An additional brief solution of the CPP limiting case of the Problem of Apollonius via Geometric Algebra (GA)

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Abstract

This document adds to the collection of GA solutions to plane-geometry problems, most of them dealing with tangency, that are presented in [1]-[7]. Reference [1] presented several ways of solving the CPP limiting case of the Problem of Apollonius. Here, we use ideas from [6] to solve that case in yet another way.

1 Statement of the problem

The CPP limiting case of the Problem of Apollonius reads,

> *"Given a circle and two points outside of it, construct the circles that are tangent to the given one, and that also pass through both of the given points" (Fig. 1).*

2 Expressing elements of the problem in ways that facilitate solution via GA

Experience gained from earlier work (especially [6]) suggests that we use the elements identified in Fig. 2. Note, especially, that the angles θ and 2θ terminate at the line that connects the centers of the two circles, and have as their vertices the point of tangency and the center of the solution circle.

We should note that instead of the angles shown, we could have used angles of rotation from $\boldsymbol{b}-\boldsymbol{t}$ to $\hat{\boldsymbol{t}}$, and from $\boldsymbol{b}-\boldsymbol{c}_2$ to $\hat{\boldsymbol{t}}$.

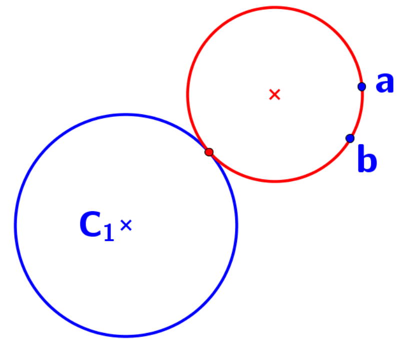

Figure 1: The CPP limiting case of the Problem of Apollonius: *Given a circle and two points outside of it, construct the circles that are tangent to the given one, and that also pass through both of the given points.*

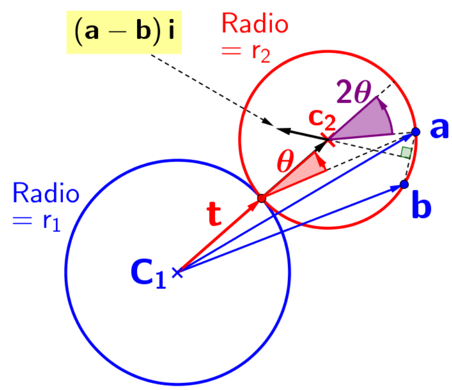

Figure 2: The point of tangency $\boldsymbol{t}$, and the geometric elements of the problem that will be used to identify $\boldsymbol{t}$ via GA.

3 Solution

We will follow [6] in beginning by deriving an expression for r_2 and r_1+r_2, which we will then use in an equation that equates two expressions for the multivector $e^{2\theta\boldsymbol{i}}$. Please see [6] for further details.

3.1 Expressions for r_2 and $r_1 + r_2$

From Fig. 2, we can derive that

$$(r_1 + r_2)\,\hat{\boldsymbol{t}} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2} + \left[\frac{(\boldsymbol{a}-\boldsymbol{b})\,\boldsymbol{i}}{\|\boldsymbol{a}-\boldsymbol{b}\|}\right]\sqrt{{r_2}^2 - \left(\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right)^2}.$$

We'll rearrange that as

$$(r_1 + r_2)\,\hat{\boldsymbol{t}} - \frac{\boldsymbol{a}+\boldsymbol{b}}{2} = \left[\frac{(\boldsymbol{a}-\boldsymbol{b})\,\boldsymbol{i}}{\|\boldsymbol{a}-\boldsymbol{b}\|}\right]\sqrt{{r_2}^2 - \left(\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right)^2},$$

then square both sides. After simplifying and solving for r_2, we find that

$$r_2 = \frac{2r_1\left(\boldsymbol{a}+\boldsymbol{b}\right)\cdot\hat{\boldsymbol{t}} - a^2 - b^2 - 2{r_1}^2}{4r_1 - 2\left(\boldsymbol{a}+\boldsymbol{b}\right)\cdot\hat{\boldsymbol{t}}}\ . \tag{1}$$

From that result, we can then derive

$$r_1 + r_2 = \frac{2{r_1}^2 - a^2 - b^2}{4r_1 - 2\left(\boldsymbol{a}+\boldsymbol{b}\right)\cdot\hat{\boldsymbol{t}}}\ . \tag{2}$$

3.2 Equating two expressions for the multivector $e^{2\theta\boldsymbol{i}}$

Still following [6], we see from Fig. 2 that

$$\underbrace{\left[\frac{\boldsymbol{a}-\boldsymbol{t}}{\|\boldsymbol{a}-\boldsymbol{t}\|}\right]\left[\hat{\boldsymbol{t}}\right]}_{=e^{\theta\boldsymbol{i}}}\underbrace{\left[\frac{\boldsymbol{a}-\boldsymbol{t}}{\|\boldsymbol{a}-\boldsymbol{t}\|}\right]\left[\hat{\boldsymbol{t}}\right]}_{=e^{\theta\boldsymbol{i}}} = \underbrace{\left[\frac{\boldsymbol{a}-\boldsymbol{c}_2}{\|\boldsymbol{a}-\boldsymbol{c}_2\|}\right]\left[\hat{\boldsymbol{t}}\right]}_{=e^{2\theta\boldsymbol{i}}},$$

Note that of the two given points, only one of them (in this case, $\boldsymbol{a}$) figures in multivector that we are using. We make no use of the other point, than to develop an expression for r_2 in Eq. (3.1).

from which

$$\left[\boldsymbol{a}-\boldsymbol{t}\right]\left[\hat{\boldsymbol{t}}\right]\left[\boldsymbol{a}-\boldsymbol{t}\right]\left[\boldsymbol{a}-\boldsymbol{c}_2\right] = \textit{ some scalar.}$$

We use the identity $\boldsymbol{uv} \equiv 2\boldsymbol{u}\wedge\boldsymbol{u} + \boldsymbol{vu}$ to rewrite that result as

$$\left(2\left[\boldsymbol{a}-\boldsymbol{t}\right]\wedge\hat{\boldsymbol{t}} + \hat{\boldsymbol{t}}\left[\boldsymbol{a}-\boldsymbol{t}\right]\right)\left[\boldsymbol{a}-\boldsymbol{t}\right]\left[\boldsymbol{a}-\boldsymbol{c}_2\right] = \textit{ some scalar.}$$

We expand that result as

$$\left(2\boldsymbol{a}\wedge\hat{\boldsymbol{t}}\right)\left[\boldsymbol{a}-\boldsymbol{t}\right]\left[\boldsymbol{a}-\boldsymbol{c}_2\right] - \left(\boldsymbol{a}-\boldsymbol{t}\right)^2\hat{\boldsymbol{t}}\left[\boldsymbol{a}-\boldsymbol{c}_2\right] = \textit{ some scalar,}$$

from which

$$\left\langle\left(2\boldsymbol{a}\wedge\hat{\boldsymbol{t}}\right)\left[\boldsymbol{a}-\boldsymbol{t}\right]\left[\boldsymbol{a}-\boldsymbol{c}_2\right] - \left(\boldsymbol{a}-\boldsymbol{t}\right)^2\hat{\boldsymbol{t}}\left[\boldsymbol{a}-\boldsymbol{c}_2\right]\right\rangle_2 = 0.$$

After further expansions and simplifications, we obtain

$$a^2 - {r_1}^2 - 2\boldsymbol{a}\cdot\boldsymbol{c}_2 + 2r_1\left(r_1+r_2\right) = 0. \tag{3}$$

Let's pause now to compare that result to

$$p^2 - {r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{c}_3 + 2\boldsymbol{t}\cdot\boldsymbol{c}_3 = 0,$$

which was obtained in [6] for the CCP limiting case. The geometric elements referred to therein are shown in Fig. 3 .

Returning now to the present (CPP) limiting case, we continue by recognizing that $c_2 = \left(r_1+r_2\right)\hat{\boldsymbol{t}}$, then substituting in Eq. (3) the expression for $r_1 + r_2$ that's given in Eq. (2):

$$a^2 - {r_1}^2 - 2\boldsymbol{a}\cdot\left\{\left[\frac{2{r_1}^2 - a^2 - b^2}{4r_1 - 2\left(\boldsymbol{a}+\boldsymbol{b}\right)\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{t}}\right\} + 2r_1\left[\frac{2{r_1}^2 - a^2 - b^2}{4r_1 - 2\left(\boldsymbol{a}+\boldsymbol{b}\right)\cdot\hat{\boldsymbol{t}}}\right] = 0$$

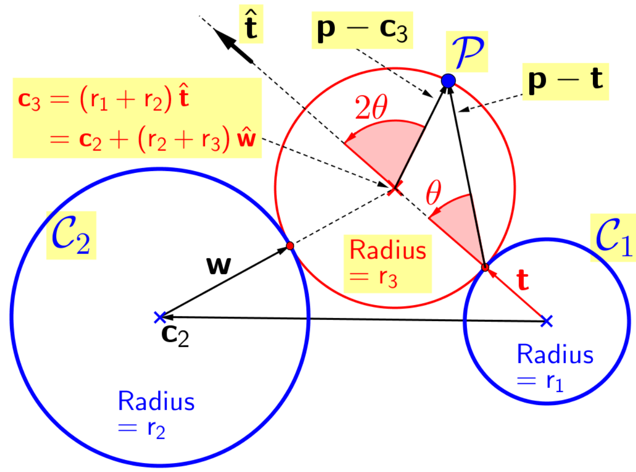

Figure 3: The geometric elements used in [6]'s solution of the CCP limiting case.

After another round of expansions, simplifications, and rearrangements, we arrive at

$$\left\{\left(b^2 - {r_1}^2\right) \boldsymbol{a} - \left(a^2 - {r_1}^2\right) \boldsymbol{b}\right\} \cdot \hat{\boldsymbol{t}} = r_1 \left(b^2 - a^2\right),$$

both sides of which we multiply by r_1, giving the result that was obtained in several ways in [1]:

$$\left\{\left(b^2 - {r_1}^2\right) \boldsymbol{a} - \left(a^2 - {r_1}^2\right) \boldsymbol{b}\right\} \cdot \boldsymbol{t} = {r_1}^2 \left(b^2 - a^2\right).$$

As we know from [1]-[7], a solution of that form means that there are two solution circles, whose points of tangency are reflections of each other with respect to the vector $\boldsymbol{w} = \left(b^2 - {r_1}^2\right) \boldsymbol{a} - \left(a^2 - {r_1}^2\right) \boldsymbol{b}$ (Fig. 4). The projections upon $\hat{\boldsymbol{w}}$ of the vectors to those points of tangency are equal, and are given by

$$\boldsymbol{P}_{\hat{\boldsymbol{w}}}\left(\boldsymbol{t}\right) = \left[\frac{{r_1}^2 \left(b^2 - a^2\right)}{\|\boldsymbol{w}\|}\right] \hat{\boldsymbol{w}}.$$

References

[1] J. Smith, "Rotations of Vectors Via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric 'Construction' Problems" (Version of 11 February 2016). Available at http://vixra.org/abs/1605.0232 .

[2] "Solution of the Special Case 'CLP' of the Problem of Apollonius via Vector Rotations using Geometric Algebra". Available at http://vixra.org/abs/1605.0314.

[3] "The Problem of Apollonius as an Opportunity for Teaching Students to Use Reflections and Rotations to Solve Geometry Problems via Geometric (Clifford) Algebra". Available at http://vixra.org/abs/1605.0233.

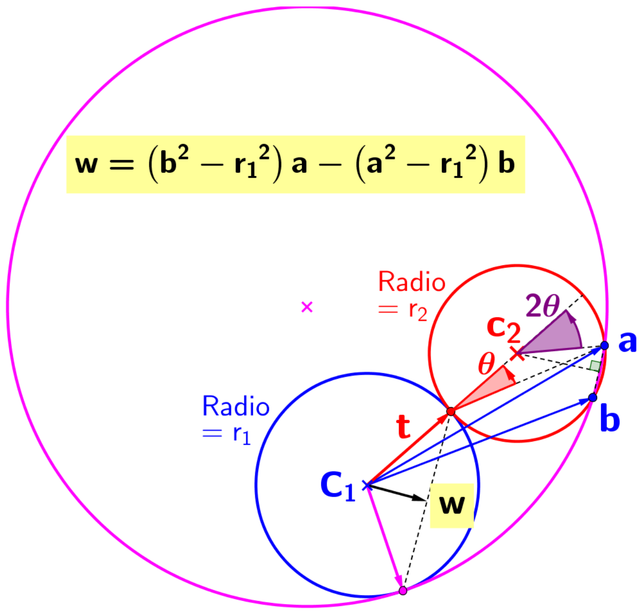

Figure 4: The two solution circles. Their points of tangency are reflections of each other with respect to the vector $\boldsymbol{w} = \left(b^2 - {r_1}^2\right)\boldsymbol{a} - \left(a^2 - {r_1}^2\right)\boldsymbol{b}$

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[4] "A Very Brief Introduction to Reflections in 2D Geometric Algebra, and their Use in Solving 'Construction' Problems". Available at http://vixra.org/abs/1606.0253.

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[6] "Simplified Solutions of the CLP and CCP Limiting Cases of the Problem of Apollonius via Vector Rotations using Geometric Algebra". Available at http://vixra.org/abs/1608.0217.

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