Electromagnetic Time Dilation and Contraction, and a Geometrodynamic Foundation of Classical and Quantum Electrodynamics

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Abstract: We summarize how the Lorentz Force motion observed in classical electrodynamics may be understood as geodesic motion derived by minimizing the variation of the proper time along the worldlines of test charges in external potentials, while the spacetime metric remains invariant under, and all other fields in spacetime remain independent of, any rescaling, i.e., re-gauging of the charge-to-mass ratio q/m. In order for this to occur, time is dilated or contracted due to repulsive and attractive electromagnetic interactions respectively, in very much the same way that time is dilated due to relative motion in special relativity and due to gravitational fields in general relativity, without contradicting the well-corroborated experimental content of standard electrodynamic theory and both special and general relativity. As such, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics in four spacetime dimensions, in which mechanical motions and objects are merely promoted into canonical motions and objects in accordance with well-established local symmetry principles. Further, when we consider the self-interactions of individual leptons understood to be responsible for the magnetic moment anomalies, and upon identifying a universal relation between time and energy whereby all forms of energy dilate (or contract) time regardless of their kinetic or interaction origin, it is shown how these magnetic moment anomalies which are quintessential hallmarks of quantum field theory, both measure and empirically validate electromagnetic time dilation, and are a direct and immediate consequence of local abelian and non-abelian gauge symmetries.

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Contents

1. Motivation, Purpose and Historical Background

The equation of motion for a test particle along a geodesic line in curved spacetime specified by the metric interval $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ with metric tensor $g_{\mu\nu}$ was first obtained by Albert Einstein in §9 of his landmark 1915 paper [1] introducing the General Theory of Relativity. The infinitesimal linear element $d\tau = ds / c$ for the proper time is a scalar invariant which is independent of the chosen system of coordinates. Likewise the finite proper time $\tau = \int_A^B d\tau$ measured along the worldline of the test particle between two spacetime events *A* and *B* has an invariant meaning independent of the choice of coordinates. Specifically, the geodesic of motion is stationary, and results from a minimization of the variational equation

$$0 = \delta \int_A^B d\tau . \tag{1.1}$$

After carrying out the well-known calculation originally given by Einstein in [1], the particle's equation of geodesic motion is found to be:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu , \tag{1.2}$$

with Christoffel connection defined (denoted "$\equiv$") by $-\Gamma^\beta{}_{\mu\nu} \equiv \tfrac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$ and the relativistic four-velocity by $u^\mu \equiv dx^\mu / d\tau$.

The geodesic (1.2) can also be viewed in alternative, yet equivalent way. In curved spacetime, $DB^\beta / D\tau \equiv \left(\partial x^\nu / \partial \tau \right) \partial_{;\nu} B^\beta$ defines the "derivative along the curve" for any contravariant vector B^β, using gravitationally-covariant derivatives $\partial_{;\nu} B^\beta = \partial_\nu B^\beta + \Gamma^\beta{}_{\sigma\nu} B^\sigma$ and the chain rule. So when $B^\beta = u^\beta$, then, in view of (1.2), we may also write:

$$\frac{Du^\beta}{D\tau} = \frac{\partial x^\alpha}{\partial \tau} \partial_{;\alpha} u^\beta = \frac{\partial x^\alpha}{\partial \tau} \left(\partial_\alpha u^\beta + \Gamma^\beta{}_{\sigma\alpha} u^\sigma \right) = \frac{\partial x^\alpha}{\partial \tau} \left(\frac{\partial}{\partial x^\alpha} \frac{dx^\beta}{d\tau} + \Gamma^\beta{}_{\sigma\alpha} u^\sigma \right) = \frac{du^\beta}{d\tau} + \Gamma^\beta{}_{\mu\nu} u^\mu u^\nu = 0 . \tag{1.3}$$

This has exactly the same content as the geodesic equation (1.2). But given that $du^\beta / d\tau = 0$ describes Newtonian inertial motion when the gravitational connection $\Gamma^\beta{}_{\mu\nu} = 0$, we may think of $Du^\beta / D\tau = 0$ above as describing *covariantly-inertial* motion in the presence of gravitation. This is what gives gravitational geodesics their colloquial characterization as "straight lines," or more precisely, "inertial lines" in curved spacetime.

Just as ordinary derivatives $\partial_\alpha = \left(\partial / \partial t, \nabla \right)$ are replaced by gravitationally-covariant derivatives $\partial_{;\alpha}$ in curved spacetime, so too in gauge theory ordinary derivatives ∂_α are replaced by gauge-covariant or "canonical" derivatives $\mathfrak{D}_\alpha \equiv \partial_\alpha - iqA_\alpha$, where q is the electric charge strength and A_α is the gauge field / vector potential, and where we use $\mathfrak{D}_\alpha$ rather than the often-employed D_α to distinguish symbolically from the *D* of gravitational motion in (1.3). Motivated by the geodesic nature of gravitationally-covariant motion for which $Du^\beta / D\tau = 0$ rather than $du^\beta / d\tau = 0$ and how this motion stems directly from the replacement of ordinary with gravitationally-covariant derivatives, the purpose of this paper is to summarize how electrodynamic Lorentz Force motion is likewise geodesic motion which is *canonically-inertial* described by $\mathfrak{D}u^\beta / \mathfrak{D}\tau = 0$, which stems directly from the canonical derivatives of gauge theory. As will be shown, this comes about as a consequence of heretofore unrecognized time dilations and contractions which occur any time two material bodies are electromagnetically interacting. It will also be shown how in quantum electrodynamics, these time dilations directly give rise to the observed lepton magnetic moment anomalies.

Finding a geometrodynamic foundation for electrodynamics limited to four spacetime dimensions has been of great interest yet defied solution for almost a century. The Special Theory of Relativity [2] together with Minkowski's famous proclamation [3] that "from now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will still stand independently on its own," first established the geometric unification of space and time that now underlies all of physics. With the General Theory [1], Einstein soon thereafter applied Riemannian geometry to introduce curvature to spacetime and found that gravitation including motion in a gravitational field could be fully explained on this entirely geometric foundation, giving birth to what Wheeler would later coin as "geometrodynamics." [4]

After the General Theory established that the Riemann curvature was simply a measurement $R^\alpha{}_{\beta\mu\nu} B_\alpha = \left[\partial_{;\nu}, \partial_{;\mu}\right] B_\beta$ of degree to which derivatives at any given spacetime event in are non-commuting when operating on any four-vector B_β, it was natural to try to explain electrodynamics in a similar way based on spacetime curvature. Hermann Weyl's gauge theory – which will be central to this paper – is perhaps the most important of these efforts, and has become the foundation for our modern understanding not only of electrodynamics, but also of the weak and strong interactions which are non-abelian extensions of electrodynamics. Although "gauge" is a historical misnomer from when Weyl first tried unsuccessfully in [5], [6] to explain electrodynamics by imposing a symmetry under local *gauge* transformations $\psi \to \psi' = e^{\Lambda(t,\mathbf{x})}\psi$ *rescaling the magnitude* of a wavefunction, Weyl did eventually find, correctly in [7], that electrodynamics is indeed the natural consequence of imposing a local *phase* symmetry under a *magnitude-preserving redirection* $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ of the wavefunction in a complex two-dimensional phase space established by the parameter $e^{i\Lambda} = \cos\Lambda + i\sin\Lambda$. Apropos to curvature, this "gauge" theory established that the electromagnetic field strength bivector $F^{\mu\nu}$ – like $R^\alpha{}_{\beta\mu\nu}$ – measures the degree to which gauge-covariant derivatives $\mathfrak{D}_\alpha \equiv \partial_\alpha - iqA_\alpha$ were non-commuting. This is why $F^{\mu\nu}$ is often referred to as the "curvature" tensor. However, the field strength only bears an imaginary relation $qF^{\mu\nu}\phi = i\left[\mathfrak{D}^\mu, \mathfrak{D}^\nu\right]\phi$ to the gauge-covariant derivatives, and so this is not a *real* curvature as is that of $R^\alpha{}_{\beta\mu\nu}$. Indeed, it was because the incorrect re-gauging of the wavefunction allowed this curvature to be real like the curvature of Riemann, that Weyl adhered so long to a "gauge" rather than a "phase" transformation. But this was not in accord with observed natural reality.

During the same era when Weyl was developing gauge theory, Kaluza [8] and Klein [9] did succeed in explaining the Lorentz force law as a type of geodesic motion owing to spacetime curvature that remained real, and even gave a geometric explanation for the electric charge itself. But this came at the cost of adding a fifth dimension to spacetime and curling that dimension into a very tiny cylinder. While theoretically-attractive as a geometrodynamic theory, Kaluza-Klein has not become universally accepted because it relies on a fifth dimension which does not appear to have been observed and likely never could be observed. For his part, Einstein also pursued a geometrodynamic theory of electrodynamics until the end of his life, but he too was never fully satisfied with his or anybody else's results.

To date, a century later, finding a geometrodynamic foundation for even classical – much less quantum – electrodynamics remains elusive, and there certainly is no theory of electromagnetism which rises to the level of pure geometry embodied in either the Special or the General Theories of Relativity. Using settled and accepted gauge theory as a foundation, the goal of this paper is to bring a century of work pursuing a geometrodynamic foundation for electrodynamics to a successful conclusion, by achieving for electrodynamics, the pure geometrization that the Special Theory of Relativity achieved for relative motion and the General Theory of Relativity achieved for gravitation.

2. A Brief Note about Signs and Sign Conventions

The dilation and contraction of time whenever a charged body is placed into an electromagnetic potential and the connection of this to electromagnetic interaction energies and to the lepton magnetic moment anomalies will be a fundamental finding of this paper. But because electromagnetic interactions can be attractive or repulsive unlike gravitation which is always attractive, a fundamental question will arise whether for Coulomb interactions between two charges, electrodynamic time dilation occurs between *like and therefore repelling charges*, or between *opposite and therefore attracting charges*. Note that one or the other but not both of these possibilities could be true, because two like-electrical-charges repel while two like-gravitational-charges (masses) attract. While one may have a preconception about which of these possibilities is true (we will find that time dilates from the interaction of two like-thus-repelling charges and so contracts for electrical attraction between unlike charges), the answer to this question depends upon, and can only be answered definitively by, whether certain interaction and energy signs are positive or negative, and by how these signs enter into the overall theoretical development. So before we begin, it is important to take a moment to review certain sign conventions and requirements.

In natural units $c=1$, the Lorentz force law which we shall study here at length, is $du^\beta / d\tau = +(q/m) F^\beta{}_\sigma u^\sigma$. Specifically, if one adopts a sign convention in which a test charge q in the mixed electromagnetic field $F^\beta{}_\sigma = g_{\sigma\alpha} F^{\beta\alpha}$ is taken to be positive and the proper potential ϕ_0 of the gauge field $A^\alpha = (\phi, \mathbf{A})$ in the field strength $F^{\beta\alpha} = \partial^\beta A^\alpha - \partial^\alpha A^\beta$ is also taken to be positive, then when using a timelike metric signature $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1)$ in the flat spacetime limit $g_{\mu\nu} = \eta_{\mu\nu}$ this Lorentz force law requires a positive overall sign. We may see this using a Coulomb interaction at rest, as follows:

Combining all of the foregoing we may write $du^\beta / d\tau = +(q/m)\left(\eta_{\sigma\alpha}\eta^{\beta\tau}\partial_\tau A^\alpha - \partial_\sigma A^\beta\right)u^\sigma$ for the Lorentz force. At rest we may set $u^k = 0$ and $A^\alpha = (\phi, \mathbf{A}) = (\phi_0, \mathbf{0})$, so for the space components we obtain $du^k / d\tau = +(q/m)\eta_{00}\eta^{k\tau}\partial_\tau \phi_0 u^0$. Then we take $\phi_0 = +k_e Q / r$ to be a positive proper Coulomb potential for a positive source charge *Q*, where $k_e = 1/4\pi\varepsilon_0 = c^2\mu_0 / 4\pi$ is Coulomb's constant. So if we place the positive *q* into this potential the electrostatic interaction energy will be $\phi_0 q = +k_e Qq / r$, which grows smaller as the separation *r* between the two positive charges is increased. Because the test charge will naturally tend toward a lower energy over a higher energy, this tells us that this interaction is repulsive. Consequently, commensurately, we must have a positively-signed $du^k / d\tau > 0$ for the acceleration, with a vector direction pointing toward greater space separation, emerge from the Lorentz force law. So let us make sure it does.

If, for example, we align the radial separation along the *z* axis so that $(x, y, z) = (0, 0, r)$, then along this radius the Lorentz force law will yield $du^3 / d\tau = +(q/m)\eta_{00}\eta^{33}\partial_3\phi_0 u^0 = -(1/m)\eta_{00}\eta^{33}u^0\left(k_e Qq / r^2\right)$, via $\partial_3(1/r) = -1/r^2$. Now, at this juncture, we will have $\eta_{00}\eta^{33} = -1$ irrespective of whether we choose a timelike signature $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1)$ or a spacelike signature $\mathrm{diag}(\eta_{\mu\nu}) = (-1,+1,+1,+1)$ for the Minkowski metric tensor, because $\eta_{00}\eta^{33} = -1$ either way. So in either case, the Lorenz force law will reduce to $du^3 / d\tau = (1/m)\left(k_e Qq / r^2\right)u^0$, with the sign now boiling down to that of $u^0 = dt / d\tau$. With this $(x, y, z) = (0, 0, r)$ alignment, the required repulsive result now becomes $du^3 / d\tau > 0$.

Now we must choose our metric tensor signature and use that consistently throughout. In all cases, the flat spacetime line element is $ds^2 = c^2 d\tau^2 = \eta_{\mu\nu}dx^u dx^\nu$. For a timelike signature with $\delta_{jk}dx^j dx^k = dx^2 + dy^2 + dz^2 = dr^2$, the time interval $ds^2 = c^2 d\tau^2 = \left(c^2 dt^2 - dr^2\right)$. If we are studying the time evolution of material bodies moving within the light cone, then it is generally preferable to use a timelike signature, because at rest, with $dx^k = 0$, the metric reduces to $d\tau^2 = dt^2$. Then, while we still have the choice of setting $d\tau / dt = \pm 1$ upon taking the square root, it makes no sense to align $d\tau$ other than with dt so that these both measure the same time progression, and we therefore set $d\tau / dt = 1$. On the other hand, if we are studying two spacelike-separated events outside the light cone at the same coordinate time, then it is preferable to use a spacelike signature, because at the same time coordinate, with $dt = 0$, the metric reduces to $ds^2 = dr^2$. Then, although we may choose either of $ds = \pm dr$ when taking the square root, we likewise align the coordinate length with the proper length so $dr / ds = 1$ and these both measure the same length in the same direction. But it will also be seen that were we to choose a spacelike signature, then at rest the metric would reduce to $d\tau^2 = -dt^2$, and that one would then have $d\tau = \pm i dt$, requiring the Lorenz force to have an imaginary $(q/m)F^{\beta}{}_{\sigma}u^{\sigma}$ term. So though optional, it is preferred to choose a timelike signature. Once we do so, all else must be done consistently with this.

So to keep all terms real in the Lorentz force, we choose a timelike signature, which means that after alignment $d\tau / dt = 1$, so that the Lorentz acceleration given all of the conventions recited here finally becomes $du^3 / d\tau = (1/m)\left(k_e Qq / r^2\right)$. For the $(x, y, z) = (0, 0, r)$ alignment, this is a radial acceleration $d^2 r / d\tau^2 = (1/m)\left(k_e Qq / r^2\right)$. Given that Q and q are both positive, this yields the required repulsive motion $d^2 r / d\tau^2 > 0$, properly corresponding with the repulsive interaction energy $\phi_0 q = +k_e Qq / r$ that lessens with increased separation between two positive (like-signed) charges Q and q. Obviously, if either Q or q but not both is a negative charge, then the interaction energy will go over to $\phi_0 q \to -k_e Qq / r$, growing smaller with reduced charge separation, while the motion becomes the attractive $d^2 r / d\tau^2 < 0$. Both of these are consistent with electrical attraction between unlike charges. Finally, because this will become important in the development to follow, we again note the widely-known and deeply fundamental empirical fact that for gravitation like charges attract, while for electromagnetism like charges repel.

PART I: CLASSICAL GEOMETRO-ELECTRODYNAMICS

3. Geometro-Electrodynamics and Time Dilation and Contraction: An Overview

To begin development, if a test particle, to which we now ascribe a mass $m > 0$, also has a non-zero net electrical charge $q \neq 0$ and the region of spacetime in which it subsists also has a nonzero electromagnetic field strength $F^{\beta\alpha} \neq 0$, then the equation of motion is no longer given by (1.2), but is supplemented by an additional term which contains the Lorentz Force law, namely, with a positive sign for the reasons and with the sign conventions already discussed:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^{\beta}{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha}\frac{dx^\sigma}{c d\tau} = -\Gamma^{\beta}{}_{\mu\nu} u^\mu u^\nu + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha}\frac{u^\sigma}{c}. \tag{3.1}$$

In the above, the field strength $F^{\beta\alpha}$ containing the electric and magnetic field bivectors **E** and **B** is defined as usual by $F^{\beta\alpha} \equiv \partial^\beta A^\alpha - \partial^\alpha A^\beta$ in relation to the gauge potential four-vector A^α. The above force law is of course a well-known, well-corroborated, well-established law of physics.

Given that the gravitational geodesic (1.2) is derived from the variational equation (1.1), the question arises whether there is a way to obtain (3.1) from the same variation as in (1.1), thus revealing electrodynamic motion to also entail particles moving along geodesic paths in four spacetime dimensions. Conceptually, it cannot be argued other than that this would be a desirable state of affairs. But physically the difficulty rests in how to accomplish this without ruining the integrity of the metric and the background fields in spacetime by making them a function of the charge-to-mass ratio q/m. This ratio is and must remain a characteristic of the test particle alone. It is not and cannot be a characteristic of the line element $d\tau$, or the metric tensor $g_{\mu\nu}$, or the gauge field A^α, or the field strength $F^{\beta\alpha}$ which define the field-theoretical spacetime background through which the test particle is moving. And, at bottom, this difficulty springs from the *inequivalence* of the "electrical mass" (a.k.a. charge) *q* and the inertial mass *m*, versus the Newtonian equivalence of gravitational and inertial mass. In (3.1), this is captured by the fact that *m* does *not* appear in the gravitational term $-\Gamma^\beta{}_{\mu\nu}u^\mu u^\nu$, while the q/m ratio *does* appear in the electrodynamic Lorentz Force term that we rewrite as $(q/m)F^\beta{}_\sigma u^\sigma$ in natural units with $c=1$.

This difficulty may also be seen very simply if we compare Newton's law with Coulomb's law. In the former case we start with a force $F=-GMm/r^2$ (with the minus sign indicating that gravitation is attractive) and in the latter $F=-k_e Qq/r^2$ (for which we choose an attractive interaction to provide a direct comparison to gravitation), where *G* is Newton's gravitational constant and the analogous k_e is Coulomb's constant. If the gravitational field is taken to stem from mass *M* and the electrical field from charge *Q*, then the test particle in those fields has gravitational mass *m* and electrical mass *q*. But the Newtonian force $F=ma$ always contains the inertial mass *m*. So in the former case, because the gravitational and inertial mass are equivalent, the acceleration $a=F/m=-GMm/mr^2=-GM/r^2$ and these two masses cancel, giving $-\Gamma^\beta{}_{\mu\nu}u^\mu u^\nu$ without any mass in (3.1). But in the latter case the acceleration $a=F/m=-k_e Qq/mr^2=-(q/m)k_e Q/r^2$ because the electrical and inertial masses are not equivalent, hence $(q/m)F^\beta{}_\sigma u^\sigma$ containing this same ratio in (3.1). Here, the motion is distinctly dependent on the electrical and inertial masses *q* and *m* of the test particle. And as a result, different charges *q* with different masses *m* starting with the exact same initial velocity at the exact same position may all be moving through the exact same background fields and yet have different observable motions.

So, were we to pursue the conceptually-attractive goal of understanding electrodynamic motion as the result of particles moving through spacetime along geodesic paths, with the variational equation (1.1) applying to electrodynamic motion just as it does to gravitational motion, the line element $d\tau$ would inescapably have to be a function $d\tau(q/m)$ of q/m. This in turn would *appear* to violate the integrity of the line element $d\tau$ as well as the metric tensor $g_{\mu\nu}$ in $c^2d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$, because these would all *seem to be* dependent upon the attributes *q* and *m* of the test particles that are moving through the spacetime background. Were this to be reality and not just seeming appearance, this would be physically impermissible.

Consequently, despite there being many known derivations of the Lorentz Force law, there does not, to date, appear to be an acceptable rooting of the Lorentz Force law in the variational equation $0=\delta\int_A^B d\tau$ which would reveal electrodynamic motion to be geodesic motion just like the familiar gravitational motion. And this is because it has not been understood how to obtain electrodynamic motion from a minimized variation while simultaneously maintaining the integrity of the field theory such that the metric and the background fields do not depend upon the attributes of the test particles which may move through these fields. This, in turn, is because electrical mass is not equivalent to the inertial mass, which causes different test particles to move differently even when they start out with the exact same positions and motions in the exact same background fields, in contrast to the Newtonian equivalence of gravitational and inertial mass from which all particles respond identically in the same gravitational background.

So, when a first test particle with electrical mass q and inertial mass m is placed in a field $F^{\beta\alpha}$, and a second test particle with electrical mass q' and inertial mass m' of a different ratio $q'/m' \neq q/m$ is placed at equipotential in the same field $F^{\beta\alpha}$ with the same initial conditions for each, there are observably-different Lorentz Force motions for these two different test charges even though they are at equipotential. Were we to try to derive this motion from (1.1) the line element $d\tau$ would have to be a mathematical function $d\tau(q/m)$ of q/m, yet to maintain the integrity of the field theory the line element $d\tau$ would also have to be physically independent of q/m, which may seem paradoxical. Nevertheless, it is possible to have a line which is a function of q/m, from which the variational equation $0=\delta\int_A^B d\tau$ does yield the combined gravitational and electrodynamic equation of motion (3.1), yet for which the line element $d\tau$, the metric tensor $g_{\mu\nu}$, the gauge field A^α, and the electromagnetic field strength $F^{\beta\alpha}$ are all independent of this q/m ratio. Specifically, close study reveals that this paradox may be resolved by recognizing that as measured by periodic signals emitted by the test charges acting as geometrodynamic clocks, *time does not flow at the same rate for these two test charges in very much the same way that time does not flow at the same rate for two reference frames in special relativity which are in motion relative to one another*.

In particular, in the absence of gravitation with $g_{\mu\nu}=\eta_{\mu\nu}$ and $\Gamma^\beta{}_{\mu\nu}=0$, the first test particle will have a Lorentz motion given by:

$$\frac{d^2x^\beta}{d\tau^2}=\frac{q}{m}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^\sigma}{cd\tau}, \tag{3.2}$$

which also contains a set of coordinates x^σ. Now usually it is assumed that for the second test particle the motion is given by this same equation (3.2), merely with the substitution of $q \to q'$ and $m \to m'$; that is, by:

$$\frac{d^2x^\beta}{d\tau^2}=\frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^\sigma}{cd\tau}. \tag{3.3}$$

The particular assumption here is that there is no change in the measurement of time, i.e., the periodicity of emitted signals, when (3.2) is replaced with (3.3); and more generally the assumption is that the coordinate interval dx^σ in (3.3) is identical to the dx^σ in (3.2). Yet, it is impossible to have both (3.2) and (3.3) emerge through the variation $0=\delta\int_A^B d\tau$ from the same metric element

$d\tau$, and simultaneously maintain the integrity of the field theory, unless the coordinates are different, wherein dx^σ in (3.2) is *not identical* to what must now be $dx^\sigma \to dx'^\sigma \neq dx^\sigma$ in (3.3).

In fact, the very physics of having electric charges in electromagnetic fields induces a change in coordinates as between these two test charges with different $q'/m' \neq q/m$, very similar to the coordinate change via Lorentz transformations induced by relative motion. As a result, the electrodynamic motion of the second test charge is given, not by (3.3), but by:

$$\frac{d^2 x'^\beta}{d\tau^2} = \frac{q'}{m'} \eta_{\sigma\alpha} F^{\beta\alpha} \frac{dx'^\sigma}{cd\tau}. \tag{3.4}$$

Here, x^σ in (3.2) and $x'^\sigma \neq x^\sigma$ in (3.4), respectively, are two different sets of coordinates. Yet, they are interrelated by a definite transformation. Most importantly, this results in *time itself* being measured differently as between these two sets of coordinates, making time dilation and contraction as fundamental an aspect of electrodynamics, as it already is of the special relativistic theory of motion and the general relativistic theory of gravitation. In fact, what is really happening – physically – is that the placement of a charge in an electromagnetic field is *inducing a physically-observable change of coordinates* $x^\sigma(q/m) \to x'^\sigma(q'/m')$ in the very same way that relative motion between the coordinate systems $x^\sigma(v)$ and $x'^\sigma(v')$ of two different inertial reference frames with velocities *v* and v' induces a Lorentz transformation $x^\sigma(v) \to x'^\sigma(v')$ that relates the two coordinate systems to one another via $c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu(v) dx^\nu(v) = \eta_{\mu\nu} dx'^\mu(v') dx'^\nu(v')$, with an invariant line element $d\tau^2 = d\tau'^2$ and the same metric tensor $\eta_{\mu\nu} = \eta'_{\mu\nu}$ in either reference frame.

As it turns out, the line element that yields (3.1) from (1.1), including electrodynamic motion, which element is quadratic in $d\tau$, is:

$$c^2 d\tau^2 = g_{\mu\nu} \left(dx^\mu + \frac{q}{mc} d\tau A^\mu \right) \left(dx^\nu + \frac{q}{mc} d\tau A^\nu \right) = g_{\mu\nu} \mathfrak{D}x^\mu \mathfrak{D}x^\nu. \tag{3.5}$$

Above, we have defined a gauge-covariant coordinate interval $\mathfrak{D}x^\mu \equiv dx^\mu + (q/mc)\, d\tau A^\mu$, again with a canonical $\mathfrak{D}$ to distinguish from the gravitational *D* in (1.3). And it will be seen that upon multiplying through by $m^2/d\tau^2$ this becomes:

$$m^2c^2 = g_{\mu\nu} \left(m\frac{dx^\mu}{d\tau} + \frac{q}{c} A^\mu \right) \left(m\frac{dx^\nu}{d\tau} + \frac{q}{c} A^\nu \right) = g_{\mu\nu} \left(p^\mu + \frac{q}{c} A^\mu \right) \left(p^\nu + \frac{q}{c} A^\nu \right) = g_{\mu\nu} \pi^\mu \pi^\nu. \tag{3.6}$$

This, it will be recognized, is the usual relationship $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ between the rest mass *m* and canonical energy-momentum $\pi^\mu \equiv p^\mu + qA^\mu/c$, with the ordinary mechanical / kinetic energy-momentum continuing to be denoted by $p^\mu = mdx^\mu/d\tau$. To make certain there is no confusion, it is to be noted that some authors continue to use p^μ to denote the canonical momentum when there are charges and gauge fields present. We find it preferable to employ the different symbol π^μ to avert confusion. Insofar as terminology, we shall consistently refer to $p^\mu = m dx^\mu / d\tau$ as the *mechanical momentum*, and to $\pi^\mu \equiv p^\mu + qA^\mu / c$ as the *canonical momentum*. The gauge interval $\mathfrak{D}x^\mu \equiv dx^\mu + (q/mc) d\tau A^\mu$ defined in (3.5) is then seen to be merely a restatement of the gauge-covariant derivatives $\mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and canonical momenta $\pi^\mu \equiv p^\mu + qA^\mu / c$ which emerge from gauge theory and relate to one another via $i\partial_\sigma \Leftrightarrow p_\sigma$ and $i\mathfrak{D}_\sigma \Leftrightarrow \pi_\sigma$, and in particular, which emerge from the mandate for local gauge (really, phase) symmetry.

Now, the line element (3.5) is clearly a function of q/m and so has the *appearance* of depending on the ratio q/m. But this is only appearance. For, when we now place the second test charge with the second ratio $q'/m' \neq q/m$ in the exact same metric measured by the invariant line element $d\tau$ and moving through the exact same fields $g_{\mu\nu}$ and A^μ, this metric gives:

$$c^2 d\tau^2 = g_{\mu\nu} \left(dx'^\mu + \frac{q'}{m'c} d\tau A^\mu \right) \left(dx'^\nu + \frac{q'}{m'c} d\tau A^\nu \right) = g_{\mu\nu} \mathfrak{D}x'^\mu \mathfrak{D}x'^\nu , \tag{3.7}$$

with $\mathfrak{D}x'^\mu = dx'^\mu + (q'/m'c) d\tau A^\mu$. Most importantly, with $d\tau' = d\tau$ and $g'_{\mu\nu} = g_{\mu\nu}$ and $A'^\mu = A^\mu$ the metric and the background fields remain completely independent of the mass and charge of the test particle. So despite $d\tau$ being a function of the q/m ratio, this $d\tau = d\tau'$ as a measured proper time element is actually *invariant* with respect to the q/m ratio. To ensure this, *the differences between different q/m and q'/m' are entirely absorbed into the coordinate transformation $x^\mu \to x'^\mu$, which as we shall see is quite analogous to the Lorentz transformation of special relativity*. So the counterpart to (3.6) now becomes:

$$m'^2 c^2 = g_{\mu\nu} \left(m' \frac{dx'^\mu}{d\tau} + \frac{q'}{c} A^\mu \right) \left(m' \frac{dx'^\nu}{d\tau} + \frac{q'}{c} A^\nu \right) = g_{\mu\nu} \pi'^\mu \pi'^\nu , \tag{3.8}$$

with an invariant $d\tau$ and unchanged background fields $g_{\mu\nu}$ and A^μ in the face of different *m* and different *q* and different q/m.

In fact, this transformation $x^\mu \to x'^\mu$ is *defined* so as to keep $d\tau = d\tau'$ invariant, and $g_{\mu\nu} = g'_{\mu\nu}$ and $A^\mu = A'^\mu$ and by implication the field strength bivector $F^{\beta\alpha} = F'^{\beta\alpha}$ all unchanged, just as Lorentz transformations are defined so as to maintain a constant speed of light for all inertial reference frames independently of their state of motion. That is, combining (3.5) and (3.7), this transformation $x^\mu \to x'^\mu$ which results in time dilations and contractions, is *defined* by:

$$c^2 d\tau^2 = g_{\mu\nu} \left(dx^\mu + \frac{q}{mc} d\tau A^\mu \right) \left(dx^\nu + \frac{q}{mc} d\tau A^\nu \right) \equiv g_{\mu\nu} \left(dx'^\mu + \frac{q'}{m'c} d\tau A^\mu \right) \left(dx'^\nu + \frac{q'}{m'c} d\tau A^\nu \right) . \tag{3.9}$$

Consequently, $d\tau = d\tau'$ is a function of charge q and mass m yet is invariant with respect to the same, and there is no inconsistency. Likewise, the fields $g_{\mu\nu} = g'_{\mu\nu}$ and $A^{\mu} = A'^{\mu}$ are independent of the charge and the mass of the test particle, because again, everything stemming from the different q/m ratios is absorbed into a coordinate transformation $x^{\mu} \to x'^{\mu}$. Thus, while "gauge" is a historical misnomer for what is really invariance under local *phase* transformations $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ applied to a wavefunction ψ, we see in (3.9) that the line element $d\tau$ truly is invariant under what can be genuinely called a *re-gauging* of the q/m ratio. And from (3.6) and (3.8), we see that this symmetry is really not new. It is merely a restatement of the usual relationship $m^2c^2 = g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ between rest mass and canonical momentum. This this way, "gauge" theory really is a "gauge" theory, but it is the q/m ratio not the wavefunction that is re-gauged.

As a result, each and every different test particle carries its own coordinates, all interrelated so as to keep $d\tau$ invariant, and $g_{\mu\nu}$, A^{μ} and $F^{\beta\alpha}$ unchanged. The coordinate transformation interrelating all the test particles causes time $x^0 = t$ to *dilate for electrical repulsion between like-charges and to contract for electrical attraction between opposite charges*. For a test particle placed at rest into a background potential $A^{\mu} = (\phi, \mathbf{A}) = (\phi_0, \mathbf{0})$ where ϕ_0 is the proper potential, this time dilation or contraction is measured by a dimensionless ratio $dt/d\tau \equiv \gamma_{em}$ that integrally depends upon the magnitude of the likewise-dimensionless ratio $q\phi_0/mc^2$ of proper electromagnetic interaction energy $q\phi_0$ to the test particle's rest energy mc^2. This in turn supplements the ratio $dt/d\tau = \gamma_v = 1/\sqrt{1-v^2/c^2}$ for motion in special relativity and $dt/d\tau = \gamma_g = 1/\sqrt{g_{00}}$ for a clock at rest in a gravitational field, and assembles them into the overall product combination $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$ governing time dilation and contraction when all of motion and gravitational and electromagnetic interactions are present.

For $q\phi_0/mc^2 << 1$, and for a repulsive Coulomb force $F = k_e Qq/r^2$, the interaction energies $E_{em} = \int_r^{\infty} F_{em} \cdot dr = +k_e Qq/r$ (see (10.14) infra) which diminish with increased separation between the charges are related to these electromagnetic time dilations in a manner identical to how the kinetic energy $E_v = \frac{1}{2}mv^2$ is contained in $mc^2\gamma_v = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \frac{1}{2}mv^2$ for nonrelativistic velocities $v << c$ in special relativity ("$\cong$" symbol denotes approximate equality). In fact, the actual expression for the electromagnetic contribution to the time dilation is

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{1}{1-q\phi_0/mc^2} \cong 1 + q\phi_0/mc^2 . \tag{3.10}$$

also shown in the $q\phi_0/mc^2 << 1$ "weak" interaction approximation. This will be explicitly derived in (10.11) infra, from the transformation defined in (3.9). And for a Coulomb proper potential $\phi_0 = +k_e Q/r$ for a repulsive electrical interaction, this is $\gamma_{em} = 1/\left(1-k_e Qq/mc^2 r\right)$, see (10.12) infra. So the combined time dilation factor $dt / d\tau = \gamma_{em}\gamma_g\gamma_v$ mentioned earlier, employing the Schwarzschild metric with $g_{00} = 1 - 2GM / c^2 r$ thus the gravitational factor $\gamma_g = 1/\sqrt{g_{00}(r)} \cong 1 + GM / c^2 r$ in the weak field Newtonian limit (where the Reissner–Nordström metric term $Gk_e Q^2 / c^4 r^2$ may clearly be neglected), produces an overall energy which, in the low velocity, weak-gravitational and weak-electromagnetic interaction limit, is derived at (10.23) infra, namely:

$$\begin{aligned} E &= mc^2 \frac{dt}{d\tau} = mc^2 \gamma_{em}\gamma_g\gamma_v \cong mc^2 \left(1 + \frac{q}{m}\frac{k_e Q}{c^2 r}\right)\left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r}v^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r}v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r}v^2 \end{aligned} . \tag{3.11}$$

What we see here, in succession, are 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the Coulomb interaction energy of the charged mass, 4) the kinetic energy of the Coulomb energy, 5) the gravitational interaction energy of the mass, 6) the kinetic energy of the gravitational energy, 7) the gravitational energy of the Coulomb energy and 8) the kinetic energy of the gravitational energy of the Coulomb energy. It is clear that this accords entirely with empirical observations of the linear limits of these same energies.

Importantly, unlike gravitational redshifts or blueshifts which are a consequence of spacetime curvature, these electromagnetic time dilations *do not stem directly from curvature*. They only affect curvature indirectly through any changes in energy to which they give rise, because gravitation still "sees" all energy. Hermann Weyl's ill-fated attempt from 1918 until 1929 in [5], [6], [7] to base electrodynamics on *real* gravitational curvature in the same way as gravitation is via $R^{\alpha}{}_{\beta\mu\nu} B_\alpha = \left[\partial_{;\nu}, \partial_{;\mu}\right] B_\beta$ made clear that electrodynamics did not originate from *real* spacetime curvature in four dimensions. This is because Weyl's initial attempt was rooted in invariance under a non-unitary local transformation $\psi \to \psi' = e^{\Lambda(t,\mathbf{x})}\psi$ which re-gauges the magnitude of a wavefunction, rather than under the correct transformation $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ with an imaginary exponent that simply redirects the phase. Specifically, the latter correct phase transformation is associated with an *imaginary*, not real, curvature that places a factor $i = \sqrt{-1}$ into the geodesic deviation $D^2\xi^\mu / D\tau^2$ when expressed in terms of the commutativity of spacetime derivatives via $qF^{\mu\nu}\phi = i\left[\mathfrak{D}^\mu, \mathfrak{D}^\nu\right]\phi$. So at best, electrodynamics can be understood on the basis of a *mathematically-imaginary spacetime curvature*. Kaluza [8] and Klein [9] do of course provide an explanation based on real curvature, but at the cost of adding a fifth dimension. And so the time dilation and contraction that we suggest here to provide a four-dimensional geometrodynamic understanding of electrodynamics, is much more akin to the time dilation of special relativity than it is to the gravitational redshifts and blueshifts of general relativity. It may transpire entirely in flat spacetime, and real spacetime curvature only becomes implicated indirectly, when the energies added to mc^2 reach sufficient magnitude beyond their linear limits shown in (3.11) to curve the nearby spacetime.

Also importantly, the similarity of the ratios $q\phi_0 / mc^2$ and v^2 / c^2 as the driving number in $\gamma_{em} = 1/\left(1 - q\phi_0 / mc^2\right)$ and $\gamma_v = 1/\sqrt{1 - v^2 / c^2}$, respectively, is more than just an analogy. Just as $v < c$ (a.k.a. $mv^2 < mc^2$) is a fundamental limit on the motion of material subluminal particles, so too, it turns out that $q\phi_0 < mc^2$ is a material limit on the strength of the interaction energy between a test charge *q* with mass *m* interacting with the sources of the proper potential ϕ_0. This transpires by requiring particle and antiparticle energies to always be positive and time to always flow forward in accordance with Feynman-Stueckelberg. Further, it turns out that when $\phi_0 = k_e Q / r$ is the Coulomb potential whereby this limit becomes $k_e Qq / r < mc^2$ (a.k.a. $r > k_e Qq / mc^2$), we find that there is a lower physical limit on how close two interacting charges can get to one another, thereby solving the long-standing problem of how to circumvent the $r = 0$ singularity in Coulomb's law.

To be sure, these electromagnetic time dilations are miniscule for everyday electromagnetic interactions, as are special relativistic time dilations for everyday motion. So testing of $dt / d\tau$ changes for electrodynamics at the classical macroscopic level may perhaps be best pursued with experimental approaches similar to those used to test relativistic time dilations. As a very simple example to establish a numeric benchmark, consider two bodies with charges $Q = q = 1\text{ C}$ (Coulomb) separated by $r = 1\text{ m}$ (meter). In this event, the Coulomb interaction energy has a magnitude $k_e Qq / r = k_e = 1/4\pi\varepsilon_0 = 8.897 \times 10^9\text{ J}$ (Joules). Yet, if the test particle which we take to have the charge *q* has a rest mass $m = 1\text{ kg}$ (kilogram), then the electrodynamic time dilation factor contained in (3.11) is $\gamma_{em} \cong 1 + k_e / c^2 = 1 + \mu_0 / 4\pi = 1 + 10^{-7} = 1.0000001$. This is a very tiny time dilation for a tremendously energetic interaction. The release of this much energy per second would yield a power of approximately 8.897 GW (gigawatts), which roughly approximates seven or eight nuclear power plants, or roughly four times the power of the Hoover Dam, or the power output of a single space shuttle launch, or the power of about seventy five jet engines, or that of a single lightning bolt. For a special relativistic comparison, consider an airplane flying one mile in six seconds, versus light which travels a bit over one million miles in six seconds. Here, $v / c \cong 10^{-6}$ and the time dilation is $\gamma_v = 1/\sqrt{1 - v^2 / c^2} \cong 1.0000000000005$. So in fact the exemplary electrodynamic time dilation is substantially less miniscule than this exemplary special relativistic dilation. However in daily experience where one encounters watts and kilowatts not gigawatts, these time dilations would be of similar magnitude.

Experimentally, to test for these electromagnetic time dilations $\gamma_{em} = 1/\left(1 - k_e Qq / mc^2 r\right)$ embedded in (3.11), one would compare the detected periodicity of otherwise identical, synchronized geometrodynamic clocks or oscillators which are then electrically charged with different q / m ratios, and then placed at rest into the proper potential ϕ_0. Or more generally, these would be measured by electrically charging otherwise identical clocks and then placing them into the potential to have differing dimensionless $q\phi_0 / mc^2$ ratios, then measuring their relative oscillatory periods. Given that $k_e Qq / mc^2 r$ is a ratio of the electromagnetic interaction energy $k_e Qq / r$, to the total rest energy mc^2 of the test particle which is dominated by nuclear energy, the time dilation or contraction is seen to be driven by what may qualitatively be thought of as the ratio of the electromagnetic energy to nuclear energy of the test charge. This is why, for example, the benchmark ratio reviewed in the last paragraph is so very small even for such a large electromagnetic interaction of billions of Joules.

While such macroscopic experiments to detect this electromagnetic time dilation are certainly of interest, the question also arises whether there are *microscopic* experiments which can be performed upon individual test charges, such as the charged leptons, to detect this time dilation. As we shall study in depth in Part III of this paper, this time dilation is directly responsible for the lepton magnetic moment anomalies a which are well-known to arise from repulsive QED self-interactions internal to an individual lepton as characterized by Feynman loop diagrams. Moreover, those anomalies in provide direct, already-available empirical validation that time is dilated as a result of repulsive electromagnetic interactions. And, as to the question whether time dilates for repulsive interactions or for attractive ones which was introduced with the note about signs and sign conventions in section 2, the fact that the observed lepton *g*-factors $g = 2 + 2a > 2$ as opposed to $g = 2 - 2a < 2$ is a direct consequence of electromagnetic time dilation occurring for repulsive interactions between like charges, not attractive interactions between unlike charges.

One may erroneously conclude that the foregoing theoretical approach, if it is to have any observable consequences, must be a proposal for an alternative to standard electromagnetism, which of course has passed many experimental tests to very high precision (e.g., the magnetic moments of the electron and muon). But it is not an alternative; it is a non-contradictory supplement which, when applied to individual leptons, actually explains the lepton magnetic moment anomalies as a consequence of the time dilation that occurs because of repulsive lepton self-interactions in QED. *These result do not contradict known observations in any way*. Rather, all of the usual results of classical and quantum electrodynamics *including the observed anomalies* may be expressed in relation to the measurement of time as observed by comparing the periods of charged geometrodynamic clocks in a variety of circumstances, to which the energies are related by $E = mc^2 \left(dt / d\tau \right)$ in (3.11). So what becomes new – but is not contradictory to known observations in any way – is this generalized classical linkage between time and energy for what turns out to be all type of energy from all sources and origins, and the connection of the time dilation to the lepton magnetic moment anomalies.

In sum, to be able to obtain equation (3.1) for gravitational and electrodynamic motion from the minimized proper time variation (1.1) in a way that preserves the integrity of the metric and the background fields independently of the q / m ratio for a given test charge and thereby achieves the conceptually-attractive goal of understanding electrodynamic motion to be geodesic motion just like gravitational motion all in four spacetime dimensions, we are required to recognize that repulsive electrodynamic interactions inherently dilate and attractive electrodynamic interactions inherently contract time itself, *as an observable physical effect*. This is identical to how relative motion dilates time, and to how gravitational fields dilate (redshift) or contract (blueshift) time. In this way, it becomes possible to have a spacetime metric which – although a function of the electrical charge and inertial mass of test particles – also remains invariant with respect to those charges and masses and particularly with respect to a re-gauging of the charge-to-mass ratio. This preserves the integrity of the field theory, and establishes that electrodynamic motion is in fact geodesic motion which satisfies the minimized proper time variation $0=\delta\int_A^B d\tau$ from (1.1). Moreover, this connects observed energies of motion, and of gravitational and electrodynamic interactions, as well as the magnetic moment anomalies, all to the geometrodynamic measurement of time. As a result, it becomes possible to lay an entirely geometrodynamic foundation for classical and quantum electrodynamics in four spacetime dimensions.

In the next section we shall review in detail exactly how (3.1), which includes gravitational and electrodynamic motion, is deductively derived from minimizing the action (1.1) using the line element (3.5) and the related equation (3.6) for the canonical energy-momentum. As we shall see in (4.4), this derivation produces an additional term in the Lorentz force that is not gauge-invariant, and thus leaves an unobservable ambiguity in the physical motion. To address this, as reviewed in section 5, it is necessary to impose two conditions on the gauge field. The first condition fixes the gauge field to the Maxwell Lagrangian in lieu of the often-imposed Lorenz gauge, but still leaves some residual ambiguity in the gauge field. The second condition fixes the additional Lorentz force term to zero, thereby removing the remaining gauge ambiguity. Then, in section 6, we reformulate the former Lagrangian-based gauge condition in terms of the Maxwell action. In sections 7 and 8, respectively, we use these gauge conditions to uncover a covariant scalar equation for power, and a scalar field equation for energy flux, in the presence of both gravitational and electrodynamic interactions and sources. In essence, sections 4 through 8 directly explicate the derivation of the Lorentz force (3.1) from the minimized variation (1.1) and the immediate consequences of this in terms of required gauge fixing conditions and resulting power and energy flux equations. Section 9 then reviews how time dilation is derived in Special and General relativity, as the basis for showing in section 10 precisely how the time dilation and contraction summarized above, as well as the time / energy relation (3.11), are derived by simply requiring the metric line element must to invariant and the background fields in spacetime to remain unchanged, under a re-gauging of the electrodynamic charge-to-mass ratio q/m.

In Part III, starting with an introduction in section 11, we turn to Quantum Electrodynamics and the lepton magnetic moment anomalies. In Section 12 we first derive several ratio relationships between canonical and mechanical objects such as the canonical momentum $\pi^\mu = p^\mu + qA^\mu / c$ and the mechanical momentum p^μ. We then show how when applied to the repulsive Coulomb interaction between two bodies with charges equal to that of the charged leptons, separated by the Compton wavelength of the body taken to be the test charge, the electromagnetic time dilation factor $\gamma_{em} = dt/d\tau \cong 1+\alpha/2\pi$ comes to depend upon Schwinger's one-loop contribution $a_S = \alpha/2\pi$ to the lepton magnetic moment anomalies, see (12.7) infra. This raises the question whether these are in fact connected. Recognizing that because of Heisenberg uncertainty one may not really talk about the "separation" between two individual leptons or the position of a single lepton in other than a statistical way, section 13 carefully and systematically demonstrates leading to (13.16) infra, how the electrodynamic self-interaction contributions a_{QED} to the lepton magnetic moment anomalies really are an *exact* measure of the electromagnetic time dilation, such that $\gamma_{em} = dt/d\tau = 1 + a_{\text{QED}} = g_{\text{QED}}/2$ for each lepton type.

But the complete observed anomaly $a = a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$ also contains comparatively-small contributions from electroweak and hadronic self-interactions. Given that $E = mc^2 dt / d\tau$ by this point has been shown to apply to energies of motion and of gravitational interaction and of electromagnetic interaction, we surmise in section 14 that this must really be a generalized, universal connection between time and energy whereby times "sees" all energy just as gravitation "sees" all energy. In other words, any change in the total energy of a material body, irrespective of the nature or origin of that energy change, must have a commensurate effect on the rate at which periodic signals used to measure time are emitted when that body is used as a geometrodynamic clock. As a result, we are then able to account for the electroweak and hadronic contributions, leading in (14.1) infra to the relation $\gamma_{em} = dt / d\tau = 1 + a = g / 2$ between the time dilation and the complete magnetic moment anomalies from all contributing self-interactions. In section 15 we use the foregoing to calculate the bare masses of each of the thee charged leptons, and review how the Compton wavelengths of the leptons establish "statistical diameters" for the lepton probability densities which may well be empirically measurable as a means for providing experimental validation of these results. Section 16 casts the electromagnetic time dilation into a recursive form, and shows that this leads to a direct connection to the 2004 DeVries formula for the fine structure constant, which formula remains valid to date within experimental errors but has heretofore never been afforded a physical explanation. Finally, section 17 contains concluding remarks.

4. Derivation of Lorentz Force Geodesic Motion from Variation Minimization

The foundational calculation to derive (3.1) including the Lorentz force from the minimized variation (1.1) begins with the spacetime metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ which is multiplied through by m^2 and turned into the free particle energy-momentum relation $m^2 c^2 = g_{\mu\nu} p^\mu p^\nu$ containing the mechanical momentum $p^\mu = m dx^\mu / d\tau$. This in turn is readily turned into Dirac's $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ for a free electron in flat spacetime making use of $\eta^{\mu\nu} = \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\}$. Then, we simply use Weyl's well-known gauge prescription [7] which transforms the mechanical momentum to the canonical momentum $p^\mu \to \pi^\mu \equiv p^\mu + qA^\mu / c$ thus the energy-momentum relation to $m^2 c^2 = g_{\mu\nu} \pi^\mu \pi^\nu$ in (3.6), and the ordinary derivatives to gauge-covariant derivatives $\partial_\sigma \to \mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and thus Dirac's equation to $\left(i\gamma^\mu \mathfrak{D}_\mu - m\right)\psi = 0$ for interacting particles. All of this emerges by requiring "gauge" symmetry under the local phase transformation $\varphi \to \varphi' = U\varphi = e^{i\Lambda(t,\mathbf{x})}\varphi$ acting generally on the scalar fields $\varphi = \phi$ of the Klein-Gordon equation and the fermion fields $\varphi = \psi$ of Dirac's equation, redirecting phase but preserving magnitude. This is all well-known, so it is not necessary to detail this further. The point is that the relation $m^2 c^2 = g_{\mu\nu} \pi^\mu \pi^\nu$ in (3.6) is easily derived from the metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ using local gauge symmetry, and that nothing more is needed to furnish the starting point to minimize the variation and arrive at the combined gravitational and electrodynamic motion (3.1).

Starting with (3.6) and dividing through by $m^2 c^2$, we form the number 1 as such:

$$1=g_{\mu\nu}\left(\frac{dx^{\mu}}{cd\tau}+\frac{q}{mc^{2}}A^{\mu}\right)\left(\frac{dx^{\nu}}{cd\tau}+\frac{q}{mc^{2}}A^{\nu}\right)=g_{\mu\nu}\left(\frac{u^{\mu}}{c}+\frac{q}{mc^{2}}A^{\mu}\right)\left(\frac{u^{\nu}}{c}+\frac{q}{mc^{2}}A^{\nu}\right)=g_{\mu\nu}\frac{U^{\mu}}{c}\frac{U^{\nu}}{c}, \quad (4.1)$$

which "1" will be useful in a variety of circumstances. The above includes the mechanical four-velocity $u^{\mu}\equiv dx^{\mu}/d\tau$ and a canonical four-velocity defined by $U^{\mu}\equiv u^{\mu}+qA^{\mu}/mc$. From here, we work in natural units $c=1$ and use dimensional rebalancing to restore *c* only after a final result.

The first place that "1" above will be useful is in (1.1), where, distributing the expression after the first equality while absorbing $g_{\mu\nu}$ into the electrodynamic term indices, we write:

$$0=\delta\int_{A}^{B}(1)\,d\tau=\delta\int_{A}^{B}d\tau\left(g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+2\frac{q}{m}A_{\sigma}\frac{dx^{\sigma}}{d\tau}+\frac{q^{2}}{m^{2}}A_{\sigma}A^{\sigma}\right)^{.5}. \quad (4.2)$$

From here, we carry out the variational calculation, which deductively culminates in:

$$0=\delta\int_{A}^{B}d\tau=\int_{A}^{B}\delta x^{\alpha}d\tau\begin{pmatrix}-g_{\alpha\nu}\dfrac{d^{2}x^{\nu}}{d\tau^{2}}+\dfrac{1}{2}\left(\partial_{\alpha}g_{\mu\nu}-\partial_{\mu}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\mu}\right)\dfrac{dx^{\mu}}{d\tau}\dfrac{dx^{\nu}}{d\tau}\\+\dfrac{q}{m}\left(\partial_{\alpha}A_{\sigma}-\partial_{\sigma}A_{\alpha}\right)\dfrac{dx^{\sigma}}{d\tau}+\dfrac{1}{2}\dfrac{q^{2}}{m^{2}}\partial_{\alpha}\left(A_{\sigma}A^{\sigma}\right)\end{pmatrix}. \quad (4.3)$$

Going from (4.2) to (4.3) is straightforward. The top line contains the same result usually obtained for gravitational geodesics, and is the result of setting $q=0$ in (4.2). This is the calculation Einstein first presented in §9 of [1], and does not need to be reviewed further. The terms on the bottom line emerge as a direct and immediate consequence of starting with the canonical energy-momentum relation $m^{2}c^{2}=g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ rather than the ordinary mechanical $m^{2}c^{2}=g_{\mu\nu}p^{\mu}p^{\nu}$, which is to say, the bottom line is a result merely of mandating local gauge symmetry. Some specific guides to note when performing the detailed calculation include: a) we assume no variation in the charge-to-mass ratio, i.e., that $\delta(q/m)=0$, over the path from *A* to *B*; b) applied to gauge field terms, the variations obtained using the chain rule are $\delta A_{\sigma}=\delta x^{\alpha}\partial_{\alpha}A_{\sigma}$ and $\delta\left(A_{\sigma}A^{\sigma}\right)=\delta x^{\alpha}\partial_{\alpha}\left(A_{\sigma}A^{\sigma}\right)$; c) we also use $dA_{\sigma}/d\tau=\partial_{\alpha}A_{\sigma}dx^{\alpha}/d\tau$; and d) there is an integration-by-parts in the calculation. This integration-by-parts produces a boundary term $\int_{A}^{B}d\left(A_{\sigma}\delta x^{\sigma}\right)=\left(A_{\sigma}\delta x^{\sigma}\right)\Big|_{A}^{B}=0$ that can be eliminated, and for the remaining term causes the sign reversal appearing in $\partial_{\alpha}A_{\sigma}-\partial_{\sigma}A_{\alpha}$.

Now, for material worldlines, the proper time $d\tau\neq 0$. And between the boundaries at *A* and *B* the variation $\delta x^{\sigma}\neq 0$. So the large parenthetical expression in (4.3) must be zero. The connection is of course given by $-\Gamma^{\beta}{}_{\mu\nu}=\tfrac{1}{2}g^{\alpha\beta}\left(\partial_{\alpha}g_{\mu\nu}-\partial_{\mu}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\mu}\right)$ and field strength by $F_{\alpha\sigma}=\partial_{\alpha}A_{\sigma}-\partial_{\sigma}A_{\alpha}=\partial_{;\alpha}A_{\sigma}-\partial_{;\sigma}A_{\alpha}$. So with *c* restored, this enables us to extract:

$$\frac{d^2x^\beta}{d\tau^2} = -\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{q}{m}F^\beta{}_\sigma\frac{dx^\sigma}{cd\tau} + \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right). \tag{4.4}$$

This clearly reproduces (3.1) and includes the Lorentz force motion alongside the gravitational geodesic, all obtained from the minimized variation (4.2). Therefore, (4.4) does represent geodesic motion, although when contrasted to the Lorentz motion it contains an additional term $\partial^\beta\left(A_\sigma A^\sigma\right)$ that we shall shortly review in depth.

As with (1.3), we may view (4.4) in an alternative albeit equivalent way that highlights how Lorentz motion plus the extra term is now merely a consequence of local gauge symmetry: It is well-known how imposing gauge symmetry spawns the heuristic rules $\partial_\sigma \to \mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and $p^\mu \to \pi^\mu \equiv p^\mu + qA^\mu / c$ for gauge-covariant derivatives and canonical momenta, and $m^2c^2 = g_{\mu\nu}p^\mu p^\nu \to m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ for the energy momentum relation. Here, referring to (1.3), we see another heuristic rule which emerges in lockstep with these others, namely:

$$\frac{Du^\beta}{D\tau} = \frac{du^\beta}{d\tau} + \Gamma^\beta{}_{\mu\nu}u^\mu u^\nu \to \mathrm{A}^\beta \equiv \frac{\mathfrak{D}u^\beta}{\mathfrak{D}\tau} \equiv \frac{Du^\beta}{D\tau} - \frac{q}{mc}F^\beta{}_\sigma u^\sigma - \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right) = 0. \tag{4.5}$$

In the absence of gravitation we may write this as:

$$\frac{du^\beta}{d\tau} \to \frac{\mathfrak{D}u^\beta}{\mathfrak{D}\tau} \equiv \frac{du^\beta}{d\tau} - \frac{q}{mc}F^\beta{}_\sigma u^\sigma - \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right) = 0. \tag{4.6}$$

In the above, $\mathfrak{D}u^\beta / \mathfrak{D}\tau$ symbolizes the gauge-covariant or *canonical acceleration*, which is rooted in the further heuristic $dx^\mu \to \mathfrak{D}x^\mu \equiv dx^\mu + \left(q/mc\right)d\tau A^\mu$ defined at (3.5). And more generally, using the boldface $\boldsymbol{\mathfrak{D}}$ notation whenever there are both gravitational and electrodynamic fields, we have used $\mathrm{A}^\beta \equiv \boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau = 0$ to denote the *gravitationally-* <u>and</u> *gauge-covariant* acceleration, which we shall refer to collectively as the "canonical acceleration." The canonical equation $\boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau = 0$ in (4.5) states that the *canonical acceleration* is gravitationally-covariant and gauge-covariant, which we shall refer to generally as "canonical covariance." Yet, when shown in terms of *mechanical* four-velocities $u^\mu = dx^\mu / d\tau$, the *mechanical acceleration* contains the geodesic motion of gravitation and the Lorentz force motion of electrodynamics. In the absence of any charge or electromagnetic potential / field (4.5) reverts back to $Du^\beta / D\tau = du^\beta / d\tau + \Gamma^\beta{}_{\mu\nu}u^\mu u^\nu = 0$ for gravitationally-covariant motion (1.3). In the absence of gravitation (4.5) reduces to (4.6) for the canonically-covariant Lorentz force alone. And in the absence of both gravitation and electromagnetism what remains is merely $du^\beta / d\tau = 0$ for the Newtonian inertial motion governed by special relativity alone. From this view, all classical physical motion is inertial and geodesic because $\boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau \equiv 0$; the motion is simply *canonically-inertial* with regard to any gravitational curvature $R^\alpha{}_{\beta\mu\nu}B_\alpha = \left[\partial_{;\nu}, \partial_{;\mu}\right]B_\beta$ and any (imaginary) gauge curvature $qF^{\mu\nu}\phi = i\left[\mathfrak{D}^\mu, \mathfrak{D}^\nu\right]\phi$. What we observe physically are canonical motions growing out of the application of local symmetry principles to the Newtonian equation $du^\beta / d\tau = 0$ for mechanical inertial motion.

All of the above provides a conceptually-compelling view of classical physical motion. However, (4.4) yields a term $\partial^\beta \left(A_\sigma A^\sigma \right)$ which is not ordinarily a part of the Lorentz force law. And in fact, this term needs to be removed for one empirical reason and two theoretical reasons: The empirical reason is that this term is not part of the well-established, well-corroborated. universally-observed Lorentz Force law (3.1). The first theoretical reason is that the motion cannot depend upon a term $\partial_\beta \left(A_\sigma A^\sigma \right)$ which in turn depends upon and changes as a function of the unobservable local phase $\Lambda(t, \mathbf{x})$. Specifically, the gauge transformation $qA_\sigma \rightarrow qA'_\sigma = qA_\sigma - \partial_\sigma \Lambda$ would introduce the phase into (4.4) and thus leave the observable motion ambiguous and in violation of gauge symmetry. The second theoretical reason is that by removing this term, (4.4) now does fully describe the Lorentz motion as geodesic motion, which is conceptually attractive. So the question arises whether there is some clear natural basis upon which this term does in fact get removed in the physical world.

A simple fix would be to modify the metric (3.5) by subtracting out the second-order term with $A_\sigma A^\sigma$, and to then start the variation of (4.2) on the basis of:

$$c^2 d\tau^2 = \mathfrak{D}x_\sigma \mathfrak{D}x^\sigma - \frac{q^2}{m^2c^2} d\tau^2 A_\sigma A^\sigma = \left(dx_\sigma + \frac{q}{mc} d\tau A_\sigma \right)\left(dx^\sigma + \frac{q}{mc} d\tau A^\sigma \right) - \frac{q^2}{m^2c^2} d\tau^2 A_\sigma A^\sigma . \qquad (4.7)$$

When turned into the number "1" as in (4.1) and then used in the variation as in (4.2), it is clear that this will result in (4.4) but without the extra term $\partial^\beta \left(A_\sigma A^\sigma \right)$ because the source of that term is subtracted out of (4.7). So the result is the Lorentz force plus gravitational motion, precisely, as desired. However, this approach loses some conceptual strength, because the Lorentz force does not emerge simply from applying local gauge symmetry and the heuristic rules which emerge from this symmetry as reviewed in equations (4.5) and (4.6). Now the rule becomes: apply gauge symmetry, *and then take the extra step* of subtracting off the $A_\sigma A^\sigma$ term to get a desired result. Occam's razor would in this circumstance compel us to see if this second step can be eliminated, and whether the term $\partial^\beta \left(A_\sigma A^\sigma \right)$ can be removed from (4.4) in some other, more natural way.

As we shall now see in sections 5 through 8, this extra term in (4.4), and the process for its prospective removal from (4.4), is intimately connected with gauge fixing, Maxwell's electric charge equation, the electrodynamic Lagrangian and action, electrodynamic and gravitational power, and the sources $T^{\mu\nu}$ in Einstein's field equation for gravitation.

5. The Lagrangian Gauge and the Geodesic Gauge, and Canonically-Inertial Motion

To study the extra term $\partial^\beta\left(A_\sigma A^\sigma\right)$ in (4.4), we start with Maxwell's equation $J^\beta=\partial_{;\alpha}F^{\alpha\beta}$ for the electric charge density. Via the usual expression $F^{\alpha\beta}=\partial^{;\alpha}A^\beta-\partial^{;\beta}A^\alpha=\partial^\alpha A^\beta-\partial^\beta A^\alpha$ for the field strength we write this in terms of the gauge fields as $J^\beta-\partial_{\alpha;}\partial^\alpha A^\beta+\partial^\beta\partial_{;\alpha}A^\alpha=0$. But we do *not* use the Lorenz condition $\partial_{;\alpha}A^\alpha=0$ to fix the gauge; rather for now we leave this term as is. We then multiply this Maxwell equation through by A_β, thus writing the scalar equation:

$$A_\beta J^\beta-A_\beta\partial_{;\alpha}\partial^\alpha A^\beta+A_\beta\partial^\beta\partial_{;\alpha}A^\alpha=0. \tag{5.1}$$

For the second term above we have $-A_\beta\partial_{;\alpha}\partial^\alpha A^\beta=\partial_{;\alpha}A_\beta\partial^\alpha A^\beta-\partial_{;\alpha}\left(A_\beta\partial^\alpha A^\beta\right)$, using the product rule. We may also form the identity $A_\beta\partial^\alpha A^\beta=\frac{1}{2}\partial^\alpha\left(A_\beta A^\beta\right)$. Using both of these in (5.1) yields:

$$A_\beta J^\beta+\partial_{;\alpha}A_\beta\partial^\alpha A^\beta-\tfrac{1}{2}\partial_{;\alpha}\partial^\alpha\left(A_\beta A^\beta\right)+A_\beta\partial^\beta\partial_{;\alpha}A^\alpha=0. \tag{5.2}$$

The second term $\partial_{;\alpha}A_\beta\partial^\alpha A^\beta=\partial_\alpha A_\beta\partial^\alpha A^\beta=\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$, and with this, the first two terms are equivalent to minus the electrodynamic Lagrangian density, $A_\beta J^\beta+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}=-\mathcal{L}_{em}$. Therefore, (5.2) is simply:

$$-\tfrac{1}{2}\partial_{;\alpha}\partial^\alpha\left(A_\beta A^\beta\right)+A^\beta\partial_\beta\partial_{;\alpha}A^\alpha=\mathcal{L}_{em}. \tag{5.3}$$

Again, this is an alternative way of saying that $A_\beta J^\beta=A_\beta\partial_{;\alpha}F^{\alpha\beta}$, which is a four-dimensional scalar product of Maxwell's charge equation with the gauge field. Note that $\partial_\beta\partial_{;\alpha}A^\alpha=\partial_{;\beta}\partial_{;\alpha}A^\alpha$ because the gravitationally-covariant derivative of any scalar is equal to the ordinary derivative of the same. As is easily seen, within $\partial_{;\alpha}\partial^\alpha\left(A_\beta A^\beta\right)$ the first term above contains the extra term $\partial^\beta\left(A_\sigma A^\sigma\right)$ that appeared in (4.4). And the second term contains $\partial_{;\alpha}A^\alpha$ which in the Lorenz gauge is fixed to $\partial_{;\alpha}A^\alpha=0$. The latter is a covariant scalar condition which removes one degree of freedom from the gauge field A^α.

Now, because photons which comprise the gauge field are massless, we are not *required* to use $\partial_{;\alpha}A^\alpha=0$ as we would be if photons were massive. Instead, we are permitted to fix the gauge directly to the physical Maxwell Lagrangian by setting:

$$A^\beta\partial_\beta\partial_{;\alpha}A^\alpha\equiv\mathcal{L}_{em}. \tag{5.4}$$

This is also a covariant scalar gauge condition which removes one degree of freedom, so it would be a suitable replacement for the Lorenz gauge. For obvious reasons we shall refer to this as the "*Lagrangian gauge*." If we were to impose this condition, then as a consequence of combining (5.4) with Maxwell's equation represented via (5.3), we would also find that:

$$\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0\,. \tag{5.5}$$

Therefore, at the very least, the *four-gradient* $\partial_{;\beta}\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$ of the term $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$ would become zero. The question now is: may we and should we adopt the Lagrangian gauge (5.4), and also, the stronger condition that $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ itself?

Were we to impose the condition $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ and thus add further constraint beyond (5.4), then (5.5) would still remain true and thus be compatible with the Lagrangian gauge condition (5.4). And all of this would remain compatible with the scalar representation (5.3) of Maxwell's equation in $A_{\beta}J^{\beta}=A_{\beta}\partial_{;\alpha}F^{\alpha\beta}$. So there is no apparent conflict or contradiction with any standard electrodynamics that arises from setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$. But it is also well-known that a covariant scalar gauge condition such as the Lorenz gauge $\partial_{;\alpha}A^{\alpha}=0$ or the Lagrangian gauge of (5.4) still leaves some residual ambiguity in the gauge field, which ambiguity still needs to be removed. The question is how we do so. Setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ would be an even stronger constraint than (5.5), because this would clearly squeeze out some further ambiguity. The question now is whether this would remove just enough ambiguity to eliminate *all* residual ambiguity, while simultaneously not over-determining the results by imposing too much constraint.

This brings us back to (4.4). As noted in the paragraph prior to (4.7), a gauge transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$ applied to (4.4) would leave the physical motion ambiguous because of the extra term $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$. Further, there is no way to completely remove this ambiguity without removing this term entirely. The weaker condition (5.5) which via (5.3) is a proxy for the Lagrangian gauge (5.4), which in turn is a substitute for the Lorenz gauge, would remove all traces of this extra term from the *third-derivative* expression that would result were we to take $\partial_{;\beta}d^{2}x^{\beta}/d\tau^{2}$ by applying $\partial_{;\beta}$ to (4.4). But there would still remain some ambiguity at the second derivative which is (4.4) because of what happens when we apply the transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$. Therefore, to remove *all* ambiguity from the physical motion, we do need to apply the stronger condition $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$. Once we do so, all of the remaining ambiguity is removed from the physical motion of (4.4), and the result is no more and no less than the Lorentz force law. And because the Lorentz force law thereafter is entirely symmetric under the gauge transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$, we are assured that not only have we removed all physical ambiguity by setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$, but also that we have not removed too much ambiguity so as to over-determine the physical result. Rather, we have precisely determined the physical result with no residual ambiguity and nothing over-determined and no inconsistency with standard electrodynamics. This includes assurance from the derivation (5.1) through (5.5) that there is no contradiction whatsoever with Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$. And, this ensures that the

Lorentz motion must be locally gauge invariant, which even standing alone with no other considerations is sufficient justification for imposing $\partial^\beta\left(A_\sigma A^\sigma\right)=0$ as a gauge condition.

Therefore, we shall now formally take the following two steps: First, to covariantly remove one degree of freedom from the gauge field, we shall fix the gauge using the Lagrangian gauge condition $A^\beta\partial_\beta\partial_{;\alpha}A^\alpha=\mathcal{L}_{em}$ of (5.4). This is *in lieu of* applying the Lorenz gauge condition $\partial_{;\alpha}A^\alpha=0$. Second, to remove any additional ambiguity from the gauge field, we shall impose the condition:

$$\partial^\beta\left(A_\alpha A^\alpha\right)\equiv 0 \tag{5.6}$$

on the four-gradient of the scalar quantity $A_\alpha A^\alpha$. The d'Alembertian of this scalar will then also be zero as shown in (5.5), which is fully compatible with Maxwell's electric charge equation $J^\beta=\partial_{;\alpha}F^{\alpha\beta}$. By imposing both conditions (5.4) and (5.6), the result in (4.4) now reduces to:

$$\frac{d^2x^\beta}{d\tau^2}=-\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+\frac{q}{m}F^\beta{}_\sigma\frac{dx^\sigma}{cd\tau}. \tag{5.7}$$

Now we arrive at is the Lorentz force law together with the gravitational geodesic equation of motion, precisely. Note, because we now have $A^\beta\partial_\beta\partial_{;\alpha}A^\alpha=\mathcal{L}_{em}$, that the additional use of the Lorenz gauge $\partial_{;\alpha}A^\alpha=0$ is *not permitted*: imposing this condition would cause $\mathcal{L}_{em}=0$ and thereby over-determine the physical results.

Now, the Lorentz force law has been derived from the minimized variation $0=\delta\int_A^B d\tau$ of (1.1) starting at (4.2) by merely requiring local gauge symmetry and, true to Occam's razor, nothing more. The extra term $\partial^\beta\left(A_\alpha A^\alpha\right)$ has been removed not by the unnatural fix of (4.7), but rather by the natural solution of fixing the gauge to entirely remove any ambiguity from the physical motion without over-determination. Following all of this, (4.5) reduces to:

$$\mathrm{A}^\beta=\frac{\mathfrak{D}u^\beta}{\mathfrak{D}\tau}\equiv\frac{Du^\beta}{D\tau}-\frac{q}{m}F^\beta{}_\sigma u^\sigma=\frac{du^\beta}{d\tau}+\Gamma^\beta{}_{\mu\nu}u^\mu u^\nu-\frac{q}{m}F^\beta{}_\sigma u^\sigma=0, \tag{5.8}$$

and the combined Lorentz and gravitational acceleration truly is geodesic motion.

Specifically, the motion (5.8) is inertial in both a gravitationally- and canonically-covariant manner. As a shorthand, we shall refer to this as "*canonically-inertial motion*." This is a generalization of Newtonian inertial motion $du^\beta/d\tau=0$ to the circumstance where gravitational and electromagnetic fields are present and the test particle has a charge q that interacts with the electromagnetic fields $F^\beta{}_\sigma$. Now, instead of a mechanical motion $du^\beta/d\tau=0$, we have a canonical motion $\mathfrak{D}u^\beta/\mathfrak{D}\tau=0$, while the mechanical motion $du^\beta/d\tau\neq 0$. Given that (5.6) produces $\mathfrak{D}u^{\beta}/\mathfrak{D}\tau=0$ in (5.8) which is an equation of gravitational and Lorentz geodesic motion, we shall refer to (5.6) as the "*geodesic gauge*" condition. Note also that the two gauge conditions (5.4) and (5.6) are necessary and sufficient to yield (5.7) a.k.a. (5.8). But, by starting with (5.6) alone, one immediately deduces (5.7) as well as (5.5). And (5.5) via (5.3) which embeds Maxwell's equation, yields (5.4). So from a simpler view, the Lagrangian gauge (5.4) is a corollary of the geodesic gauge (5.6) combined with Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$. So by imposing the geodesic gauge (5.6), and by simply having Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$, we also have the Lagrangian gauge. Finally, note again that the Lagrangian gauge (5.4) precludes the Lorentz gauge because that would force $\mathcal{L}_{em}=0$ and so over-determine the physical results.

The foregoing (5.8) is yet another example of the general heuristic rule that when gauge fields and charges are present, mechanical motions and objects are promoted to canonical motions and objects, with the canonical motions and objects behaving in the same way as their counterpart mechanical motions and objects do in the absence of the gauge fields and charges. Thus, for example, the mechanical $\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0$ is inherited by Dirac's canonical $\left(i\gamma^{\mu}\mathfrak{D}_{\mu}-m\right)\psi=0$; while the mechanical energy relation $m^2c^2=g_{\mu\nu}p^{\mu}p^{\nu}$ is inherited by the canonical $m^2c^2=g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ of (3.6). Here, in the absence of gravitation, the mechanical $du^{\beta}/d\tau=0$ is inherited by the canonical $\mathfrak{D}u^{\beta}/\mathfrak{D}\tau=0$ for the Lorentz force, which is to say, from (4.6) in geodesic gauge:

$$\frac{du^{\beta}}{d\tau}\rightarrow\frac{\mathfrak{D}u^{\beta}}{\mathfrak{D}\tau}\equiv\frac{du^{\beta}}{d\tau}-\frac{q}{mc}F^{\beta}{}_{\sigma}u^{\sigma}=0. \tag{5.9}$$

This is simply (5.8) without gravitational fields.

Now, let us explore some further significant results which arise from the Lagrangian gauge (5.4) and the geodesic gauge (5.6). As noted at the end of the previous section, these results relate to the electrodynamic Lagrangian and action [the former already seen in the Lagrangian gauge $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\mathcal{L}_{em}$ of (5.4)], electrodynamic and gravitational power, and the sources $T^{\mu\nu}$ in Einstein's equation.

6. The Electrodynamic Action in Lagrangian Gauge

It is very illustrative to rewrite the Lagrangian gauge (5.4) using the product rule as

$$\mathcal{L}_{em}=A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\partial_{\beta}\left(A^{\beta}\partial_{;\alpha}A^{\alpha}\right)-\partial_{\beta}A^{\beta}\partial_{;\alpha}A^{\alpha}, \tag{6.1}$$

and then obtain the electrodynamic action $S_{em}=\int d^4x\mathcal{L}_{em}$. Once inside the action integral, we may set $\int d^4x\partial_{\beta}\left(A^{\beta}\partial_{;\alpha}A^{\alpha}\right)=0$ via the boundary condition $A_{\beta}(t,\mathbf{x})=0$ at the extremum $t,\mathbf{x}=\pm\infty$. What we then end up with is an action:

$$S_{em}=\int d^4x\mathcal{L}_{em}=-\int d^4x\partial_\beta A^\beta \partial_{;\alpha}A^\alpha=-\int d^4x\left(\partial_\beta A^\beta \partial_\alpha A^\alpha+\Gamma^\alpha{}_{\sigma\alpha}A^\sigma\partial_\beta A^\beta\right), \tag{6.2}$$

noting also that $\Gamma^\alpha{}_{\sigma\alpha}=\partial_\sigma\sqrt{-g}\big/\sqrt{-g}$ where g is the metric tensor determinant. In flat spacetime, with $\partial_\sigma\sqrt{-g}=0$, this becomes the very simple action:

$$S_{em}=\int d^4x\mathcal{L}_{em}=-\int d^4x\left(\partial_\alpha A^\alpha\right)^2. \tag{6.3}$$

It will be seen that (6.3), containing a $-\left(\partial_\alpha A^\alpha\right)^2$, is analogous to the R_ξ gauge conditions, which are ordinarily written as $\delta\mathcal{L}=-\left(\partial_\alpha A^\alpha\right)^2/2\xi$. However, (6.2) and (6.3) are not local conditions; they are global because they represent an integral over the entire volume of the four-dimensional spacetime.

Once working with the action, we are but a step away from Quantum Electrodynamics, which is generated through the path integration $Z_{em}=\int DA^\alpha \exp\left(iS_{em}/\hbar\right)$. As usual, we may obtain the electrodynamic action $S_{em}=\int d^4x\left(\frac{1}{2}A_\mu\left(g^{\mu\nu}\partial_\sigma\partial^\sigma-\partial^\mu\partial^\nu\right)A_\nu-J_\mu A^\mu\right)$ starting with $A_\beta J^\beta+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}=-\mathcal{L}_{em}$. Note that this has no expressly-appearing gravitationally-covariant derivatives, because of the cancellations that occur via $F^{\alpha\beta}=\partial^{;\alpha}A^\beta-\partial^{;\beta}A^\alpha=\partial^\alpha A^\beta-\partial^\beta A^\alpha$. However, there is an implicit gravitational term, because $J^\beta=\partial_{;\alpha}F^{\alpha\beta}$. This is the exact origin starting at (5.1) of the $\partial_{;\alpha}$ appearing in (6.1) and (6.2). Then we use Gaussian integration to path integrate as usual. But the upshot of (6.2) is to tell us that:

$$S_{em}=\int d^4x\left(\frac{1}{2}A_\mu\left(g^{\mu\nu}\partial_\alpha\partial^\alpha-\partial^\mu\partial^\nu\right)A_\nu-J_\alpha A^\alpha\right)=-\int d^4x\left(\left(\partial_\alpha A^\alpha\right)^2+\Gamma^\alpha{}_{\sigma\alpha}A^\sigma\partial_\beta A^\beta\right). \tag{6.4}$$

This provides a second expression for the action based on employing the Lagrangian gauge (5.4), which as pointed out after (5.8) is a corollary of the geodesic gauge plus Maxwell's equation.

So (6.4) above is a direct consequence of the Lagrangian gauge (5.4). But the more general condition of which (5.4) thus (6.4) is a corollary, is the geodesic gauge condition $\partial^\beta\left(A_\alpha A^\alpha\right)=0$ of (5.6) to which we now turn. This condition leads to a relation for electrodynamic and gravitational power, and to a direct connection with the sources $T^{\mu\nu}$ in Einstein's equation.

7. The Electro-Gravitational Power Equation

We now study the effect of the geodesic gauge condition (5.6) on the canonical energy-momentum relation (3.6). We first return to (3.6), which, with indices summed and with $c=1$, we expand without commuting the left-right ordering of the momenta and the gauge fields, to obtain $m^2=p_\sigma p^\sigma+qA_\sigma p^\sigma+qp_\sigma A^\sigma+q^2A_\sigma A^\sigma$. The reason we refrain from commuting is to highlight that were we to combine the two middle terms into $qA_\sigma p^\sigma + qp_\sigma A^\sigma = 2qA_\sigma p^\sigma$ we would need to commute p_σ and A^σ which needs to be done with care given the Heisenberg commutation relation $\left[p_j, B\right] = -i\hbar\partial_j B$ for any field $B(t,\mathbf{x})$ which is a function of the spacetime coordinates. And as to the time component, we would also want to be mindful of the Heisenberg equation of motion $\left[H_0, O\right] = -i\hbar\partial_0 O$ for an operator O with no explicit time dependence, together with relationship $H_0\left|\psi\right\rangle = p_0\left|\psi\right\rangle$ between the Hamiltonian H_0 operator and the observable energy $p_0 = E$ which contains its eigenvalues. Thus, even if we were to commute the energy with the time component of the potential $A^0 = \phi$ thus setting $\left[p_0, A^0\right] = 0$, we would still have to recognize that $p_j A^j = A_j p^j - i\hbar\partial_j A^j$ and thus include a term of the form $-i\hbar\partial_j A^j$, if not $-i\hbar\partial_\sigma A^\sigma$, if it was our desire to move beyond classical physics and account for the quantum mechanical non-commutativity.

For present purposes, to be completely general, let us use the relationship $\left[p_\sigma, A^\sigma\right] = -i\hbar\partial_\sigma A^\sigma$ a.k.a. $p_\sigma A^\sigma = A_\sigma p^\sigma - i\hbar\partial_\sigma A^\sigma$ covariantly extended into the time dimension, recognizing that we may always restrict this to its space components by setting $\left[p_0, A^0\right] = 0$, thus $\partial_0 A^0 = 0$, and may additionally ignore quantum effects entirely by setting $\left[p_j, A^j\right] = 0$, thus the space divergence $\partial_j A^j = \nabla \cdot \mathbf{A} = 0$. Therefore, we start by writing (3.6), with $\hbar = c = 1$, as:

$$m^2 = p_\sigma p^\sigma + 2qA_\sigma p^\sigma + q^2 A_\sigma A^\sigma - iq\partial_\sigma A^\sigma . \tag{7.1}$$

The final term $\partial_\sigma A^\sigma$ arises from the commutativity just discussed, and may be removed or ignored under the circumstances just discussed.

Now, let us take the covariant spacetime gradient $\partial_{;\beta}$ of the above. The rest mass is invariant, so its four-gradient $\partial_{;\beta} m = \partial_\beta m = 0$. Therefore, after reduction we obtain:

$$0 = p_\sigma \partial_{;\beta} p^\sigma + q\partial_{;\beta} A_\sigma p^\sigma + qA_\sigma \partial_{;\beta} p^\sigma + \tfrac{1}{2} q^2 \partial_{;\beta}\left(A_\sigma A^\sigma\right) - \tfrac{1}{2} iq\partial_{;\beta}\partial_\sigma A^\sigma . \tag{7.2}$$

Now we apply the geodesic gauge (5.6), so the term $\partial_{;\beta}\left(A_\sigma A^\sigma\right) = \partial_\beta\left(A_\sigma A^\sigma\right) = 0$ is removed. We may also use the field strength to replace $\partial_{;\beta} A_\sigma = F_{\beta\sigma} + \partial_{;\sigma} A_\beta$. Additionally, $p^\sigma = mu^\sigma$ is the ordinary mechanical momentum, so we can divide out *m*, whereby $p^\sigma \to u^\sigma$ throughout the contravariant momentum terms in the above. Thus, segregating the field strength term on the left, (7.2) becomes:

$$qF_{\beta\sigma} u^\sigma = -p_\sigma \partial_{;\beta} u^\sigma - qA_\sigma \partial_{;\beta} u^\sigma - q\partial_{;\sigma} A_\beta u^\sigma + \tfrac{1}{2} i\left(q/m\right)\partial_{;\beta}\partial_\sigma A^\sigma . \tag{7.3}$$

We of course recognize $qF_{\beta\sigma}u^{\sigma}$ as a variant of the Lorentz force term in (3.1). The net effect of the geodesic gauge (5.6), is to have removed the terms $A_{\sigma}A^{\sigma}$ of second order in the gauge field.

Now, we wish to express the terms on the right in relation to the passage of proper time, that is, as derivatives along the curve, see (4.5) and (4.7). For the next-to-last term in (7.3) we may substitute $\partial_{;\sigma}A_{\beta}u^{\sigma}=dA_{\beta}/d\tau-\Gamma^{\tau}{}_{\sigma\beta}A_{\tau}u^{\sigma}$ derived using the gravitationally-covariant derivative and the chain rule. So also with $\partial_{;\beta}\partial_{\sigma}A^{\sigma}=\partial_{\beta}\partial_{\sigma}A^{\sigma}$, (7.3) advances to:

$$qF_{\beta\sigma}u^{\sigma}=-p_{\sigma}\partial_{;\beta}u^{\sigma}-qA_{\sigma}\partial_{;\beta}u^{\sigma}-q\frac{dA_{\beta}}{d\tau}+q\Gamma^{\tau}{}_{\sigma\beta}A_{\tau}u^{\sigma}+\frac{1}{2}i\left(q/m\right)\partial_{\beta}\partial_{\sigma}A^{\sigma}. \tag{7.4}$$

As to the remaining terms, we now multiply by $u^{\beta}=dx^{\beta}/d\tau$ throughout, giving us a $u^{\beta}\partial_{;\beta}u^{\sigma}$ in the first two terms after the equality. Then we may similarly derive and then substitute $u^{\beta}\partial_{;\beta}u^{\sigma}=du^{\sigma}/d\tau+\Gamma^{\sigma}{}_{\beta\tau}u^{\beta}u^{\tau}$. Also writing $p_{\sigma}=mu_{\sigma}$ for the remaining mechanical momentum, and seeing that the terms with $\Gamma^{\tau}{}_{\sigma\beta}A_{\tau}u^{\beta}u^{\sigma}$ cancel identically, and using the chain rule in the final term $u^{\beta}\partial_{\beta}\partial_{\sigma}A^{\sigma}=\left(d/d\tau\right)\partial_{\sigma}A^{\sigma}/=\partial_{\sigma}dA^{\sigma}/d\tau$, with renamed indices and $\hbar=c=1$ restored, we now have:

$$\frac{q}{c}F_{\mu\nu}u^{\mu}u^{\nu}=-\left(mu_{\sigma}+\frac{q}{c}A_{\sigma}\right)\frac{du^{\sigma}}{d\tau}-\frac{q}{c}u^{\sigma}\frac{dA_{\sigma}}{d\tau}-m\Gamma^{\sigma}{}_{\mu\nu}u_{\sigma}u^{\mu}u^{\nu}+\frac{1}{2}i\hbar\frac{q}{mc}\partial_{\sigma}\frac{dA^{\sigma}}{d\tau}. \tag{7.5}$$

This $\left(q/c\right)F_{\mu\nu}u^{\mu}u^{\nu}$ term on the left is a scalar number, and it has dimensions of power. So this is an expression for electrodynamic and gravitational power. However, because $F_{\mu\nu}$ is an antisymmetric tensor, this term vanishes identically. Therefore, moving all of the mechanical and gravitational terms to the left and keeping the electrodynamic terms on the right, we may consolidate to:

$$mu_{\sigma}\left(\frac{du^{\sigma}}{d\tau}+\Gamma^{\sigma}{}_{\mu\nu}u^{\mu}u^{\nu}\right)=-\frac{q}{c}\frac{d}{d\tau}\left(A_{\sigma}u^{\sigma}\right)+\frac{1}{2}i\hbar\frac{q}{mc}\partial_{\sigma}\frac{dA^{\sigma}}{d\tau}. \tag{7.6}$$

It is easily seen that when the right hand side becomes zero in the absence of electrodynamics, the left hand side contains the gravitational geodesic motion (1.1). The final term may also be vanished by setting $\hbar=0$, i.e., in the classical limit. In terms of spacetime coordinates with all terms expanded, and isolating all the acceleration terms on the left, another way to express this is:

$$\left(m\frac{dx_{\sigma}}{d\tau}+\frac{q}{c}A_{\sigma}\right)\frac{d^{2}x^{\sigma}}{d\tau^{2}}=-\left(m\Gamma^{\sigma}{}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}-\frac{q}{c}\frac{dA^{\sigma}}{d\tau}\right)\frac{dx_{\sigma}}{d\tau}+\frac{1}{2}i\hbar\frac{q}{mc}\partial_{\sigma}\frac{dA^{\sigma}}{d\tau}. \tag{7.7}$$

In the absence of gravitation, we merely set $\Gamma^{\sigma}{}_{\mu\nu}=0$. And if we neglect the non-commutativity discussed in the first paragraph of this section, then we may set $\hbar=0$ to vanish the final term.

Now let us see how the Lagrangian gauge (5.4) connects to Einstein's equation and gravitational curvature.

8. The Electro-Gravitational Energy Flux Field Equation

As already reviewed, by fixing to the Lagrangian gauge $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}\equiv\mathcal{L}_{em}$ of (5.4) in lieu of the Lorenz gauge $\partial_{;\alpha}A^{\alpha}=0$, Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$ also constrains us to require the relation $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$ of (5.5). The stronger geodesic gauge $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0$ of (5.6) was used to remove the remaining gauge ambiguity from the equation of motion (4.4), or (4.5), thereby producing the combined gravitational and Lorentz force law of motion (5.7). This raises an interesting question: if we want to explore the impact on the equation of motion of the weaker condition $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$ which is required for compatibility with Maxwell's equation, then it is clear that this can be done by taking the covariant gradient $\partial_{;\beta}$ of the original equation of motion (4.4) from before we imposed the stronger condition of (5.6). What makes this interesting is that this ties together the sources in both the Einstein equation for gravitation and Maxwell's equation for electric charges, as we shall now see.

Mindful that $A_{\beta}J^{\beta}+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}=-\mathcal{L}_{em}$, we start by taking the covariant gradient $\partial_{;\beta}$ of (4.5), and then applying (5.3) which stems from Maxwell's charge equation, to obtain:

$$\partial_{;\beta}\mathrm{A}^{\beta}=\partial_{;\beta}\frac{\mathfrak{D}u^{\beta}}{\mathfrak{D}\tau}=\partial_{;\beta}\frac{Du^{\beta}}{D\tau}-\frac{q}{m}\partial_{;\beta}\left(F^{\beta}{}_{\sigma}u^{\sigma}\right)+\frac{q^{2}}{m^{2}}\left(\mathcal{L}_{em}-A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}\right)=0. \tag{8.1}$$

To be clear, the above, via the development laid out from (4.2) to (4.5), is a direct deductive consequence of taking the variation $0=\delta\int_{A}^{B}d\tau$ based on the canonical mass-energy-momentum relation $m^{2}c^{2}=g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ of (3.6) in combination with Maxwell's charge equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$. No additional assumptions are used to obtain (8.1), and in particular, no gauge conditions have yet been imposed on (8.1).

First, let us focus on the term $\partial_{;\beta}Du^{\beta}/D\tau$. Using the expression $R^{\alpha}{}_{\beta\mu\nu}B_{\alpha}=\left[\partial_{;\nu},\partial_{;\mu}\right]B_{\beta}$ which relates the Riemann tensor to the degree to which gravitationally-covariant derivatives do not commute when operating on an arbitrary vector B_{α}, from which we deduce $R^{\alpha}{}_{\nu}u_{\alpha}=R^{\alpha\beta}{}_{\beta\nu}u_{\alpha}=\left[\partial_{;\nu},\partial_{;\beta}\right]u^{\beta}$ for the velocity four-vector u^{β}, it is easily seen that:

$$\partial_{;\beta}\frac{Du^{\beta}}{D\tau}=\partial_{;\beta}\left(\frac{\partial x^{\nu}}{\partial\tau}\partial_{;\nu}u^{\beta}\right)=\partial_{;\beta}\frac{\partial x^{\nu}}{\partial\tau}\partial_{;\nu}u^{\beta}+\frac{\partial x^{\nu}}{\partial\tau}\partial_{;\beta}\partial_{;\nu}u^{\beta}=\partial_{;\beta}u^{\nu}\partial_{;\nu}u^{\beta}+u^{\nu}\partial_{;\nu}\partial_{;\beta}u^{\beta}-R_{\mu\nu}u^{\mu}u^{\nu}. \tag{8.2}$$

So the Ricci tensor which is part of the Einstein equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and thus related to the energy tensor $T_{\mu\nu}$ which is the source of gravitation, is seen to be contained in (8.1). This is especially direct using the inverse form $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$.

Next let us insert (8.2) into (8.1) and also expand terms while applying Maxwell's $J_\sigma = \partial_{;\beta} F^{\beta}{}_{\sigma}$. With some index renaming, this now yields a scalar equation:

$$\begin{gathered}\partial_{;\beta} \mathrm{A}^{\beta} = \partial_{;\beta} \frac{\mathfrak{D} u^{\beta}}{\mathfrak{D}\tau} = -R_{\mu\nu} u^{\mu} u^{\nu} - \frac{q}{m} J^{\beta}{}_{\sigma} u^{\sigma} + \partial_{;\beta} u^{\nu} \partial_{;\nu} u^{\beta} + u^{\nu} \partial_{;\nu} \partial_{;\beta} u^{\beta} - \frac{q}{m} F^{\beta}{}_{\sigma} \partial_{;\beta} u^{\sigma} \\ + \frac{q^2}{m^2}\left(\mathcal{L}_{em} - A^{\beta} \partial_{\beta} \partial_{;\alpha} A^{\alpha}\right) = 0\end{gathered} \quad . \tag{8.3}$$

Here, we find both gravitational sources in $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$ and electric charge sources $\mu_0 J_\sigma = \partial_{;\beta} F^{\beta}{}_{\sigma}$ (with $\mu_0 = 1/\varepsilon_0 c^2$ balancing dimensionality) all as part of the same dynamical equation. Now, to eliminate the entire second line of (8.3), we impose the Lagrangian gauge condition $A^{\beta} \partial_{\beta} \partial_{;\alpha} A^{\alpha} \equiv \mathcal{L}_{em}$ of (5.4) which covariantly removes just as much freedom from this equation as does the Lorenz gauge $\partial_{;\alpha} A^{\alpha} = 0$. Again, the Lagrangian gauge is a corollary of the geodesic gauge and Maxwell's charge equation. We may also write $\partial_{;\mu} \partial_{;\nu} u^{\nu} = \partial_{\mu} \partial_{;\nu} u^{\nu}$ because $\partial_{;\nu} u^{\nu}$ is a scalar. We also multiply the above through by *m*, while noting that $m R_{\mu\nu} u^{\mu} u^{\nu}$ has dimensions of energy per area i.e. energy flux. We then restore *c* so as to give all terms this same dimensionality, while mindful that $\kappa = 8\pi G / c^4$ and $\mu_0 \varepsilon_0 c^2 = 1$. And, we make explicit use of $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$ while isolating all sources on the left. With all of this, these sources are now seen to bring about motion via the differential equation:

$$-\kappa T_{\mu\nu} m u^{\mu} u^{\nu} + \tfrac{1}{2} \kappa T m u_{\sigma} u^{\sigma} + \mu_0 q J_{\sigma} u^{\sigma} = m \partial_{;\nu} u^{\mu} \partial_{;\mu} u^{\nu} + m u^{\mu} \partial_{\mu} \partial_{;\nu} u^{\nu} - (q/c) F^{\tau}{}_{\sigma} \partial_{;\tau} u^{\sigma} . \tag{8.4}$$

This is a combined differential equation for the gravitational and electrodynamic motion of material bodies with a four-velocity u^{ν}. Because all terms have dimensions of energy per area, i.e. energy flux, we recognize this to be a scalar energy flux equation.

In general one may find it helpful to keep this equation in the form of (8.4). To the extent one wishes to be more explicit about the derivatives involved in (8.4), we may expand using $\partial_{;\nu} u^{\mu} = \partial_{\nu} u^{\mu} + \Gamma^{\mu}{}_{\sigma\nu} u^{\sigma}$ and the like. So the first term after the equality is:

$$m \partial_{;\nu} u^{\mu} \partial_{;\mu} u^{\nu} = m \partial_{\nu} u^{\mu} \partial_{\mu} u^{\nu} + 2\Gamma^{\mu}{}_{\sigma\nu} m u^{\sigma} \partial_{\mu} u^{\nu} + \Gamma^{\nu}{}_{\tau\mu} \Gamma^{\mu}{}_{\sigma\nu} m u^{\sigma} u^{\tau} . \tag{8.5}$$

For the next term, $\partial_\nu u^\nu = 0$ by the chain rule, so we have $\partial_{;\nu} u^\nu = \Gamma^\nu{}_{\sigma\nu} u^\sigma$. Noting as well that $\Gamma^\nu{}_{\sigma\nu} = \partial_\sigma \sqrt{-g} / \sqrt{-g} = \tfrac{1}{2}(1/g)\partial_\sigma g$, this next term in (8.4) is:

$$\begin{aligned} mu^\mu \partial_\mu \partial_{;\nu} u^\nu &= mu^\mu \partial_\mu \left(\Gamma^\nu{}_{\sigma\nu} u^\sigma\right) = m\frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\left(\frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^\sigma}\frac{dx^\sigma}{d\tau}\right) \\ &= \frac{1}{2}m\left(-\frac{1}{g^2}\left(\frac{dg}{d\tau}\right)^2 + \frac{1}{g}\frac{d^2 g}{d\tau^2} + \frac{1}{g}\frac{\partial g}{\partial x^\sigma}\frac{d^2 x^\sigma}{d\tau^2}\right) \end{aligned} . \tag{8.6}$$

Placing (8.5) and (8.6) into (8.4) and also expanding the $F^\tau{}_\sigma \partial_{;\tau} u^\sigma$ term, we then obtain the final expanded form of the energy flux equation:

$$\begin{aligned} &-\kappa T_{\mu\nu} mu^\mu u^\nu + \tfrac{1}{2}\kappa T m u_\sigma u^\sigma + \mu_0 q J_\sigma u^\sigma \\ &= m\partial_\nu u^\mu \partial_\mu u^\nu + 2\Gamma^\mu{}_{\sigma\nu} m u^\sigma \partial_\mu u^\nu + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} m u^\mu u^\nu \\ &-\frac{1}{2}m\frac{1}{g^2}\left(\frac{dg}{d\tau}\right)^2 + \frac{1}{2}m\frac{1}{g}\frac{d^2 g}{d\tau^2} + \frac{1}{2}m\frac{1}{g}\frac{\partial g}{\partial x^\sigma}\frac{d^2 x^\sigma}{d\tau^2} - \frac{q}{c}F^\tau{}_\sigma \partial_\tau u^\sigma - \frac{q}{c}\Gamma^\sigma{}_{\alpha\tau} F^\tau{}_\sigma u^\alpha \end{aligned} . \tag{8.7}$$

In regions of spacetime where there is no gravitating matter, i.e., *in vacuo*, we set $T_{\mu\nu} = 0$ and $T = 0$ above, and then solve for the motion, given only the probability density contained in the time component of $J_\sigma = \rho_0 u_\sigma = \overline{\psi}\gamma_\sigma \psi$. In the further absence of electrodynamic sources we set $J_\sigma = 0$ so the entire top line of the above equation becomes zero.

One interesting way to use (8.7) is to remove all energy sources except for the Maxwell-Poynting electromagnetic field energy tensor which is $4\pi\mu_0 c^2 T_{\mu\nu} = -F_{\sigma\mu}F^\sigma{}_\nu + \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$ with dimensional balancing, with $\mu_0 \varepsilon_0 c^2 = 1$. This tensor of course has no trace, which is related to why electromagnetic fields travel at the speed of light and photons are massless. So when this is the only energy present – and recognizing that this energy still gravitates and thus affects the metric and the spacetime curvature – then, with the source term $\mu_0 c q J_\sigma u^\sigma$ isolated on the left, and with the constants reorganized via $\kappa / 4\pi\mu_0 c^2 = G / 2\pi c^4 k_e$ to display the embedded ratio G / k_e of Newton's to Coulomb's constant, (8.7) becomes:

$$\begin{aligned} &\mu_0 q J_\sigma u^\sigma \\ &= m\partial_\nu u^\mu \partial_\mu u^\nu + 2\Gamma^\mu{}_{\sigma\nu} m u^\sigma \partial_\mu u^\nu + \left(\Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} - \frac{G}{2\pi c^4 k_e} F_{\sigma\mu} F^\sigma{}_\nu\right) m u^\mu u^\nu + \frac{G}{8\pi c^4 k_e} F_{\alpha\beta} F^{\alpha\beta} m u_\sigma u^\sigma \\ &-\frac{1}{2}m\frac{1}{g^2}\left(\frac{dg}{d\tau}\right)^2 + \frac{1}{2}m\frac{1}{g}\frac{d^2 g}{d\tau^2} + \frac{1}{2}m\frac{1}{g}\frac{\partial g}{\partial x^\sigma}\frac{d^2 x^\sigma}{d\tau^2} - \frac{q}{c}F^\tau{}_\sigma \partial_\tau u^\sigma - \frac{q}{c}\Gamma^\sigma{}_{\alpha\tau} F^\tau{}_\sigma u^\alpha \end{aligned} . \tag{8.8}$$

An equation free of electrodynamic source charges then results from setting $J_\sigma = 0$ in the above.

It is important to keep in mind that (8.7) may be derived directly from the known Lorentz force law (3.1) as represented in (5.8), even had we not obtained this from the minimization of the action (1.1). This is because (8.7) is simply the spacetime gradient $\partial_{;\beta}$ applied to (5.8) as starting at (8.1), and because (5.8) is true whether or not we obtain it from a variation. But the motivation to operate on the Lorentz force law in this way comes from the fact that when we do obtain the Lorentz force from a variation, Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$ together with the Lagrangian gauge $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\mathcal{L}_{em}$ of (5.4) mandate the gauge condition $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$, which is a corollary of the geodesic gauge $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0$ of (5.6). So when we study the impact of this weaker corollary $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$ on the Lorentz force, the result is the energy flux field equation (8.7). When we impose the stronger condition $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0$, the result is the Lorentz force itself. What is important about (8.7) and (8.8) is that they put the energy source tensor $T_{\mu\nu}$ or the spacetime curvature $R_{\mu\nu}$ (as chosen for best convenience in any given calculation), directly into the dynamical equation for energy flux right alongside of the electrodynamic sources J_{σ}.

Having now reviewed how the combined gravitational and Lorentz motion (3.1) is derived from the variational equation (1.1), and the required gauge conditions and the immediately-consequent power and energy flux equations, we now show how to derive the electrodynamic time dilation and contraction summarized in section 3, including how the time dilation (3.10) and the key energy relation (3.11) are derived. Again, as a reminder, this is all premised on requiring the line element to remain invariant and the background fields in spacetime to remain unchanged, under a re-gauging of the electrodynamic charge-to-mass ratio q/m.

PART II: DERIVATION OF ELECTRODYNAMIC TIME DILATION

9. Review of Time Dilation in Special and General Relativity

As a comparative baseline for deducing the effects of electromagnetic time dilation and contraction, we begin in this section by briefly reviewing the connection between time dilation and kinetic and potential energies in the Special and General Theories of Relativity, paying close attention to the signs of various terms. In the next section we then extend this known development to demonstrate a heretofore unknown electrodynamic connection to time dilation and contraction.

In Special Relativity, where the metric tensor is that of flat spacetime, $g_{\mu\nu}=\eta_{\mu\nu}$, we begin with the metric $c^{2}d\tau^{2}=\eta_{\mu\nu}dx^{\mu}dx^{\nu}$. Using a timelike signature $\eta_{\mu\nu}=\left(1,-\mathbf{1}\right)$ consistent with the conventions reviewed in section 2, and given $dx^{\mu}=\left(cdt,d\mathbf{x}\right)$ and the squared velocity $v^{2}=\left(dx^{k}/dt\right)\left(dx^{k}/dt\right)$, this is easily restructured using the chain rule into:

$$1=\frac{1}{c^2}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\frac{1}{c^2}\left(\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}-\frac{dx^k}{d\tau}\frac{dx^k}{d\tau}\right)=\frac{1}{c^2}\left(\frac{dt}{d\tau}\right)^2\left(c^2-\frac{dx^k}{dt}\frac{dx^k}{dt}\right)=\left(\frac{dt}{d\tau}\right)^2\left(1-\frac{v^2}{c^2}\right). \tag{9.1}$$

Then, selecting the positive square root whereby $dt/d\tau=1$ at rest in accord with the conventions reviewed in section 2, this yields the time dilation factor:

$$\gamma_v\equiv\frac{dt}{d\tau}=\frac{1}{\sqrt{1-v^2/c^2}}\cong 1+\frac{1}{2}\frac{v^2}{c^2}\geq 1, \tag{9.2}$$

also shown in the $v\ll c$ non-relativistic limit. Physically, we associate *t* with the time coordinate measured by an *observer* in her or her own reference frame, and we associate τ with the proper time of an *observed* reference frame moving at velocity *v* relative to the observer. Operationally, the observer will measure the time coordinate *t* using a geometrodynamic clock at rest in his or her "laboratory" which clock "ticks" at some periodic rate, and also will measure by τ, a second set of "ticks" coming from an identical clock situated in the frame that is moving at relative velocity *v*. Because the successive ticks of any clock may occur with great rapidity in the case of a very accurate clock but will never be infinitesimally-separated, for all practical purposes (9.2) will be measured by $\Delta t/\Delta\tau=1/\sqrt{1-v^2/c^2}>1$, which will always be greater than 1 for any relative velocity $v>0$. Because the elapsed Δt of the observer's clock at rest will always exceed the observed elapsed $\Delta\tau$ coming from the identical albeit moving clock, there will be more ticks tolled by the rest clock than from the moving clock over any given interval. Therefore, time in the moving frame will be observed to be in "slow motion," i.e., dilated, i.e., redshifted, in relation to time measured in the rest frame.

As to energy, we multiply (9.2) through by mc^2 to obtain the total energy:

$$E=mc^2\gamma_v=mc^2\frac{dt}{d\tau}=\frac{mc^2}{\sqrt{1-v^2/c^2}}=mc^2+E_v=E_0+E_v\cong mc^2+\frac{1}{2}mv^2. \tag{9.3}$$

In a central result of the Special Theory of Relativity [2] by which the rest energy $E_0=mc^2$ of any material body with mass *m* was first discovered in [10], this is equal at low velocities to the rest energy $E_0=mc^2$ plus the Newtonian kinetic energy $E_v\cong\frac{1}{2}mv^2$. It is important to note that although any periodic signals emitted by objects in the moving frame will be observed via (9.2) to have redshifted toward lower energies, the objects in that frame will likewise increase their total energy by supplementing their rest energy with a kinetic energy via (9.3). One may understand this somewhat counterintuitive result by thinking of the moving body as having "stolen" energy from the signals it emits and plowed that into its own increased kinetic energy. One may also understand this more mundanely by the simple experiential fact that a faster-moving body can do more work with its kinetic energy than a slower-moving body or a body at rest.

Turning to the General Theory of Relativity, one starts with the line element $c^2d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$, with the gravitational field represented by the metric tensor $g_{\mu\nu}$. At any infinitesimal location in spacetime the Minkowski tangent space is of course $\eta_{\mu\nu}=(1,\mathbf{-1})$, using the sign conventions previously reviewed. To isolate the effect of gravitation from that of motion upon the measurement of time, we place a geometrodynamic clock at rest in the gravitational field, $dx^k = 0$, so the line element becomes $c^2 d\tau^2 = g_{00} dx^0 dx^0 = g_{00} c^2 dt^2$, which easily rearranges to $dt^2 / d\tau^2 = 1/g_{00}$. When taking the square root we continue to use the positive root, so that $dt / d\tau = 1/\sqrt{g_{00}} \to 1$ in the limit where $g_{\mu\nu} \to \eta_{\mu\nu}$, again consistent with sign conventions previously reviewed. Then, we make use of the Schwarzschild solution for a static, spherically-symmetric gravitational field, for which $g_{00} = 1 - 2GM / c^2 r$ in the vicinity of a gravitating mass *M*. Showing also the weak-field limit for which the Newtonian potential $\Phi / c^2 = (g_{00} - 1)/2 = -GM / c^2 r$, the gravitational time dilation factor γ_g for $M > 0$ is:

$$\gamma_g \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}}} = \frac{1}{\sqrt{1 - 2GM / c^2 r}} \cong 1 + \frac{GM}{c^2 r} = 1 - \frac{\Phi}{c^2} > 1 . \tag{9.4}$$

As in (9.2) *t* is measured in the frame of an observer outside the gravitational field (or in a field that is negligible in comparison to the field of *M*), and τ originates from a clock under observation (e.g., from a geometrodynamic body that emits a periodic light signal) which is situated in the gravitational field of *M*. Once again, now with $\Delta t / \Delta \tau = 1 + GM / c^2 r > 1$, over any finite interval there will be more ticks emitted from the observer's clock than from the clock in the gravitational field of *M*. This is why spectral lines of oscillators near the sun or near distant stars are redshifted.

As to energy, starting now with (9.4), we again multiply through by mc^2, where *m* is the mass of a test particle placed into the field of *M*. This yields a total energy:

$$\begin{aligned} E &= mc^2 \gamma_g = mc^2 \frac{dt}{d\tau} = \frac{mc^2}{\sqrt{g_{00}}} = \frac{mc^2}{\sqrt{1 - 2GM / c^2 r}} = mc^2 - E_g = E_0 - E_g \cong mc^2 \left(1 + \frac{GM}{c^2 r} \right) \\ &= mc^2 + \frac{GMm}{r} = mc^2 - \Phi m > E_0 \end{aligned} . \tag{9.5}$$

In the top line above, we have a similar result as we do in the case of motion. Although the body in the gravitational field emits redshifted light with reduced energy via (9.4), that energy is again "stolen" and plowed into what is now an increased gravitational potential energy of that body. However, the Newtonian gravitational potential $\Phi = c^2 (g_{00} - 1)/2 = -GM / r$ is a negative number, with opposite sign from the kinetic energy. This means that a test particle of mass *m* placed into this potential, naturally moving toward a state of lower energy, will seek to get closer to *M*, consistent with gravitation always being attractive. This results in an interaction energy $\Phi m = -GMm / r$ which does diminish as the separation *r* grows smaller. So even though the mass in (9.5) gains energy in the gravitational field, the negatively-signed gravitational potential energy $E_g \cong \Phi m = -GMm / r$ is *subtracted* from the rest energy in (9.3), yielding an overall increase in the actual observed, usable energy of the test particle. To make sense of this, consider for example that a body near the surface of the sun weighs about 27 times as much as it does near the surface of the earth. Therefore, it carries 27 times as much gravitational potential energy at a given separation from the sun's surface as it does at the same separation from the earth's surface. So, notwithstanding that any light signals from the body near the sun are redshifted, this is a real, measurable, increase in the energy available for that body to do work.

An important point of distinction between gravitation and electromagnetism already previewed in the various signs in (9.5), is that gravitational interactions are *always* attractive, which attractive interaction occurs between *like charges*. In contrast, electromagnetism can be attractive or repulsive, with attraction between *unlike charges* and repulsion between like charges. So it is also helpful in preparation for examining electrodynamic time dilation and contraction, to express all of the above in terms of the attractive radial Newtonian gravitational "force" $F_g = -GMm/r^2$, in which the minus sign is responsible for the vector direction of gravitational attraction. This is consistent with the positive sign used in the section 2 discussion of sign conventions to express that the Coulomb repulsion between two like-charges must yield a positively signed $d^2r/d\tau^2 > 0$ for the Lorentz acceleration. Relating the definite integral over r of this attractive force to the term $+GMm/r$ appearing in (9.5) yields:

$$\int_\infty^r F_g \cdot dr = \int_\infty^r -\frac{GMm}{r^2} dr = \frac{GMm}{r}\bigg|_{r=\infty}^{r=r} = +\frac{GMm}{r} = -\Phi m = -E_g = -\int_r^\infty F_g \cdot dr \,. \tag{9.6}$$

Note the offsetting reversal of the integration boundaries and sign in the final expression above.

Finally, before proceeding to study electrodynamic time dilation and contraction, when the test charge is in a gravitational field and is also in motion, we may simply multiply the two time dilation factors γ_v and γ_g together, whereby from (9.2) and (9.4), we obtain:

$$\frac{dt}{d\tau} = \gamma_v \gamma_g = \frac{1}{\sqrt{1-v^2/c^2}\sqrt{g_{00}}} \cong \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(1+\frac{GM}{c^2 r}\right) = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{GM}{c^2 r}+\frac{1}{2}\frac{v^2}{c^2}\frac{GM}{c^2 r} \,. \tag{9.7}$$

If we again multiply through by the rest energy mc^2 of the test particle, we obtain:

$$E = mc^2\frac{dt}{d\tau} = mc^2\gamma_v\gamma_g = \frac{mc^2}{\sqrt{1-v^2/c^2}\sqrt{g_{00}}} \cong mc^2+\frac{1}{2}mv^2+\frac{GMm}{r}+\frac{1}{2}\frac{GMm}{c^2 r}v^2 = E_0+E_{kin}-E_g-\frac{1}{2}\frac{E_g}{c^2}v^2 \,. \tag{9.8}$$

In successive terms above, in the linear limit. we see: 1) the rest energy of the test mass, 2) the kinetic energy of the test mass, 3) the potential energy gained by the test mass because it is in the gravitational field, which is equal to minus the negative gravitational potential energy, and 4) the kinetic energy of this gravitational interaction energy. It will be seen that these are the first, second, fifth and sixth terms of (3.11). Because this accords with what is empirically observed, and in particular yields the kinetic energy of the gravitational energy which is required to completely account for all energy when there is both motion and gravitation, this validates the technique of multiplying these two time dilations together to obtain a combined time dilation, and of then multiplying through by mc^2 to obtain the overall energy and the various types of energy that contribute to this overall energy. This also establishes a direct connection between time and energy whether kinetic or potential, which will become very important in the development to follow.

Now we turn to deducing the electrodynamic time dilation and contraction reviewed in section 3, and to deriving the relationship (3.11) between time dilation and energy. To begin, we return to (3.9) which states that the line element $d\tau$ must be invariant, and the metric tensor $g_{\mu\nu}$ and the gauge field A^μ [the latter now subject to the Lagrangian and geodesic gauge conditions (5.4) and (5.6)] must be unchanged, under a rescaling i.e., re-gauging of $q/m \to q'/m'$. Thus, it is (3.9) which *defines* the coordinate transformation $x^\mu \to x'^\mu$ which leads to electrodynamic time dilation and contraction. Now we show exactly how this occurs.

10. Electrodynamic Time Dilation and Contraction, and Time-Energy Relations in Special and General Relativity and Electrodynamics

As noted earlier, the number "1" constructed in (4.1) is useful in a variety of circumstances. One of those circumstances was to derive the Lorentz force from a variation starting at (4.2). Another such circumstance is for the derivation of electrodynamic time dilation and contraction. The starting point for this derivation is (3.9) which maintains the invariance of $d\tau = d\tau'$ and leaves the background fields $g_{\mu\nu}$, A^μ and $F^{\mu\nu}$ unchanged under a re-gauging $q/m \to q'/m' \neq q/m$ of the charge-to-mass ratio. We then turn (3.9) into the same "1" which appears in (4.1) by dividing through $c^2 d\tau^2$ thus obtaining:

$$\begin{aligned} 1 &= g_{\mu\nu}\left(\frac{dx^\mu}{cd\tau}+\frac{q}{mc^2}A^\mu\right)\left(\frac{dx^\nu}{cd\tau}+\frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\left(\frac{dx'^\mu}{cd\tau}+\frac{q'}{m'c^2}A^\mu\right)\left(\frac{dx'^\nu}{cd\tau}+\frac{q'}{m'c^2}A^\nu\right) \\ &= g_{\mu\nu}\left(\frac{u^\mu}{c}+\frac{q}{mc^2}A^\mu\right)\left(\frac{u^\nu}{c}+\frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\left(\frac{u'^\mu}{c}+\frac{q'}{m'c^2}A^\mu\right)\left(\frac{u'^\nu}{c}+\frac{q'}{m'c^2}A^\nu\right) \quad . \\ &= g_{\mu\nu}\frac{U^\mu}{c}\frac{U^\nu}{c} = g_{\mu\nu}\frac{U'^\mu}{c}\frac{U'^\nu}{c} \end{aligned} \tag{10.1}$$

This shows how the invariant number "1" in (4.1) transforms under a $q/m \to q'/m'$ re-gauging, and also includes both the mechanical four-velocity $u^\mu = dx^\mu / d\tau$ and the canonical four-velocity $U^\mu = u^\mu + (q/mc)A^\mu$ and their "primed" counterparts. Note that we may infer $U^\mu = U'^\mu$ from the final line, which means the canonical velocity is invariant under a $q/m \to q'/m'$ rescaling. Only the mechanical velocity is changed.

Now, let us turn to the Lorentz contraction factor $\gamma_v = 1/\sqrt{1-v^2/c^2}$ and the ordinary four-velocity $v^\mu / c = (1, \mathbf{v}/c)$ used to describe motion in special relativity. With $g_{\mu\nu} = \eta_{\mu\nu}$ and $\mathrm{diag}(\eta_{\mu\nu}) = (1, \mathbf{-1})$ in accord with sign conventions review in section 2, it is easily shown and well-known that $\eta_{\mu\nu}(\gamma_v v^\mu)(\gamma_v v^\nu)/c^2 = 1$ *by mathematical identity*, which is another "1." In special relativity without any electromagnetic interactions, the mechanical relativistic four-velocity $u^\mu = \gamma_v v^\mu$ so that this "1" in natural units is given by the familiar $\eta_{\mu\nu} u^\mu u^\nu = u_\sigma u^\sigma = c^2 = 1$ using $c = 1$ units. But as we see in (10.1), when there are electromagnetic interactions, the "1" in flat spacetime is instead formed by the scalar product $\eta_{\mu\nu} U^\mu U^\nu = U_\sigma U^\sigma = c^2 = 1$ which employs the *canonical* four-velocity U^μ. Comparing the identity $\eta_{\mu\nu}\left(\gamma_v v^\mu\right)\left(\gamma_v v^\nu\right) = c^2$ with $\eta_{\mu\nu} U^\mu U^\nu = c^2$ contained within (10.1), we are able to infer:

$$U^\mu = \gamma_v v^\mu . \tag{10.2}$$

And if we make use of the fact that $U^\mu = u^\mu + \left(q/mc\right) A^\mu$ and $u^\mu = dx^\mu / d\tau$ then this may be extended to:

$$U^\mu = u^\mu + \frac{q}{mc} A^\mu = \frac{dx^\mu}{d\tau} + \frac{q}{mc} A^\mu = \gamma_v v^\mu , \tag{10.3}$$

which may be conversely rewritten in terms of the ordinary mechanical velocity as:

$$u^\mu = \frac{dx^\mu}{d\tau} = U^\mu - \frac{q}{mc} A^\mu = \gamma_v v^\mu - \frac{q}{mc} A^\mu . \tag{10.4}$$

Now we turn to the gauge potential $A^\mu = \left(\phi, \mathbf{A}\right)$. Ordinarily, with $u^\mu = \gamma_v v^\mu$, this is written in terms of the proper (rest) potential ϕ_0 as $A^\mu = \phi_0 u^\mu / c = \phi_0 \gamma_v v^\mu / c$, employing the mechanical four-velocity u^μ, because at rest with $\gamma_v = 1$ and $v^\mu / c = \left(1, \mathbf{0}\right)$ this will produce $A^\mu = \left(\phi_0, \mathbf{0}\right)$. However, as we see in (10.2), in the presence of electromagnetism, $U^\mu = \gamma_v v^\mu$, and from (10.4) $u^\mu = \gamma_v v^\mu - \left(q/mc\right) A^\mu$. So to ensure that we continue to have $A^\mu = \left(\phi_0, \mathbf{0}\right)$ at rest in the potential, we must now relate the gauge potential to the proper potential and to the motion using the *canonical* four-velocity, such that :

$$A^\mu = \phi_0 U^\mu / c = \phi_0 \gamma_v v^\mu / c . \tag{10.5}$$

Were we to continue to use $A^\mu = \phi_0 u^\mu / c$, then we would have $A^\mu = \phi_0 \gamma_v v^\mu / c - \left(q\phi_0 / mc^2\right) A^\mu$ which is a recursive expression in A^μ and which becomes $A^\mu = \left(\phi_0, \mathbf{0}\right) - \left(q\phi_0 / mc^2\right) A^\mu$ at rest, rather than simply $A^\mu = \left(\phi_0, \mathbf{0}\right)$. Note also that $A^\mu = \phi_0 U^\mu / c = A'^\mu = \phi_0 U'^\mu / c$ because $U^\mu = U'^\mu$ as pointed out at (10.1). So $A^\mu = \phi_0 U^\mu / c$ in (10.5) *must* be the relation between A^μ and motion relative to the proper potential, to enforce the essential requirement that the background field $A^\mu = A'^\mu$ must be unchanged under a $q/m \to q'/m'$ rescaling. This would *not* occur were we to have $A^\mu = \phi_0 u^\mu / c$. So using (10.5) and $U^\mu = U'^\mu$, we arrive from the middle line of (10.1) at:

$$1 = g_{\mu\nu}\left(\frac{u^\mu}{c} + \frac{q}{mc^2}\frac{\phi_0 U^\mu}{c}\right)\left(\frac{u^\nu}{c} + \frac{q}{mc^2}\frac{\phi_0 U^\nu}{c}\right) = g_{\mu\nu}\left(\frac{u'^\mu}{c} + \frac{q'}{m'c^2}\frac{\phi_0 U^\mu}{c}\right)\left(\frac{u'^\nu}{c} + \frac{q'}{m'c^2}\frac{\phi_0 U^\nu}{c}\right). \quad (10.6)$$

So when $q \to q'$ and $m \to m'$, the only other object that changes is $dx^\mu \to dx'^\mu$ in the mechanical four velocity $u^\mu = dx^\mu / d\tau \to u'^\mu = dx'^\mu / d\tau$. Everything else is unchanged including the line element $d\tau$, the gravitational field $g_{\mu\nu}$, and the gauge field $A^\mu = \phi_0 U^\mu / c$. With these preliminaries, we are now ready to derive the electromagnetic time dilation and contraction.

Generally, we will wish to compare periodic signals emitted by a geometrodynamic clock which has a net charge of zero and so is neutral, in relation to signals from a geometrodynamic clock with a nonzero net charge. So working from (10.6), we shall set $q = 0$ to represent electrical neutrality, and leave q' as it is to maintain a charged body, and thereby obtain:

$$1 = g_{\mu\nu}\frac{u^\mu}{c}\frac{u^\nu}{c} = g_{\mu\nu}\frac{U^\mu}{c}\frac{U^\nu}{c} = g_{\mu\nu}\left(\frac{u'^\mu}{c} + \frac{q'}{m'c^2}\frac{\phi_0 U^\mu}{c}\right)\left(\frac{u'^\nu}{c} + \frac{q'}{m'c^2}\frac{\phi_0 U^\nu}{c}\right). \quad (10.7)$$

The relationship $g_{\mu\nu}u^\mu u^\nu = g_{\mu\nu}U^\mu U^\nu$ is true, because $U^\mu = u^\mu + (q/mc)A^\mu$, so that when we have a neutral body $q = 0$, the mechanical and canonical velocities are synonymous, $U^\mu = u^\mu$. From the final equality in (10.7) we may infer that $U^\mu = u'^\mu + \left(q'/m'c^2\right)\phi_0 U^\mu$. Rearranged to isolate U^μ and also using $U^\mu = u^\mu$ for the neutral body as well as $u^\mu = dx^\mu / d\tau$ and likewise $u'^\mu = dx'^\mu / d\tau$ for the "primed" body, we deduce:

$$\frac{dx^\mu}{d\tau} = u^\mu = U^\mu = \frac{1}{1 - q'\phi_0 / m'c^2}u'^\mu = \frac{1}{1 - q'\phi_0 / m'c^2}\frac{dx'^\mu}{d\tau}. \quad (10.8)$$

The time component with $x^\mu = (ct, \mathbf{x})$ and *c* divided out of the above is then seen to be:

$$\frac{dt}{d\tau} = \frac{1}{1 - q'\phi_0 / m'c^2}\frac{dt'^\mu}{d\tau}. \quad (10.9)$$

Now, as with (9.2) for special relativistic motion and (9.4) for general relativistic gravitation, we associate *t* with temporal oscillations from a neutral clock used to measure the time coordinate for the *observer*. We also associate τ with oscillations such as the spectra of periodic light signals coming from an *observed* test particle with mass m' and charge $q' \neq 0$ and time element dt', in the potential $\phi_0 \neq 0$. So as a consequence of the latter association we may set $d\tau = dt'^\mu$, that is, $dt'/d\tau = 1$. *This is an extremely important step, and the reader should carefully review (9.2) and (9.4) to become convinced that it is in fact correct and consistent in (10.9) to associate periodic signals from the neutral body and periodic signals from the charged body with* dt *and* $d\tau = dt'^\mu$ *respectively.* We also posit for the moment that the charged body with m' and charge q' is not in any gravitational fields, because $dt'^{\mu} / d\tau$ would then deviate from 1 as a result of the gravitational fields. Later, we shall remove this restriction.

As a result of the foregoing, also with the weak field $q'\phi_0 << m'c^2$ limit, (10.9) becomes:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{1}{1 - q'\phi_0 / m'c^2} \cong 1 + \frac{q'\phi_0}{m'c^2} . \tag{10.10}$$

This is how we derive $dt / d\tau$ and then define the factor γ_{em}, first introduced prior to (3.11), to be the rate at which time ticks from a positively-signed charge q' placed in a positively-signed potential ϕ_0 and emitting periodic signals, in relation to how time ticks from a neutral clock of an observer, in accordance with the sign conventions reviewed in section 2. As a notational convenience, since $dt / d\tau$ specifies time measurements taken using signals from the neutral body with $q = 0$, we drop the primes from the mass and charge, and re-denote the above simply as:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{1}{1 - q\phi_0 / mc^2} \cong 1 + \frac{q\phi_0}{mc^2} . \tag{10.11}$$

This now specifies how a neutral geometrodynamic clock used by an observer to measure a time coordinate *t*, will "tick" in relation to an otherwise-identical oscillator with charge *q* and mass *m* in a proper potential ϕ_0. As with (9.2) and (9.4), time measurements can never be infinitesimally small, so $\Delta t / \Delta\tau = 1/\left(1 - q\phi_0 / mc^2\right)$ is the practical operational representation of (10.11).

Now, let us consider the special case of a Coulomb proper potential $\phi_0 = k_e Q / r$, thus an electromagnetic potential energy $E_{em} = q\phi_0 = k_e Qq / r$. As noted when we reviewed sign conventions in the introduction, this describes an electrically-repulsive interaction because the energy of the test charge will diminish as the separation *r* grows larger. Employing this in (10.11) for $Q > 0$ and $q > 0$ now yields:

$$\gamma_{em} \equiv \frac{dt}{d\tau} = \frac{1}{1 - k_e Qq / mc^2 r} \cong 1 + \frac{k_e Qq}{mc^2 r} > 1 . \tag{10.12}$$

Following the analysis at (9.2) and (9.4) for special and general relativistic time dilation, the above predicts that *time will dilate and signals will redshift for <u>electrically-repulsive</u> interactions between like-charges*, which will become detectable when the electromagnetic potential energy $E_{em} = k_e Qq / r$ grows sufficiently large in relation to the rest energy mc^2 of the test charge. If we flip the sign of *Q* or *q* but not both to represent electrical attraction, then we will have $\gamma_{em} < 1$, which means that time will contract and that light emitted from *q* will blueshift for electrically-attractive interactions between like charges.

Very consequentially, although it is well-known that *time dilates for gravitational attraction* as reviewed at (9.4), in a striking contrast that will be explored at length in the development to follow, (10.12) reveals that *time contracts for electrical attraction*. Another way of saying the same thing is that time dilates for interactions between *like-charges*, for both gravitation and electromagnetism, but because like-gravitational-charges (masses) attract and like-electrical-charges repel, the *time effects are opposite as between attractive gravitation and attractive electromagnetism*. This all is a result of the positive sign of $+k_e Qq/mc^2 r$ in the linear limit of (10.12), and this is the main reason why we have been so carefully-attentive to signs and sign conventions from the start of this paper. A very important consequence of this result, will be a connection to and empirical confirmation by the "anomalous" lepton magnetic moments, to be developed in part III of this paper.

Also of significance, if we apply Feynman-Stueckelberg to require that proper time for the test particle always flows forward for particles and antiparticles alike such that $d\tau > 0$, and we require that the measurement of time in any other frame flows in the same direction whereby we require $d\tau/dt > 0$, then from (10.11) this means that $1/\left(1 - q\phi_0/mc^2\right) > 0$. Because if any real number $x > 0$ its reciprocal $1/x > 0$ also, this also means that $1 - q\phi_0/mc^2 > 0$ a.k.a. $q\phi_0/mc^2 < 1$ a.k.a. $q\phi_0 < mc^2$. This now becomes a material limit on the strength of electromagnetic interactions and particularly states that the interaction energy of a test charge in an electromagnetic potential is always less than the rest energy of the test charge itself. And for a Coulomb interaction, this becomes $1 - k_e Qq/mc^2 r > 0$ which algebraically restructures into $k_e Qq/mc^2 r < 1$ a.k.a. $k_e Qq/r < mc^2$ and then into $r > k_e Qq/mc^2$, thereby establishing a lower physical limit on how close two interacting charges can come to one another. As mentioned in the section 3 overview, this solves the long-standing problem of how the $r = 0$ singularity in Coulomb's law is circumvented in the physical world. This limit also bars $\gamma_{em} = dt/d\tau = 1 - k_e Qq/mc^2 r < \infty$ from ever growing to infinity for material particle electromagnetic interactions, which is highly analogous to the limitation $v < c$ for the motion of material particles.

As we did at (9.3) and (9.5), let us now multiply (10.12) through by the rest energy $E_0 = mc^2$ of the test charge to obtain the total energy:

$$E = mc^2\gamma_{em} = mc^2\frac{dt}{d\tau} = \frac{mc^2}{1 - k_e Qq/mc^2 r} = mc^2 + E_{em} = E_0 + E_{em} \cong mc^2\left(1 + \frac{k_e Qq}{mc^2 r}\right) = mc^2 + \frac{k_e Qq}{r} . \quad (10.13)$$

In the lowest order $k_e Qq/mc^2 r << 1$ limit this reveals the Coulomb interaction energy $E_{em} \cong k_e Qq/r$. Also, similarly to what we did at (9.6), we start with a repulsive electrostatic radial force $F_r = +k_e Qq/r^2$, and relate this to $E_{em} \cong k_e Qq/r$ via:

$$\int_r^\infty F_{em} \cdot dr = \int_r^\infty + \frac{k_e Qq}{r^2} dr = -\frac{k_e Qq}{r}\bigg|_{r=r}^{r=\infty} = +\frac{k_e Qq}{r} = +\phi q = +E_{em} = -\int_\infty^r F_{em} \cdot dr . \quad (10.14)$$

This likewise expresses the lowest order interaction energy $E_{em} \cong k_e Qq/r$ as an integral of the Coulomb force. Note the reversal of the integration boundaries and sign in the final expression.

Very importantly, (10.13) reveals heretofore unrecognized non-linear behaviors in *classical* electrodynamic interactions which parallel the non-linear behaviors already well-known in special relativity for motion and general relativity for gravitation: In special relativity, as seen in (9.3), the Newtonian kinetic energy $E_v = \frac{1}{2}mv^2$ is merely the lowest-order term added to the rest energy mc^2 in the low-velocity $v << c$ limit where $E = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \frac{1}{2}mv^2$. In general relativity, as seen in (9.5) for the Schwarzschild metric, the (negative) Newtonian gravitational interaction energy $-E_g = GMm/r$ is merely the lowest-order term added to mc^2 in the weak field $2GMm/r << mc^2$ limit where $mc^2/\sqrt{1-2GM/c^2r} \cong mc^2 + GMm/r$. And in (10.14) for electrodynamics, we now similarly reveal that the Coulomb interaction energy $E_{em} = k_e Qq/r$ is merely the lowest-order term added to mc^2 in the weak field $k_e Qq/r << mc^2$ limit where $mc^2/\left(1-k_e Qq/mc^2r\right) \cong mc^2 + k_e Qq/r$. But as the interactions grow stronger, even in classical electrodynamics, we have non-linear behaviors also. Let us review these more closely:

Mathematically, the linear Coulomb interaction energy arises from the fact most directly seen in (10.11) and (10.12), that $\gamma_{em} = 1/\left(1-x\right) \cong 1+x$ for $x << 1$ with $x = k_e Qq/mc^2r$. So when we multiply by mc^2 in (10.13) we find $mc^2\gamma_{em} = mc^2/\left(1-x\right) = mc^2 + E_{em} \cong mc^2 + k_e Qq/r$. But this is just the lowest order limit. With the complete series $1/\left(1-x\right) \cong 1+x+x^2+x^3+x^4+\ldots$ which converges for $-1<x<1$, we now deduce from (10.13) that:

$$\begin{aligned} E_{em} &= mc^2\left(\frac{dt}{d\tau}-1\right) = \frac{k_e Qq/r}{1-k_e Qq/mc^2r} = mc^2\frac{x}{1-x} = mc^2\frac{E_{em0}}{mc^2-E_{em0}} \\ &= \frac{k_e Qq}{r}\left(1+\frac{k_e Qq}{mc^2r}+\left(\frac{k_e Qq}{mc^2r}\right)^2+\left(\frac{k_e Qq}{mc^2r}\right)^3+\left(\frac{k_e Qq}{mc^2r}\right)^4+\ldots\right) = \frac{k_e Qq}{r}\sum_{n=0}^{\infty}\left(\frac{k_e Qq}{mc^2r}\right)^n, \end{aligned} \tag{10.15}$$

where $E_{em0} \equiv mc^2x = k_e Qq/r$ denotes the first order linear term in the $E_{0em} << mc^2$ "weak" interaction limit. Again *this is a form of non-linear <u>classical</u> electrodynamic behavior that appears to have heretofore been unrecognized.* At present, the only non-linear electrodynamic behaviors which are known, are those arising in *quantum* electrodynamics as a result of Feynman "loop" diagram calculations which cause the abelian dimensionless interaction strength $\alpha = e^2/4\pi\varepsilon_0\hbar c$ (which approaches the numerical value of $\alpha = 1/137.035999139$ [11] at low probe energies) to "run" toward increased magnitude as two charges move closer together. These loop diagram calculations are also used to explain the "anomalous" lepton magnetic moments. It will be helpful for when we begin to consider these anomalies in part III to represent (10.15) wholly in terms of dimensionless energy ratios, using the relation:

$$\frac{E_{em}}{mc^2}=\frac{E_{em0}}{mc^2-E_{em0}}=\frac{E_{em0}/mc^2}{1-E_{em0}/mc^2}=\frac{1}{1-E_{em0}/mc^2}-1 \tag{10.16}$$

between the lowest order, linear interaction energy E_{em0} and the total nonlinear energy E_{em}.

We also note as pointed out before (10.13) that $k_e Qq/r<mc^2$ is a material limit on electromagnetic interaction strength, and is analogous to the limit $v<c$ a.k.a. $mv^2<mc^2$ for material particle motion in special relativity. As such, the non-linear series obtained from $mc^2\gamma_{em}=mc^2/(1-x)$ with $x=k_e Qq/mc^2 r$ is *naturally convergent* because of the natural limit $x<1$ just reviewed for repulsion, while for attraction which merely flips the sign, it is $x>-1$. As we shall show in Part III of this paper, the non-linear behaviors in (10.15), (10.16) may also be used to explain, and are confirmed by, the lepton magnetic moment anomalies.

Next, similarly to what we did at (9.7), let us multiply together the special relativistic time dilation factor reviewed at (9.2) with the electrodynamic factor found at (10.12) and also show the low-velocity limit to obtain the combined time dilation factor for a test charge in relative motion:

$$\frac{dt}{d\tau}=\gamma_v\gamma_{em}=\frac{1}{\sqrt{1-v^2/c^2}}\frac{1}{1-k_e Qq/mc^2 r}\cong\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(1+\frac{k_e Qq}{mc^2 r}\right). \tag{10.17}$$

Then, multiplying through by mc^2, we obtain the total energy of motion and electrodynamic interaction:

$$\begin{aligned}E&=mc^2\frac{dt}{d\tau}=mc^2\gamma_v\gamma_{em}=mc^2\frac{1}{\sqrt{1-v^2/c^2}}\frac{1}{1-k_e Qq/mc^2 r}\cong mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(1+\frac{k_e Qq}{mc^2 r}\right)\\&=mc^2+\frac{1}{2}mv^2+\frac{k_e Qq}{r}+\frac{1}{2}\frac{k_e Qq}{c^2 r}v^2\end{aligned}. \tag{10.18}$$

Here, in succession, we see 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the electrical interaction energy of the charged mass, and 4) the kinetic energy of the electrical energy, which are precisely the first four terms of the key energy relationship (3.11).

Now let's turn to gravitation. One would surmise based on (9.7) that all we need to do to include gravitation is extend the time dilation factor to be $dt/d\tau=\gamma_{em}\gamma_g\gamma_v$, which is in fact correct. But because the gravitational dilation (9.4) starts with a metric $c^2 d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$ while the metric used for electrodynamics is (3.5) in the form $c^2 d\tau^2=g_{\mu\nu}\mathfrak{D}x^\mu\mathfrak{D}x^\nu$ with the gauge-coordinate interval $\mathfrak{D}x^\mu\equiv dx^\mu+(q/mc)d\tau A^\mu$, we need to be careful. Because the charged mass has the associated coordinates x'^μ, we write the metric as $c^2 d\tau^2=g_{\mu\nu}\mathfrak{D}x'^\mu\mathfrak{D}x'^\nu$, with $\mathfrak{D}x'^\mu\equiv dx'^\mu+(q'/m'c)d\tau A^\mu$. So using $A^0=\phi$ and dividing out *c* the canonical time interval is

$\mathfrak{D}t' \equiv dt' + \left(q'/m'c^2\right) d\tau\phi$. Then, taking the charged mass to be at rest in the gravitational field, the metric becomes $c^2 d\tau^2 = g_{00}\mathfrak{D}x'^0\mathfrak{D}x'^0$, a.k.a. $\mathfrak{D}t'/d\tau = 1/\sqrt{g_{00}}$. And, inverting the general-case relation (10.9) prior to the specific-case imposing of $dt'/d\tau = 1$, we have:

$$\frac{dt'}{d\tau} = \left(1 - \frac{q'\phi_0}{m'c^2}\right)\frac{dt}{d\tau} = \frac{dt}{d\tau} - \frac{q'\phi_0}{m'c^2}\frac{dt}{d\tau}. \tag{10.19}$$

Therefore, using (10.19) in $\mathfrak{D}t'/d\tau = 1/\sqrt{g_{00}}$ just obtained yields:

$$\frac{1}{\sqrt{g_{00}}} = \frac{\mathfrak{D}t'}{d\tau} = \frac{dt'}{d\tau} + \frac{q'\phi}{m'c^2} = \frac{dt}{d\tau} - \frac{q'\phi_0}{m'c^2}\frac{dt}{d\tau} + \frac{q'\phi}{m'c^2} = \frac{dt}{d\tau} - \frac{q'\phi_0}{m'c^2}\frac{dt}{d\tau} + \frac{q'\phi_0}{m'c^2}\frac{dt}{d\tau} = \frac{dt}{d\tau} \equiv \gamma_g. \tag{10.20}$$

In the above, because $U^\mu = u^\mu$ in the neutral frame, we have also used the fact that $A^\mu = \phi_0 U^\mu / c = \phi_0 u^\mu / c$ for which the time component is $A^0 = \phi = \phi_0 U^0 / c = \phi_0 u^0 / c = \phi_0 dt / d\tau$. So the electrodynamic terms offset and cancel, leaving the usual time dilation relationship $dt/d\tau = 1/\sqrt{g_{00}} \equiv \gamma_g$ for a particle at rest in a gravitational field.

So now, combining (10.20) for gravitational interactions together with (10.11) for electromagnetic interactions and (9.3) for motion, the complete time dilation for all three is given most generally by:

$$\frac{dt}{d\tau} = \gamma_{em}\gamma_g\gamma_v = \frac{1}{1 - q\phi_0/mc^2}\frac{1}{\sqrt{1 - v^2/c^2}}\frac{1}{\sqrt{g_{00}}}. \tag{10.21}$$

For the special case of the Schwarzschild metric $g_{00} = 1 - 2GM/c^2 r$ and a Coulomb proper potential $\phi_0 = k_e Q/r$, and in the weak-field $2GM/c^2 r << 1$ and $(q/m)k_e Q/c^2 r << 1$, low velocity $v << 1$ linear limits, this becomes:

$$\frac{dt}{d\tau} = \gamma_{em}\gamma_g\gamma_v = \frac{1}{1 - k_e Qq/mc^2 r}\frac{1}{\sqrt{1 - v^2/c^2}}\frac{1}{\sqrt{1 - 2GM/c^2 r}} \cong \left(1 + \frac{q}{m}\frac{k_e Q}{c^2 r}\right)\left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right). \tag{10.22}$$

If we then multiply through by the rest energy $E_0 = mc^2$ of the test charge, we find a total energy:

$$\begin{aligned} E = mc^2\frac{dt}{d\tau} &= mc^2\gamma_{em}\gamma_g\gamma_v \cong mc^2\left(1 + \frac{q}{m}\frac{k_e Q}{c^2 r}\right)\left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r}v^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r}v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r}v^2 \end{aligned}. \tag{10.23}$$

This precisely reproduces, and is how we derive, the central energy relation (3.11) presented in the overview of section 3.

Keeping in mind that the electrostatic interaction in (10.23) is a repulsive interaction between like-charges, we further represent the linear limit in terms of the potentials $\Phi = -GM / r$ and $\phi = k_e Q / r$ and the potential energies $E_g = \Phi m$ and $E_{em} = \phi q$ as well as the kinetic energy $E_v = \frac{1}{2} m v^2$, also using the definite integrals in (9.6) and (10.14), as follows:

$$\begin{aligned} E = mc^2 \frac{dt}{d\tau} &= mc^2 \gamma_{em} \gamma_g \gamma_v \cong mc^2 \left(1 + \frac{q}{m} \frac{k_e Q}{c^2 r}\right)\left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 \left(1 + \frac{q}{m}\frac{\phi}{c^2}\right)\left(1 - \frac{\Phi}{c^2}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 \left(1 + \frac{E_{em}}{mc^2}\right)\left(1 - \frac{E_g}{mc^2}\right)\left(1 + \frac{E_v}{mc^2}\right). \\ &= mc^2 \left(1 - \frac{1}{mc^2}\int_\infty^r F_{em} \cdot dr\right)\left(1 + \frac{1}{mc^2}\int_\infty^r F_g \cdot dr\right)\left(1 + \frac{E_v}{mc^2}\right) \end{aligned} \tag{10.24}$$

These expressions (10.23) and (10.24) – and in particular the combination of signs in these expressions – are of fundamental interest, for reasons that we shall now review in depth. Most importantly, these lead upon close study to a direct connection with the lepton magnetic moment anomalies, and a showing of how these anomalies constitute direct empirical evidence of electromagnetic time dilation.

PART III: QUANTUM GEOMETRO-ELECTRODYNAMICS AND THE LEPTON MAGNETIC MOMENT ANOMALIES

11. Electrodynamic Time Dilation and the Magnetic Moment Anomalies: Introduction

It has been known since 1784 and thoroughly validated over the more than two centuries since, that Coulomb's inverse-square law for electrostatic interaction between two "electrical masses" a.k.a. charges is entirely analogous to Newton's inverse-square law for the interaction between two gravitational masses, with one important difference in sign: As between two like-signed gravitational masses the Newtonian force $F_g = -GMm / r^2$ is attractive while as between two like-signed electrical masses the Coulomb force $F_{em} = +k_e Qq / r^2$ is repulsive. Likewise, it has now been known for a full century that both motion and gravitation dilate time, but also with one important difference in sign: Referring to (10.24) in the nonrelativistic limit of low velocity and the Newtonian limit of weak gravitational fields, although the time dilations $dt / d\tau = 1 + \frac{1}{2} v^2 / c^2 > 1$ for motion and $dt / d\tau = 1 + GM / c^2 r > 1$ for gravitation both contain a positive sign when expressed in terms of velocity and mass-over-distance respectively, they have *opposite* signs when expressed in terms of kinetic and potential energies as $dt / d\tau = 1 + E_v / mc^2$ in the former case and as $dt / d\tau = 1 - E_g / mc^2$ in the latter. This of course is because kinetic energy $E_v = \frac{1}{2}mv^2$ always has a positive sign, but gravitational potential energy $E_g = \Phi m = -GMm/r$ always has a negative sign so that lower energy states result from two relatively-static masses moving toward one another, i.e., attracting, rather than moving apart, i.e., repelling. Simply put: kinetic energy is positive energy, but masses in a gravitational field fall down not up and so the gravitational interaction energy is negative.

Given the foregoing, even without deriving the Lorentz force law from the variational minimization (1.1) as has been done here, one might extrapolate on general principle from special and general relativity that perhaps there is time dilation occurring when two *electrical* masses interact, which dilation follows the form of gravitational time dilation that we now write as $dt/d\tau = 1 + GMm/mc^2r$, where the *m* in the numerator is the same as the *m* in the denominator due to the equivalence of gravitational and inertial mass. So in the same way that Coulomb's law follows Newton's law, up to a sign we extrapolate this to $dt/d\tau = 1 \pm k_e Qq/mc^2r$, which includes the inequivalence of electrical and inertial mass. But, were we to extrapolate this, there would still be an important question to answer as indicated by the $\pm$ sign in the latter relation: Does time dilate for the electromagnetic interaction *between two like-charges* as it does for gravitation? Or, does time dilate for the electromagnetic interaction *between two attracting-charges* as is does for gravitation? *This is a critical question, because the answer can only be one or the other but not both.* And this question may be reframed simply by asking whether $dt/d\tau = 1 + k_e Qq/mc^2r$ with a positive sign that dilates time, applies to electrical attraction, or to electrical repulsion.

Now, one might have the preconception – as did the author at first – that this time dilation should occur in the presence of electrical attraction just it does for gravitational attraction. But this is a bias, and the question raised above can only be properly answered by following the mathematics carefully from start to finish. And in fact, a very careful and deliberate study (which is why we have paid great attention to signs throughout) reveals that *this preconception is incorrect*. In fact, for electromagnetism, as deduced at (10.12), time is dilated for interactions between two like-charges as it is for gravitation, which means *time is dilated for electromagnetic repulsion and contracted for electromagnetic attraction*. This is not a trivial result: to incorrectly answer this question about the vector *direction* of the time-dilating interaction, even if everything else is correct as to vector *magnitude*, would be akin to predicting that mass will fall up rather than down in a gravitational field. Even if one could predict the correct magnitude of the acceleration, predicting that objects fall up would still be a wrong answer. This is a sign that must be gotten right. So let us review:

The prediction in (10.12) that time dilates for electrostatic repulsion is rooted directly in the derivation of the Lorentz force from the variation (1.1): by starting with the metric (3.5) which easily becomes (3.6) and (4.1), we derive a Lorentz force $d^2x^\beta/d\tau^2 = +(q/m)F^\beta{}_\sigma dx^\sigma/cd\tau$ contained in (4.4) which becomes (5.7) once the geodesic gauge of (5.6) is applied. As reviewed in section 2, for an electrostatic Coulomb interaction this will describe electrical repulsion when *Q* and *q* both have the same sign. But as was also shown starting at (10.1) which uses the same metric (3.5), and working to (10.12), this repulsive interaction will dilate time. The bridge between these two results – repulsive Coulomb force and dilated time – is the variation $0 = \delta\int_A^B d\tau$ of (1.1).

So this time dilation for electrical repulsion, not attraction, becomes an important prediction of the present theory, and one should look for ways in which it can be verified, such as were discussed at the macroscopic level following (3.11). But even better, *one should look for ways in which this electromagnetic time dilation between repelling charges might already be validated by what is well-known*. And this brings us to the magnetic moments of the leptons, i.e., the electron, and the mu and tau leptons.

Specifically, as it turns out, these questions about interaction sign and time dilation are directly tied to quantum field theory because gravitational interactions between like-charges are attractive as a result of the propagators for spin-2 gravitons, while electromagnetic interactions between like-charges are repulsive as a result of the propagators for spin-1 photons, see, e.g., section 1.5 of [12]. And, as it also turns out, and as we shall demonstrate in the development to follow, the reason the charged leptons in one-loop Schwinger order [13] have a magnetic moment g-factor $g = 2(1+\alpha/2\pi) > 2$ rather than a g-factor $g = 2(1-\alpha/2\pi) < 2$ is precisely because the time dilation $dt/d\tau = 1 + k_e Qq/mc^2 r > 1$ applies to electrical repulsion rather than electrical attraction. If the time dilation occurred for electrical attraction as it does for gravitational interactions, then the first-order g-factor would be $g = 2(1-\alpha/2\pi)$ and therefore less than 2. (Again, $\alpha = e^2/4\pi\varepsilon_0\hbar c$ is the running fine structure coupling which approaches the numerical value of $\alpha = 1/137.035999139$ [11] at low probe energies.)

As now embark upon proving this connection, we start with the well-established understanding that the magnetic moment "anomaly" arises from lepton self-interaction, and that when a lepton self-interacts whether one considers the problem classically or in quantum field theory, *this self-interaction is necessarily repulsive*. To this understanding we add (10.12), which tells us that two repelling charges dilate time. This means that lepton self-interactions will dilate time, so that this time dilation should be able to be used in some way to measure the lepton self-interaction which in turn gives rise to the magnetic moment anomalies.

Classically, one would view a lepton as a negative charge and consider different parts of the charge interacting with one another, and of course each part will repel every other part. In Quantum Electrodynamics, one extracts terms from the path integral and associates each term with a Feynman diagram which includes one or more self-interaction "loops." In the process the magnetic moment anomaly is explained, yet one can only do an exact calculation up to the loop orders that can be enumerated then calculated. Beyond three or at most four loops, exact analytical calculations become intractable. So if a direct mapping can be developed between the quantum approach and the classical approach, then the classical approach yields an advantage, because one can use ordinary calculus to do exact analytical calculations that cannot be done exactly in quantum field theory.

With all of this in mind, because lepton self-interaction is inherently repulsive and so based on (10.12) should cause time dilation, and because lepton self-interaction is also intimately and inextricably intertwined with magnetic moment anomalies, the task upon which we now embark is to show that having a g-factor $g > 2$ rather than a $g < 2$ is also directly and inextricably tied to time dilation being a result of electrically-repulsive rather than attractive interactions. This is the electrodynamic equivalent of making certain we theoretically predict that masses will fall down not up, and it ties together classical and quantum electrodynamics and reveals a universal relation between time dilation and energy of all types and origins in a way that has not heretofore been recognized.

12. The Canonical-to-Mechanical Ratio and the Lepton Magnetic Moment Anomalies

We begin by combining (10.3) with (10.5) and rearranging $U^{\mu}=u^{\mu}+\left(q\phi_0/mc^2\right)U^{\mu}$ to isolate U^{μ}, also using the definition of $\gamma_{em}=dt/d\tau$ in (10.11), to write the canonical velocity in terms of the mechanical velocity as:

$$U^{\mu}=u^{\mu}+\frac{q}{mc}A^{\mu}=u^{\mu}+\frac{q\phi_0}{mc^2}U^{\mu}=\frac{1}{1-q\phi_0/mc^2}u^{\mu}=\gamma_{em}u^{\mu}, \tag{12.1}$$

most importantly, $U^{\mu}=\gamma_{em}u^{\mu}$. Likewise, multiplying through by m we may relate the canonical momentum $\pi^{\mu}=mU^{\mu}$ to the mechanical momentum $p^{\mu}=mu^{\mu}$ by:

$$\pi^{\mu}=mU^{\mu}=mu^{\mu}+\frac{q}{c}A^{\mu}=p^{\mu}+\frac{q}{c}A^{\mu}=\frac{1}{1-q\phi_0/mc^2}p^{\mu}=\gamma_{em}p^{\mu}, \tag{12.2}$$

most importantly, $\pi^{\mu}=\gamma_{em}p^{\mu}$. This in turn means that the relativistic energy-momentum relation (3.6) may be written in terms of the time dilation factor γ_{em} and the scalar product $p_{\sigma}p^{\sigma}$ of the mechanical momentum as:

$$m^2c^2=g_{\mu\nu}\left(p^{\mu}+\frac{q}{c}A^{\mu}\right)\left(p^{\nu}+\frac{q}{c}A^{\nu}\right)=g_{\mu\nu}\pi^{\mu}\pi^{\nu}=\pi_{\sigma}\pi^{\sigma}=\gamma_{em}{}^{2}p_{\sigma}p^{\sigma}. \tag{12.3}$$

Now, in natural $c=1$ units, the relation $\pi^{\mu}=p^{\mu}+qA^{\mu}$ between the canonical momentum π^{μ} and mechanical momentum p^{μ} in (12.2) is well-known, wherein an extra term qA^{μ} is *added* to p^{μ} to arrive at π^{μ}. What is important about (12.1) through (12.3) is that this relation can also be expressed by taking the mechanical objects u^{μ} and p^{μ} and simply *multiplying* through by the single time dilation factor γ_{em} to obtain $U^{\mu}=\gamma_{em}u^{\mu}$ and $\pi^{\mu}=\gamma_{em}p^{\mu}$.

In fact, we can summarize the ratio of the canonical to the mechanical objects in both (12.1) and (12.2), and also the ratio E/mc^2 of total energy to rest energy from (10.13), also showing the $q\phi_0<<mc^2$ limit and the special case of Coulomb interactions, by the chain of ratio relations:

$$\gamma_{em}=\frac{dt}{d\tau}=\frac{\pi^{\mu}}{p^{\mu}}=\frac{U^{\mu}}{u^{\mu}}=\frac{\mathfrak{D}^{\mu}}{\partial^{\mu}}=\frac{E}{mc^2}=\frac{\text{canonical}}{\text{mechanical}}=\frac{1}{1-q\phi_0/mc^2}=\frac{1}{1-k_eQq/mc^2r}\cong 1+\frac{q\phi_0}{mc^2}=1+\frac{k_eQq}{mc^2r}. \tag{12.4}$$

In the above we have also used the heuristic relations $\pi^{\mu} \Leftrightarrow i\mathfrak{D}^{\mu}$ and $p^{\mu} \Leftrightarrow i\partial^{\mu}$. So from this view, the electromagnetic time dilation factor $\gamma_{em} = dt / d\tau$ is seen to equal the ratio of the *canonical* objects U^{μ}, π^{μ} and $E = mc^2 + E_{em}$, to the respective *mechanical* objects u^{μ}, p^{μ} and $E_0 = mc^2$. And it is seen that these ratios are spawned simply by applying a local gauge transformation which causes $\partial^{\mu} \to \mathfrak{D}^{\mu}$, which are also shown in a similar ratio $\mathfrak{D}^{\mu} / \partial^{\mu}$. It is also important to see that the term $+k_e Qq / mc^2 r$ for the Coulomb energy contained after the final equality above is the lowest order term in γ_{em}, for the limiting case where the Coulomb energy $k_e Qq / r << mc^2$. Keep in mind also from after (10.12), that $q\phi_0 < mc^2$ and for Coulomb interactions $k_e Qq / r < mc^2$ imposes a natural material limit on the strength of electromagnetic interactions between two charges, which is analogous to the upper limit established by the speed of light for material motion.

The possibility of a connection to lepton magnetic moments first comes into view when we ask how these time dilations manifest for individual charge quanta with the charge $\mp e$ of an electron or a proton and related quanta such as the mu and tau leptons, where $\alpha = e^2 / 4\pi\varepsilon_0 \hbar c$ is the running fine structure coupling. Of course, as soon as we start to talk about individual charge quanta, e.g., electrons, it is not possible even in principle to specify an exact position or momentum owing to Heisenberg uncertainty, which will be subjected to deep examination in the next section. Specifically, if we set $Q = q = -e$ in (12.4) so each of the Coulomb charges has the charge of an electron and they are thereby repelling, and using $k_e = 1/4\pi\varepsilon_0$ and $\hbar = h/2\pi$ and the standard Compton wavelength $\lambda = h/mc$ of the test particle, we first find that the key dimensionless ratio:

$$\frac{q\phi_0}{mc^2} = \frac{k_e Qq}{mc^2 r} = \frac{e^2}{4\pi\varepsilon_0 mc^2 r} = \frac{\alpha\hbar}{mcr} = \frac{h}{mcr}\frac{\alpha}{2\pi} = \frac{\alpha}{2\pi}\frac{\lambda}{r} = a_S \frac{\lambda}{r} \cong a\frac{\lambda}{r} = \left(\frac{g-2}{2}\right)\frac{\lambda}{r}. \tag{12.5}$$

In particular, an appearance is made by $a_S = \alpha / 2\pi = .00116140973242$ which is Schwinger's (subscript S) one-loop contribution to the anomalous magnetic moment of the electron, mu and tau leptons [13]. And we then also make use of the approximate fact that $a_S \cong a = (g-2)/2$, where g is the lepton g-factor and a is the empirically-observed anomaly, whether $a = a_e, a_\mu, a_\tau$ is for the electron or for the mu and tau leptons.

Then, if we insert (12.5) into (12.4) we obtain:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{\text{canonical}}{\text{mechanical}} = \frac{1}{1 - \frac{k_e Qq}{mc^2 r}} = \frac{1}{1 - \frac{\alpha}{2\pi}\frac{\lambda}{r}} \cong 1 + \frac{\alpha}{2\pi}\frac{\lambda}{r} = 1 + a_S \frac{\lambda}{r} \cong 1 + a\frac{\lambda}{r} = 1 + \left(\frac{g-2}{2}\right)\frac{\lambda}{r}. \tag{12.6}$$

Then, in the circumstance (to be reviewed in detail in the next section) where the Coulomb separation between the two lepton charges (which again raises uncertainty issues that we shall also need to consider) is equal to the Compton wavelength, i.e., when $r = \lambda$, this simplifies to

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{\text{canonical}}{\text{mechanical}} = \frac{1}{1-\alpha/2\pi} \cong 1+\frac{\alpha}{2\pi} = 1+a_S \cong 1+a = \frac{g}{2} = \frac{g}{g_0}, \tag{12.7}$$

where $g_0 = 2$ is the Dirac *g*-factor. So for an individual charge quantum, the electromagnetic time dilation – which is also the canonical-to-mechanical ratio in (12.4) – turns out when we set $r = \lambda$, to be approximately equal to one-half of the *g*-factor of all of the leptons, $\gamma_{em} = dt/d\tau \cong g/2$, up to the Schwinger one-loop order. The means that for each of the electron and the mu and tau leptons (subscript *l*), there exists a set of $r_l \cong \lambda_l$ very close to the Compton wavelength which will make $\gamma_{em} = dt/d\tau = g/2$ for each lepton, exactly. This observation that $\gamma_{em} = dt/d\tau \cong g/2$ when the separation of two, e.g., electrons is equal to the Compton wavelength of the electron (again which separation is subject to uncertainty as we shall review), raises the question whether there might be a fundamental relation among the observed lepton magnetic moments, electromagnetic time dilation, the canonical-to-mechanical ratio, and lepton self-interaction.

So the question we shall now study in depth, is whether the observed lepton magnetic moment anomalies can in fact be understood as arising directly from electromagnetic time dilation which is equal to the canonical-to-mechanical ratio. In particular, we ask whether the ratio of the observed *g*-factors $g_l = 2+2a_l$ which contain the non-zero lepton anomalies $a_l = a_e, a_\mu, a_\tau$, to the Dirac *g*-factor $g_0 = 2$, with the former regarded as canonical and the latter as mechanical objects, is in fact a direct measure and empirical confirmation of a time dilation factor $dt/d\tau$ intrinsic to the repulsive self-interactions of the electron and the mu and tau leptons. As we shall now show, the answer to this question appears to be affirmative.

13. "Canonical Co-Scaling" Directly Connecting Electromagnetic Time Dilation, Lepton Self-Interaction Energies, and Lepton Magnetic Moment Anomalies

The Particle Data Group at [14] provides a very thorough review of the muon anomalous magnetic moment. Although the numeric data developed in this review applies specifically to the muon, the exposited theoretical principles for analysis apply equally to the electron and the tau lepton. In the standard model, for a given lepton, the complete anomaly denoted in [14] as a_{SM} which we simply denote as $a = (g-2)/2$ is generally divided into three parts, namely, QED contributions, electroweak contributions, and hadronic contributions. These are then summed whereby $a = a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$, see equation 4 and Figure 1 in [14]. This may also be written in terms of the *g*-factor as $g/2 = 1+a = 1+a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$. Additionally, although a has these three contributions, it is a_{QED} which dominates the other two contributions by five or six orders of magnitude. So for the muon, as reviewed in equations 6, 9, 11 and 13 which are added to arrive at equation 15 of [14], without showing the error bars, we have $a_{\mathrm{QED}} = 116584718.95 \times 10^{-11}$, while $a_{\mathrm{EW}} = 153.6 \times 10^{-11}$ and $a_{\mathrm{Had}}[\mathrm{LO}+\mathrm{NLO}] = 7022 \times 10^{-11}$ are very much smaller. So up to parts per 10^5, one may use the very close approximation $a \cong a_{\mathrm{QED}}$. In the regard, we may denote component electromagnetic contribution to the *g*-factor as $g_{\mathrm{QED}}/2 \equiv 1 + a_{\mathrm{QED}}$. The same qualitative considerations – though not the exact same numbers – apply to the electron and the tau lepton.

Now, three quark and lepton generations are of course *empirically* observed in nature, as are the mediating bosons $W^{\pm\mu}$ and Z^{μ} of electroweak interactions, as are protons and neutrons and other hadrons. All of these are required ingredients in the anomaly calculations. But from a *theoretical* standpoint, these ingredients only arise following three developments: First, while electrodynamics which is being studied in this paper is an *abelian* interaction, the weak and strong interactions are *non-abelian*. So before we can even talk about electroweak interactions or hadrons or their effects upon lepton self-interaction theoretically, we must introduce non-abelian interactions generally. This is ordinarily done by way of Yang-Mills gauge theory [15]. Second, once we have introduced non-abelian interactions, we must know the specific non-abelian GUT gauge group and the manner of its symmetry breaking that leads to the specific ensemble of quarks and leptons and hadrons that are empirically observed in nature. Finally, relatedly, in deference to Isadore Rabi's famous quip "who ordered that?" following the discovery of the muon, we must also answer the still-unanswered question as to why nature replicates quarks and leptons into three fermion generations distinguishable only by rest mass. These latter two questions have been studied, for example, in [16] by this author.

Because this paper has focused on electrodynamics to the exclusion of weak and strong or hadronic interactions, in this section we will connect seek to connect the dominant electrodynamics-based anomaly component a_{QED} to the canonical-to-mechanical time dilation ratio $\gamma_{em} = dt / d\tau$ in the manner shown in (12.7). Based on what has just been reviewed, this result then would apply very closely because $a \cong a_{\mathrm{QED}}$ up to our neglect of the further terms $a_{\mathrm{EW}} + a_{\mathrm{Had}}$. In the following section we shall review how these electroweak and hadronic contributions may be likewise included in this connection to the time dilation $\gamma_{em} = dt / d\tau$.

The electron, and the mu and tau leptons, are all observed and understood to be indivisible elementary "point" particles without internal substructure. More precisely, insofar as we have been able to discern to date using experimental equipment capable of resolving lengths on the order of 1 Fermi and smaller, the leptons are indeed "point" particles, and there is no empirical evidence that they have any substructure. This is important, because in the early days of quantum theory the notion was entertained that an electron might be distributed with a *charge density* ρ just like the classical charge distribution contained in the current four-vector $J^{\mu} = (\rho, \mathbf{J})$ sourcing Maxwell's charge equation $J^{\mu} = \partial_{\sigma} F^{\sigma\mu}$. But it has long since been recognized that electrons and other leptons are observed as structureless point particles, and that ρ is a *probability density* for finding the structureless lepton at a given spatial position when an experiment is performed to detect the lepton. This ρ is the time component of a conserved (continuous) Dirac current

$J^{\mu} = \overline{\psi}\gamma^{\mu}\psi = (\rho, \mathbf{J})$ with $\partial_{;\mu} J^{\mu} = 0$, and specifically, as is well-known, is $J^0 = \rho = \psi^{\dagger}\psi$, given $\overline{\psi} = \psi^{\dagger}\gamma^0$ and $\eta^{00} = \gamma^0\gamma^0 = 1$. It is often said for linguistic communication, that the lepton wavefunction ψ has an associated probability density $\rho = \psi^{\dagger}\psi$, but that by the very act of observing the lepton, we "collapse" the wavefunction so as to be found at a specific point position observed somewhere in that density region. So in the discussion to follow, we must regard ρ as the *probability density* of the lepton, rather than as a classical *charge density*, with the foregoing understanding in mind.

The important point for the discussion to follow, is that even though ρ is a probability density and not a charge density, we can still use ordinary calculus to analyze self-interactions between various "pieces" of the probability distribution prior to collapse in exactly the same way we would analyze self-interactions between various "pieces" of a classical charge density. And, by doing so, we are able to arrive at results that are entirely consistent with the QED understanding of magnetic moment anomalies reviewed in [14], but which can be calculated to infinite order using the limit-taking techniques of ordinary calculus.

Specifically, in classical theory, the self-interaction of a given charge density ρ may be studied by calculating the electromagnetic interactions between and among different "portions" of that density, and by using ordinary calculus to ascertain the limit as each portion grows infinitesimally small and the combinatorial number of pairwise interactions approaches infinity. But for a quantum particle, such as an electron or the other leptons, we do not have a charge density, we have a probability density. Yet, for a probability distribution that has not yet been collapsed by an observation, we can treat the distribution no differently than we would treat a classical charge density: we simply use ordinary calculus to calculate the self-interaction energy *between and among different parts of the probability distribution*, rather than different parts of the charge density. Thus, in the same way as is done for the classical density, we use ordinary calculus to take the limit as each element of the distribution approaches infinitesimal volume while the number of pairwise interaction combinations among these elements grows infinitely large. So let us begin this calculation.

Classically, the Coulomb energy is $E_{em0} = q\phi_0 = k_e Qq / r$ between two like-signed charges separated by a distance *r*. But as (10.15) and (10.16) makes clear, this is only the lowest order term in the non-linear energy $E_{em} / mc^2 = 1/\left(1 - E_{em0} / mc^2\right)$. For the moment, let us neglect these non-linear behaviors and use the lowest-order interaction energy $E_{em0} = k_e Qq / r$ which arises from the internal self-repulsion of a lepton with a charge $-e$, where *e* is a positive number given by the running coupling $\alpha = k_e e^2 / \hbar c$ with asymptotic value $\alpha = 1/137.035999139$ at low energies.

To start, as a crude estimate of E_{em0}, we assume a spherically-symmetric lepton probability density, and we divide this density into two halves *A* and *B* each with $Q = q = -\frac{1}{2}e$, so that $E_{em0} = k_e\left(-\frac{1}{2}e\right)\left(-\frac{1}{2}e\right)/r = \frac{1}{4}k_e e^2 / r$ were we to treat this classically. But we need to proceed with care because the leptons do not have a position in the classical sense, but rather, have a probability density $J^0 = \rho = \psi^{\dagger}\psi$, and at rest, $J^0 = \rho_0 = \psi^{\dagger}\psi$. So rather than view this classically as a charge density divided into two halves, we engage in a series of observations whereby we "collapse" the wavefunction, and then record where we have observed the electron each time. Suppose that in one of our observations the lepton is detected in the *A* half, and in another one of our observations it is detected in the *B* half. We may then talk about the distance *r* between the observation that lands in *A* and the one that lands in *B*. This employs a form of conditional probability: if $\mathbf{x}_1 \,|\, A$ designates a first position $\mathbf{x}_1$ where a lepton is observed in one trial to be in the *A* half of the density, and if $\mathbf{x}_2 \,|\, B$ designates a second position $\mathbf{x}_2$ where a lepton is observed in a second trial to be in the *B* half of the density, then the magnitude $r = \left|\left(\mathbf{x}_1 \,|\, A - \mathbf{x}_2 \,|\, B\right)\right|$ tells us the radial distance between $\mathbf{x}_1$ found in *A* and $\mathbf{x}_2$ found in *B*. But this conditional separation distance is a statistical number, and it will not be the same from one trial to the next. Rather, after repeating many trials we will ascertain an *expected* value $\langle r \rangle$ representing the average distance between observations "found to land in *A*" (denoted $|\,A$) and those "found to land in *B*" (denoted $|\,B$). Statistically, $\langle r \rangle$ is the "average independent draw" separation for the detectable positions of a lepton when detected one time in half *A* and another time in half *B*. Equivalently, $\langle r \rangle$ is the weighted average separation between the two halves of the probability distribution ρ, and is mathematically (but not physically) the same as the average weighted separation between two equal halves of a classical charge density ρ.

So as a consequence, given the position uncertainty of the lepton, for any two random draws which land in different halves, we cannot talk directly about $E_{em0} = \tfrac{1}{4} k_e e^2 / r$, but – taking the expected value of each side of the foregoing – only about the expectation value $\langle E_{em0} \rangle = \left\langle \tfrac{1}{4} k_e e^2 / r \right\rangle = \tfrac{1}{4} k_e e^2 \langle 1/r \rangle$ of the interaction energy of that draw pair, which is dependent upon this positon uncertainty. However, following a very large number of draws, or, for a probability density which is not collapsed and is teeming with huge numbers of self-interactions such as those shown by the loops diagrams in Figure 1 of [14] and higher orders thereof, we may remove the expectation value brackets from the energy and simply write $E_{em0} = \tfrac{1}{4} k_e e^2 \langle 1/r \rangle$. This is because although $\langle 1/r \rangle$ is obtained from probabilistic events, it is still a number with a definite value just as is the standard deviation of a probability distribution. Let us discuss why this is so from both a statistical and a physical viewpoint.

Statistically, a probability distribution has a standard deviation, and the standard deviation is a fixed number which has a definite relation to the average weighted separation between two randomly-selected points in that distribution. But *it makes no sense to speak of the expected value of the standard deviation* for a given distribution. It is just a standard deviation. In fact, the variance (squared standard deviation) of a distribution is *defined* as the average of the squared spread about the expected value, $\sigma_x{}^2 \equiv \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2$. And for a mean $\langle x \rangle = 0$ the standard deviation is simply $\sigma_x = \sqrt{\langle x^2 \rangle}$, which is a Pythagorean relationship with a dimensionality akin to the very large number of samples that comprise $\langle x^2 \rangle$. For example, for a

Gaussian distribution represented along a single dimension labelled *x*, it is well-known that $\langle x\rangle = 2\sigma_x / \sqrt{\pi} \cong 1.128379\sigma_x$ is the weighted average draw separation, and is directly related to σ_x as such. Consequently, $E_{em0} = \frac{1}{4}k_e e^2 \langle 1/r\rangle$ – just like $\sigma_x = \sqrt{\langle x^2\rangle}$ when centered about zero – will also be a direct function of σ_x and so be a definite number defined by an expectation value.

Another way to see this – physically rather than statistically – is to keep in mind that in QED the observed rest energy $E = mc^2$ of a lepton is equal to the lepton's bare energy denoted $E_0 = m_0 c^2$ plus its electromagnetic self-interaction energy E_{em} (neglecting electroweak and hadronic contributions). That is, $mc^2 = m_0 c^2 + E_{em}$ with E_{em} arising from tremendous numbers of self-interaction events such as those shown in Figure 1 of [14] (the very large number of samples that comprise $\langle x^2\rangle$ in the discussion of the prior paragraph). One does not add $\langle E_{em}\rangle$ to $m_0 c^2$ to obtain what would then be an *expected value* of the mass in $\langle m\rangle c^2 = m_0 c^2 + \langle E_{em}\rangle$. Rather, the observed rest mass *m* is a definite number observed with high precision to be unchanged from one observation to the next, not a number with an expected $\langle m\rangle$ that varies from one observation to the next and so has some standard deviation σ_m about the mean. The electron rest mass $m_e = 0.510998928\text{ MeV}$ [17] is not an *average or expected* detected mass of the electron; it is the *always-detected* mass of the electron. For a physical rest mass $mc^2 = m_0 c^2 + E_{em}$ such as that of a lepton, $\sigma_m = 0$, and E_{em} is a fixed number connected to an expected draw separation connected to a standard deviation all of which are fixed numbers. So $E_{em0} = \frac{1}{4}k_e e^2 \langle 1/r\rangle$ says that the first order self-interaction energy E_{em0} itself – not the expected value of that energy – between two halves of a lepton probability density which has not been collapsed by an observation is equal to the numeric coefficient $\frac{1}{4}k_e e^2 = \frac{1}{4}\alpha\hbar c$ times the expected value of the inverse separation, $\langle 1/r\rangle$.

It is also important to note that in general, $\langle 1/r\rangle \neq 1/\langle r\rangle$ are not the same. This is because for a probability distribution $\rho(r)$, the expectation value $\langle r\rangle = \int_{-\infty}^{\infty} r\rho(r)\,dr$ thus $1/\langle r\rangle = 1/\int_{-\infty}^{\infty} r\rho(r)\,dr$, while the expectation value of the inverse is $\langle 1/r\rangle = \int_{-\infty}^{\infty} (1/r)\rho(r)\,dr$ which is not equal to $1/\langle r\rangle$. Also, in general, as is well known in statistics, $\langle 1/r\rangle > 1/\langle r\rangle$ for positive *r*. The only distribution for which $\langle 1/r\rangle = 1/\langle r\rangle$ is a Dirac delta $\rho(r) = \delta(r)$; as soon as there is *any finite spread* in the distribution, that is, a standard deviation greater than zero, we will always have $\langle 1/r\rangle > 1/\langle r\rangle$. As a result, $E_{em0} = \frac{1}{4}k_e e^2 \langle 1/r\rangle > \frac{1}{4}k_e e^2 / \langle r\rangle$, which is to say that the self-interaction energy E_{em0} between the two halves of the density will always be greater than $\frac{1}{4}k_e e^2 / \langle r\rangle$, where $\langle r\rangle = \int_{-\infty}^{\infty} r\rho(r)\,dr$ and $\langle 1/r\rangle = \int_{-\infty}^{\infty} (1/r)\rho(r)\,dr$ are both determined by the precise nature of the probability distribution, with the latter $\langle 1/r\rangle$ determining the precise value of the self-interaction energy via $E_{em0} = \frac{1}{4}k_e e^2 \langle 1/r\rangle$.

With all of the foregoing in mind, having split the probability density into two halves, as a second step, let us now split the probability density into three equal thirds, with the same average draw separation $\langle r\rangle$ relative to one another. Now, we will have three pairwise interactions that all need to be summed, so $E_{em0} = 3k_e\left(\tfrac{1}{3}e\right)\left(\tfrac{1}{3}e\right)\langle 1/r\rangle = \tfrac{1}{3}k_e e^2\langle 1/r\rangle > \tfrac{1}{3}k_e e^2/\langle r\rangle$ is the self-interaction energy. This is still approximate, but less so than the split into halves. For a third step, we split the distribution into four equal quarters, still with $\langle r\rangle$ separating the quarters one from another. There are now $C(4,2) = 4\cdot 3/2 = 6$ pairwise interactions, so the self-interaction energy is now $E_{em0} = 6k_e\left(\tfrac{1}{4}e\right)\left(\tfrac{1}{4}e\right)\langle 1/r\rangle = \tfrac{3}{8}k_e e^2\langle 1/r\rangle > \tfrac{3}{8}k_e e^2/\langle r\rangle$. One can visualize this relationship via a pyramid with its center at the center of the (spherically-symmetric) distribution and one quarter of the distribution centered at each vertex of the pyramid. This is closer still to the exact energy, than were the two- or three-part divisions. But to really be exact, we now need to keep going with more and more splits, and we now need calculus. Specifically:

We may generalize the above to any number of "pieces" but for the following matter: The physical space is three dimensional, so were we to split the distribution into 5 or more equally-charged, equally-spaced portions, we would need to visualize this using a hyper-pyramid to locate the vertices. Let us momentarily ignore this matter, and split the distribution into N equal parts, each separated by $\langle r\rangle$, in an $N-1$-dimensional space. Now the number of pairwise interactions is $C(N,2) = N\cdot(N-1)/2 = \left(N^2-N\right)2$, and so the overall expected interaction energy will be $E_{em0} = C(N,2)k_e\left(\tfrac{1}{N}e\right)\left(\tfrac{1}{N}e\right)\langle 1/r\rangle = \left(\tfrac{1}{2}\left(N^2-N\right)/N^2\right)k_e e^2\langle 1/r\rangle > \left(\tfrac{1}{2}\left(N^2-N\right)/N^2\right)k_e e^2/\langle r\rangle$. Finally, we may take the calculus limit as $N\rightarrow\infty$, to find that:

$$E_{em0} = \lim_{N\rightarrow\infty}\left(C(N,2)k_e\left(\tfrac{1}{N}e\right)\left(\tfrac{1}{N}e\right)\left\langle\frac{1}{r}\right\rangle\right) = \lim_{N\rightarrow\infty}\left(\frac{1}{2}\left(\frac{N^2-N}{N^2}\right)k_e e^2\left\langle\frac{1}{r}\right\rangle\right) = \frac{1}{2}k_e e^2\left\langle\frac{1}{r}\right\rangle > \frac{1}{2}\frac{k_e e^2}{\langle r\rangle}. \quad (13.1)$$

Although this is derived in what has become an abstract infinite-dimensional space, the fact that we are using a statistical average inverse separation $\langle 1/r\rangle$ allows us to regard (13.1) in the calculus limit as an *exact* expression in three space dimensions for the (lowest order, per (10.15)) interaction energy arising from an infinite number of pairwise interactions between infinitesimally-small charge elements, on one condition: We must now regard $\langle 1/r\rangle$ to be the weighted expected value of the inverse separation of any two independent "draws" of a lepton *from anywhere in the undivided distribution*, *without the conditional probabilities* required when we artificially subdivide the lepton into two or three or four etc. discrete and not-infinitesimal charges e/N as was used to construct the calculus limit in (13.1). Put differently, once we no longer subdivide a lepton into N parts, we longer need to think about an $N-1$ dimensional space, because that space is built upon the artificial partitioning of the lepton probability density into N pieces and the measurement of the expected separation between any two pieces conditioned on a draw from those two pieces. Now, we just think about the average separation between two independent draws taken from anywhere in the lepton probability density, unconditionally. And as earlier noted, $\langle 1/r\rangle > 1/\langle r\rangle$ a.k.a. $\langle r\rangle > 1/\langle 1/r\rangle$ for other than a Dirac delta. So with $\langle r\rangle$ representing the unconditional expected draw separation, which is now also the mean average separation between any two randomly-selected points of the probability density ρ, this $\langle r\rangle$ will bear a direct relation to the statistical standard deviation σ of the probability density, again, with the exact relation dependent upon the exact type of distribution. Again, as noted, for example, $\langle x\rangle = 2\sigma_x/\sqrt{\pi}$ for a one-dimensional Gaussian. So (13.1) tells us the result that the lowest order Coulomb self-energy of the lepton $E_{em0} = \frac{1}{2}k_e e^2\langle 1/r\rangle$, exactly. The inverse $\langle 1/r\rangle$ is a definite number; so too therefore is E_{em0}.

Next, turning to higher orders, referring to (10.16), we emphasize that (13.1) was calculated using the *linear* Coulomb interaction energy $E_{em0} = q\phi_0 = k_e Qq/r = k_e e^2/r$ for two charge quanta $Q = q = -e$. An important finding in (10.15) which is represented in terms of energy by (10.16) is that E_{em0} is merely the lowest order energy just as $E_v \cong \frac{1}{2}mv^2$ in (9.3) is the lowest order term in the special relativistic energy of motion and $E_g \cong -GMm/r$ in (9.5) is the lowest-order term in the gravitational energy for the Schwarzschild solution of general relativity. In this sense, (10.15) and (10.16) are very important results, because they introduce heretofore-unrecognized non-linear behaviors into classical electrodynamics when the interaction energy grows large in relation to the test particle rest energy, and they also introduce a material limit $q\phi_0/mc^2 < 1$ in the strength of electromagnetic interactions which becomes $k_e Qq/mc^2 r < 1$ for Coulomb interactions and $k_e e^2/mc^2 r < 1$ for Coulomb interactions with $Q = q = -e$. So if we now use the first order self-energy (13.1) in the full expression of (10.16) we may obtain the complete self-interaction energy of the lepton probability density including all of the non-linear terms, namely:

$$\frac{E_{em}}{mc^2} = \frac{1}{1-\dfrac{k_e e^2}{mc^2}\left\langle \dfrac{1}{2r}\right\rangle} - 1 = \frac{\dfrac{k_e e^2}{mc^2}\left\langle \dfrac{1}{2r}\right\rangle}{1-\dfrac{k_e e^2}{mc^2}\left\langle \dfrac{1}{2r}\right\rangle} = \frac{k_e e^2}{mc^2}\left\langle \frac{1}{2r}\right\rangle \sum_{n=0}^{\infty}\left(\frac{k_e e^2}{mc^2}\left\langle \frac{1}{2r}\right\rangle\right)^n . \tag{13.2}$$

Then, given $k_e = 1/4\pi\varepsilon_0$, we may multiply the third and sixth expressions in (12.5) through by r to obtain the relation $k_e e^2/mc^2 = (\alpha/2\pi)\lambda$. Using this in (13.2) and adding 1 throughout, we obtain the selected terms:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{E}{mc^2} = \frac{mc^2 + E_{em}}{mc^2} = 1 + \frac{E_{em}}{mc^2} = \frac{1}{1-\dfrac{\alpha}{2\pi}\lambda\left\langle \dfrac{1}{2r}\right\rangle} = \sum_{n=0}^{\infty}\left(\frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{2r}\right\rangle\right)^n = \frac{\text{canonical}}{\text{mechanical}}, \tag{13.3}$$

where $E = mc^2 + E_{em}$ is the total rest energy of the lepton, and where we have also made use of $\gamma_{em} = dt/d\tau = E/mc^2$ from (12.4) to relate everything to the electromagnetic time dilation as well as the canonical-to-mechanical ratio.

Now, given the context of the lepton self-energy $E = mc^2 + E_{em}$ contained in the above, let us focus on the meaning of the rest energy mc^2 in (13.3). In general, classically, when a test particle is placed into an electromagnetic field, mc^2 is the rest energy of the test particle when there is no electromagnetic interaction (formally, prior to imposing local gauge symmetry). Once that test particle is given a charge q and placed into a proper electromagnetic potential ϕ_0, the interaction energy becomes $E_{em} = q\phi_0$ (formally, after imposing gauge symmetry). And in the special case of a Coulomb potential, this energy is $E_{em} = +k_e Qq / r$ for two like-charges, which signifies a repulsive interaction for which the energy diminishes as the two charges are separated. Therefore, mc^2 is the "starting energy" neglecting electrodynamics and $E = mc^2 + E_{em}$ is the total energy including the energy of electrodynamic interaction.

In (13.3) for an individual lepton self-interacting, mc^2 is likewise the "starting energy" neglecting electrodynamics, while E_{em} is the self-interaction energy. Therefore, $E = mc^2 + E_{em}$ is the total energy of the lepton including self-interactions. But the only lepton energy we ever observe in an experiment is the so called "dressed" energy mc^2 with the self-interaction energy already "baked in." This is central to the use of the Ward –Takahashi identities in QED. Specifically, the "dressed" energy mc^2 of a lepton is equal to its "bare" energy denoted $m_0 c^2$ plus its self-interaction energy E_{em}, that is, $mc^2 = E = m_0 c^2 + E_{em}$ (neglecting comparatively-small electroweak and hadronic loop contributions). So given that $mc^2 + E_{em}$ in (13.3) is the sum of a "starting" energy plus self-interaction energy, we must *reinterpret* mc^2 in (13.3) as the unobserved *bare energy*, and therefore replace $mc^2 \to m_0 c^2$ in (13.3). Likewise, we must reinterpret $E = m_0 c^2 + E_{em}$ as the observed rest energy mc^2, and so also replace the total $E \to mc^2$. With these replacements (13.3) becomes:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{m}{m_0} = \frac{m_0 c^2 + E_{em}}{m_0 c^2} = \frac{1}{1 - \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{2r} \right\rangle} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{2r} \right\rangle \right)^n = \frac{\text{canonical}}{\text{mechanical}}. \tag{13.4}$$

Now, the canonical-to-mechanical time dilation ratio first seen in (12.4) and then (12.6) and (12.7) also represents *the ratio of the total dressed mass to the bare mass*, in the form of $\gamma_{em} = dt / d\tau = m / m_0$.

Next we consider the special case where $\lambda\langle 1/2r \rangle = 1$, which because $\langle 1/r \rangle > 1/\langle r \rangle$ from basis statistics, implies that $\lambda\langle 1/2r \rangle = 1 > \lambda / \langle 2r \rangle$ a.k.a. $\langle 2r \rangle > \lambda$ with the exact $\langle r \rangle$ dependent on the precise form of the density ρ. In this special case, (13.4) will reduce to:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{m}{m_0} = \frac{1}{1 - \alpha / 2\pi} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{2\pi} \right)^n = 1 + \frac{\alpha}{2\pi} + \ldots = \frac{\text{canonical}}{\text{mechanical}} \cong 1 + a_{\text{QED}} = \frac{g_{\text{QED}}}{2} \cong 1 + a = \frac{g}{2}. \tag{13.5}$$

So in this special case where $\lambda\langle 1/2r\rangle = 1$, the canonical-to-mechanical ratio of m/m_0 of the dressed mass to the bare mass is approximately equal to $g_{\text{QED}}/2 \cong g/2$, as is $\gamma_{em} = dt/d\tau$.

This approximation of $\gamma_{em} = dt/d\tau = m/m_0$ to $g/2$ comes about in the special case where $\lambda\langle 1/2r\rangle = 1$, exactly. This means, in turn, that there exists some dimensionless ratio $\mathrm{P} \cong 1$ *defined for each lepton* by the approximation $1/\mathrm{P} \equiv \lambda\langle 1/2r\rangle \cong 1$ *such that* $m/m_0 = g_{\text{QED}}/2$ *exactly*. In other words, if $\lambda\langle 1/2r\rangle = 1$ leads to $\gamma_{em} = dt/d\tau = m/m_0 \cong g_{\text{QED}}/2$, then there is some approximate $\mathrm{P} \cong 1$ such that the canonical-to-mechanical $\gamma_{em} = dt/d\tau = m/m_0 = g_{\text{QED}}/2$ *exactly*. And it also means that there is a slightly different ratio $1/\mathrm{P}' \equiv \lambda\langle 1/2r'\rangle \cong 1$ which may be defined for each lepton *such that* $\gamma_{em} = dt/d\tau = m/m_0 = g/2$. Given $g/2 = 1 + a = 1 + a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$ and $a_{\text{EW}} + a_{\text{Had}} << a_{\text{QED}}$, the very slight difference between each lepton's P and P' will be driven directly by the very slight difference between a and a_{QED} based on the electroweak and hadronic contributions to a, as will be reviewed in the next section. So defining $\lambda\langle 1/2r\rangle \equiv 1/\mathrm{P} \cong 1$ *such that* $\gamma_{em} = dt/d\tau = m/m_0 = g_{\text{QED}}/2$ exactly, and using this in (13.4), yields:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{m}{m_0} = \frac{m_0c^2 + E_{em}}{m_0c^2} = \frac{1}{1-\alpha/2\pi\mathrm{P}} = \sum\nolimits_{n=0}^{\infty}\left(\frac{\alpha}{2\pi\mathrm{P}}\right)^n = \frac{\text{canonical}}{\text{mechanical}} \equiv \frac{g_{\text{QED}}}{2} = 1 + a_{\text{QED}}. \quad (13.6)$$

This $1/\mathrm{P}$ may be calculated by algebraically rearranging the relation between the fifth and eighth expressions above and by using $g_{\text{QED}}/2 = 1 + a_{\text{QED}}$,and using $\langle 1/r\rangle > 1/\langle r\rangle$, into:

$$\frac{1}{\mathrm{P}} \equiv \lambda\left\langle\frac{1}{2r}\right\rangle = \frac{2\pi}{\alpha}\frac{g_{\text{QED}}-2}{g_{\text{QED}}} = \frac{4\pi}{\alpha}\frac{a_{\text{QED}}}{g_{\text{QED}}} = \frac{2\pi}{\alpha}\frac{a_{\text{QED}}}{1+a_{\text{QED}}} > \frac{\lambda}{\langle 2r\rangle}. \quad (13.7)$$

This relationship applies independently to each of the three leptons; that is, just as there are three distinct $\lambda_e, \lambda_\mu, \lambda_\tau$ and g_e, g_μ, g_τ and related anomalies, so too there are three distinct $\mathrm{P}_e, \mathrm{P}_\mu, \mathrm{P}_\tau$ and three distinct $\langle r_e\rangle, \langle r_\mu\rangle, \langle r_\tau\rangle$.

Given that $1/\mathrm{P} \equiv \lambda\langle 1/2r\rangle > \lambda/\langle 2r\rangle$, this also means that $\langle 2r\rangle > \mathrm{P}\lambda$ establishes a lower bound on the expected draw separation $\langle r\rangle$. For example, using the five-loop magnitude for the muon anomaly $a_{\text{QED}\mu} = 116584718.95(0.08)\times 10^{-11}$ from equation 6 of [14] which is also based on using $\alpha = 1/137.035999049$, we may use (13.7) to deduce that $\mathrm{P}_\mu = 0.9973552022323$ and:

$$\langle 2r_\mu\rangle > \mathrm{P}_\mu\lambda_\mu = \frac{\alpha}{2\pi}\frac{1+a_{\text{QED}}}{a_{\text{QED}}}\lambda_\mu = 0.9973552022323\cdot\lambda_\mu, \quad (13.8)$$

where $\lambda_\mu = h / m_\mu c$ is the Compton wavelength of the muon. In other words, the expected value of twice the radial draw separation (which one may roughly think of as a "statistical diameter" $\langle d\rangle \equiv \langle 2r\rangle > \mathrm{P}\lambda$ of the probability density) has a lower bound which is very slightly less than the Compton wavelength of the muon. Recalling that we would have $\langle 2r\rangle = \mathrm{P}\lambda$ only for a density $\rho(r) = \delta(r)$ which is a Dirac delta, the observed $\langle d\rangle \equiv \langle 2r\rangle$ for a lepton probability density with any significant spatial spread i.e., standard deviation $\sigma_{\mathbf{x}}$ is thus expected to be somewhat larger than the Compton wavelength of that lepton.

Next, let us generally write a_{QED} for each lepton as a Maclaurin series for $a_{\mathrm{QED}} = f(x)$, with coefficients $C_n = f^{(n)}(0)$ representing the n^{th} derivatives of $f(x)$ at $x = 0$, thus:

$$a_{\mathrm{QED}} = \sum\nolimits_{n=1}^{\infty} \frac{C_n}{n!}\left(\frac{\alpha}{2\pi}\right)^n . \tag{13.9}$$

Here, each lepton has its own set of coefficients C_n, but with $C_1 = 1$ for all three leptons whereby the first, dominant term in the series for all leptons is $a_{\mathrm{QED}} = \alpha / 2\pi = a_S$ from Schwinger [13]. Then also with $C_0 = 1$ for all leptons, the P ratio from (13.7) which establishes the numeric coefficient of the Compton wavelengths in the form $\langle d\rangle = \langle 2r\rangle > \mathrm{P}\lambda$ as in (13.8), takes on the form (below, we invert (13.7) then apply (13.9)):

$$\mathrm{P} = \frac{\alpha}{2\pi}\left(\frac{1+a_{\mathrm{QED}}}{a_{\mathrm{QED}}}\right) = \frac{\frac{C_0}{0!}\frac{\alpha}{2\pi} + \frac{C_1}{1!}\left(\frac{\alpha}{2\pi}\right)^2 + \frac{C_2}{2!}\left(\frac{\alpha}{2\pi}\right)^3 + \dots}{\frac{C_1}{1!}\frac{\alpha}{2\pi} + \frac{C_2}{2!}\left(\frac{\alpha}{2\pi}\right)^2 + \frac{C_3}{3!}\left(\frac{\alpha}{2\pi}\right)^3 + \dots} = \frac{\sum_{n=0}^{\infty} \frac{C_n}{n!}\left(\frac{\alpha}{2\pi}\right)^n}{\sum_{n=0}^{\infty} \frac{C_{n+1}}{(n+1)!}\left(\frac{\alpha}{2\pi}\right)^n} < \frac{\langle d\rangle}{\lambda} . \tag{13.10}$$

We may also use (13.9) in (13.6) to write (13.6) as:

$$\gamma_{em} = \frac{dt}{d\tau} = \frac{m}{m_0} = \frac{1}{1-\alpha / 2\pi\mathrm{P}} = \sum\nolimits_{n=0}^{\infty}\left(\frac{\alpha}{2\pi\mathrm{P}}\right)^n = \frac{g_{\mathrm{QED}}}{2} = 1 + a_{\mathrm{QED}} = \sum\nolimits_{n=0}^{\infty} \frac{C_n}{n!}\left(\frac{\alpha}{2\pi}\right)^n . \tag{13.11}$$

This includes a direct relation $\sum_{n=0}^{\infty}(C_n / n!)(\alpha / 2\pi)^n = \sum_{n=0}^{\infty}(\alpha / 2\pi\mathrm{P})^n = 1/(1-\alpha / 2\pi\mathrm{P})$ between the series with the QED loop coefficients C_n and the series with P which is equal to the closed mathematical function $1/(1-\alpha / 2\pi\mathrm{P})$.

In addition to $C_0 = C_1 = 1$, from equation 5 in [14] it is possible to arithmetically obtain the Maclaurin-style C_2 through C_5 as defined in (13.9), for the muon. These are computed for the muon to be $C_2 = 2! \cdot 2^2 \cdot .765857425 = 6.12685940$, and likewise, without showing the detailed computation, $C_3 = 1154.424478$, $C_4 = 50,257.77$, and $C_5 = 2,892,672$. From the leading terms in (13.10) this means that:

$$\mathrm{P}_\mu = \frac{\frac{\alpha}{2\pi} + \left(\frac{\alpha}{2\pi}\right)^2 + \ldots}{\frac{\alpha}{2\pi} + \frac{C_2}{2!}\left(\frac{\alpha}{2\pi}\right)^2 + \ldots} = \frac{\frac{\alpha}{2\pi} + \left(\frac{\alpha}{2\pi}\right)^2 + \ldots}{\frac{\alpha}{2\pi} + 3.06342970\left(\frac{\alpha}{2\pi}\right)^2 + \ldots} < \frac{\langle d \rangle}{\lambda}. \tag{13.12}$$

Here, it will be noticed that the ratio P from (13.10) which sets the lower bound $\langle 2r \rangle > \mathrm{P}\lambda$ on the weighted expected separation between independent draws from the lepton probability density (and because of the proportionality $\sigma_r \propto \langle r \rangle$, also on the standard deviation of the density), will change as the abelian running electromagnetic coupling α increases for deeper probes of the lepton probability density. For example, from (13.8) this ratio has the numeric value $\mathrm{P}_\mu = 0.9973552022323$ for the muon, but only for the asymptotic $\alpha = 1/137.035999049$. From (13.12) it is clear that as one probes more deeply into the muon density with the abelian coupling α becoming larger, the denominator will grow faster than the numerator, so P_μ will be detected to grow smaller, and therefore, the lower bound $\sigma_r \propto \langle 2r \rangle > \mathrm{P}\lambda$ will also grow smaller. Thus, the lower bound on the standard deviation of the muon probability density will diminish. And it is clear that it is the two-loop (current density J^4) coefficient $C_2/2!$ which determines this behavior.

Given the above, we may also study how P_e behaves for the electron. The coefficients of (13.9) for the electron have been calculated by many authors, and are well-summarized in equations 3, 5 and 6 of [18]. These are computed in the Maclaurin-style of (13.9) to be $C_2 = -2.62783172463354$, $C_3 = 56.69958989$, $C_4 = -733.6704$ and $C_5 = 35174.4$. We may first use these in (13.9) along with $\alpha = 1/137.035999049$ to calculate that

$$a_{\mathrm{QED}} = \frac{\alpha}{2\pi} + \frac{C_2}{2!}\left(\frac{\alpha}{2\pi}\right)^2 + \frac{C_3}{3!}\left(\frac{\alpha}{2\pi}\right)^3 + \frac{C_4}{4!}\left(\frac{\alpha}{2\pi}\right)^4 + \frac{C_5}{5!}\left(\frac{\alpha}{2\pi}\right)^5 = 0.00115965217732. \tag{13.13}$$

This should also be contrasted to the empirical $a = a_{\mathrm{QED}} + a_{\mathrm{EW}} + a_{\mathrm{Had}} = 0.00115965218076(27)$ that includes electroweak and hadronic contributions and is slightly different starting at the 10^{-11} position. We may then use (13.13) and $\alpha = 1/137.035999049$ in (13.10) to calculate that:

$$\mathrm{P}_e = \frac{\alpha}{2\pi}\left(\frac{1 + a_{\mathrm{QED}}}{a_{\mathrm{QED}}}\right) = 1.00267699854089 < \frac{\langle d \rangle}{\lambda}. \tag{13.14}$$

Further, using (13.13) in the leading terms of (13.10), contrast (13.12) for the muon, we find that:

$$\mathrm{P}_e = \frac{\frac{\alpha}{2\pi} + \left(\frac{\alpha}{2\pi}\right)^2 + \ldots}{\frac{\alpha}{2\pi} + \frac{C_2}{2!}\left(\frac{\alpha}{2\pi}\right)^2 + \ldots} = \frac{\frac{\alpha}{2\pi} + \left(\frac{\alpha}{2\pi}\right)^2 + \ldots}{\frac{\alpha}{2\pi} - 1.31391586231677\left(\frac{\alpha}{2\pi}\right)^2 + \ldots} < \frac{\langle d \rangle}{\lambda} . \tag{13.15}$$

From (13.14) we learn that the lower bound $\langle d_e \rangle > 1.00267699854089 \cdot \lambda_e$ for the electron density spread is actually slightly *larger* than the electron Compton wavelength, in contrast to (13.8) which shows that for the muon this is smaller than the Compton wavelength. And from (13.15) we learn that unlike the muon, the electron lower bound $\langle 2r \rangle > \mathrm{P}\lambda$ will grow *larger* as the running α grows larger. This means that for a deeper probe of the electron the standard deviation $\sigma_r \propto \langle r \rangle$ of the probability density will therefore also be detected to grow larger. Contrasting the electron with the muon, it will be understood that the behaviors of both (13.14) and (13.15) emanate from the fact that for the electron the empirical $a_e < \alpha / 2\pi = a_S$, while for the muon the behaviors of (13.8) and (13.12) stem from the empirical $a_\mu > \alpha / 2\pi = a_S$. And from this we may infer that because $a_\tau > \alpha / 2\pi$ for the tau lepton just like for the muon, the expected draw separation from the tau density will have a minimum value that is *smaller* than the tau Compton wavelength, and will be seen to *decrease* its standard deviation as it is probed more deeply.

In sum, as a result of defining the ratio $\mathrm{P} \cong 1$ in (13.6) *such that* $m_0 / m = g_{\mathrm{QED}} / 2$ *exactly*, which ratio is deduced in (13.7) and used in (13.8) for the muon and (13.14) for the electron, we may now combine (13.11) with (12.4) and use $\alpha = k_e e^2 / \hbar c$ to write, *for the electromagnetic self-interaction of leptons*:

$$\begin{aligned}\gamma_{em} &= \frac{dt}{d\tau} = \frac{\pi^\mu}{p^\mu} = \frac{U^\mu}{u^\mu} = \frac{\mathfrak{D}^\mu}{\partial^\mu} = \frac{m}{m_0} = \frac{m_0 c^2 + E_{em}}{m_0 c^2} = 1 + \frac{E_{em}}{m_0 c^2} = 1 + a_{\mathrm{QED}} = \frac{g_{\mathrm{QED}}}{2} = \frac{g_{\mathrm{QED}}}{g_{\mathrm{D}}} = \frac{\text{canonical}}{\text{mechanical}} \\ &= \frac{1}{1 - \alpha / 2\pi \mathrm{P}} = \frac{1}{1 - k_e e^2 / hc\mathrm{P}} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{2\pi \mathrm{P}}\right)^n = \sum_{n=0}^{\infty} \frac{C_n}{n!}\left(\frac{\alpha}{2\pi}\right)^n\end{aligned} . \tag{13.16}$$

This very important ratio chain describes the repulsive self-interaction of a single lepton, where $mc^2 = m_0 c^2 + E_{em}$, where m is the dressed mass and m_0 is the bare mass and E_{em} is the electromagnetic self-interaction energy of the lepton. And it must be emphasized that (13.16) applies to the particular circumstance where we are considering lepton self-interaction *separately from any external potential*, so that E_{em} is the internal electromagnetic interaction energy which contributes to the dressed mass $mc^2 = m_0 c^2 + E_{em}$. This means that in particular, π^μ / p^μ above must be understood as a ratio $\pi^\mu / p^\mu \to p^\mu / p_0{}^\mu = mu^\mu / m_0 u^\mu = m / m_0$ of the dressed mechanical momentum to the bare mechanical momentum. Or, to put a point on it, as the ratio of the canonical-mechanical momentum to the mechanical-mechanical momentum.

So as we surmised might be the case in (12.7) – at least to the degree that we neglect the electroweak and hadronic contributions to $a = a_{\mathrm{QED}} + a_{\mathrm{EW}} + a_{\mathrm{Had}}$ – (13.16) teaches that the ratio $g_{\mathrm{QED}}/2$ is indeed a direct measure of a time dilation factor $\gamma_{em} = dt/d\tau$ intrinsic to the self-interaction of each of the leptons, and in turn is also a direct measure of the canonical-to-mechanical ratio first uncovered at (12.4). Conversely, obtaining the anomalous magnetic moment from theoretical first principles boils down to ascertaining the ratio of canonical objects to mechanical objects, and equivalently, to ascertaining a time dilation factor $dt/d\tau$ intrinsic to the electromagnetic self-interactions within the lepton which give rise to the self-interaction energy E_{em}. The result within (13.17) that $m/m_0 = 1 + a_{\mathrm{QED}}$, is as direct a statement as can be made that the anomalous magnetic moments arise from and directly measure the electromagnetic energy of lepton self-interactions, and provide a simple way to calculate the unobserved bare masses directly from the observed masses and the associated anomalies.

We shall use the term "canonical co-scaling" to describe this very significant result whereby the lepton g-factors $g_{\mathrm{QED}} : g_{\mathrm{D}} = \text{canonical} : \text{mechanical}$ co-scale in direct proportion to the ratio of all of the other canonical-to-mechanical quantities in (13.16), and most importantly, in direct proportion to the electromagnetic time dilation factor $\gamma_{em} = dt/d\tau$. Very importantly, as has been demonstrated throughout this paper, these canonical quantities all arise from their mechanical counterparts simply from the requirement for local gauge symmetry which spawns the ratio $\mathfrak{D}^\mu/\partial^\mu$ of gauge-covariant to ordinary derivatives. Simply put: the anomalous magnetic $g > 2$ (or so far, at least $g_{\mathrm{QED}} > 2$) themselves arise out of the Dirac $g_0 = 2$, also simply from applying local gauge symmetry. So these are just another consequence of how ordinary derivatives become gauge-covariant derivatives, $\partial_\sigma \to \mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$, as a consequence of requiring local gauge symmetry. No more, and no less.

Also, in (13.16) *the magnetic moment anomalies provide direct empirical validation of the time dilation predicted in (10.12), and establish that in the physical world, time is in fact dilated for interactions between repelling like-charges, rather than between attracting thus unlike-charges*. Referring back to the discussion at the start of section 11, this is because if $dt/d\tau = 1 + k_e Qq/mc^2 r$ in lowest order was for electrical attraction and not repulsion, which was the author's own initial misconception due to using the analogous $dt/d\tau = 1 + GMm/mc^2 r$ from gravitational theory, then between two repelling charges time would contract as $dt/d\tau = 1 - k_e Qq/mc^2 r$ in lowest order, and the same analysis which then led to (13.5) would have instead yielded $dt/d\tau \cong 1 - \alpha/2\pi \cong 1 - a_{\mathrm{QED}} \cong g_{\mathrm{QED}}/2$. So given that $a \cong a_{\mathrm{QED}}$ up to the comparatively-minor $a_{\mathrm{EW}} + a_{\mathrm{Had}}$ contributions, this would have meant that $g \cong 2 - \alpha/\pi < 2$ which is contradicted by observation, versus $g \cong 2 + \alpha/\pi > 2$ which is observed. Consequently (13.16), which contains what is now an *exact* connection $\gamma_{em} = dt/d\tau = m/m_0 = 1 + a_{\mathrm{QED}}$ between the electromagnetic contribution to time dilation and the lepton magnetic moment anomaly by way of the self-interaction energy contained in $mc^2 = m_0 c^2 + E_{em}$, provides direct empirical evidence that repulsive electromagnetic interactions between *like-electrostatic charges* dilate time just as do attractive gravitational interactions between (what always are) *like-gravitational charges* a.k.a.

masses. And as earlier noted, this is a quantum field theory consequence of the reversed propagator sign for spin-1 photons versus spin-2 gravitons which causes like-gravitational-masses to attract but like-electrical-masses to repel. This, again, is why we have paid such close attention to signs and sign conventions throughout.

To this point, however, we have only used $a = a_{\text{QED}}$, neglecting the relatively tiny, albeit still detectable, electroweak and hadronic contributions through which $a = a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$. As stated at the start of this section, this is because this paper has thus far developed electrodynamics to the exclusion of weak and strong or hadronic interactions. Now, we turn to these electroweak and hadronic contributions, which as we shall show, are most simply accounted for by a very profound and universal connection between time and energy.

14. Time "Sees" all Energy: Why the Magnetic Moment Anomalies are an Exact Consequence of Local Abelian and non-Abelian Gauge Symmetries

It is often said that gravitation "sees" all energy. What is meant by this statement is that any two masses / energies in proximity to one another will attract one another, no matter what the source of those energies. And the quantum mediator of these interactions, of course, is the spin-2 graviton. A good example of this is (10.24), in which not only do the rest masses gravitate, but so too do the energies of motion and the energies of electromagnetic interaction and even, in a self-feeding non-linear way governed by Einstein's equation, the energies of gravitation. In this respect, gravitation is unique among all interactions: For electromagnetism, the spin-1 photon mediators will only "see" particles with electrical charges. And so, for example, they will miss the neutrinos. For electroweak interactions, the spin-1 *W* bosons will only "see" particles with weak isospin such as left-chiral quarks and leptons, and so will miss the right handed fermions. And the spin-1 *Z* boson will miss right-chiral neutrinos which carry neither charge nor isospin. For strong interactions, gluons will only "see" quarks, not leptons. And for hadronic interactions which employ spin-0 Yukawa mesons to mediate attractive short-range interactions between baryons such as protons and neutrons which are composite entities comprising confined quarks and the gluon-mediated interactions among those quarks, the mesons will only "see" baryons but not leptons.

A fundamental finding of this paper is that <u>time "sees" all energy</u>, just as does gravitation: In (9.3) we reviewed how time "sees" the kinetic energy E_v of motion because $E = mc^2\gamma_v = mc^2 dt / d\tau = mc^2 + E_v$. In (9.5) we reviewed how time "sees" gravitational interaction energy E_g because $E = mc^2\gamma_g = mc^2 dt / d\tau = mc^2 - E_g$. In (10.13) we discovered that time "sees" electromagnetic interaction energy E_{em} via $E = mc^2\gamma_{em} = mc^2 dt / d\tau = mc^2 + E_{em}$. And in (10.24) we saw how when all of kinetic and gravitational and electromagnetic energies are present simultaneously, because of the compound relation $E = mc^2\gamma_{em}\gamma_g\gamma_v = mc^2 dt / d\tau$, time "sees" all of these energies and all of the nonlinear compositions of these energies. So it is natural to believe that this pattern will continue, whereby when an energy E_{EW} is produced as a result of electroweak interactions mediated by *W* and *Z* bosons, or an energy E_{QCD} is produced as a result of strong interactions between quarks within a baryon which are mediated by gluons, or an energy

E_{Had} is produced as a result of hadronic interactions between baryons mediated by Yukawa mesons, time will "see" all of these energies as well, just as we already know that that gravitation does "see" all of these energies. Specifically, time should be expected to dilate or contract in proportion with these other energies as well, just as gravitation will act on all of these energies. Any time there is some motion or interaction which gives rise to measurable energy that is "seen" by gravitation, the very same motion or interaction simultaneously gives rise to a dilation or contraction seen in the measurement of time.

In fact, from this viewpoint, if one were create a field map for any region of the universe – whether macroscopic or microscopic – in terms of the energies E that exist at each event point in that region, one could equivalently map out that very same region in terms of the $dt/d\tau = E/mc^2$ ratio at each event point with the total energy E having a variety of origins from a variety of interactions and motions. The coordinates for measuring all events in this "$dt/d\tau$ field" are then established by a laboratory clock of the observer observing this field, for which clock $dt/d\tau = 1$ exactly, by definition. And all other observed events (except for extremely-large motions or extremely-strong interactions) will have a $dt/d\tau \cong 1$ differing from 1 only by parts per million or billion or trillion or higher. But there will still be a difference from 1 that establishes a measurable $dt/d\tau = E/mc^2$ field which serves a proxy for the energy field. We shall refer to this as the "time dilation field," recognizing that in some instances time will dilate negatively, i.e., contract.

So from the person running at six miles per hour to the car driving at 60 miles per hour to the plane flying at 600 miles per hour to the rocket travelling at 6 miles per second, to a mass in a gravitational field, to an electron in the electromagnetic field of a nucleus, to a proton or neutron bound into an atomic nucleus, to a quark interacting inside a proton or a neutron, to a lepton or a quark or a baryon that is weakly interacting, each will have its own unique total $dt/d\tau$ to go along with its unique total energy $E/mc^2 = dt/d\tau$ from whatever origin or origins. In short, from this viewpoint, $dt/d\tau = E/mc^2$ is a *universal relation* between the total energy of a material body from all sources and origins of its energy. In this relation, $d\tau$ for all practical purposes is a tick interval $\Delta\tau$ at which periodic signals are emitted by that body acting as a geometrodynamic clock, while dt is the tick interval Δt at which periodic signals are emitted by a like-clock in the observer's laboratory. It is with this understanding, that we now turn to the electroweak and hadronic contributions to the lepton magnetic moments contained in the standard model relation $a = a_{\mathrm{QED}} + a_{\mathrm{EW}} + a_{\mathrm{Had}}$.

Using the generalized energy/time relation $dt/d\tau = E/mc^2$ to consider the contributions to $a = a_{\mathrm{QED}} + a_{\mathrm{EW}} + a_{\mathrm{Had}}$, the key relation within (13.16) is $1 + E_{em}/m_0c^2 = 1 + a_{\mathrm{QED}}$. From this we see that the QED contribution to the anomaly $a_{\mathrm{QED}} = E_{em}/m_0c^2$ directly measures the ratio of the electromagnetic self-interaction energy E_{em} to the bare energy m_0c^2. This suggests that $a_{\mathrm{EW}} = E_{\mathrm{EW}}/m_0c^2$ will measure the electroweak self-interaction energy and $a_{\mathrm{Had}} = E_{\mathrm{Had}}/m_0c^2$ will measure the hadronic contribution to the self-interaction energy. We summarize each distinct contribution (subscript $\mathrm{C} = \mathrm{QED, EM, Had}$) to the magnetic anomaly by $a_{\mathrm{C}} = E_{\mathrm{C}}/m_0c^2$. So with a total self-interaction (SI) energy $E_{\mathrm{SI}} = E_{em} + E_{\mathrm{EW}} + E_{\mathrm{Had}}$ from all sources, and given

$a = a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$, and taking the total dressed energy to be $mc^2 = m_0 c^2 + E_{\text{SI}}$, and – most importantly – taking $dt / d\tau = E / mc^2$ to be universally-true for all energies irrespective of source or origin whereby time "sees" all energy, we may immediately advance (13.16) to:

$$\begin{aligned}
\gamma_{em} &= \frac{dt}{d\tau} = \frac{\pi^\mu}{p^\mu} = \frac{U^\mu}{u^\mu} = \frac{\mathfrak{D}^\mu}{\partial^\mu} = \frac{m}{m_0} = \frac{m_0 c^2 + E_{\text{SI}}}{m_0 c^2} = \frac{\text{canonical}}{\text{mechanical}} \\
&= 1 + \frac{E_{\text{SI}}}{m_0 c^2} = 1 + \frac{E_{em} + E_{\text{EW}} + E_{\text{Had}}}{m_0 c^2} = 1 + a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}} = 1 + a = \frac{g}{2} = \frac{g}{g_{\text{D}}} . \qquad (14.1) \\
&\cong \frac{1}{1 - \alpha / 2\pi \text{P}} = \frac{1}{1 - k_e e^2 / hc\text{P}} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{2\pi \text{P}} \right)^n = \sum_{n=0}^{\infty} \frac{C_n}{n!} \left(\frac{\alpha}{2\pi} \right)^n
\end{aligned}$$

The middle line in (14.1) shows the result of time "seeing" all energy contributions, which, when applied to lepton self-interactions, takes the form $dt / d\tau = E / E_0 = mc^2 / m_0 c^2$ of a dressed-to-bare ratio. The only expressions in (13.16) which will now become approximate in relation to all other expressions by virtue of $a \cong a_{\text{QED}}$ are the those shown on the bottom lines of both (13.16) and (14.1). This is because although $1 + a_{\text{QED}} = 1 / \left(1 - \alpha / 2\pi \text{P}\right) = \sum_{n=0}^{\infty} \left(C_n / n!\right)\left(\alpha / 2\pi\right)^n$ by itself is related *exactly* to the electromagnetic running coupling $\alpha = k_e e^2 / \hbar c$, one must expect a_{EW} to be related to the electroweak charges g_W and g_Z. And further, as stated on page 3 of [14], a_{Had}, the "hadronic (quark and gluon) loop contributions to a_μ^{SM} give rise to the main theoretical uncertainties. At present, those effects are not calculable from first principles, but such an approach, at least partially, may become possible as lattice QCD matures." In all events, however, neither can one expect these hadronic contributions to be an exclusive function of the electromagnetic $\alpha = k_e e^2 / \hbar c$, because these will involve the strong charges g_{QCD} and possibly short-range Yukawa couplings between hadrons as well.

But what is critically important, via time "seeing" all energy, is the result in (14.1) that the time dilation $dt / d\tau = m / m_0 = 1 + a = g / 2$ is now a direct, exact measurement of the lepton magnetic moment anomaly, including the electroweak and the hadronic and any other possible contributions not part of the standard model that may go into the observed anomaly *a*. The only requirement for some self-interaction to contribute to the magnetic moment anomaly is that it contribute some energy to the total energy mc^2 in $mc^2 / m_0 c^2 = 1 + a$. That $a = a_{\text{QED}} + a_{\text{EW}} + a_{\text{Had}}$ with $a \cong a_{\text{QED}}$ in the standard model, is a statement that the only self-interactions which measurably-contribute to the dressed lepton energy are those involving electrodynamic interaction loops which are dominant, and electroweak and hadronic loops which offer small corrections. Now, given that for lepton self-interactions, the complete $g / 2 = \mathfrak{D}^\mu / \partial^\mu$ is exactly equal to the canonical-to-mechanical ratio of the gauge-covariant to the ordinary spacetime derivatives, with the canonical co-scaling of (13.6) carrying through with exactitude to (14.1), we see how the anomalous lepton magnetic moments are all a direct and immediate theoretical consequence of local gauge symmetry, *and that no other theoretical basis is necessary*. Of course, the gauge symmetry we must now speak of in (14.1), includes the non-abelian symmetries of Yang-Mills theory [15] which underlies the theories of weak and strong interactions. But what is so powerful about the principle that time "sees" all energy, is that even without going into the details of electroweak and strong and hadronic interaction theory, it is possible to precisely account via (14.1) for the electroweak and hadronic and any other possible contributions to the lepton magnetic moment anomalies.

15. Lepton Bare Masses and Probability Density Standard Deviations for Possible Experimental Validation, and the Observable Physical Meaning of Lepton Compton Wavelengths

As a direct result of (14.1) it becomes possible to calculate an *exact* value for the bare masses of each of the three leptons, via the simple relation $m_0 = 2m/g$. Using the empirical values $g_e = 2.00231930436152$ and $g_\mu = 2.0023318418$ deduced from [17] and $g_\tau = 2.00235442$ from [19], together with the rest energies $m_e c^2 = 0.5109989280\text{ MeV}$; $m_\mu c^2 = 105.6583715\text{ MeV}$ and $m_\tau c^2 = 1776.86\text{ MeV}$ from [17], we compute that the bare masses (note GeV units for tau):

$$m_{e0}c^2 = 0.5104070334\text{ MeV};\quad m_{\mu 0}c^2 = 105.5353257\text{ MeV};\quad m_{\tau 0}c^2 = 1.77477\text{ GeV}. \tag{15.1}$$

Consequently, the total self-energy of each lepton, from all contributions, is easily calculated via $E_{\mathrm{SI}} = mc^2 - m_0 c^2$ to be (note KeV units for the first two leptons):

$$E_{\mathrm{SI}e} = 0.5918946\text{ KeV};\quad E_{\mathrm{SI}\mu} = 123.0458\text{ KeV};\quad E_{\mathrm{SI}\tau} = 2.09\text{ MeV}. \tag{15.2}$$

As to the ratio P first deduced in (13.7) to which the expected draw separation is related by $\langle 2r\rangle = \langle d\rangle > \mathrm{P}\lambda$, (14.1) makes clear as discussed that this is now only an approximate relationship, up to the contributions from electroweak and hadronic loops, and their dependency on couplings other than $\alpha = k_e e^2/\hbar c$. So restructuring (13.7) to include this approximation, including $g \cong g_{\mathrm{QED}}$, we may write:

$$\langle d\rangle = \langle 2r\rangle > \mathrm{P}\lambda \cong \frac{\alpha}{2\pi}\frac{g}{g-2}\lambda. \tag{15.3}$$

Then using the successive signs "$>\cong$" to mean "greater than approximately," we find that the approximate lower bounds on the expected draw separations, i.e., on the average weighted separations between spatial points within each lepton probability density, using $\alpha = 1/137.035999049$ and the foregoing g-factors, are computed to be:

$$\langle d_e\rangle >\cong 1.002676995571\cdot\lambda_e;\quad \langle d_\mu\rangle >\cong 0.9972922220\cdot\lambda_\mu;\quad \langle d_\tau\rangle >\cong 0.98773962\cdot\lambda_\tau. \tag{15.4}$$

Because these are proportional to the standard deviation of each probability density, $\sigma \propto \langle d \rangle$, (15.4) provides us with approximate relative ratios for the standard deviations of each of the lepton probability densities, on the assumption that the form (e.g., Gaussian) of each lepton density is the same. So what we learn from (15.4) – which might be accessible to experimental testing – is that in relation to the Compton wavelength of each lepton, the probability densities are more densely packed for each successive higher generation. If the statistical standard deviation for the tau lepton probability density is scaled to 1, then the ratios of these standard deviations is easily computed to be $\sigma_\tau : \sigma_\mu : \sigma_e = 1 : 1.00967118 : 1.01512279$. In other words, in relation to the Compton wavelength of each lepton, the standard deviation of the muon probability density is about 1% larger than that of the tau lepton, while the standard deviation of the electron probability density is about 1.5% larger than that of the tau lepton, assuming their underlying distribution types are all the same and differ only by their standard deviations. Again, it may perhaps be possible to experimentally test the approximate "spread" ratios contained in (15.4).

The foregoing also gives us a very direct physical explanation of the Compton wavelengths $\lambda = h / mc$ of the three leptons. Often, it is pointed out that the Compton wavelength of a lepton (or other particle) is the wavelength of a photon with an energy equal to the rest energy of that lepton. And this in turn makes this a scale at which the uncertainty difficulties of localizing the position of a lepton come into play, whereby the Compton wavelength is a sort of qualitative boundary between circumstances under which one may or may not use classical physics. But these are all roundabout understandings, because leptons are structureless point particles as has been discussed early in section 13, and so the Compton wavelength most certainly cannot be not the "size" of a lepton in any classical sense. However, because leptons prior to "collapse" do have an associated probability density $J^0 = \rho = \psi^\dagger \psi$ with an associated standard deviation σ (or standard deviations about each axis and / or in radial and angular directions), there is a "size" of the spatial region within which it is most likely to detect a lepton when its wavefunction is "collapsed." So if the average separation between any two independent draws from the probability density is $\langle r \rangle$, then one may readily think of $\langle d \rangle = \langle 2r \rangle$ as a sort of "statistical diameter" for the lepton probability density. Therefore, what (15.4) teaches is that the minimum statistical diameter of a lepton is very close to the Compton wavelength of the lepton, with the exact relation dependent on the specific nature of ρ. Further, because the standard deviation is also proportional to the average draw separation (again, $\langle x \rangle = 2\sigma_x / \sqrt{\pi} \cong 1.128379\sigma_x$ for a one-dimensional Gaussian), $\langle d \rangle = \langle 2r \rangle$ will also establish a minimum value for the standard deviations $\sigma_e, \sigma_\mu, \sigma_\tau$ of the lepton probability distributions, again with the exact σ value(s) dependent upon the particulars of the distribution.

So the Compton wavelength now has a very clear and direct and satisfying and most importantly, *empirically-verifiable* meaning, at least for the leptons: One-half of the Compton wavelength of a lepton sets an approximate lower bound on the expected separation of any two independent draws obtained by detecting the lepton at some position within the probability density, with the exact expected independent draw separation determined by the specific character of the probability density. Because by basic statistics the standard deviation of a probability distribution is directly related to the expected independent draw separation, the Compton wavelengths also establish the standard deviations of the lepton probability densities. In short: the Compton wavelength of a lepton, up to a constant factor not too far from 2 (representing a diameter $d=2r$ not a radius), measures the spatial standard deviation $\sigma_{\mathbf{x}}$ of the probability density $\rho(\mathbf{x})=\psi^\dagger\psi$ for that lepton. This is a much more satisfactory and direct explanation than those which describe the Compton wavelength as that of a photon energy commensurate with particle rest mass, or as the distance at which uncertainty becomes a factor. The Compton wavelengths are a directly-measurable *statistical size*, not of the leptons, but *of the probability densities of the leptons,* and they bear a direct relationship to expected probabilistic draw separation and to the standard deviations of the lepton probability densities. Again, it should be quite possible to measure this, and also to measure $\sigma_\tau:\sigma_\mu:\sigma_e=1:1.00967118:1.01512279$ which are the approximate statistical spread ratios contained within (15.4).

16. Electromagnetic Time Dilation and DeVries' Formula for the Fine Structure Constant

Before concluding, it is of significant interest to demonstrate how the electromagnetic time dilation (10.11) and the consequent non-linear electromagnetic interactions discussed at (10.15) and (10.16) and thereafter can alternatively be expressed using recursive mathematics, because as we shall now see the DeVries formula [20] for the fine structure constant is recursive in a very similar way to the non-linear behaviors developed here. As such, this may lay the foundation for providing a physical explanation of the DeVries formula which remains accurate within experimental errors more than a decade after it was first published, but still has not been afforded a physical explanation.

For this demonstration we start with the electromagnetic time dilation factor first obtained in (10.11), namely $\gamma_{em}=dt/d\tau=1/\left(1-q\phi_0/mc^2\right)$. At (10.15) and (10.16) this was seen to be at the heart of the non-linear terms in the interaction energy stemming from the mathematical series $\gamma_{em}=1/(1-x)\cong 1+x$ with $x=q\phi_0/mc^2=k_eQq/mc^2r$ for a Coulomb interaction. And at (12.4) this was first shown to also be equal to several other canonical-to-mechanical ratios which at (13.16) were shown to include the electromagnetic contribution to the g-factor via $\gamma_{em}=g_{\mathrm{QED}}/2$ and at (14.1) upon application of the generalized energy/time relation $dt/d\tau=E/mc^2$, to the complete $\gamma_{em}=g/2$ including weak and hadronic contributions.

But this time dilation factor may also be easily written in the alternative form:

$$\gamma_{em}=1+\frac{q\phi_0}{mc^2}\gamma_{em}=\frac{dt}{d\tau}=1+\frac{q\phi_0}{mc^2}\frac{dt}{d\tau}=\frac{1}{1-q\phi_0/mc^2}\,. \tag{16.1}$$

This is clearly recursive because γ_{em} is expressed as a function $1+\left(q\phi_0/mc^2\right)\gamma_{em}$ of itself, which repeats *ad infinitum*. Yet γ_{em} may also be isolated and written directly as $1/\left(1-q\phi_0/mc^2\right)$, the way we have done previously. But since the recursion is what interests us here, let us expand the recursion by simply placing γ_{em} from (16.1) into itself several times to identify and solve for the infinite series, as such:

$$\begin{aligned}\gamma_{em} &= 1+\frac{q\phi_0}{mc^2}\gamma_{em} = 1+\frac{q\phi_0}{mc^2}\left(1+\frac{q\phi_0}{mc^2}\left(1+\frac{q\phi_0}{mc^2}\left(1+\frac{q\phi_0}{mc^2}\left(1+\frac{q\phi_0}{mc^2}(1+...)\right)\right)\right)\right)\\ &= 1+\frac{q\phi_0}{mc^2}+\left(\frac{q\phi_0}{mc^2}\right)^2+\left(\frac{q\phi_0}{mc^2}\right)^3+\left(\frac{q\phi_0}{mc^2}\right)^4+\left(\frac{q\phi_0}{mc^2}\right)^5+... \\ &= \sum_{n=0}^{\infty}\left(\frac{q\phi_0}{mc^2}\right)^n = \frac{1}{1-\left(q\phi_0/mc^2\right)} = \gamma_{em}\end{aligned} \qquad (16.2)$$

This is the same series that underlies the non-linear energy (10.15), but recursively expressed.

What makes this of special interest is that the DeVries formula [20] for the fine structure constant, namely:

$$\alpha = \Gamma^2 \exp\left(-\frac{\pi^2}{2}\right); \quad \Gamma = 1+\frac{\alpha}{(2\pi)^0}\left(1+\frac{\alpha}{(2\pi)^1}\left(1+\frac{\alpha}{(2\pi)^2}\left(1+\frac{\alpha}{(2\pi)^3}\left(1+\frac{\alpha}{(2\pi)^4}(1+...)\right)\right)\right)\right), \qquad (16.3)$$

has a the parameter Γ which is structurally very similar to the top line of (16.2). Given that $\exp\left(-\pi^2/2\right)=1/139.0456366606...$, it is clear that $\Gamma^2>1$ for which $\Gamma^2 \cong 1$ is a dimensionless number that turns out to be slightly larger than 1 such that $\alpha=\Gamma^2\exp\left(-\pi^2/2\right)$ equals the empirically-observed fine structure constant with the value $\alpha=1/137.035999139(31)$ reported in [11] or the value $\alpha=1/137.035999049(90)$ used in [14], to within present-day experimental errors. It is for this reason that even though it has not yet been afforded a physical interpretation, the DeVries relationship must be regarded very seriously. Note also that the only two independent numbers in (16.3) are α and π. So while the fine structure constant $\alpha(\pi)$ is a function of only the number π, we can also say that $\pi(\alpha)$ is a function only of α, which suggests and extremely close connection between the physics of electromagnetism and the pure mathematics of circles, spheres, etc. All of this motivates us to see if some connection can be established between (16.2) and (16.3).

Starting with (16.2), let us now consider a Coulomb potential $\phi_0 = k_e Q/r$ for which the linear interaction energy ratio $E_{em0}/mc^2 = q\phi_0/mc^2 = k_e Qq/mc^2 r$, see (10.15). Then as we did to arrive at (13.1), let us consider a single charged self-interacting lepton probability density prior to wavefunction "collapse," and divide this into two halves each with $Q=q=-\frac{1}{2}e$. Thus, $E_{em0}=\frac{1}{4}k_e e^2\langle 1/r\rangle$, with $\langle 1/r\rangle > 1/\langle r\rangle$ being the expected value of the inverse draw separation on the condition that the two draws come from different halves of the probability density. What we then learned leading up to (13.1) is that as we subdivide the lepton further and further ad infinitum then take the ordinary calculus limit, the linear self-interaction energy ends up being given by $E_{em0} = q\phi_0 = k_e e^2 \langle 1/2r \rangle$, which is (13.1). Then using the relation $k_e e^2 / mc^2 = (\alpha / 2\pi)\lambda$ as we did at (13.3), and also using the statistical diameter $\langle d \rangle = \langle 2r \rangle$, from (13.1) we obtain:

$$\frac{E_{em0}}{mc^2} = \frac{q\phi_0}{mc^2} = \frac{k_e e^2}{mc^2}\left\langle \frac{1}{2r} \right\rangle = \frac{\alpha}{2\pi}\lambda \left\langle \frac{1}{2r} \right\rangle = \frac{\alpha}{2\pi}\lambda \left\langle \frac{1}{d} \right\rangle. \tag{16.4}$$

Although this is the linear expression for the electromagnetic self-interaction energy, we see from (13.3) that the complete non-linear energy is obtained by setting $x = (\alpha\lambda / 2\pi)\langle 1/2r \rangle$ in the mathematical function $\gamma_{em} = 1/(1-x)$. But the recursive (16.2) is just an alternative way of representing this function. So to obtain a recursive expression for γ_{em} which contains all of the non-linear terms of (10.15) built in, we may substitute (16.4) into the top line of (16.2) to obtain:

$$\gamma_{em} = 1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d} \right\rangle (1+...)\right)\right)\right)\right). \tag{16.5}$$

This is simply an equivalent, albeit recursive way to write (13.3). This may also be written in terms of the dimensionless ratio parameter $1/\mathrm{P} = \lambda\langle 1/d \rangle$ defined in (13.7).

Contrasting this with (16.3) we now see an extremely close resemblance with the dimensionless DeVries number Γ. This resemblance is not just structural; now it is directly physically-substantive because the driving number in each is α. In fact, the *only* difference between γ_{em} in (16.5) and Γ in (16.3) is that in (16.5) the denominator of α is 2π at all recursive orders, while in (16.3) the denominator at n^{th} order is $(2\pi)^n$. Yet, (16.5) also has the term $\lambda\langle 1/d \rangle$ multiplying $\alpha / 2\pi$ at each order, while in (16.3) a $1/(2\pi)^n$ multiplies α at each order. Thus, it seems there is a link to be found between $\lambda\langle 1/d \rangle$ and $1/(2\pi)^n$, which we now explore.

In $\lambda\langle 1/d \rangle$, λ is the Compton wavelength of the lepton, and thus a single, definite number just like the mass $m = h/c\lambda$. But treating the inverse statistical dimeter $\langle 1/d \rangle$ as a free variable at each order, let us now define an infinite sequence of statistical diameters that we denote by $\langle d_n \rangle$ with $0 \le n \le \infty$ where each such diameter is associated with each recursive order in (16.5). So changing $d \to d_n$ in (16.5) and setting n at each order to be the number of that order, (16.5) becomes:

$$\gamma'_{em} = 1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d_0} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d_1} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d_2} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d_3} \right\rangle \left(1 + \frac{\alpha}{2\pi}\lambda\left\langle \frac{1}{d_4} \right\rangle (1+...)\right)\right)\right)\right). \tag{16.6}$$

By the substitution $d \to d_n$ we did change the nature of γ_{em} in (16.6) because in (16.5) d was the same number from one order to the next while just above it is different. So because of this change we have denoted this modified γ_{em} as γ'_{em}. We may also generalize (13.7) be defining a succession of ratio parameters $1/\mathrm{P}_n = \lambda\langle 1/d_n\rangle$.

Next, contrasting (16.6) with (16.3), let us impose (define) the condition:

$$\frac{1}{\mathrm{P}_n} = \lambda\left\langle \frac{1}{d_n}\right\rangle \equiv \frac{1}{(2\pi)^{n-1}} > \frac{\lambda}{\langle d_n\rangle}, \tag{16.7}$$

with $0 \le n \le \infty$, where the inequality holds for any distribution other than a Dirac delta as has been previously reviewed prior to (13.1). This also means that:

$$\mathrm{P}_n \equiv (2\pi)^{n-1}. \tag{16.8}$$

The inequality (16.7) then restructures easily to:

$$\langle d_n\rangle > (2\pi)^{n-1}\lambda = \mathrm{P}_n\lambda, \tag{16.9}$$

and is dependent upon the exact nature, e.g., Gaussian or otherwise, of the probability density $\rho = \psi^\dagger\psi$. If we then insert (16.7) into (16.6) we obtain:

$$\begin{aligned}\gamma'_{em} &= 1+\frac{\alpha}{(2\pi)^0}\left(1+\frac{\alpha}{(2\pi)^1}\left(1+\frac{\alpha}{(2\pi)^2}\left(1+\frac{\alpha}{(2\pi)^3}\left(1+\frac{\alpha}{(2\pi)^4}(1+...)\right)\right)\right)\right) \\ &= 1+\frac{\alpha}{2\pi\mathrm{P}_0}\left(1+\frac{\alpha}{2\pi\mathrm{P}_1}\left(1+\frac{\alpha}{2\pi\mathrm{P}_2}\left(1+\frac{\alpha}{2\pi\mathrm{P}_3}\left(1+\frac{\alpha}{2\pi\mathrm{P}_4}(1+...)\right)\right)\right)\right) = \Gamma\end{aligned}. \tag{16.10}$$

It is to produce this exact equality that we chose the conditions (16.7). This means that obtaining a physical understanding of DeVries' Γ factor, boils down to understanding (16.7) through (16.9).

Toward this objective, let us now write the recursive relation (16.6) in "inductive form," by which we mean a form in which a function at a given recursive order is defined in relation to the same function at the next adjacent order. A good example is the definition $n! = n\cdot(n-1)!$ with terminal condition $0! = 1$ for factorials. In this form, working from (16.6) and (16.7), we define:

$$\gamma_{em\ n} \equiv 1+\frac{\alpha}{2\pi}\lambda\left\langle\frac{1}{d_n}\right\rangle\gamma_{em\ n+1} = 1+\frac{\alpha}{(2\pi)^n}\gamma_{em\ n+1} = 1+\frac{\alpha}{\mathrm{P}_{n+1}}\gamma_{em\ n+1}, \tag{16.11}$$

where $0 \le n \le \infty$. Then, it is clear that (16.6) is compactly written as $\gamma'_{em} = \gamma_{em\ 0}$, and that with the condition (16.7) the DeVries number is actually $\Gamma = \gamma_{em\ 0}$. So with the foregoing, the DeVries relationship (16.3) simplifies to the trilogy:

$$\alpha = \left(\gamma_{em\ 0}\right)^2 \exp\left(-\frac{\pi^2}{2}\right); \quad \Gamma = \gamma_{em\ 0}; \quad \gamma_{em\ n} = 1 + \frac{\alpha}{\left(2\pi\right)^n}\gamma_{em\ n+1}. \tag{16.12}$$

Here, $\Gamma = \gamma_{em\ 0}$ is the terminal relation analogous to $0! = 1$, while $\gamma_{em\ n} = 1 + \left(\alpha / \left(2\pi\right)^n\right)\gamma_{em\ n+1}$ is the inductive form of the recursion analogous to $n! = n \cdot \left(n-1\right)!$.

So, what might one make of (16.7) through (16.9), physically? To gain some clues, it is probably simplest to work from (16.9), which we write as the equality $\left\langle d_n \right\rangle_{\min} = \left(2\pi\right)^{n-1}\lambda$ of a set of *minimum* statistical diameters in relation to the Compton wavelength. For the first several orders, using the reduced Compton wavelength $\lambda\!\!\!^{-} = \lambda / 2\pi$, these statistical diameter minima are:

$$\left\langle d_0 \right\rangle_{\min} = \lambda\!\!\!^{-}; \quad \left\langle d_1 \right\rangle_{\min} = \lambda; \quad \left\langle d_2 \right\rangle_{\min} = 2\pi\lambda; \quad \left\langle d_3 \right\rangle_{\min} = \left(2\pi\right)^2 \lambda; \quad \left\langle d_4 \right\rangle_{\min} = \left(2\pi\right)^3 \lambda. \tag{16.13}$$

Another way to write this is as the recursive kernel and terminal condition:

$$\left\langle d_n \right\rangle_{\min} = 2\pi \left\langle d_{n-1} \right\rangle_{\min}; \quad \left\langle d_0 \right\rangle_{\min} = \lambda\!\!\!^{-}. \tag{16.14}$$

In this vein, $2\pi = C / r$ is of course the ratio of the circumference of a circle to its radius, so at each order, the statistical diameter is promoted from a radial length to a circumferential length.

Further, while each of the $\left\langle d_n \right\rangle_{\min}$ arise from a different order of the recursion, it is perfectly reasonable to think of the n in $\left\langle d_n \right\rangle_{\min}$ as some type of quantum number. Additionally, because $\gamma_{em} = dt / d\tau$ is the electromagnetic contribution to time dilation, and because $dt \to \Delta t$ measures the "tick" interval of a clock in a laboratory, the existence of a succession of $\gamma_{em\ n}$ in (16.12) is suggestive of a relation $\gamma_{em\ n} = dt / d\tau_n$ containing a collection of proper time "tick" rates $d\tau_n \to \Delta\tau_n$ emanating from the lepton. At the same time, the "tick" rate of a signal is related to its frequency by $f = 1 / \Delta\tau$, so that a collection of tick rates is related to a collection $f_n = 1 / \Delta\tau_n$ of frequencies. And this in turn means that there is some collection of time dilation factors:

$$\gamma_{em\ n} = \frac{dt}{d\tau_n} \to \gamma_{em\ n} = \frac{\Delta t}{\Delta\tau_n} = \Delta t\, f_n \tag{16.15}$$

which are equal to f_n multiplied by the Δt ticks of the laboratory clock.

So in this vein, the intuitive sense that one may gain by studying (16.13) and how it interrelates to the DeVries formula is not unlike how one might think about the quantum number that Planck first found in $E = nhf$ to describe the energy oscillations in a blackbody radiation spectrum. Or with $\lambda = h / p$, this is not unlike the Bohr / deBroglie relation $n\lambda = 2\pi r$ first used to model the hydrogen atom. In fact, the deBroglie relation may be rewritten as $r_n / \lambda = n / 2\pi$ and then likened to $\langle d_n \rangle_{\min} / \lambda = (2\pi)^{n-1}$, keeping in mind that the former is for electron orbits in (hydrogen) atoms while the latter is for the self-interaction of electrons standing alone. So in the same way that $r_n / \lambda = n / 2\pi$ was the point of entry to modelling atoms, $\langle d_n \rangle_{\min} / \lambda = (2\pi)^{n-1}$ would seem to be the point of entry for modelling individual electrons. Likewise, (16.15) contains an infinite collection of frequencies, and we know that any time we have such a collection of frequencies, these may be approached using Fourier analysis. Further, it is well-known that Heisenberg's matrix mechanics arose from considering the Fourier analysis of an infinite collection of frequencies. So one can envision that (16.15), properly advanced, could perhaps lead to an even deeper understanding matrix mechanics. And most certainly, the relations ascertained in the forgoing do seem to be rife with quantum mechanical information that needs to be closely studied and deciphered.

17. Conclusion

From ancient times, through those of Galileo, Huygens, Newton, Bernoulli, du Châtelet, Joule, Carnot and Einstein, the principle of energy conservation and the understanding that energy is a universal form of currency or liquidity of the natural world out of which everything is made and which can be transformed from one form to another but can never be created or destroyed, has evolved into perhaps the most universal, overarching, unifying principle of theoretical physics. Likewise, the manipulation of material objects to convert energy from one form to another, be it chemical, solar, nuclear, mechanical, electromagnetic, heat or other types of energy, has been the foundation of humanity's technological advancement insofar as being able to perform important and necessary work without the use of human or animal physical labor.

The Special and General theories of Relativity gave the first inkling of a similarly-deep and universal connection between energy and time. Not only was energy understood to be time component $E = p^0$ of an energy-momentum vector in spacetime, but more importantly it was understood that for an object in relative motion time dilates in relation to the total energy by $E = mc^2\gamma_v = mc^2 dt / d\tau \cong mc^2 + \frac{1}{2}mv^2$ which includes a rest energy mc^2 plus to lowest order the Newtonian kinetic energy $\frac{1}{2}mv^2$ as in (9.3). And it was understood that a mass in a gravitational field also dilates time in relation to the total energy for the Schwarzschild solution in the Newtonian limit according to $E = mc^2\gamma_g = mc^2 dt / d\tau \cong mc^2 + GMm / r$ where $E_g = -GMm / r$ is the gravitational interaction energy as in (9.5). Moreover, when there is both motion and gravitation, the total energy continues to be related to the overall time dilation by $E = mc^2 dt / d\tau$, but with a compounded effect whereby $dt / d\tau = \gamma_v \gamma_g$ as shown in (9.8). The present paper similarly establishes, consistent with the well-validated Lorentz force law at (5.7) as obtained from the metric (3.5) via the variation minimization (1.1), that when an electromagnetic charge q with mass

m is placed into a proper potential ϕ_0, there is also a time dilation given by $\gamma_{em} \equiv dt/d\tau = 1/\left(1-q\phi_0/mc^2\right)$ at (10.11) which likewise obeys the relation $E = mc^2 dt/d\tau$ as seen in (10.13). And this likewise compounds with gravitational and kinetic energy according to $E = mc^2\gamma_{em}\gamma_g\gamma_v$ as seen in (10.23) a.k.a. (3.11).

So if there is any single result of paramount importance here, it is the finding at (14.1) of a universal relation between time and energy whereby *all forms of energy dilate (or contract) time regardless of their origin, and that this is not only a classical feature of nature, but that this carries through to the lepton magnetic moment anomalies which are the quintessential hallmarks of the success of quantum field theory.* In short, just as gravitation "sees" all energy, so too does time "see" all energy, not just macroscopically, but even at the microscopic level of individual quantum particles, via the universal relation $dt/d\tau = E/mc^2$. Any time a material body of whatever character gains or loses energy of whatever form from whatever origin, the rate at which a geometrodynamic clock associated with that body will tick is altered, and therefore, so too is the measurement of time when that body is used as a clock. *One cannot change the energy of a particle or a system without simultaneously changing how time is measured when that particle or system is used as a geometrodynamic clock.* So, as a simple example from Special Relativity, when a person is stuck by a ball moving at, say, 60 miles per hour, one can and does say that the impact is the result of the kinetic energy of motion of that ball relative to the person. But one can equally say that the impact is the result of time being different for the ball than for the person, albeit with a miniscule difference of parts per quadrillion (10^{15}). And when lightning strikes, one can say that it is nature trying to bring a large potential difference into equipotential, or one can say that nature is trying to bring different rates of time into equilibrium. Any talk of energy, has a parallel and equivalent talk of time.

The other present result of underlying, unifying importance, is that the motions of material bodies in nature and many of the observed numeric objects observed in nature, are fundamentally "canonical" motions and objects growing from "mechanical" motions and objects as a result of local symmetry principles. Thus, gravitational motion has been known for a century to simply be the canonical motion $du^\beta/d\tau \to Du^\beta/D\tau = 0$ of (1.3) obtained by promoting ordinary spacetime derivatives to gravitationally-covariant derivative $\partial_\alpha \to \partial_{;a}$ governing parallel transport in curved spacetime with $R^\alpha{}_{\beta\mu\nu}A_\alpha = \left[\partial_{;\nu}, \partial_{;\mu}\right]A_\beta$. And based on this present work, the Lorenz force motion of classical electrodynamics is seen to be simply the canonical motion $du^\beta/d\tau \to \mathfrak{D}u^\beta/\mathfrak{D}\tau = 0$ of (5.9) obtained via the variation (1.1) and using the geodesic gauge (5.6), of promoting spacetime derivatives to gauge-covariant derivatives $\partial_\alpha \to \mathfrak{D}_\alpha = \partial_\alpha - iqA_\alpha$ governing parallel transport in an abstract space first developed by Hermann Weyl in [5], [6], [7] in which the field strength $qF^{\mu\nu}\phi = i\left[\mathfrak{D}^\mu, \mathfrak{D}^\nu\right]\phi$ defines an imaginary form of curvature.

Then, at the same time mechanical motion is promoted to canonical motion $\mathfrak{D}u^\beta/\mathfrak{D}\tau = 0$ as a consequence of the derivative promotion $\partial_\alpha \to \mathfrak{D}_\alpha$ of gauge symmetry, so too a number of mechanical objects are simultaneously promoted into canonical objects as shown in (12.4), such as the four-velocity $u^\mu \to U^\mu$, the four-momentum $p^\mu \to \pi^\mu$ and the energy $mc^2 \to E$, canonically co-scaling directly with the electromagnetic contribution $\gamma_{em} = dt / d\tau$ to the time dilation. And although (12.4) applies to classical objects, when we study the behavior of individual charged lepton quanta, we find at (13.16) that this carries through to the quantum level, whereby the "mechanical" Dirac g-factor g_{D} also co-scales via $g_{\mathrm{D}} \to g_{\mathrm{QED}}$ into the canonical $g_{\mathrm{QED}} = 2 + 2a_{\mathrm{QED}}$ which includes the electromagnetic contribution to the lepton magnetic moment anomalies a_{QED} that are a hallmark of Quantum Electrodynamics. At the same time, the bare lepton masses co-scale into dressed masses via $m_0 \to m$ also in step with the time dilation $\gamma_{em} = dt / d\tau$. This ties the electromagnetic time dilation together to the modern understanding that the magnetic moment anomalies arise from the same lepton self-interactions that turn the bare masses into dressed masses in accordance with the Ward-Takahashi identities.

Then, when we turn again to the universal relation $dt / d\tau = E / mc^2$ between time and energy whereby the time "sees" all energy, we find at (14.1) that even the electroweak and hadronic anomaly contributions may be accounted for. Now, $g_{\mathrm{D}} = 2$ co-scales into $g = 2 + 2a$ containing the complete, observed anomaly with all contributions, because electroweak and hadronic interactions also produce energies which directly affect time. So the physics of all interactions – gravitational, electromagnetic, weak and strong – and the hadronic interactions which this author has studied in depth at [21], [22], [16], [23] and [24], all enjoy the unifying thread whereby mechanical motions and objects grow into canonical motions and objects, and measurements of time are affected by any and all energies of whatever form from whatever origin. In this way it becomes possible to establish a geometrodynamic foundation for classical and quantum electrodynamics centered about time dilation and contraction and a universal time-energy relation, and lay out the path by which this is extended via non-abelian gauge theories to weak and strong and hadronic interactions.

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