On the analytical demonstration of Planck-Einstein relation

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Abstract

In this paper a possible analytical demonstration of Planck's Law for the spectral distribution of the electromagnetic energy radiated by hot bodies is presented, but in this case, the solutions of Maxwell's equations for the electromagnetic radiation problem of Hertz's dipole are used. The concepts of quantum energy and of photon are redefined from the classical point of view, relating them to the possible electronic nature of electromagnetic waves and the electromagnetic field in general. Both the physical analysis and the concepts proposed respect the law of conservation of energy and allow to finally express the quantic constant, which is obtained here as a perfect combination of other fundamental constants of nature. The classic interpretation of the law obtained could be considered as the meeting point between Classical Physics and Quantum Mechanics, which suggests a new review of the theoretical basis of the latter is needed.

Keywords: *Planck-Einstein relation, quantum of energy, photon, Hertz dipole, Maxwell's equations*

In 1901 the German scientist *Max Planck* published a proposal of a thermodynamic model to describe the spectral distribution of the electromagnetic energy radiated by hot bodies, [1]. In his research he obtained for the first time two fundamental constants of nature, $\boldsymbol{k}$, called *Boltzmann's constant* in honor to the Austrian scientist *Ludwig Eduard Boltzmann*, who is considered the father of statistical mechanics, and $\boldsymbol{h}$, which is named *Planck's* constant by the scientific community in honor to him. Thus, he could propose the concept of quantum of energy, which in turn allowed *Albert Einstein* in 1905 to introduce the concept of photon, [2]. Both concepts are an important part of the theoretical foundations of quantum mechanics as we know today and its empirical mathematical expression, which is $\boldsymbol{E = nh\upsilon}$, also called, *Planck-Einstein relation*. *Planck* used a statistical thermodynamic model because at that time, apparently, it was not possible to use appropriately the system of basic equations of electromagnetism of *James Clerk Maxwell* to explain the case of the black body radiation.

In this paper, an electrodynamic model which could be accepted as an analytical demonstration of *Planck-Einstein relation* is proposed, but using instead the solutions of *Maxwell's* equations for electromagnetic radiation of *Hertz's* dipole. This demonstration was presented for first time as part of the Doctoral Thesis of this author. Also, a novel classical reinterpretation of concepts such as quantum of energy and photon is proposed here. These deductions could constitute the solution of one of the most notable problems of Physics, which has been unsolved for more than a century, and could change the way we understand nature nowadays, opening the door to obtain classical and simple solutions of scientific problems without a clear explanation until today. Moreover, all the explanations and the analysis method used in this research comply strictly with the law of conservation of energy.

This paper is organized in four sections; the first is the present *introduction*, the second is the *formulation*, in which the analytical demonstrations are performed. The third corresponds to the *discussion* of the main results. Finally, in the *conclusions* section the possibility of being at the junction of *Classical Physics* and *Quantum Mechanics* is considered among other things. At the end of the paper the *references* used are listed.

Formulation

Figure 1 shows the equivalent circuit of an antenna-generator system in which the generator is represented by an alternating voltage source $\boldsymbol{V_g}$ with internal resistance $\boldsymbol{R_g}$, that delivers a signal of frequency $\boldsymbol{\upsilon}$. The antenna used is a *Hertz* dipole radiator type and is represented by its radiation resistance $\boldsymbol{R_{Rad}}$.

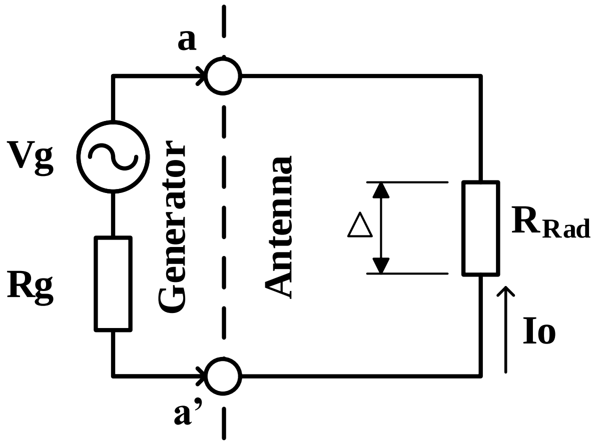

Figure. 1. Antenna-generator system.

The radiation resistance of Hertz's dipole is obtained from the solutions of *Maxwell's* equations for the problem of electromagnetic radiation, [3]:

$$R_{Rad} = 80\pi^2 \left(\frac{\Delta}{\lambda}\right)^2 \quad [\Omega] \tag{1}$$

In equation (1), Δ is the radiator's length or distance between the charges accumulated in its ends, and $\boldsymbol{\lambda}$ represents the wavelength of the electromagnetic waves emitted by the dipole. On the other hand, it is known that:

$$\lambda = \frac{c}{\upsilon} \quad [m] \tag{2}$$

In which $\boldsymbol{c}$ is the speed of the electromagnetic waves in vacuum (speed of light of about $\boldsymbol{3 \cdot 10^8\ m/s}$). Then, substituting equation (2) in (1), the radiation resistance is obtained as a function of the frequency $\boldsymbol{\upsilon}$:

$$R_{Rad}(\upsilon) = 80\pi^2 \Delta^2 \frac{\upsilon^2}{c^2} \quad [\Omega] \tag{3}$$

Moreover, the power of the radiation emitted as a function of frequency is:

$$P(\upsilon) = I^2 R_{Rad}(\upsilon) \quad [W] \tag{4}$$

Where $\boldsymbol{I} = \left[\boldsymbol{V_g} / \left(\boldsymbol{R_g} + \boldsymbol{R_{Rad}(v)}\right)\right]$ is the current flowing through the circuit of Figure 1, and the spectral power distribution given by equation (4) is shown in Figure 2.

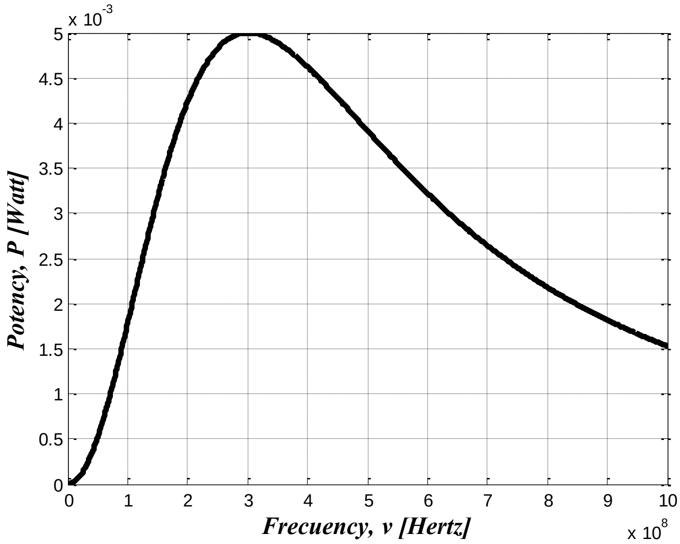

Figure.2. Power spectral distribution radiated by the *Hertz* dipole.

Under these conditions, the energy of the electromagnetic radiation is given by equation (5):

$$E(v) = \frac{d[P(v)]}{dv} = I^2 \cdot \frac{d[R_{Rad}(v)]}{dv} \qquad [J] \tag{5}$$

Its unit analysis is as follows:

$$[Ampere^2] \cdot \frac{[\,Ohm\,]}{[Hertz\,]} \rightarrow \frac{[\,Watt\,]}{[Hertz\,]} \rightarrow \frac{[Joule/\ second]}{[Hertz]}$$

$$\rightarrow \left[\frac{Joule}{Hertz}\right] \cdot [\,Hertz] \rightarrow [\,Joule\,]$$

Substituting equation (4) into (5) and operating, the following result is obtained:

$$E(v) = I^2 \cdot 160\pi^2\Delta^2 \frac{v}{c^2} \ \ [J] \tag{6}$$

Figure 3 shows the energy spectral distribution according to equation (6).

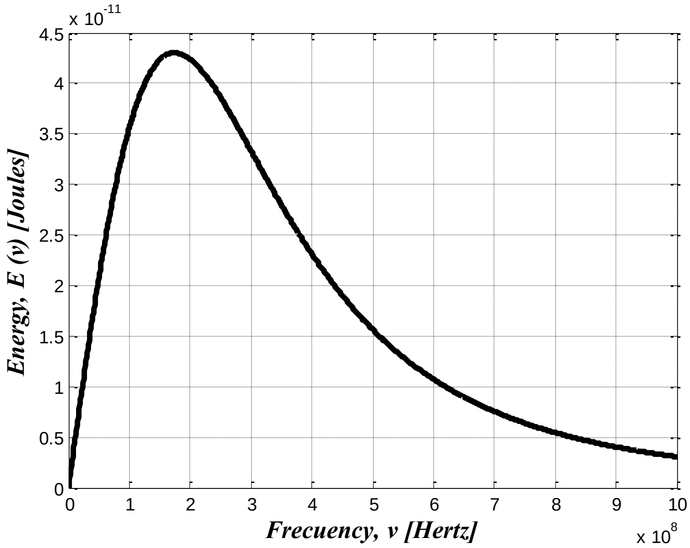

Figure.3. Energy spectral distribution radiated by the *Hertz* dipole.

Now, grouping conveniently the terms in equation (6), and multiplying the numerator and denominator by the charge of the electron $\boldsymbol{e}$, the equation is not altered:

$$E(v) = \frac{I^2}{e}\left(160\Delta^2 \frac{\pi^2}{c^2} e\right) v, \quad [J] \tag{7}$$

Exchanging the $Ampere^2$ of the current to the square on the first factor in equation (7) with the $Coulomb$ associated with the charge of the electron inside the brackets, the first factor would remain dimensionless, given that the $Coulombs$ of the numerator and the denominator are cancelled.

Note that the ratio $\left[\mathbf{1}[\boldsymbol{C}]/\boldsymbol{e}\,[\boldsymbol{C}]\right] = [\mathbf{1/1.602} \ldots \cdot \mathbf{10^{-19}}] = \mathbf{6.24} \ldots \cdot \mathbf{10^{18}}$, and it would represent the number of electrons corresponding to one $Coulomb$ of electric charge. Then, the dimensionless factor $\boldsymbol{n} = \boldsymbol{I^2}/\boldsymbol{e}$ will be designated as the quantic number in equation (7). Remember that the $Ampere^2$ was transferred inside the brackets.

The rest of equation (7) would have $Ampere^2 \cdot Ohm/Hertz$ units, which is the same as $Watt/Hertz$, and this in turn can be expressed in the form of $Joule \cdot Hertz/Hertz$. Given this, all that is inside the brackets will be called electrodynamic quantic constant, with $Joule/Hertz$ units and it is designated as $\boldsymbol{h_e}$ as an analogy with $Planck's$ constant. The remaining $Hertz$ in equation (7) corresponds to the frequency $\boldsymbol{v}$ outside of the brackets. Finally, equation (7) in compact notation will remain as follows:

$$E = n\, h_e v, \quad [J] \tag{8}$$

Analysis of the constant $\boldsymbol{h_e}$

As it can be seen in Figure 2, the maximum of the power spectrum always coincides with a frequency value for which $\Delta = \boldsymbol{\lambda/4}$, or expressed in electrical lengths as $\Delta/\boldsymbol{\lambda} = \mathbf{0.25}$. This guarantees that the product $\mathbf{160\Delta^2}$ in equation (7) always results in a constant value of $\mathbf{10}$. Given this, $\boldsymbol{h_e}$ will be:

$$h_e = 10 \cdot \frac{\pi^2}{c^2} \cdot e, \quad \left[\frac{J}{H_z}\right] \tag{9}$$

Whose value is:

$$h_e \approx 1.75 \cdot 10^{-34}, \quad \left[\frac{J}{H_z}\right] \tag{10}$$

On the other hand, in the case of the energy radiated by *Hertz's dipole*, in equation (8), the quantic number $\boldsymbol{n}$ is also a function of frequency $\boldsymbol{v}$, since the current flowing through the circuit depends on it; see Figure.1.

$$n(v) = \frac{I^2}{e} = \frac{1}{e}\left(\frac{V_g}{R_g + R_{Rad}(v)}\right)^2, \quad [dimensionless] \tag{11}$$

Under these conditions, the energy distribution shown in Figure 3 can be arranged in two functions. According to equation (9), one would correspond to the quantic number function $\boldsymbol{n}(\boldsymbol{v})$ given by equation (11), and the other would correspond to the linear function $\boldsymbol{h_e v}$, which already includes the electrodynamic quantic constant $\boldsymbol{h_e}$. Both variations are shown below in Figure 4.

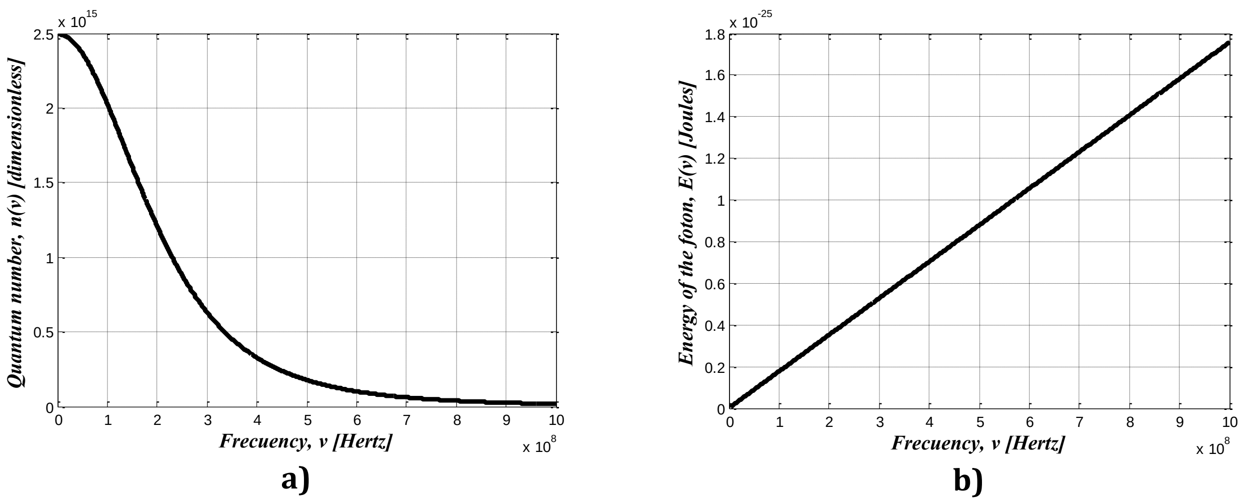

Figure. 4. Spectrum associated with functions: a) *quantic number* and b) *photon energy*.

As shown in equation (9), constant $\boldsymbol{h_e}$ results in a perfect combination of other fundamental constants of nature, being the first constant $\mathbf{10}$, which is the basis of our current numeric system, which in turn is composed of the unit and the zero, two of the most important numbers in mathematics; the second constant is the number $\boldsymbol{\pi}$, of extraordinary beauty and value given its presence in nature, the third constant is $\boldsymbol{c}$, which is the speed of light in vacuum and sets the upper limit of speed that can be achieved in nature. Finally, the charge of the electron $\boldsymbol{e}$ *is the fourth constant*, which is the elementary mass particle that will carry the electromagnetic energy, and it will be discussed in more depth later.

Interpretation proposal

In the case of the electromagnetic radiation emitted by the *Hertz* dipole and generally by artificial antennas developed by man, a possible classical interpretation of the equation (8) could be the following:

The quantic number $\boldsymbol{n}$ represents the number of electromagnetic energy quanta (electrons) that are emitted by the radiator in a second, which is the matter that constitutes the electromagnetic field radiated as electromagnetic waves; and $\boldsymbol{v}$ would set the amount of photons (electromagnetic waves) radiated by the dipole in a second as well.

Then, operating with constant $\boldsymbol{h_e}$ from the equation of the quantic law (8), the result is:

$$h_e = \frac{E}{nv}, \quad \left[\frac{J}{H_Z}\right] \tag{12}$$

Which states that $\boldsymbol{h_e}$ is proportional to the energy carried by each photon (electromagnetic wave) radiated by the dipole. This interpretation is reasonable, since the numerator is the energy radiated in a second and the denominator represents the amount of quanta and photons that are emitted in the same time unit.

The physical units of this constant $[Joules/Hertz]$, would lead to the same argumentation presented before, since the constant would be proportional to the amount of energy supplied by the signal generator to the radiator in each oscillation or cycle of the electric current. Then, $\boldsymbol{n_v = n/v}$ would be the number of energy quanta that compose each photon emitted by the radiator.

Relation and quantitative differences with Planck's constant

The experimental value officially reported for *Planck's* constant is associated with the quantic *Hall* effect, [4], whose value is $\boldsymbol{h = 6.62606957 \cdot 10^{-34}}$ $[\boldsymbol{J/H_z}]$, so that the relation between $\boldsymbol{h}$ and $\boldsymbol{h_e}$ would be as follows:

$$\alpha^2 = \frac{h}{h_e} = \frac{6.62 \ldots \cdot 10^{-34}}{1.75 \cdot 10^{-34}} \approx 3.78$$

This indicates that the constant $\boldsymbol{h}$ is $\mathbf{3.78}$ times greater than $\boldsymbol{h_e}$. Now, an important question would be how to justify scientifically the multiplication of $\boldsymbol{h_e}$ by the coefficient $\boldsymbol{\alpha^2}$ in order to demonstrate that this would be, in fact, *Planck's constant*.

It would be reasonable to think that the constant $\boldsymbol{h}$ is calculated using the energy that is measured experimentally from the radiation emitted by the surface of a hot body, so that the radiators that constitute this surface would emit in a semi-space or an exterior half-sphere, characterized by a certain directivity. On the other hand, the quantic constant $\boldsymbol{h_e}$ presented here has been obtained analytically from the solutions of *Maxwell's* equations for the radiation of a *Hertz* dipole isolated in free space, and radiating in a $\boldsymbol{4\pi}$ steradian sphere (full space). This implies that when the dipole is in the surface of the body, the intensities of their fields in the external half-space of interest would be those resulting from the superposition of the direct ray and the reflected ray in the surface of the body itself. In this case, the resulting magnitudes are:

$$E_R = E_{inc} + E_{ref} = E_{inc} + |\Gamma| E_{inc} = E_{inc}(1 + |\Gamma|) \tag{13}$$

and

$$H_R = H_{inc} + H_{ref} = H_{inc} + |\Gamma| H_{inc} = H_{inc}(1 + |\Gamma|) \tag{14}$$

Where $\boldsymbol{\Gamma}$ would be the reflection coefficient on the surface of the radiant body. On the other hand:

$$\alpha \approx \sqrt{3.78} \approx 1.94$$

This allows associating the coefficient $\boldsymbol{\alpha}$ with the factor $[\mathbf{1} + |\boldsymbol{\Gamma}|]$ of the resulting fields of equations (13) and (14):

$$\alpha = 1 + \Gamma \approx 1 + 0.94$$

This represents a reflection coefficient of $\boldsymbol{\Gamma} \approx \mathbf{0.94}$, which is quite reasonable, since the condition $\boldsymbol{\Gamma} \leq \mathbf{1}$ must always be met. Moreover, the solution of the radiation problem for the case of *Hertz's* dipole is composed of the following expressions [3]:

$$\vec{E} = -\frac{i\,I\,\Delta\,k^2}{4\pi}\eta\left\{2\left[\frac{1}{(kr)^3}+\underbrace{\frac{i}{(kr)^2}}_{\cdot}\right]cos(\theta)\,\vec{\imath}_r + \left[\frac{1}{(kr)^3}+\frac{i}{(kr)^2}-\underbrace{\frac{1}{(kr)}}_{Rad}\right]sin(\theta)\,\vec{\imath}_\theta\right\}e^{-ikr}, \quad [V/m] \qquad (15)$$

and

$$\vec{H} = -i\frac{I\,\Delta\,k^2}{4\pi}\left[\frac{i}{(kr)^2}-\underbrace{\frac{1}{(kr)}}_{Rad}\right]e^{-ikr}\,sin(\theta)\,\vec{\imath}_\varphi. \qquad [A/m] \qquad (16)$$

So that the radiation fields extracted from equations (15) and (16) are respectively:

$$\vec{E} = i\frac{I\,\Delta}{4\pi}\eta\left[\frac{k}{r}\right]e^{-ikr}\,sin(\theta)\;\vec{\imath}_\theta, \qquad [V/m] \qquad (17)$$

And

$$\vec{H} = i\frac{I\,\Delta}{4\pi}\left[\frac{k}{r}\right]e^{-ikr}\,sin(\theta)\;\vec{\imath}_\varphi. \qquad [A/m] \qquad (18)$$

Equations (17) and (18) represent the fields that predominate far from the radiator (radiated fields); note that the amplitudes of these fields depend only on the inverse of the distance $\boldsymbol{r}$. Furthermore, they correspond to the incidents fields $\boldsymbol{E_{inc}}$ and $\boldsymbol{H_{inc}}$ of equations (13) and (14) in the half-space of interest, thus, the radiation resistance of the *Hertz*'s dipole will be affected by the presence of the surface of the body. This argumentation allows introducing the coefficient $\boldsymbol{\alpha}$ in the expression of the radiation resistance as follows:

$$R_{Rad}(\lambda) = \frac{1}{I^2}\int_0^{2\pi}\int_0^{\pi/2}\left[\underbrace{\alpha\vec{E}}_{\vec{E}_R}\times\underbrace{\alpha\vec{H}^*}_{\vec{H}_R}\right]sin(\theta)\cdot r^2\,d\theta\,d\varphi = \alpha^2 80\pi^2\left(\frac{\Delta}{\lambda}\right)^2. \qquad [\Omega] \qquad (19)$$

The introduction of the correction factor $\boldsymbol{\alpha^2}$ will be finally justified as follows:

$$h = \alpha^2\cdot h_e = 3.78\cdot 1.75\cdot 10^{-34} \approx 6.62\ldots\cdot 10^{-34}\;[J/H_Z]$$

Discussion

Note that a reflection coefficient $\boldsymbol{\Gamma}$ is implicit in $\boldsymbol{\alpha^2}$, which may vary slightly depending on the material properties of the radiating body. This could explain the existing differences in the value that is obtained for $\boldsymbol{h}$ for different experiments regarding the official value reported, so that the values obtained in different situations could be correct; therefore, the so called *Planck's constant* would not be a universal constant of nature. Meanwhile the composed constant $\boldsymbol{h_e}$ is a universal constant of nature.

On the other hand, the fact that the charge of the electron $\boldsymbol{e}$ can be found in $\boldsymbol{h_e}$, could be a sign that the electron is not only the elementary unit of electric charge, but the elementary particle constituting the electromagnetic field and therefore, the material carrier of electromagnetic energy. Then, the set of electromagnetic energy quanta (electrons), governed by $\boldsymbol{n}(\boldsymbol{v})$ in equation (11), would be present in nature, forming flows of electromagnetic fields; and they would be governed by the laws of classical electromagnetism of *Maxwell*.

It is interesting to note that in the equations (15) and (16) governing the radiated electromagnetic field, the only terms involving matter are those of the current $\boldsymbol{I}$, and that this is reduced to periodical electronic flows. It is also known that electronic flows have ondulatory and particle properties, and also electrical and mechanical properties, such as electric charge, mass and movement. Now it could be accepted that these flows also have magnetic properties and polarization characteristics depending on the orientation of these, so that the electromagnetic radiation would be only the periodic emission of discrete packets of energy (vortex electronic and quantum mechanical flows). Therefore, the electromagnetic radiation would be quantized like the electromagnetic field, which has been verified here by obtaining analytically the constant $\boldsymbol{h_e}$ (quantic electrodynamic action) and the demonstration of *Planck's* Law from the laws of classical electromagnetism of *Maxwell*, strictly meeting the law of conservation of energy.

This would show that the constant $\boldsymbol{h_e}$ is proportional to the energy contained in each electromagnetic wave (vortex flow) emitted by a radiator isolated in free space, for each oscillation of the feeding current supplied by the generator; therefore, its units are $[\boldsymbol{J}/\boldsymbol{H_Z}]$.

Conclusions

An electrodynamic model that makes use of the solutions of *Maxwell's* equations for the problem of electromagnetic radiation is proposed, and with this model, *Planck's* Law for the spectral distribution of the electromagnetic energy radiated by hot bodies could be demonstrated. The concept of quantum of energy was redefined, relating it to the electron as the fundamental particle of mass that constitutes the electromagnetic field. The concept of photon was associated with the electronic vortex flow radiated in the form of an electromagnetic wave, and the electrodynamic quantic constant was obtained as a perfect combination of other fundamental constants of nature, also fulfilling other purely aesthetic mathematical criteria.

Other conclusions that could be drawn from this research are, for example, the extension of the applicability range of the system of basic equations of electromagnetism of *Maxwell*. The possible electronic nature of the electromagnetic waves and the electromagnetic field was also considered. On the other hand, the treatment of electromagnetic waves as particles and their massive nature could be justified in certain situations given that the electrons (energy quanta) have mass. The massive nature of the energy quanta (electrons) also ensures that the laws of conservation of mass, charge, energy and mechanical impulse are met in the electromagnetic field at the same time, something that would constitute a reasonable explanation of the intrinsic natural unity of the electromagnetic field, which was proposed by Maxwell himself and it is accepted nowadays; but it could never be fully demonstrated, since the electron was discovered by *J. J. Thomson* in 1897 [5], and its properties must still be investigated.

Finally, the demonstration of the *Planck-Einstein* relation presented in this research introduces a meeting point between classical physics and quantum mechanics and perhaps, it could prove in the future that the current statements of the latter are based on the true material laws of nature.

Methods

A purely analytical method based on the solutions of Maxwell's equations for the problem of electromagnetic radiation was followed. This method can be easily verified given its simplicity and the closed form of its results.

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