technique is an observational domain of the whole neuro-scientific discipline, each one assessing an anatomical or functional scale different from the others. Some techniques assess brain activity at gross-grained levels of observation, such as EEG and lesion studies (Buszaki and Watson, 2012; Jensen et al., 2014). Others take into account a meso-level of observation, e.g., localized brain areas and sub-areas, such as diffuse tensor imaging, MEG analysis and fMRI resting state functional connectivity. Other approaches allow the assessment of more coarse-grained levels, e.g., microlumns (Opris and Casanova, 2014), or single-neuron function and structure. Further, more reductionist approaches focus on the molecular levels of brain activity: see, for example, Jacobs et al. (2007), Stankiewicz et al. (2013), Ekstrand et al. (2014), Gárate et al. (2014). The last, but not the least, other techniques favour an approach that involves more than a single functional and anatomical level, tackling the issue of brain functions in terms of non-boundary wall domains spanning over every observational dimensions and scales

(Friston, 2010; Sporns, 2013; Tozzi, 2015). Consciousness, for example, does not seem to be confined to a single level (Koch et al., 2016). This means that far apart levels must interact one each other too (Touboul, 2012).

Here we explore the possibility to assess dimensional scales as well as multilevel brain activity in terms of algebraic topology. We show, based on novel topological findings, how brain activities with matching descriptions embedded in higher anatomical or –functional levels map to single activities in lower scales. Therefore, activities encompassed in higher dimensional levels necessarily display a counterpart in the lower ones, and vice versa. This leads to a novel scenario, where different scales of brain activity are able to scatter, collide and combine, merging together in an assessable way. Our framework holds for all the types of approaches to the brain, independent of their peculiar features, resolution, magnitude and boundaries. This means that all the neuro-techniques are *dual* under topological transformation. The term *dual* refers to a situation where two seemingly different physical systems turn out to be equivalent. If two techniques or phenomena are related by a duality, it means that one can be transformed into the other, so that one phenomenon ends up looking just like the other one (Zwiebach and Barton, 2009). We also show how such duality permits brain activity assessment in the general terms of particle trajectories taking place on donut-like manifolds. Indeed, every brain model can be described in the guise of multi-dimensional tori, projecting and mapping among levels which display different spatio-temporal features.

TOPOLOGY AND BRAIN ACTIVITIES AT DIFFERENT ANATOMICAL/FUNCTIONAL LEVELS

Here we assess brain activities in terms of geometric structures. Brain physical spaces model regions, areas or shapes on the surface of abstract geometric spaces. Nervous activity are then modelled as paths followed by particles traveling through brain micro-, meso- or macro- areas, or through different levels. Lower-level brain activity can be depicted as a structure equipped with n-dimensions (e.g., a two-dimensional disk), while the higher-level one as a $n+1$ - dimensional structure (e.g., a three-dimensional sphere). Various continuous mappings from higher to lower dimensional structures lead to the Borsuk-Ulam theorem (BUT) (Borsuk 1933;Krantz, 2009; Tozzi and Peters, 2016a). BUT states states that a single point on a circumference maps to two antipodal points on a sphere, both characterized by the same description. Points on the sphere are *antipodal*, provided they are diametrically opposite (Weisstein, 2016). If we simply evaluate nervous activities instead of *points*, BUT leads naturally to the possibility of a region-based brain geometry. Indeed, the two opposite *points* could stand not just for the description of simple topological points (Marsaglia, 1972), but also of the signals detected by different neuro-techniques, such as spatial or temporal patterns, vectors, particle trajectories, entropies, free-energies (Peters, 2016). Therefore, we can describe brain features as antipodal points on $n+1$ –dimensional structures. Brain signals of different scales can be compared, because the two antipodal points can be assessed at higher-dimensional scales of observation (brain macro-levels), which can be pulled back to single points at lower-dimensional scales (brain micro-levels) (Tozzi and Peters, 2016b). The two points (or regions, or activities) do not need necessarily to be antipodal, in order to be described together (Peters, 2016). Indeed, BUT can be generalized also for the assessment of non-antipodal features, provided there are a pair of regions, either adjacent or far apart, with the same feature vector (Peters and Tozzi, 2016a). Even though BUT was originally described just for convex spheres, it is also possible for us to look for antipodal points on structures equipped with other shapes (Mitroi-Symeonidis, 2015; Tozzi, 2016). This means that, whether brain activity displays concave, convex or flat geometry, it does not matter, because we may always find the points with matching description predicted by BUT. Furthermore, an $n+1$ –dimensional structure might map straight to a corresponding lower-dimensional surface, instead of another n-dimensional structure. In other words, a brain area may map also just to itself (Weeks, 2002), so that the mapping of two antipodal points to a single point in a dimension lower becomes a projection internal to the same structure. For technical readers, see also: Dodson and Parker (1997), Matousek (2003), Crabb and Jaworowski (2013).

In the evaluation of brain activities at every spatio-temporal scale, here we assess antipodal *regions* instead of antipodal *points* (Peters and Naimpally, 2012; Lentzen, 1939; Disalle, 1995; Henderson, 1996; Di Concilio, 2013; Di Concilio and Gerla, 2006). A brain region is called a *worldsheet*, if everyone of its sub-regions contains at least one brain activity modelled as a string that describes a path followed by a moving particle. The term *worldsheet* designates a region of the brain space completely covered by nervous activities (Peters and Tozzi, 2016b) (**Figure 1**). A 2D plane worldsheet can be rolled up to form the lateral surface of a 3D cylinder, termed a *worldsheet cylinder* (Peters 2016b). Further, a worldsheet cylinder maps to a *worldsheet torus*, formed by bending the former until the ends meet. In sum, a flattened worldsheet maps to a worldsheet torus. This means that every brain activity can be described in terms of particle movements along the surface of multi-dimensional, donut-like toruses (**Figure 1**). Hence, BUT provides a way to evaluate changes of information among different anatomical and functional brain levels in a topological space, which is distinguished from purely functional or thermodynamical perspectives.

Next, consider Brouwer's fixed point theorem (FPT) (Volovikov and Yu, 2008). Su (1997) gives a nice illustration of the FPT: no matter how you continuously slosh the coffee around in a coffee cup, some point is always in the same position that it was before the sloshing began. And if you move this point out of its original position, you will eventually we move some other point in the sloshing coffee back into its original position. In BUT terms, this means

that not only we can always find a brain region containing an activity, but also that every activity comes together with another one, termed a *wired friend*. These observations lead to a *wired friend theorem*: every occurrence of a wired friend activity on the $n+1$-dimensional structure maps to a fixed description, e.g. to another activity that belongs to an n-dimensional structure (**Figure 1**) (Peters and Tozzi, 2016b). The significance of this is that we can always find an activity, embedded in a higher-dimensional brain macro-level, which is the topological description of another activity, embedded in a lower dimensional brain micro-level. And vice versa.

Plane worldsheet   Worldsheet torus
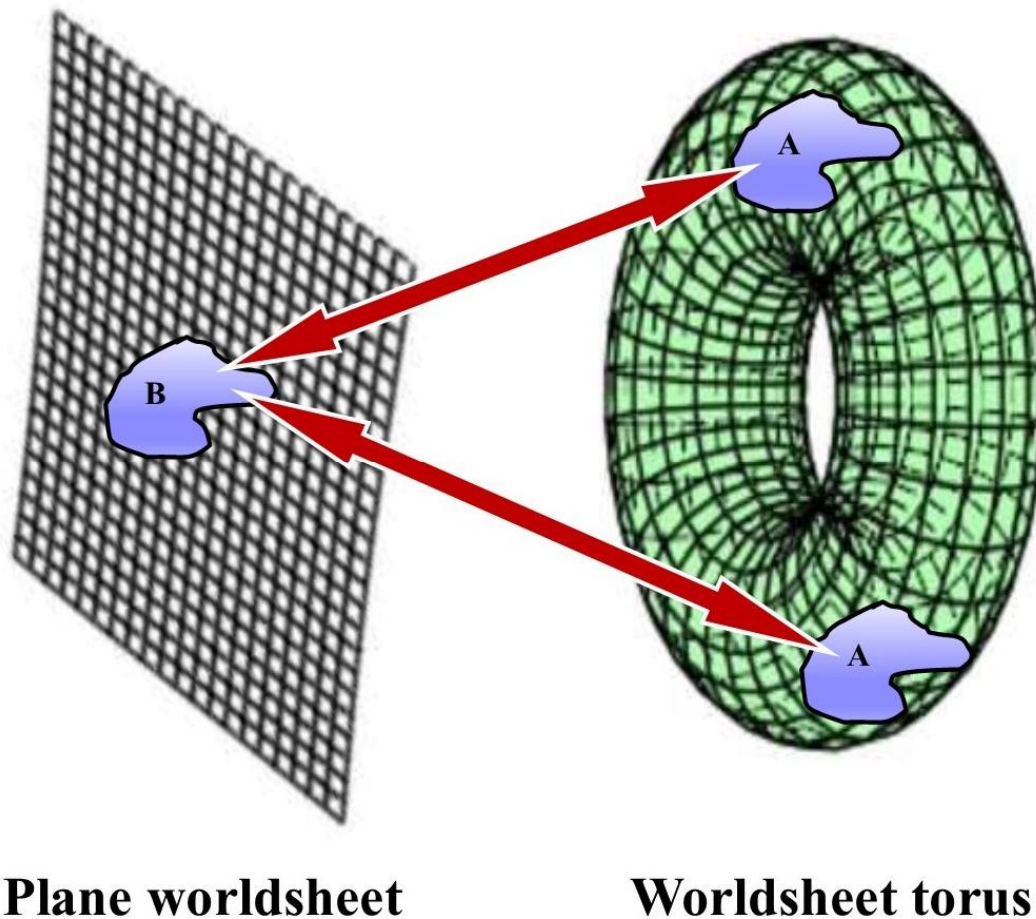
Figure 1. Samples of brain activities, illustrated in guise of geometrical shapes, are represented by the regions A and B on different structures with diverse dimensions. Regions A and B are examples of antipodal activities which lie on structures with various dimensions. See the text for further details.

THE MERGING OF BRAIN ACTIVITIES

In the previous paragraph, we showed how different brain activities, depicted in guise of geometrical shapes, necessarily have at least a feature in common. In topological terms, nervous activities/shapes are continually transforming into new homotopically equivalent nervous activities/shapes. They might influence each other by scattering, colliding and combining, to create bounded regions in the brain. Hence, it is possible for brain activities to

stick together to become *condensed*, e.g., worldsheets (operationally assessable in terms of the torus described above), which are portrayed as a collection of interacting elements of geometrical shapes.

Eventually brain activity's shapes will deform into another, as a result of the collision of a pair of separate shapes. Let a brain activity be represented in **Fig. 2.A**. This brain activity evolves over time as it twists and turns through the outer reaches of another brain activity. An inkling twisting brain activity appearing in the neighbourhood of the first one is illustrated in **Fig. 2.B**. The two activities begin interacting, so that the first now has a region of space in common with the second (**Fig. 2.C**). In effect, as a result of the interaction, they are partially stitched together. The partial absorption of one brain activity in another is shown in **Fig. 2.D**. Here, a very large region of the total brain space occupied by the first activity is absorbed by the second. Therefore, we have the birth of a condensed brain activity. The two brain activities become at first concentric in **Fig. 2.E.**, then a complete condensed shape is formed in Fig. **2.F**. When the two brain activities are completely transformed into a new one, we have instance of their homotopy of equivalence, with the second that has completely absorbed the first. To make an example, the idea of *cat* arises from the perception of many single cats of different size, colour, and so on. This is a further instance of the duality principle in brain theories. That is, one brain activity is the dual of another, provided the first can be deformed into the second there is no longer any difference between the cat I see and the cat I imagine.

Here we provide a short mathematical treatment for technical readers. Let $f, g : X \rightarrow Y$ be a pair of continuous maps. For example, let $f(X)$ and $g(X)$ be two localized brain activities. A *homotopy* (Cohen, 1973) between f and g is a continuous map $H : X \times [0,1] \rightarrow Y$ so that $H(x,1) = g(x)$ and $H(x,0) = f(x)$. It is possible to deform (transform) one brain activity with a particular shape into another with a different shape. It means that the birth of novel brain activities, that evolve out of the interaction of initially disjoint activities, is allowed. Let $id_X : X \rightarrow X$ denote an identity map defined by $id_X(x) = x$. Similarly, $id_Y : Y \rightarrow Y$ is defined by $id_Y(t) = y$. The composition $f \circ g(X)$ is defined by $f \circ g(X) = f(g(X))$. Similarly, $g \circ f(X)$ is defined by $g \circ f(X) = g(f(X))$. The sets X and Y are homotopically equivalent, provided there are continuous maps so that $g \circ f \:\square\: id_X(x)$ and $f \circ g \:\square\: id_Y(y)$. The sets X and Y are the same homotopy type, provided X and Y are homotopically equivalent (Peters and Inan, 2016; Peters and Naimpally, 2012). This leads to a comparison of brain activities, with seemingly varying shapes and sizes, that are homotopically equivalent. The brane in **Fig. 2.E** is an example of a Edelsbrunner-Harer nerve (Peters and Inan, 2016), which is a collection $Nrv\mathfrak{I}$ such that all nonempty subcollections of $Nrv\mathfrak{I}$ have a non-void common intersection, i.e.,

$$Nrv\mathfrak{I} = \left\{ X \in \mathfrak{I} : \bigcap X \neq \varnothing \right\}.$$

**Lemma 1**. Brain activities of the same homotopy type are stitched together to form a condensed brain activity.
Proof. Let G, H be brain activities with the same homotopy type. Then brain activity G can be deformed into brain activity H. In effect, shape H absorbes G. By definition, brain activity H is a condensed brain activity. This phenomenon is true in general for homotopically equivalent brain activity that have the potential to be stitched together. To make a psychological example, the prolonged view of different cars gives rise to idea of traffic. Hence, the desired result follows.

Theorem 1. A condensed brain activity is an instance of an Edelsbrunner-Harer nerve.
Proof. From Lemma 1, a condensed brain activity is an instance of an Edelsbrunner-Harer nerve.

Theorem 2. Every brain activity is a Edelsbrunner-Harer nerve.
Proof. Let G be a macro-level brain activity. Every G is a collection of micro-level shapes, so that each of the shape has brain activity G in common. That is, all the micro-levels brain activities occurring in G have nonempty intersection. Consequently, G is an Edelsbrunner-Harer nerve. Hence, the desired result follows.
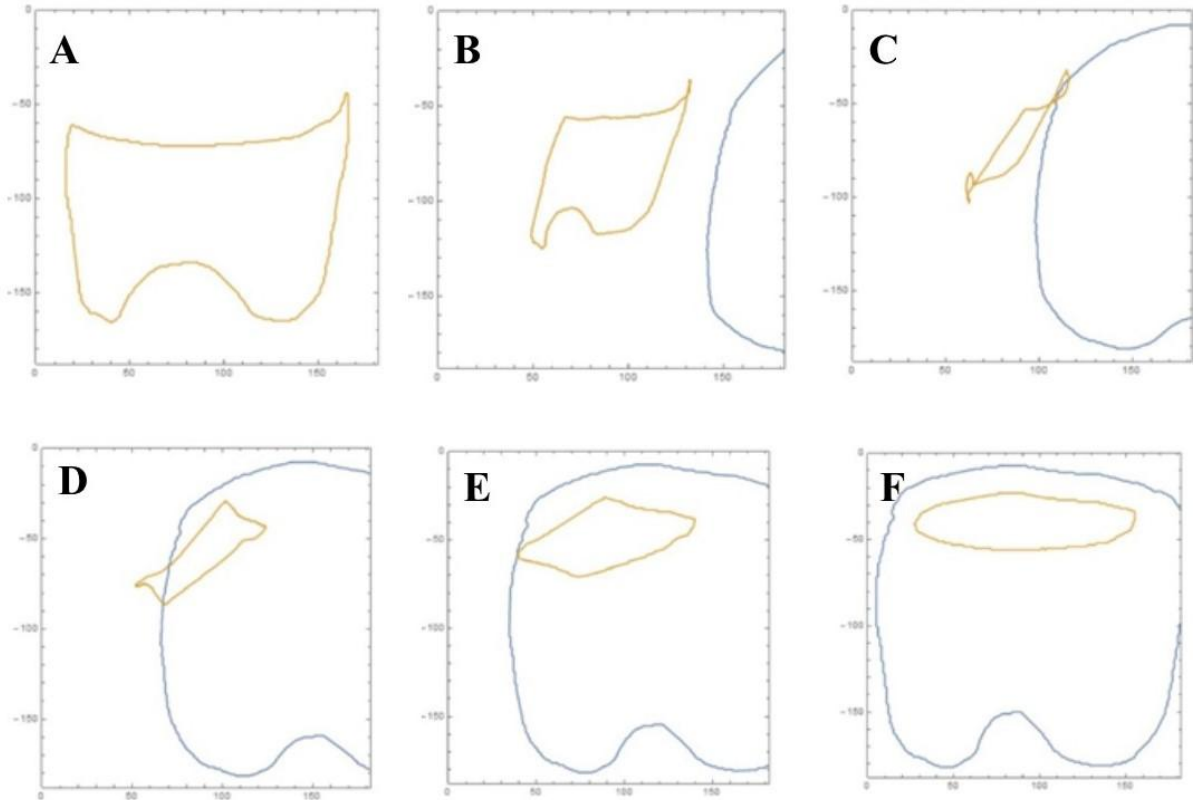
Figure 2. Homotopically equivalent shapes, standing for brain activities. **A**: a brain activity at a given spatiotemporal level. Such level could be differently coarse-grained, e.g., might stand for every micro-, meso- or macro-level. **B**: two brain activities at two different levels. **C**: interacting brain activities. **D**: dual brain activities. **E**: concentric brain activities. **F**: condensed brain activity. See the text for further details.

DISCUSSION

Neuro-techniques evaluate different types and forms of brain activity, displayed at micro-, meso-, or macro-levels of observation. Here we evaluate topological concepts that makes it possible for us to correlate the features of different neurotechniques' setups. We achieve generalizations that allow the assessment of every possible brain activity, independent on its scale, magnitude, specific features and local boundaries. Brain activities, equipped either with antipodal or non-antipodal matching description and embedded in higher-dimensional nervous structures, map to a single activity in lower-dimensional ones, and vice versa. In other words, there exists an assessable and quantifiable correspondence between micro-, meso- and macro-levels of brain activities. We could conceive brain activities that are too far apart ever to communicate with one another, so that activities bounded on distant brain regions would never have direct contact: for example, two apparently opposite brain activities such as emotions and abstraction have apparently very few in common. However, our topological investigation reveals that this scenario is unfeasible, because there must be at least one element in common also among brain activities that are very distant one each other. Brain activities will always have some element in common: they do not exist in isolation, rather they are part of a large interconnected whole, with which they interact. Whether you experience pain or pleasure, or chomp on an apple, or compute a mathematical expression, or quote a proverb, or remember your childhood, or read Wittgenstein's *Tractatus*, it does not matter: the large repertoire of your brain functions can be described in the same topological fashion. The distinction among different coarse-grained levels of nervous activity does not count anymore, because nervous function at small, medium and large scales of neural observation turn out to be topologically equivalent.

In our brain scenario, particle movements are described as functions occurring on structures displaying different possible geometric curvatures, either concave, convex or flat. Such generalization encompasses the brain models that claim for different curvatures of the brain phase space (see, for example, Sengupta et al., 2016). This means that such

theories are DUAL: e.g., their topological description is the same, despite the huge difference in the subtending hypothesized curvature. It allows a useful simplification in the assessment of brain activity. Furthermore, the level of observation is not significant in the evaluation of brain activities, because such levels are fully interchangeable. Because projections between dimensions describe neural phenomena spanning from the smallest to the highest scales, the distinction among different coarse-grained scales does not count anymore, because nervous activity is topologically the same, at small, medium and large scales of observation. This means, for example, that that all the types of cognitive studies assess the same topological activity, regardless of their different protocols and procedures.

Brain theories predict the existence of long-range connexions, e.g., paths equipped with two ends lying on physical separated brain zones. An appropriate projection mapping shows that, if the two ends have matching features (e.g., the intensity, or length, or pairwise entropy), the two activities are the same. In sum, two activities with matching description, embedded in two brain zones of different levels, display the same features. Activities with matching ends (regions) in different cortical areas might also help to throw a bridge, for example, between sensation and perception. In the same guise, the relationships between the spontaneous and the evoked activity of the brain (Tozzi et al., 2016b) take now a new significance: they become just two sides of the same coin, made of topologically-bounded dynamics.

Our results also entail that every kind of high-dimensional brain activity can be described in terms of donut-like structures (Peters and Tozzi, 2016b). Lower-dimensional features can be assessed in the generic terms of particle trajectories traveling on higher-dimensional, donut-like toruses. This methodological advance, already used in limited trials (Stemmler et al., 2015) could be useful in order to achieve a unvarying operationalization of the countless theories describing brain function (Marijuan et al., 2013). In sum, the paths described by BUT and FPT variants elucidate how the tight coupling among different neural activities gives rise to brains that are in charge of receiving and interpreting signals from other cortical zones, in closely intertwined relationships at every spatio-temporal level. Therefore, topology becomes one of the central information processing strategies of the nervous system.

REFERENCES

1) Borsuk M. 1933. DreiSatze Uber Die N-Dimensionale EuklidischeSphare.FundamentaMathematicae XX: 177–90.
2) Buzsáki G, Watson BO (2012) Brain rhythms and neural syntax: implications for efficient coding of cognitive content and neuropsychiatric disease. Dialogues ClinNeurosci 4, 345-367
3) Cohen MM. 1973. A Course in Simple Homotopy Theory. New York-Berlin: Springer-Verlag.
4) Crabb MC, Jaworowski J. 2013. Aspects of the Borsuk-Ulam Theorem. J. of Fixed Point Theory and Applications 13 (2013), 459–488, DOI 10.1007/s11784-013-0130-7, MR3122336.
5) Di Concilio A. 2013.Point-free geometries: Proximities and quasi-metrics, Math. in Comp. Sci. 7, 1, 31–42, MR3043916.
6) Di Concilio A, Gerla G. 2006.Quasi-metric spaces and point-free geometry. Math. Structures Comput. Sci. 16, 1, 115137, MR2220893.
7) Disalle R. 1995. Spacetime theory as physical geometry. Erkenntnis, 42(3):317–337.
8) Dodson CTJ, Parker PE. 1997. A User's Guide to Algebraic Topology. Dordrecht, Netherlands: Kluwer Academic Publishers.
9) Edelsbrunner H. 2014. A Short Course in Computational Geometry and Topology. Berlin: Springer-Verlag.
10) Ekstrand MI, Nectow AR, Knight ZA, Latcha KN, Pomeranz LE, Friedman JM. 2014. Molecular profiling of neurons based on connectivity. Cell 157(5):1230-1242. doi: 10.1016/j.cell.2014.03.059
11) Friston K (2010) The free-energy principle: a unified brain theory? Nat Rev Neurosci 11(2):127-138. doi: 10.1038/nrn2787
12) Gárate I, García-Bueno B1, Madrigal JL, Caso JR, Alou L, et al. 2014. Toll-like 4 receptor inhibitor TAK-242 decreases neuroinflammation in rat brain frontal cortex after stress. J Neuroinflammation 11:8. doi: 10.1186/1742-2094-11-8.
13) Gazzaniga MS. 2009. The Cognitive Neurosciences, Fourth Edition. MIT Press. ISBN: 9780262013413
14) Henderson DW. 1996. Experiencing Geometry on Plane and Sphere. New York: Prentice Hall.
15) Jacobs GH, Williams GA, Cahill H, Nathans J. 2007. Emergence of Novel Color Vision in Mice Engineered to Express a Human Cone Photopigment. Science 315(5819):1723-1725
16) Jensen O, Gips B, Bergmann TO, Bonnefond M (2014) Temporal coding organized by coupled alpha and gamma oscillations prioritize visual processing. Trends Neuroscipii: S0166-2236(14)00060-5. doi: 10.1016/j.tins.2014.04.001
17) Koch C, Massimini M, Boly M, Tononi G. 2016. Neural correlates of consciousness: progress and problems. Nat Rev Neurosci. Apr 20;17(5):307-21. doi: 10.1038/nrn.2016.22.
18) Krantz SG. 2009. A Guide to Topology. Edited by The Mathematical Association of America. Washington DC.

19) Lenzen VF. 1939.Physical geometry, Amer. Math. Monthly 46, 324–334, MR0000214.

20) Marijuán PC, del Moral R, Navarro J. 2013. On eukaryotic intelligence: signaling system's guidance in the evolution of multicellular organization. Biosystems, 114(1):8-24. doi: 10.1016/j.biosystems.2013.06.005. Epub 2013 Jul 12.

21) Marsaglia G. 1972. "Choosing a Point from the Surface of a Sphere." Annals of Mathematical Statistics 43 (2): 645–46. doi:10.1214/aoms/1177692644.

22) Matoušek J. 2003. Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry. Berlin Heidelberg: Springer-Verlag.

23) Mitroi-Symeonidis FC. 2015. Convexity and Sandwich Theorems. European Journal of Research in Applied Sciences 1 (March): 9–11.

24) Naimpally SA, Peters JF. 2013. Topology with applications. Topological spaces via near and far. World Scientific, Singapore, Amer. Math. Soc. MR3075111.

25) Opris I, Casanova MF. 2014. Prefrontal cortical minicolumn: from executive control to disrupted cognitive processing. Brain. 2014 Jul;137(Pt 7):1863-75. doi: 10.1093/brain/awt359.

26) Peters JF. 2016. Computational Proximity. Excursions in the Topology of Digital Images. Edited by Intelligent Systems Reference Library. Berlin: Springer-Verlag. doi:10.1007/978-3-319-30262-1.

27) Peters JF. 2016b. Proximal physical geometry. arXiv 1608, no. 06208.

28) Peters JF, Guadagni, C. 2015. Strongly near proximity and hyperspace topology, arXiv 1502, no. 05913.

29) Peters JF, Guadagni, 2015. C. Strongly proximal continuity & strong connectedness, arXiv 1504, no. 02740.

30) Peters JF, Inan E. 2016. Strongly proximal Edelsbrunner-Harer nerves. Proc. Jangjeon Math. Soc. 19(3):563-582.

31) Peters JF, Naimpally SA. 2012.Applications of near sets, Notices of the Amer. Math. Soc. 59(4):536–542, DOI: http://dx.doi.org/10.1090/noti817.

32) Peters JF, Tozzi A. 2016a. Region-Based Borsuk-Ulam Theorem. arXiv.1605.02987

33) Peters JF, Tozzi A. 2016b. String-Based Borsuk-Ulam Theorem. arXiv:1606.04031

34) Sengupta B, Tozzi A, Coray GK, Douglas PK, Friston KJ. 2016. Towards a Neuronal Gauge Theory. PLOS Biology 14 (3): e1002400. doi:10.1371/journal.pbio.1002400.

35) Sporns O (2013) Network attributes for segregation and integration in the human brain. CurrOpinNeurobiol. 23(2):162-171. doi: 10.1016/j.conb.2012.11.015

36) Stankiewicz AM, Swiergiel AH, Lisowski P. 2013. Epigenetics of stress adaptations in the brain. Brain Res Bull 98:76-92. doi: 10.1016/j.brainresbull.2013.07.003

37) Stemmler M, Mathis A, Herz AVM 2015 Connecting multiple spatial scales to decode the population activity of grid cells. Sci. Adv. 1:e1500816.

38) Su FE. 1997.Borsuk-Ulam implies Brouwer: A direct construction. Amer. Math. Monthly 104 (9): 855–859, MR1479992.

39) Tozzi A. 2015. Information Processing in the CNS: A Supramolecular Chemistry? Cognitive Neurodynamics 9 (5): 463–477.

40) Tozzi A. 2016. Borsuk-Ulam Theorem Extended to Hyperbolic Spaces. In: Computational Proximity. Excursions in the Topology of Digital Images, edited by J F Peters, 169–171. doi:10.1007/978-3-319-30262-1.

41) Tozzi A, Peters JF. 2016a. Towards a Fourth Spatial Dimension of Brain Activity. Cognitive Neurodynamics 10 (3): 189–199. doi:10.1007/s11571-016-9379-z.

42) Tozzi A, Peters JF. 2016b. A Topological Approach Unveils System Invariances and Broken Symmetries in the Brain. Journal of Neuroscience Research 94 (5): 351–65. doi:10.1002/jnr.23720.

43) Tozzi A, Fla Tor, Peters JF. 2016a. Building a minimum frustration framework for brain functions in long timescales. J Neurosci Res. 94(8): 702–716.

44) Tozzi A, Zare M, Benasich AA. 2016b. New Perspectives on Spontaneous Brain Activity: Dynamic Networks and Energy Matter. Frontiers in Human Neuroscience. doi:10.3389/fnhum.2016.00247.

45) Touboul J. 2012. Mean-field equations for stochastic firing-rate neural fields with delays: Derivation and noise-induced transitions. Physica D: Nonlinear Phenomena. 241 (15):1223–1244. doi:10.1016/j.physd.2012.03.01Volovikov, Yu A. 2008. Borsuk-Ulam Implies Brouwer: A Direct Construction Revisited. American Mathematical Monthly 115 (6): 553–56. doi:10.2307/2975293.

46) Weeks JR. 2002. The shape of space, IInd edition. Marcel Dekker, inc. New York-Basel.

47) Weisstein EW. 2016.Hyperplane, Mathworld. A Wolfram Web Resource. http://mathworld.wolfram.com/Hyperplane.html.

48) Zwiebach, Barton (2009). A First Course in String Theory. Cambridge University Press. ISBN 978-0-521-88032-9