Article

Informational Entropies for Non-ergodic Brains

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Abstract: Informational entropies, although proved to be useful in the evaluation of nervous function, are suitable just if we assume that nervous activity takes place under ergo dic conditions. However, widespread claims suggest that the brain operates in a non-ergodic framework. Here we show that a topological concept, namely the Borsuk-Ulam theorem, is able to wipe away this long-standing limit of both Shannon entropy and its generalizations, such as Rényi's. We demonstrate that both ergodic and non-ergodic informational entropies can be evaluated and quantified through topological methods, in order to improve our knowledge of central nervous system function.

Keywords: Enropy; Ergodicity; Borsuk-Ulam theorem; Rényi

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1. Introduction

The most successful entropy-based theories of brain function – e.g., the free-energy principle [1] – require that the brain activity take place in an ergodic phase space. In physics and thermodynamics, the ergodic hypothesis states that, over long periods, the time spent by a system in a region of the microstates' phase space with the same energy is proportional to the region's volume, so that all accessible microstates are equiprobable over a long period of time [2]. In other words, ergodicity is a random process characterized by the time average of one sequence of events being the same as the ensemble average [3,4]. It also means that, in case of a Markov chain, as one increases the steps, there exists a positive probability measure at step n that is independent of the probability distribution at initial step 0 [5].

The Shannon informational entropy (1948), [6], is able to link choice, uncertainty and thermodynamic entropy in a coherent picture able to explain macroscopic systems' behavior such as the brain, if one just knows the statistical properties of the microscopic constituents. In the context of nervous function, it has been shown that variations in entropy are correlated with different psychological and cognitive states. As an example, analysis performed on emotionally online dialogues demonstrated the tendency towards a growing entropy [7]. Further, ensemble of supervised maximum entropy classifiers can accurately detect and identify sentiments expressed in notes [8], perceptual functions are correlated with thermodynamical entropy and free energy [9] and Shannon entropy is able to predict task performance [10]. Finally, the entropy has been recently proposed as a measure of semantic and syntactic information of multidimensional discrete phenomena [11]. Shannon entropy H requires some properties to be applied, because it should be: a) continuous in the p_i, b) a monotonic increasing function of n and c) the weighted sum of the individual values of H. The most important limitation to the use of Shannon entropy is the need to operate under ergodic conditions. However, widespread claims suggest that the brain is not ergodic. Many authors suggest that the properties of brain fluctuations are inconsistent with the Markovian approximation [12], the mean-square distance travelled by brain particles displays anomalous diffusion [13] and the brain is weakly non-ergodic, as some phase space region may take extremely long times to be visited [14]. This paper aims to answer to the question whether it is possible to use informational entropies for the evaluation of non-ergodic systems [15] and questions the validity of informational entropies under non-ergodic conditions. In such a framework, an underrated theorem from algebraic topology comes in help: The Borsuk-Ulam theorem (BUT) and its variants. The theorem states that two opposite points on a sphere, when projected on a one-dimension lower circumference, give rise to a single point displaying a matching description [16]. We here show how Shannon and its generalized variants, both ergodic and non-ergodic, may be treated in terms of algebraic topology. We will discuss the mechanisms and the consequences for brain studies of such an "unification" between concepts from far-flung branches.

2. Results

2.1. Shannon entropy on a circle

Shannon entropy and its links with thermodynamical entropy. Shannon entropy (denoted by $H(X)$, X a random variable with values $(x_1, x_2, ...x_n)$ is a measure of the unpredictability of information content [6]. Entropy is defined by:

$$H(X) = \sum_{i=1}^{n} P(x_i) log_2(P(x_i)), \quad X = x_1, x_2, ...x_n, \tag{1}$$

$$H(x) = \begin{cases} 0, & \text{if } P(x) = 1, \\ 0, & \text{if } P(x) = 0, \quad \text{by definition} \end{cases}$$

Shannon entropy states that, under ergodic conditions, if we know the values of p, we may obtain the values of $S(p)$. In other words, $S(p)$ is a function(al) of a generic probability distribution p such that, if we modify p, we achieve a different value of entropy on the Shannon's curve. The connection with informational entropies' thermodynamical counterpart - i.e. the Boltzmann-Gibbs entropy - is given by a standard procedure of Maximum Entropy (MaxEnt) distribution and thermodynamical limit $(N \to \infty)$, which leads to the relation: $S(P) = k_B H(P)$, where k_B is the Boltzmann constant.

2.2. The Borsuk-Ulam theorem (BUT)

The (BUT) is a remarkable finding by K. Borsuk about Euclidean n-spheres and antipodal points. It states that [17] Every continuous map $f : S^n \to R^n$ must identify a pair of antipodal points. In other words, the sphere S^n maps to the Euclidean space R^n, which stands for an n-dimensional Euclidean space. Note that the function needs to be continuous and that n must be a natural number (although we will see that it is not completely true) [18–20].

The notation S^n denotes an n-sphere, which is a generalization of the circle [21]. A n-sphere is a n-dimensional structure embedded in a $n + 1$ space. For example, a 2-sphere (S^2) is the 2-dimensional surface of a 3-dimensional ball (a beach ball is a good example). An n-sphere is formed by points which are constant distance from the origin in $(n + 1)$-dimensions [22]. For example, a 3-sphere (also called glome or hypersphere) of radius r (where r may be any positive real number) is defined as the set of points in $4D$ Euclidean space at distance r from some fixed center point c (which may be any point in the $4D$ space) [23].

A 3-sphere is a simply connected 3-dimensional manifold of constant, positive curvature, which is enclosed in a Euclidean 4-dimensional space called a 4-ball. A 3-sphere is thus the surface or

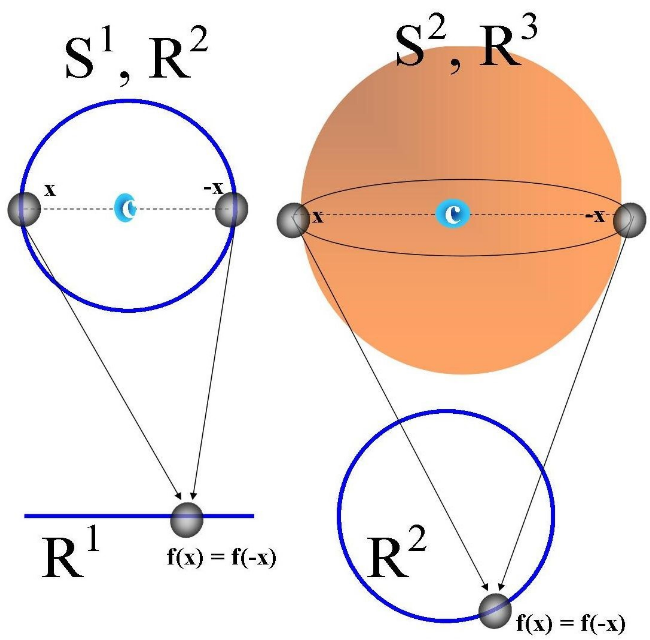

Figure 1. The Borsuk-Ulam theorem for different values of S_n. Two antipodal points in S_n project to a single point in R_n, and vice versa. Remind that every S_n is embedded in a $n+1$-ball, and thus every S_n is one-dimension higher than the corresponding R_n.

boundary of a 4-dimensional ball, while a 4-dimensional ball is the interior of a 3-sphere. From a geometer's perspective, we have different n-spheres, starting with the perimeter of a circle (S^1) and advancing to S^3, which is the smallest hypersphere, embedded in a 4-ball (Figure 1). Points on S^n are antipodal, provided they are diametrically opposite [24]. Examples of antipodal points are the poles of a sphere. Further, every continuous function from an n-sphere S^n into Euclidean n-space R^n maps some pair of antipodal points of S^n to the same point of R^n. To make an example, if we use the mapping $f : S^3 \rightarrow R^3$, then $f(x)$ in R^3 is just a signal value (a real number associated with x in S^3) and $f(x) = f(-x)$ in R^3. Furthermore, when $g : S^2 \rightarrow R^2$, the $g(x)$ in R^2 is a vector in R^2 that describes the x embedded in S^2. In other words, a point embedded in a R^n manifold is projected to two opposite points on a S^{n+1}-sphere, and vice versa.

2.3. Application of BUT to signal analysis: shapes and homotopies.

In terms of activity, a feature vector $x \in R^n$ models the description of a signal. To elucidate the picture in the application of the BUT in signal analysis, we view the surface of a manifold as a n-sphere and the feature space for signals as finite Euclidean topological spaces. The BUT tells us that for description $f(-x)$ for a signal x, we can expect to find an antipodal feature vector $f(-x)$ that describes a signal on the opposite (antipodal) side of the manifold S^n. Thus, the pair of antipodal signals have matching descriptions on S^n. Let X denote a nonempty set of points on the surface of the manifold. A topological structure on X (called a topological space) is a structure given by a set of subsets τ of X, having the following properties:
(Str.1) Every union of sets in τ is a set in τ.
(Str.2) Every finite intersection of sets in τ is a set in τ.

The pair (X, τ) is called a topological space. Usually, X by itself is called a topological space, provided it has a topology τ on it. Let X, Y be topological spaces. Recall that a function or map $f : X \rightarrow Y$ on a set X to a set Y is a subset $X \times Y$ so that for each $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$ (usually written $y = f(x)$). The mapping f is defined by a rule that tells us how to find $f(x)$. For a good introduction to mappings, see [25].

A mapping $f : X \to Y$ is continuous, provided $A \subset Y$ is open, then the inverse $f^-1 \subset X$ is also open. For more about this, see [26]. In this view of continuous mappings from the signal topological space, X on the manifold's surface to the signal feature space R^n, we can consider not just one signal feature vector $X \in R^n$, but also mappings from X to a set of signal feature vectors $f(X)$. This expanded view of signals has interest, since every connected set of feature vectors $f(X)$ has a shape. The significance of this is that signal shapes can be compared.

A consideration of $f(X)$ (set of signal descriptions for a region X) instead of $f(X)$ (description of a single signal x) leads to a region-based view of signals. This region-based view of the manifold arises naturally in terms of a comparison of shapes produced by different mappings from X (object space) to the feature space R^n. An interest in continuous mappings from object spaces to feature spaces leads into homotopy theory and the study of shapes. Let be continuous mappings from X to Y. The continuous map $H : X \times [0, 1] \to Y$ is defined by:

$$H(x, 0) = f(x), H(x, 1) = g(x), foreveryx \in X. \tag{2}$$

The mapping H is a homotopy, provided there is a continuous transformation (called a deformation) from f to g. The continuous maps f, g are called homotopic maps, provided $f(X)$ continuously deforms into $g(X)$ (denoted by $f(X) \to g(X)$). The sets of points $f(X)$, $g(X)$ are called shapes. For more about this, see [27,28].

For the mapping $H : X \times [0, 1] \to R^n$, where $H(X, 0)$ and $H(X, 1)$ are homotopic, provided $f(X)$ and $g(X)$ and have the same shape. That is, $f(X)$ and $g(X)$ are homotopic, provided:

$$\parallel f(X) - g(X) \parallel < \parallel f(X) \parallel, forallx \in X \tag{3}$$

There are natural ties between Borsuk's result for antipodes and mappings called homotopies. The early work on n-spheres and antipodal points eventually led Borsuk to the study of retraction and homotopic mappings [29–31], paving the way to the geometry of shapes and shapes of space [32]. A pair of connected planar subsets in Euclidean space R^2 have equivalent shapes, provided the planer sets have the same number of holes. For example, the letters e, O, P and numerals 6, 9 belong to the same equivalence class of single-hole shapes. In terms of signals, it means that the connected graph for $f(X)$ with, for example, an e shape, can be deformed into the 9 shape.

This suggests yet another useful application of Borsuk's view of the transformation of a shape into another, in terms of signal analysis: sets of signals not only will have similar descriptions, but also dynamic character. Moveover, the deformation of one signal shape into another occurs when they are descriptively near [33]. It means that we are allowed to embed the Shannon entropy onto a n-sphere and to treat its values in terms of antipodal points. Therefore, we can deduce an optimization scheme that enables us to transport the two Shannon's antipodal points x and $-x$ from S^n onto a S^{n-1} abstract manifold. The next two paragraphs will be devoted to illustrate how the Shannon entropy can be embedded in a n-sphere, both in ergodic and non-ergodic conditions.

2.4. *Shannon entropy under ergodic conditions*

For random numbers in the range from 0 to 1, we obtain the Shannon plot (Figure 2A). By embedding the Shannon plot in a hypersphere S^1 (the perimeter of a circle) with diameter F^1 , a continuous function $\pi : S^1 \to R^1$ maps the BUT antipodal points x and $-x$ to the same extreme entropy value, namely, $H(1) = 0$. In other words, both antipodal points have the same information content, since both are mapped to the same Shannon value, namely, 0. The center of the straight line segment $\overline{x(-x)}$ between the antipodal points (at the center of S^1) is mapped to the highest entropy, namely, $H(0.5) = 1$ (Figure 2B). The intermediate points on either side of the center of $\overline{x(-x)}$ are mapped to intermediate entropy values between 0 and 0.5. It is easy to observe that the projection

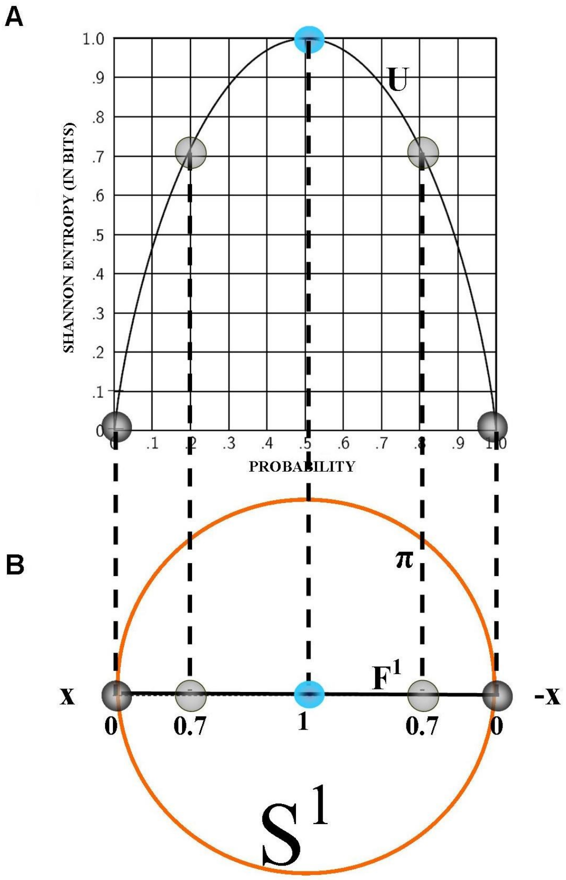

Figure 2. **A**. Shannon entropy for probability distribution $P = (p, 1-p)$, under ergodic conditions (from the original Shannon's graph). The entropy is plotted as a function of the random variablep, in the case of two possibilities with probabilities p and $(1-p)$. **B**. Shannon (ergodic) entropyin the framework of the BUT theorem. Note that the entropy follows the circle diameter, e.g., the line which connects (maps to) the two antipodal points on S^1.

$\pi : (x_1, ..., x_{n+1}) \to (x_1, ..., x_n)$ is a homeomorphism from the Shannon curve U to S^1 with diameter F^1 (Figure 2B). In such a vein, the points along one of the S^1 circles are homeomorphic to the Shannon entropy, under ergodic conditions. As a result, BUT provides a model for the computation of Shannon entropy, by evaluating the divergence of the probability of an event from the antipodal points on a hypersphere.

2.5. Shannon entropy under non-ergodic conditions

What happens in the case of non-ergodic informational entropy? For example, take the case of a point λ on the S^1 sphere illustrated in Figure 1, which lies on a 1- circle forming the angle ϕ with the diameter F^1 of the ergodic Shannon entropy (Figure 3). If we find its antipodal point $-\lambda$, we achieve their homeomorphisms which can be projected on the Shannon plot, where we obtain the two points τ and $-\tau$. We can easily calculate the values of entropy and probability of τ and $-\tau$ (which are outside the classical Shannon curve), thus achieving the values of Shannon entropy in non-ergodic conditions. A bundle of lines (through the center of S^1) with different values of ϕ cover all the possibilities of non-ergodic entropy, just by embedding them in our n-sphere S^1. (Figure 4). In such a way, we achieve a circle equipped with countless diameters: one of them displays the Shannon entropy under ergodic conditions, the others display instead the informational entropies

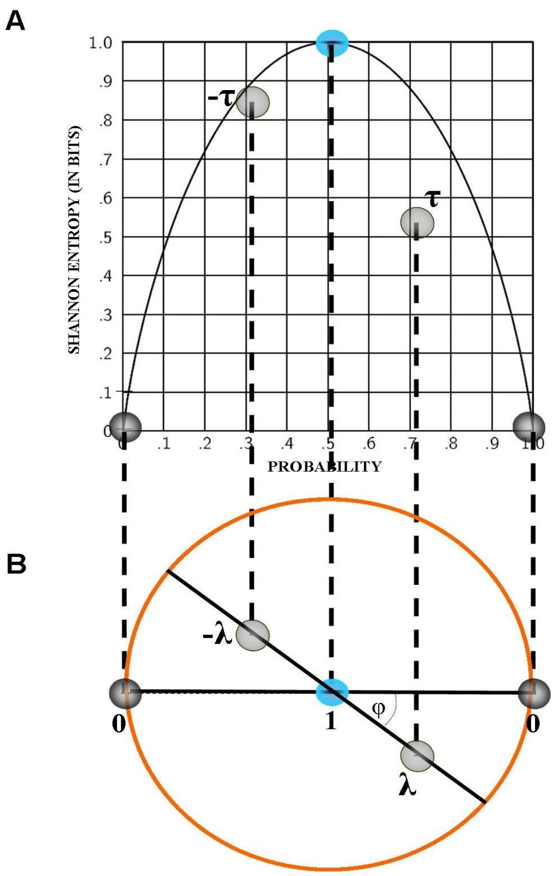

Figure 3. Non-ergodic probabilistic entropies in the framework of the BUT theorem. **A**. Shannon entropy under non-ergodic conditions. **B**. Note that, while the ergodic Shannon entropy follows the diameter which connects the two antipodal points on S^1, non-ergodic entropies follow other quantifiable diameters along the "circumference" of S^1 . Into the circle S^1, the points external to the Shannon entropy's diameter display all the possible values of non-ergodic entropy, i.e. the possibilities which does not fall into the Shannon plot's entropy curves. See text for further details.

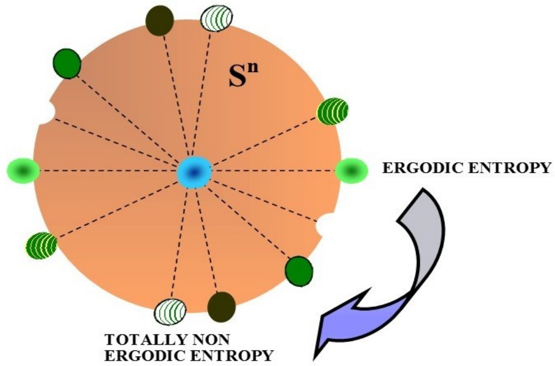

Figure 4. Bundle of lines on a n-sphere which illustrates the position of antipodal points for different non-ergodic conditions. The bundle can be used to evaluate the corresponding values of p and informational entropy on the Shannon plot.

under non-ergodic condition. This simple operation allows a quantitative evaluation of informational entropies under non-ergodic assumptions.

2.6. Rényi entropy on a circle

The Shannon entropy is just a case of a family of more generalized entropies, such as Tsallis, and in particular, Rényi entropies, which also work just in an ergodic context. Indeed, the basic thermodynamic properties of many systems (i.e., multifractals) may be discussed by extending the notion of the information Shannon entropy into the more general framework of the Rényi entropy. Let X be a random variable with values in the range from 1 to n. Rényi entropy of order n [34] is defined by

$$H_n(X) = \frac{logC(n,X)}{1-n}, \quad where \quad C(n,x) = \sum_{i=1}^{m} P^n(x_i) \tag{4}$$

The Rényi entropy approaches the Shannon entropy as β approaches 1. By now, for sake of simplicity, we will term the Rényi entropy order n with the greek letter β, so that $\beta = 1$ (i.e., the limit for $\beta \to 1$) is defined to be the Shannon entropy:

$$lim_{\beta\to 1} H_\beta(X) = \sum_i P_i ln P_i \tag{5}$$

The Rényi entropy is also closely related to the thermodynamical Gibbs entropy via the thermodynamic free energy F, through the formula: $F = (1-T)H_\beta(X)$ in which T is the temperature. Mathematically, it is expressed as follows: the Rényi entropy of a system is minus the "$1/\beta$-derivative" of its free energy with respect to a quantity. Because of its built–in predisposition to account for self–similar systems, the Rényi entropy is an effective tool to describe multifractal systems [35]. It has been demonstrated that the Rényi entropy and generalized fractal dimension α are interchangeable: the Rényi's parameter β is connected via a Legendre transformation with the multifractal singularity spectrum α. It means that, from the maximum entropy point of view, the power law exponent n and Rényi's parameter β exhibit a straight relation (see [35] for further details) and changes in power law exponents n lead to changes in Rényi's parameter β, and vice versa [36]. In Materials and Methods section, we describe two different ways to embed the Rényi entropy on a n-sphere, both in touch with BUT dictates.

3. Discussion

We demonstrated that ergodic and non-ergodic informational entropies can be solved in terms of algebraic topology, by embedding the Shannon's plot in a n-sphere and by applying the BUT. Further, we provided an effort to insert informational entropies in the framework of group theory, by considering probabilities in guise of permutations on a n-sphere. The question now is: which are the advantages of such a "treatment"? For example, what does the use of Hausdorff-Borsuk-Ulam Theorem give us for different applications of Rényi entropy? The BUT and its variants display very useful general features which help us to explain a wide-range of phenomena, including brain activity. When a single point is embedded in just one dimension higher, it gives rise to two antipodal points. Thus, by adding just a further dimension to a biological system, we are allowed to study it in terms of antipodal points [20]. Furthermore, the two antipodal points on a S^n-sphere display homotopy and have matching descriptions. If we evaluate biological dynamics instead of "signals", BUT leads naturally to the possibility of a region-based, not simply point-based, geometry. In such a vein, a collections of brain signals could be viewed as surface shapes (or functions, or signals), where one shape maps to another antipodal one. We are also allowed to use the parameter $n - a$ versatile tool which can be used both for integer and rational numbers - not just for the description of topological manifolds, but of biological systems too.

Why Rényi entropy? Rényi entropy and generalized diversity functions have shown to be proper indicators to quantify systems over time: from plant communities [37] to urban mosaics [38]. The use of Rényi entropy– unlike the many diversity measures for summarizing landscape structure based on Shannon entropy [39] – allows the description of the ecosystem status at a specific moment and its trend over time [40]. Rényi entropy offers a "continuum of possible diversity measures" [38] at diverse spatial scales, which differ in their sensitivity to rare and abundant picture indexes, becoming increasingly regulated by the commonest when β gets higher. The change in β exponent can be regarded as a scaling operation that takes place not in the real, but in the data space [41]. The aim of using the Rényi entropy does not consist in selecting the most appropriate parameter, rather in constructing 'diversity profiles': the Rényi's parameter β is particularly important, since it is not redundant and allows us to consider several measures at a time, by varying just the β parameter therein. The opportunity to treat Rényi entropies as topological structures gives rise to the possibility to evaluate brain phenomena with novel analytical tools, such as algebraic topology, combinatory, hereditary set systems [18], simplicial complexes, homology theory, functional analysis and with generalizations of the BUT, such as, for example, the Bourgin-Yang-type theorems [42] and the Grassmann manifolds [43]. If the dimension in which the sphere is embedded takes into account also non-ergodic conditions, we have a tool which is feasible for a calculation of both ergodic and non-ergodic entropies. In conclusion, we provided a very general topological mechanism which solveslong-standing problems of non-ergodic and general informational entropies, casted in a physical/biological fashion which has the potential of being operationalized and experimentally tested in the evaluation of brain dynamics.

4. Materials and Methods

4.1. The first way: Rényi entropy-Based Friendship Theorem

The correlation between Rényi entropy and thermodynamic free energy can be explained via the Friendship Theorem introduced by Rényi and his coauthors in terms of the vertices of a particular graph [44,45]. It is C. Huneke's simplified version of the Friendship Theorem [46] that we give next.

4.1.1. Friendship Theorem

if G is a graph in which any two vertices have exactly one common neighbor, then G has a vertex joined to all other vertices in the graph. This theorem can be reformulated in terms of points

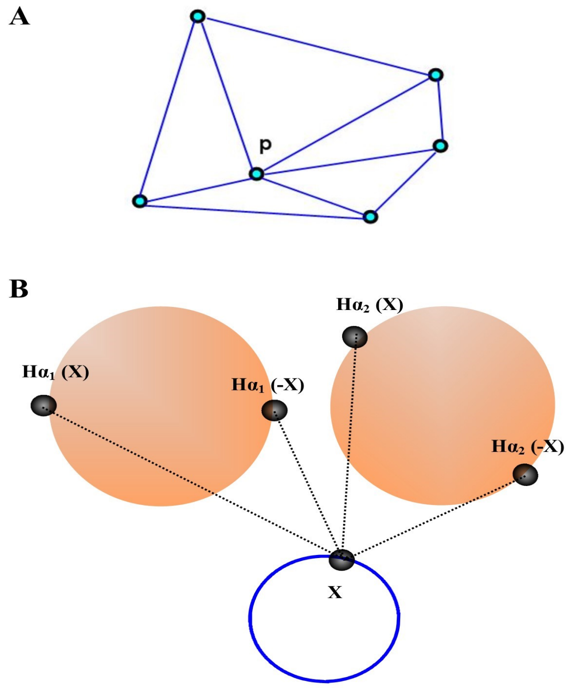

Figure 5. A.Point-Based Friendship Theorem. See the main text for further details.B. Rényi entropy-Based Friendship Theorem.

and regions in the following way. In this reformulation, points x,y are connected, provided there is a straight edge whose endpoints are x, y. Point-Based Friendship Theorem: if X is a nonempty set of points in which any two points are connected to a common point, then X has a point p that is connected to every other point in the set. This situation is illustrated in Figure 5A. It is now a straightforward step to obtain a Rényi entropy-based Friendship Theorem: if X maps to is a nonempty set of points in which any two points are connected to a common point, then X has a point p that is connected to every other point in the set. The situation described by the Rényi entropy-Based Friendship Theorem is illustrated in Figure 5B. If X is a region on an n-sphere and H_{α_1}, H_{α_2} are Rényi entropies of region X with respect to parameters α_1 and α_2 , respectively. In addition, it is assumed that X is a smooth manifold and $f : H_\alpha(X) \to X \in 2^{Rn}$ is a homeomorphism that maps H_{α_1}, H_{α_2} to a region X in Euclidean space R_n. The vectors in X represent observations such as cortical temperatures that give rise to the n-sphere entropies shown in Figure 2B. We know from BUT that, whenever there is a continuous function f on n-sphere, a pair of antipodal points is mapped by f to a value in R_n, which has a region-based extension [47]. In particular, given the homeomophism f which is a continuous function on an n-sphere S^n whose surface values are Rényi entropies, then we know there is a pair of Rényi entropies that are mapped by f to $X \in 2^{Rn}$. In effect, we arrive at a reversal of Rényi entropies.

4.2. *The second way: a fractional dimension*

In the framework of Rényi entropy, we introduce a quantity called the entropy difference:

$$\Delta_{(\alpha,\alpha_0)}(P) = H_\alpha(P) - H_{\alpha_0}(P) \tag{6}$$

If we assume that $\alpha \to \alpha_0$, we can use the expansion of $H_\alpha(P)$ around α_0, obtaining the differential entropy difference, which is proportional to entropy derivative with respect to Rényi parameter:

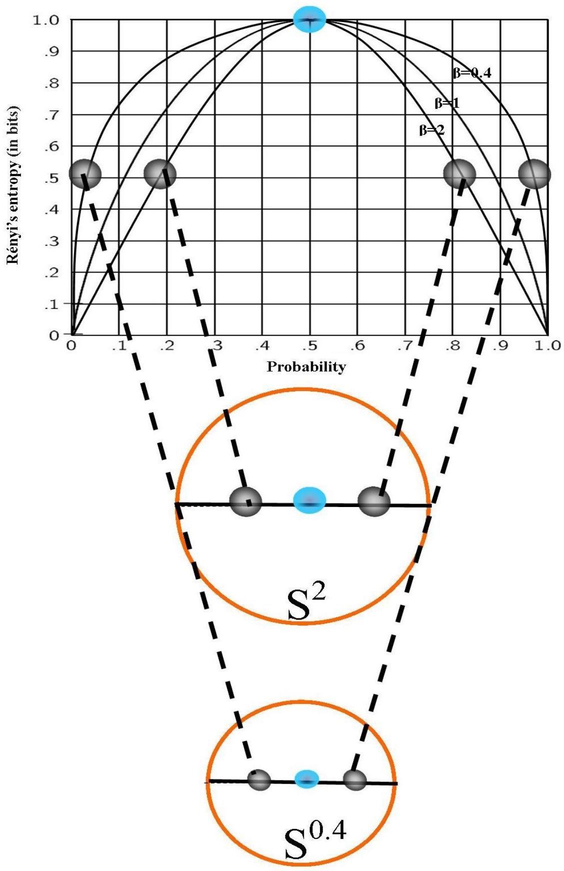

Figure 6. Generalized Renyi entropy. Note that $S^{0.4}$ has a smaller diameter than S^2 and the dotted lines are tangent to S^2. We achieve a mapping of antipodal points in $S^{0.4}$ to the Rényi entropy values associated with antipodal points on S^2.

$$\Delta_{\alpha,\alpha_0}(P) \approx \frac{dH_{\alpha_0(P)}}{d\alpha_0}(\alpha - \alpha_0) \tag{7}$$

Rényi entropy $H_n(X)$ of order n correlates with a hypersphere S^n and, applying BUT, we can predict entropy values associated antipodal points on each n-sphere. That is, for a range of Rényi entropy orders $1, ..., n$, we map each n-sphere to Rényi entropy values in R^n. See Figure 6 for an example. Indeed, hyperspheres of order n can to extended to fractional values of n, giving rise to an enlarged of hyperspheres susceptible to treatment by BUT. To see this, consider the following introduction to the Hausdorff dimension.

1. Metric space: Let X be a metric space with the metric $\mu_d(X)$ defined on it. This means that $\mu_d(X) \geq 0$ and μ_d has the usual symmetry and triangle inequality properties for all subsets of X.
2. Hausdorff measure: Let d be either 0 or a positive real number in R_0^+. The Hausdorff measure $\mu_d(X)$ equals a real number for each number d in $X = R^d$.
3. Hausdorff dimension (informal): The threshold value of d denoted by $dim_H(X)$ is the Hausdorff dimension of X, provided $\mu_d(X) = 0$, if $d > dim_H(X)$, and $\mu_d(X) = \infty$, if $d < dim_H(X)$.

Hausdorff Dimension- To arrive at the Hausdorff (fractional) dimension of a subset X in a metric space, we need to consider the Hausdorff measure of X.

Definition 1. Hausdorff measure. Let X be a subset of a metric space M and let d any real number in $R_0^+ \epsilon \in R_0^+$ (a real number that is either positive or zero) a nonempty subset of X, U_i, $i \in 1, ..., n$ is a cover of X, i.e. X is a subset of $X \subseteq C_i$ for all i [48]. Here n is any positive integer. Also, let $diam(U_i) < \epsilon$ be the diameter of the cover U_i. The d-dimensional Hausdorff measure $\mu_d(X)$ is defined by:

$$\mu_d(X) = lim_{\epsilon \to 0} \left[\inf_{U_i \supseteq X} \sum_{i=1}^{n} n \left(diam \left(U_i \right) \right)^d \right] \tag{8}$$

The basic idea is to cover X with sets U_i with small diameters and estimate the d-measure of X as the sum of the $(diam(U_i))^d$, i.e., the sum of the U_i diameters raised to the power d.

Lemma 1. Schleicher Lemma. Let d be any real number in R_0^+. For every bounded set X in a metric space, there is a unique value of $d := dim_H(X)$ in $R_0^+ U\{\infty\}$ such that:

$$\mu_d^I(X) = \begin{cases} 0, & \text{if} \quad d^I > d, \\ \infty, & \text{if} \quad d^I < d \end{cases}$$

Definition 2. Hausdorff dimension. The value of $d = dim_H(X)$ in R_0^+ called the Hausdorff dimension of X. With $d = dim_H(X)$, the Hausdorff measure $\mu_d(X)$ may be zero, positive or infinite.

Lemma 2. Schleicher Boundedness Lemma. Let d be any real number in R_0^+ and let Y be a metric space. If , then: $dim_H(X) \leq dim_H(Y)$.

Proof. Immediate from the definition of the Hausdorff dimension of a nonempty set. Assume that X is a nonempty subset (inner sphere) of an n-sphere and having the same center as S^n with the Hausdorff measure $\mu_d(X)$ defined on it and assume that $\mu_d(X)$ satisfies the Schleicher Lemma 1 conditions. The inner sphere S_d of an n-sphere S^n can be any sub-sphere in S^n, including S^n itself. Then, the inner sphere S^d has dimension $d = dim_H(X)$, $d \leq n$. In addition, assume that R^d is a d-dimensional space which is a subset of the n-dimensional Euclidean space R^n, $d \leq n$. This gives us new form of the BUT.

Theorem 1. Hausdorff-Borsuk-Ulam Theorem. Let S^d with Hausdoff dimension d be an inner sphere of an n-sphere and let $f : S^d \to R^d$ be a continuous map. There exists a pair of antipodal points on S^d that are mapped to the same point in R^d.

Proof. A direct proof of this theorem is symmetric with the proof of the BUT is given by [49], since we assume that S^d is an inner sphere of S^n symmetric about the center of S^n and, from the Schleicher Boundedness Lemma 2, $dim_H(S^d) \leq dim_H(S^n)$. We can thus evaluate how changes of Rényi parameter influence the structure of information measures in the probability space (Figure 6). To make an example, starting from a value of Rényi exponent (i.e., 0.4), it is possible to calculate the entropy values in case one wants to evaluate other exponents (Figure 6). The Figure 6 shows that the antipodal points corresponding to the exponent 2 are indeed in an exact position on the diameter corresponding to the 0.4 exponent. If one knows the position of all the Rényi exponents projected on a n-sphere, it is feasible to achieve all the corresponding values of Rényi entropies.

References

1. Friston, Karl. The Free-Energy Principle: A Unified Brain Theory? *Nat. Rev. Neurosci.* **2010** *11 (2)*.127–38.
2. Barth A. Vorlesungen über Gastheorie. Leipzig: 1898. OCLC 01712811. ('Ergoden' on p.89 in the 1923 reprint.)
3. Alaoglu, L.; Birkhoff, G. General ergodic theorems. *Ann. Math.* **1940** , *41(2)*, 293-309.
4. Walters, P. An Introduction to Ergodic Theory. *Springer***1982**, ISBN 0-387-95152-0.
5. Feller, W. An introduction to probability theory and its applications. *Wiley* **1971**.
6. Shannon, C.E. A Mathematical Theory of information. *Bell Syst. Tech. J.* **1948**, *27*, 379-423 & 623-656.
7. Sienkiewicz, J.; Skowron, M.; Paltoglou, G.; Holyst, J.A. Entropy growth in emotional online dialogues. *JPCS 410 012096* **2013** .

8. Wicentowski, R., Sydes, M.R. Emotion Detection in Suicide Notes using Maximum Entropy Classification. *Biomed Inform Insights* **2012**,*5*, 51–60.
9. Freeman, W. J.; Livi, R.; Obinata, M.; Vitiello, G. Cortical phase transitions, non-equilibrium thermodynamics and the time-dependent Ginzburg-Landau equation. *Int. J. Mod. Phys. B* **2006**, *26*, 1-29.
10. Guastello, S.J.; Gorin, H.; Huschen, S.; Peters, N.E.; Fabisch, M. The minimum entropy principle and task performance. *Nonlinear Dynamics, Psychology, and Life Science***2013**, *17(3)*, 405-423.
11. Štys, D., Korbel, J., Rychtáriková, R., Soloviov, D., Císař, P., Urban, J. Point information gain, point information gain entropy and point information gain entropy density as measures of semantic and syntactic information of multidimensional discrete phenomena. *arXiv:1501.02891* **2015**,1-8.
12. Fraiman, D.; Chialvo, D.R. What Kind of Noise Is Brain Noise: Anomalous Scaling Behavior of the Resting Brain Activity Fluctuations. *Front Physiol.***2012** , *3 (307)* , 1–11.
13. Bianco, S.; Ignaccolo, M.; Rider, M.S.; Ross, M.J.; Winsor, P.; Grigolini, P. Brain, music, and non-Poisson renewal processes. *Phys Rev E Stat Nonlin Soft Matter Phys.* **2007** *75* , 061911.
14. Papo, D. Functional Significance of Complex Fluctuations in Brain Activity: From Resting State to Cognitive Neuroscience. *Front Syst Neurosci.* **2014**, *8(112)*, 1-7.
15. Rebenshtok, A. Barkai, E. Weakly non-ergodic statistical physics. *J. Stat. Phys.* **2008**, *133(3)*, 565-586.
16. Borsuk, M.; Drei sätze über die n-dimensionale euklidische sphäre, *Fund, Math.* **1933** *20*, 177–190.
17. Dodson, C.T.J.; Parker, P.E. In *A user's guide to algebraic topology*, Kluwer, Dordrecht, Publishing House: Netherlands, 1997, pp 405 pp.
18. Matoušek, J. *Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*. Springer-Verlag Berlin Heidelberg, 2003.
19. Tozzi, A.; Peters, J.F. Towards a Fourth Spatial Dimension of Brain Activity. *Cognitive Neurodynamics* **2016a**, *10(3)*, 1–11.
20. Tozzi, A.; Peters J.F. A Topological Approach Unveils System Invariances and Broken Symmetries in the Brain. *J. Neurosci. Res.***2016b**, *94 (5)*, 351–65.
21. Weeks, J.R. The shape of space, IInd edition. Marcel Dekker, inc. New York-Basel. 2002.
22. Marsaglia, G. Choosing a Point from the Surface of a Sphere. *Ann. Math. Stat.* **1972** *43 (2)*, 645–646.
23. Henderson, DW. *Experiencing Geometry on Plane and Sphere*. Publishing House: New York: Prentice Hall, USA. 1996.
24. Weisstein E.W. antipodal points, http://mathworld.wolfram.com/AntipodalPoints.html. **2015**.
25. Willard, S. In *General topology*, Dover Pub., Inc., Mineola, NY, USA, 1970;pp 369.
26. Krantz, S.G. In *A guide to topology*; The Mathematical Association of America, Publishing House: Washington, D.C., USA, 2009: pp 107.
27. Manetti, M. *Topology*, Springer, Publication House: Heidelberg, Germany. 2015, pp 309.
28. Cohen, M.M. *A course in simple homotopy theory*, Springer-Verlag, Publishing House: New York-Berlin. 1973, pp 144.
29. Borsuk, M. Concerning the classification of topological spaces from the standpoint of the theory of retracts, XLVI (1958-1959), 177–190.
30. Borsuk, M. Fundamental retracts and extensions of fundamental sequences. *Fund, Math.* **1969** *64*, 55–85.
31. Borsuk, M.; Gmurczyk, A. On homotopy types of 2-dimensional polyhedral, *Fund, Math.* **1980**, *109 (2)*, 123–142.
32. Collins, G.P. The shapes of space,*SciAm* **2004** *291*. 94–103.
33. Peters, J.F. *Topology of Digital Images. Visual Pattern Discovery in Proximity Spaces, Intelligent Systems Reference Library*, vol. 63, Springer, 2014; pp. 1-342.
34. Bialas, A.; Czyz W.; Zalewski K. The Rényi entropy as a rigorous, measurable lower bound for the entropy of the interaction region in multi-particle production processes, *EPJ C* **2006**, *48*, 215-221.
35. Jizba, P.; Arimitsu, T. The world according to Renyi: thermodynamics of fractal systems. *AIP Conf. Proc.* **2001** *597*, 341-348.
36. Słomczynski, W.; Kwapien, J.; Zyczkowski, K. Entropy computing via integration over fractal measures. *Chaos* **2000**, *10(1)*, 180-188.
37. De Luca, E.; Novelli, C.; Barbato, F.; Menegoni, P.; Iannetta, M.; Nascetti, G. (2011). Coastal dune systems and disturbance factors: monitoring and analysis in central Italy. Environmental Monitoring and Assessment. *Environ Monit Assess***2011** *183 (1)*, 437-450.

38. Carranza, M. L.; Acosta, A.; Ricotta, C. Analyzing landscape diversity in time: the use of Rényi's generalized entropy function. *Ecological Indicators* **2007**, *7*, 505-510.
39. Patil, G. P., Taillie, C. A multiscale hierarchical Markov transition matrix model for generating and analyzing thematic raster maps. *Environ. Ecol. Stat.***2001**,*8*, 5-20.
40. Möller, F., Hoffmann-Kroll, R., Wiggering, H. Indicating ecosystem integrity e theoretical concepts and environmental requirements. *Ecol. Model.***2000***130*, 13-23.
41. Podani, J. Space series analysis: processes reconsidered. *Abstr. Bot.***1992***16*, 25–29.
42. Yang, C.T. On Theorems of Borsuk–Ulam, Kakutani–Yamabe-Yujobo and Dynson, I. *Ann. Math***1954***60*, 262–82.
43. Dol'nikov, V.L. A Generalization of the Ham Sandwich Theorem. *Math. Notes* **1992**, *52*. 771–79.
44. Havrda, J.; Charvat, F. Quantification method of classification processes: Concept of structural entropy.*Kibernetika***1967**,*3*, 30-35.
45. Rényi, A. Calcul des probabilites [French], Éditions Jacques Gabay, Sceaux, 1966,1992 Ed., vii+311 pp.
46. Huneke, C. The Friendship Theorem. *Am. Math. Monthly* **2002***109 (2)*, 192–194.
47. Peters, J.F. Computational Proximity. Excursions in the Topology of Digital Images. Edited by Intelligent Systems Reference Library. Berlin: Springer-Verlag. 2016.
48. Schleicher, D. Hausdorff dimension, its properties, and its surprises. *Am. Math. Monthly*. **2007**, *114(6)*, 509-28.
49. Su, F. E. Borsuk-Ulam implies Brouwer: a direct construction. *Amer. Math. Monthly* **1997***104*, 855–859.