Proving Grimm's Conjecture by Stepwise Forming Consecutive Composite Numbers' Points at the Number Axis

Zhang Tianshu

Zhanjiang city, Guangdong province, China
chinazhangtianshu@126.com
china.zhangtianshu@tom.com

Introduction: Grimm's Conjecture named after Carl Albert Grimm.

It was first published in American Mathematical Monthly, 76 (1969)

1126-1128. Yet it is both unproved and un-negated a conjecture hitherto.

AMS subject classification: 11A××

Abstract

Let us consider positive integers which have a common prime factor as a

kind, then the positive half line of the number axis consists of infinite

many recurring line segments of same permutations of χ kinds of

integers' points, where $\chi \geq 1$. In this article we proved Grimm's conjecture

by stepwise change symbols of each kind of composite numbers' points at

the number axis, so as to form consecutive composite numbers' points

under the qualification of proven Legendre-Zhang conjecture as the true.

Keywords: consecutive composite numbers' points, number axis,

distinct prime factors, Legendre-Zhang conjecture.

Preparations and the Proof

Grimm conjectured that if n+1, n+2 … n+k are all composite numbers,

then there are distinct primes p^{ij} such that $p^{ij} \mid (n+j)$ for $1 \le j \le k$.

For example, for consecutive composite numbers between prime 523 and prime 541, take out their one another's distinct prime factors as follows.

Composite numbers: **524,   525,   526,   527,   528,   529,   530,**

Resolve into prime factors: $2^2 \times 131$, $3 \times 5^2 \times 7$, 2×263，17×31，$2^4 \times 3 \times 11$，23^2，$2 \times 5 \times 53$，

Distinct prime factors: **131,   3,   263,   17,   11,   23,   53,**

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531,   532,   533,   534,   535,   536,   537,   538,   539,   540

$3^2 \times 59$, $2^2 \times 7 \times 19$, 13×41, $2 \times 3 \times 89$, 5×107, $2^3 \times 67$, 3×179, 2×269，$7^2 \times 11$, $2^2 \times 3^3 \times 5$

59,   19,   41,   89,   107,   67,   179,   269,   7,   2

First, we need to quote proven Legendre-Zhang's conjecture as the true, because it and the proof of the conjecture are closely related. Legendre-Zhang's conjecture asserts that there is at least an odd prime between n^2 and $n(n+1)$, and there is at least an odd prime between $n(n+1)$ and $(n+1)^2$, where n $\ge$ 2. A proof for Legendre-Zhang's conjecture comes from Zhang Tianshu, *"Legendre-Zhang's conjecture & Gilbreath's conjecture and proofs thereof"*, General Mathematics, ISSN 1221-5023, Volume 20, Issue 2-3, 2012, pp. 86-100. Or see also preprint Vixra.org: *http://vixra.org/abs/1401.0127.*

Thus it can be seen, there are at least two primes between n^2 and $(n+1)^2$ where n $\ge$ 1, then the length of consecutive composite numbers is shorter than from n^2 to $(n+1)^2$ undoubtedly.

In addition, we need to define a terminology. Namely we use DPF (C_χ) to express the set of all distinct prime factors of any composite number C_χ. After that, we shall detail a straightaway way of doing to prove Grimm's conjecture thereinafter.

Everyone knows that each and every integer's point at the positive half line of the number axis expresses a positive integer, and that infinite many a distance between two adjacent integers' points equals one another.

First let us use symbol • to denote an integer's point, whether it is in a formulation or is at the positive half line of the number axis.

Then, the positive half line is marked with symbols of integers' points into its original status. Please, see also first illustration as follows.
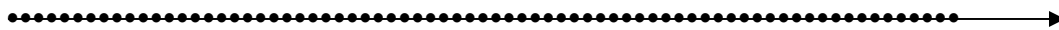
First Illustration

We use also symbol •s to denote at least two integers' points in formulations. Besides, thereinafter, the positive half line of the number axis is called the half line for short.

We consider smallest prime 2 as №1 prime, and consider prime P_χ as №χ prime, where $\chi \geq 1$. Then, prime 2 is written as P_1 as well.

And then, we consider positive integers which have common prime factor P_χ as №χ kind of integers.

Primes which P_χ expresses will be getting greater and greater along with which χ be getting greater and greater.

Since №χ kind's integers are infinitely many products which multiply

each and every positive integer by P_χ, thus there is one №χ kind's integer's point per consecutive P_χ integers' points at the half line.

When a positive integer contains many distinct prime factors, the positive integer must belong to each of the many kinds of integers concurrently.

There is unique prime P_χ within №χ kind's integers, yet let us regard others except for P_χ as №χ kind of composite numbers.

Since there are infinitely many primes, thus there are infinitely many kinds of composite numbers as well. Correspondingly there are infinitely many kinds of composite numbers' points at the half line.

By now, what we need is to find a law that take out distinct prime factors from consecutive composite numbers, so we are planning to start from stepwise forming consecutive composite numbers' points at the half line.

For this purpose, first let us define the symbol of integer's point after its prime/composite attribute is decided.

If an integer's point • at the half line is defined as a composite number's point, then change symbol ∘ for its original symbol •. Withal, use symbol ∘s to denote at least two composite numbers' points in formulations.

If an integer's point • at the half line is defined as a prime point, then change symbol ♣ for its original symbol •.

Next, we shall seriatim change ∘s for •s at places of №χ kind's composite numbers' points according as χ is from small to large, where $\chi \geq 1$.

By this token, one another's permutations of χ kinds' integers' points plus

other integers' points assume always infinite many recurrences on same pattern at the half line, irrespective of their prime/composite attribute. We seriatim resolve №χ kind of integers' points at the half line according as χ be from small to large. Please, see also below a schematic illustration.

1,2,3, 5, 7, 11,13, 17,19, 23,   29,31,   37, 41,43, 47,   53,   59,61,   67, 71,73,   79,
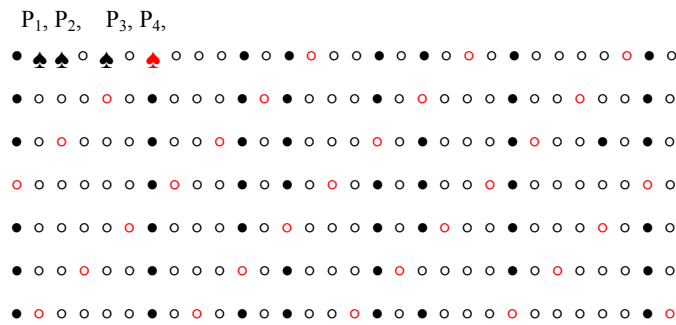

Second Illustration

Let us regard one another's equivalent shortest line segments at the half line according to same permutations of χ kinds' integers' points plus other integers' points as recurring line segments of permutations of the χ kinds of integers' points. And that use character string "RLS$_{№1\sim№\chi}$" to express a recurring line segment of $\sum$№χ kind of integers' points, where $\chi \geq 1$. Besides, at least two such recurring line segments are expressed by RLSS$_{№1\sim№\chi}$.

№1 RLS$_{№1\sim№\chi}$ begins with 1. Also, there are $\prod P_\chi$ integers' points at each of RLSS$_{№1\sim№\chi}$, where $\chi \geq 1$, and $\prod P_\chi = P_1 P_2 \dots P_\chi$. Thus one RLS$_{№1\sim№(\chi+1)}$ consists of consecutive $P_{\chi+1}$ RLSS$_{№1\sim№\chi}$, and that they link one by one. Evidently one another's permutations of χ kinds of integers' points plus other integers' points at each of RLSS$_{№1\sim№\chi}$ are one and the same.

Number the ordinals of integers' points at each of seriate $RLSS_{№1\sim№\chi}$ by consecutive natural numbers ≥ 1. Namely from left to right each and every integer's point at each of seriate $RLSS_{№1\sim№\chi}$ is marked with from small to great a natural number ≥ 1.

Then, there is one $№(\chi+1)$ kind's integer's point within $P_{\chi+1}$ integers' points which share an ordinal at $P_{\chi+1}$ $RLSS_{№1\sim№\chi}$ of any $RLS_{№1\sim№\chi+1}$.

Moreover, there is one $№(\chi+1)$ kind's composite number's point within $P_{\chi+y}$ integers' points which share an ordinal at $P_{\chi+1}$ $RLSS_{№1\sim№\chi}$ of each of seriate $RLSS_{№1\sim№\chi+1}$ on the right side of $№1$ $RLS_{№1\sim№\chi+1}$.

Prime points $P_1, P_2 \ldots P_{\chi-1}$ and P_χ are at $№1$ $RLS_{№1\sim№\chi}$. Yet, there are χ composite numbers' points at places of ordinals of $P_1, P_2, P_3\ldots$ and P_χ at each of seriate $RLSS_{№1\sim№\chi}$ on the right side of $№1$ $RLS_{№1\sim№\chi}$. Thus $№1$ $RLS_{№1\sim№\chi}$ is a special $RLS_{№1\sim№\chi}$ as compared with each of others.

For example, the one another's permutation of many kinds of integers' points at $№1$ $RLS_{№1\sim№4}$, please, see also following third illustration.

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2,3, 5, 7,   11,13, 17,19,   23,   29,31,   37, 41,43,   47,   53,   59,61,   67,   71,

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73,   79,   83,   89,   97, 101,103,107,109,113,   127, 131, 137,139,

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149,151,157,   163, 167,   173,   179,181,   191,193, 197,199,   210

Third Illustration
$№1$ $RLS_{№1}$ ends with integer's point 2; $№1$ $RLS_{№1\sim№2}$ ends with integer's point 6; $№1$ $RLS_{№1\sim№3}$ ends with integer's point 30; $№1$ $RLS_{№1\sim№4}$ ends with odd point 210.

But, for such a seriation, we are difficult to distinguish differentiae of №1 $RLS_{№1\sim№\chi}$ in comparison with each of $RLSS_{№1\sim№\chi}$ on the right side of №1 $RLS_{№1\sim№\chi}$, likewise difficult to find out the same of two $RLSS_{№1\sim№\chi}$. Thus, $P_{\chi+1}$ $RLSS_{№1\sim№\chi}$ of one $RLS_{№1\sim№(\chi+1)}$ may be folded at an illustration, for example, P_4 $RLSS_{№1\sim№3}$ of №1 $RLS_{№1\sim№4}$ and P_4 $RLSS_{№1\sim№3}$ of №n $RLS_{№1\sim№4}$ are as listed below, where n ≥2.

$P_1, P_2, \quad P_3, P_4,$

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At №1 $RLS_{№1\sim№4}$

№1,№2,№3,№4,

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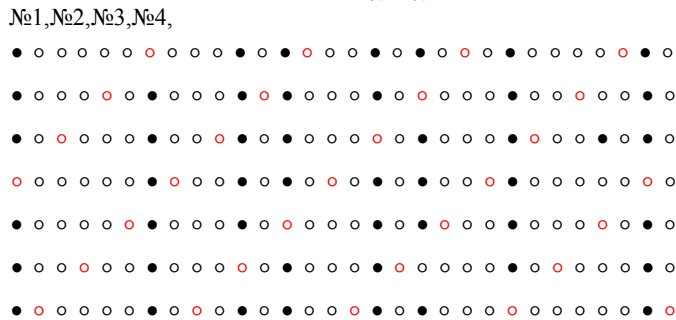
At №n $RLS_{№1\sim№4}$, where n≥2

Fourth Illustration

By now, we set about to prove Grimm's conjecture by stepwise forming consecutive composite numbers' points on the basis of the original half line, ut infra.

After change ○s for •s at places of №1 kind of composite numbers' points, there is one ○ at each of $RLSS_{№1}$ on the right side of №1 $RLS_{№1}$. That is to say, there is one ○ between adjacent two •s.

Thus a way of doing which takes out distinct prime factors from №1 kind of composite numbers is as follows.

Composite numbers like P_1^α is taken out prime factor P_1 as a distinct prime, where $\alpha \geq 2$;

Composite numbers like $P_1^\alpha P_2^\beta$ is taken out prime factor P_2 as a distinct prime, where α and $\beta \geq 1$;

Composite numbers like $P_1^\alpha P_2^\beta P_3^\eta$ is taken out prime factor P_3 as a distinct prime, where α, β and $\eta \geq 1$ etc.

Overall, for each and every №1 kind's composite number such as C_1, it is taken out greatest one within DPF (C_1) as a distinct prime.

Since there is one №2 kind's composite number's point in P_2 integers' points which share an ordinal at P_2 RLSS$_{№1}$ of any RLS$_{№1～№2}$, therefore, after successively change ∘s for •s at places of №2 kind of composite numbers' points, it is coincided one №2 kind's composite number's point on the neighborhoods and itself of №1 kind's composite number's point, thus there are one ∘ or two ∘s between adjacent two •s.

Then, each and every new increased №2 kind's composite number's point contains at least two identical or distinct prime factors $\geq P_2$.

Yet each and every №2 kind's composite number's point which contains prime factor P_1 is merely a composite number's point which repeats one №1 kind's composite number's point, hence we ignore them.

Thereupon, a way of doing which takes out distinct prime factors from

new- increased №2 kind of composite numbers is as follows.

Composite numbers like P_2^β is taken out prime factor P_2 as a distinct prime, where $\beta \geq 2$;

Composite numbers like $P_2^\beta P_3^\eta$ is taken out prime factor P_3 as a distinct prime, where β and $\eta \geq 1$;

Composite numbers like $P_2^\beta P_3^\eta P_4^\mu$ is taken out prime factor P_4 as a distinct prime, where β, η and $\mu \geq 1$, etc.

Overall, for each and every new- increased №2 kind's composite number such as C_2, it is taken out greatest one within DPF (C_2) as a distinct prime. The others may be deduced by analogy.

To sum up, after successively change ∘s for •s at places of №χ kind of composite numbers' points, composite numbers like P_χ^α is taken out prime factor P_χ as a distinct prime, where $\alpha \geq 2$; for other new- increased each and every №χ kind's composite number such as C_χ, it is taken out greatest one within DPF (C_χ) as a distinct prime.

Of course, for the aforesaid way of doing that takes out a distinct prime factor from any composite number, we can summarize those into a sentence. Namely for any composite number C_χ including P_χ^α, it is always taken out greatest one within DPF (C_χ) as a distinct prime.

Now that begin with $\chi=2$, after successively change ∘s for •s at places of №χ kind of composite numbers' points, each and every composite number which contains both P_χ and at least one prime factor$<P_\chi$ is a

repetitive composite number, then prime factors which each new-increased №χ kind's composite number contains are all greater than or equal to P_χ.

As previously mentioned that the length of consecutive composite numbers is shorter than from n^2 to $(n+1)^2$, then when $n=P_\chi$, it has $n^2= P_\chi^2$ and $(n+1)^2 =(P_\chi+1)^2 < P_\chi^3$, thus the length of consecutive composite numbers is shorter than from P_χ^2 to P_χ^3. Without doubt, the length of consecutive composite numbers is shorter than from P_χ^α to $P_\chi^{\alpha+1}$, because

$P_\chi^{\alpha+1}- P_\chi^\alpha= P_\chi^\alpha (P_\chi-1) \geq P_\chi^3- P_\chi^2 = P_\chi^2(P_\chi-1)$, where $\alpha \geq 2$.

Now that the length of consecutive composite numbers is shorter than from P_χ^α to $P_\chi^{\alpha+1}$, therefore, for composite numbers' points which belong to identical a kind, no matter which positive integer that χ equals, composite numbers' points P_χ^α and $P_\chi^{\alpha+1}$ coexist not within a strand of consecutive composite numbers' points, though they are taken out identical a prime factor P_χ respectively as several distinct prime.

Since a distinct prime factor which each and every new- increased composite number's point is taken out is greater than each of pre-existing distinct prime factors, therefore, for consecutive composite numbers' points which belong within different kinds, they can only be taken out one another's- distinct prime factors as distinct primes.

To sum up, pursuant to the aforesaid way of doing took out prime factors from consecutive composite numbers, one and each of others are no alike.

After change ∘s for •s at places of №χ kind of composite numbers' points where χ ≥2, strands of consecutive composite numbers' points on the left side of P_χ were fixed. That is to say, every strand of consecutive composite numbers' points on the left side of P_χ will increase not any composite number's point any longer.

For the aforesaid way of doing which takes out distinct prime factors from consecutive composite numbers, let us illustrate with an example.

For example, integers between prime 1327 and prime 1361 all are composite numbers, then, we orderly find out their distinct prime factors as distinct primes according to the aforesaid way of doing.

№1 kind of composite numbers between 1327 and 1361, all are even numbers. They and resolutions of prime factors thereof are as follows.

$1328=2^4×83$, $1330=2×5×7×19$, $1332=2^2×3^2×37$, $1334=2×23×29$, $1336=2^3×167$, $1338=2×3×223$, $1340=2^2×5×67$, $1342=2×11×61$, $1344=2^6×3×7$, $1346=2×673$, $1348=2^2×337$, $1350=2×3^3×5^2$, $1352=2^3×13^2$, $1354=2×677$, $1356=2^2×3×113$, $1358=2×7×97$ and $1360=2^4×5×17$.

Distinct prime factors which №1 kind's composite numbers are taken out are: 83, 19, 37, 29, 167, 223, 67, 61, 7, 673, 337, 5, 13, 677, 113, 97 and 17 in order of above-listed even composite numbers.

New-increased №2 kind's composite numbers between 1327 and 1361 and resolutions of prime factors thereof are: $1329=3×443$, $1335=3×5×89$, $1341=3^2×149$, $1347=3×449$, $1353=3×11×41$ and $1359=3^2×151$.

Distinct prime factors which new-increased №2 kind's composite numbers are taken out are: 443, 89, 149, 449, 41 and 151 in order of the six odd composite numbers.

New-increased №3, №4, №5, №6, №7, №8, №9 and №11 kinds' composite numbers between 1327 and 1361 and resolutions of prime factors thereof are: 1345=5×269, 1355=5×271, 1337=7×191, 1351=7×193, 1331=11^3, 1339=13×103, 1343=17×79, 1349=19×71, 1357= 23×59 and 1333=31×43.

Distinct prime factors which the new-increased eight kinds' composite numbers are taken out are: 269, 271, 191, 193, 11, 103, 79, 71, 59 and 31 in order of above-listed ten odd composite numbers.

In order to watch conveniently such a cluster of distinct prime factors, we renewedly arrange them in order of from small to large: 5, 7, 11, 13, 17, 19, 29, 31, 37, 41, 59, 61, 67, 71, 79, 83, 89, 97, 103, 113, 149, 151, 167, 191, 193, 223, 269, 271, 337, 443, 449, 673 and 677. Evidently, there are not at least two with the same in such a cluster of primes.

Thus it can be seen, use such a way of doing be able to get one another's-distinct prime factors for consecutive composite numbers.

In other words, consecutive composite numbers are able to be divided exactly by one another's- distinct primes respectively.

The proof was thus brought to a close. As a consequence, Grimm's conjecture is tenable.

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